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Yang Chang and Erik Schlögl

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Yang Chang[†] and Erik Schlögl[‡]

Quantitative Finance Research Centre, University of Technology, PO Box 123, Broadway, Sydney, NSW 2007, Australia

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Abstract

The phenomenon of the frequency basis (i.e. a spread applied to one leg of a swap to exchange one floating interest rate for another of a different tenor in the same currency) contradicts textbook no–arbitrage conditions and has become an important feature of interest rate markets since the beginning of the Global Financial Crisis (GFC) in 2008. Empirically, the basis spread cannot be explained by transaction costs alone, and therefore must be due to a new perception by the market of risks involved in the execution of textbook "arbitrage" strategies. This has led practitioners to adopt a pragmatic "multi–curve" approach to interest rate modelling, which leads to a proliferation of term structures, one for each tenor. We take a more fundamental approach and explicitly model liquidity risk as the driver of basis spreads, reducing the dimensionality of the market for the frequency basis from observed spread term structures for every frequency pair down to term structures of two factors characterising liquidity risk. To this end, we use an intensity model to describe the arrival time of (possibly stochastic) liquidity shocks with a Cox Process. The model parameters are calibrated to quoted market data on basis spreads, and the improving stability of the calibration suggests that the basis swap market has matured since the turmoil of the GFC.

Keywords: tenor swap, basis, frequency basis, liquidity risk, swap market
JEL Classfication: C6, C63, G1, G13

† Email: yang.chang@uts.edu.au
‡ Email: erik.schlogl@uts.edu.au

1. Introduction and Motivation

1.1 Basis Spreads in the Market

The Global Financial Crisis (GFC), which started from August 2007 and reached its peak around the collapse of Lehman Brothers in September 2008, has caused a number of changes in the behaviour of interest rate markets, in particular in over–the–counter (OTC) interest rate derivatives. Many of the standard "textbook arbitrage"[1] relationships between spot, forward and swap markets no longer hold even in approximation. Market instruments, such as single–currency interest rate swaps and cross–currency swaps, have been quoted with substantially higher basis spreads than before the GFC. Other changes include the emergence of large and positive spreads between London Interbank Offered Rate (LIBOR)[2] and Overnight Index Swap (OIS)[3] rate of the same maturity. Forward Rate Agreement (FRA)[4] rates observed in the market also significantly diverge from the rates implied by the replication via two deposits at spot LIBORs of different maturities.

We will focus on the phenomenon that floating rates, e.g. LIBOR, of different tenor indices are quoted with varying magnitude of basis spreads in tenor swap contracts (this is also called the "frequency basis"). A tenor swap exchanges two floating rate payments of the same currency based on different tenor indices, such as swapping the 3–month (3M) USD LIBOR and the 6–month (6M) USD LIBOR. Only interest payments are exchanged and no notional is exchanged. A tenor swap can be used to hedge basis risk, due to the widening or narrowing spread between the two indices. According to the classic no–arbitrage pricing principle, two floating rates of different tenors should trade flat in a swap contract because floating–rate bonds are always worth the par value at initiation, regardless of the tenor length (e.g. Hull 2008). Thus in this case the frequency basis spread always should be zero to avoid arbitrage profit. Before the crisis, a small spread (in general several basis points) was usually added to the shorter tenor rate. After controlling for transaction costs such as bid–ask spreads, such a small spread did not constitute an opportunity for arbitrage profit.

[1]For examples of such relationships, see e.g. Hull 2008.

[2]LIBOR is a daily reference rate published by the British Banker Association (BBA) based on the interest rates at which panel banks borrow unsecured funds from each other in the London interbank market.

[3]An OIS is an interest rate swap where the floating leg of the swap is equal to the geometric average of the overnight cash rate over the swap period. Overnight lending involves little default or liquidity risk, hence the LIBOR–OIS spread is an important measure of risk factors in the interbank market.

[4]A FRA is a contract which is initiated at current time t and allows the holder to exchange, at maturity S, a fixed payment (based on the fixed rate K) for a floating payment based on the spot rate $L(T, S)$ resetting at T with maturity S, with $t \leq T \leq S$. The FRA rate is the value of K which renders the contract value 0 (i.e. fair) at t. See e.g. Brigo and Mercurio (2006).

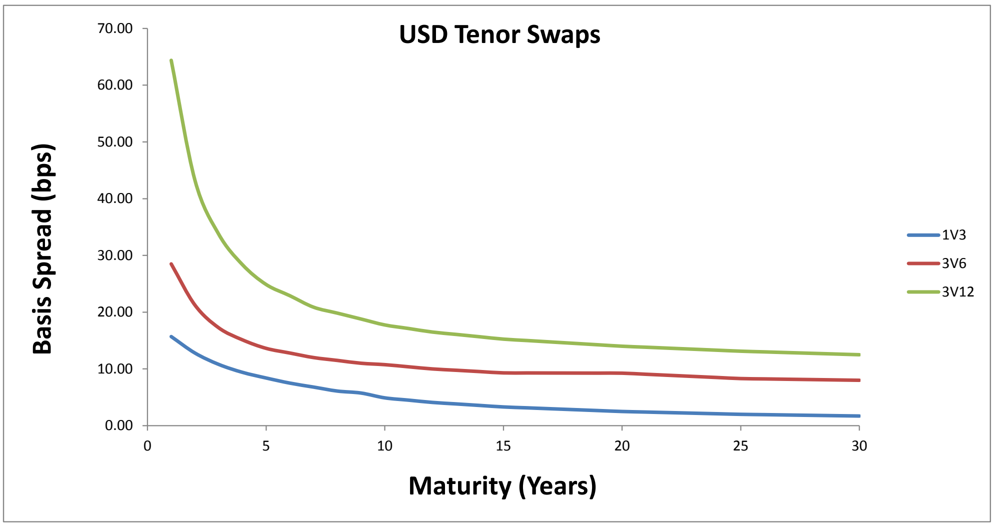

Figure 1: **USD tenor swap basis spread curves on 16/02/2009. (Data source: Bloomberg)**

After the crisis, tenor swaps have displayed a persistent and unambiguous pattern. In general, the shorter tenor floating rate is quoted with a large and positive spread in exchange for the longer tenor rate. The magnitude of the spread tends to increase as the tenor difference increases. Figure 1 shows the USD frequency basis spread curves as at 16th of February, 2009, corresponding to 1M, 3M, 6M and 12M USD LIBOR. Swap maturity ranges from 1 year to 30 years. We see at the 1–year (1Y) maturity end, the spread increased from 16 basis points (bps) for 1M vs 3M swaps to 65 bps for 3M vs 12M swaps.

The observed large spreads in tenor swaps would seem to present textbook arbitrage opportunities. However, since the crisis they have persisted, implying that such opportunities have not been fully exploited. We present an arbitrage strategy to exploit such large spreads. Assume that for a given currency, the current market quote is 3M LIBOR + 50 bps exchanging 6M LIBOR flat for 6 months. The notional amount is 1 unit and there are no transaction costs. An arbitrageur, which we assume is a LIBOR counterparty, such as an AA–rated bank, can then make arbitrage profit by,

(1) Enter the tenor swap in which the arbitrageur pays 6M LIBOR and receives 3M LIBOR + 50 bps.

(2) Roll over 3M borrowing at 3M LIBOR for 6 months, with unit notional.

(3) Deposit the notional at 6M LIBOR.

The net cash flows are summarised in Table 1. In Table 1, $L^{3m}(0)$ refers to the 3M LIBOR fixed at time 0. $L^{3m}(0.25)$ is the 3M LIBOR fixed at the end of 3 months and 0.25 is the year fraction. $L^{6m}(0)$ refers to the 6M LIBOR fixed at time 0.

Table 1: Arbitrage Strategy for Tenor Swap Basis Spreads

Strategy	$t = 0$	$t = 0.25$	$t = 0.5$
Loan	1	-$L^{3m}(0) \cdot 0.25$	-$L^{3m}(0.25) \cdot 0.25 - 1$
Deposit	-1	0	$L^{6m}(0) \cdot 0.5 + 1$
Swap	0	$(L^{3m}(0) + 50bps) \cdot 0.25$	$(L^{3m}(0.25)) + 50bps) \cdot 0.25 - L^{6m}(0) \cdot 0.5$
Net Cash Flow	0	12.5 bps	12.5 bps

The notional is canceled by the loan and deposit at time 0 and at the end of 6 months. The floating payment of the loan is canceled by the receipt from the tenor swap. The payment of 6M LIBOR in the tenor swap is financed by the interest income of the deposit. All cash flows are netted out except the spread of the tenor swap, which becomes the profit every 3 months. Because the arbitrageur has zero initial cost, this is clearly an arbitrage.

If the arbitrage strategy in Table 1 is practical, we would expect that arbitrageurs take large positions to make risk-less profit. The standard theory in finance, such as Arbitrage Pricing Theory (Ross 1976), assumes that arbitrageurs exploit such opportunities and no-arbitrage equilibria should be quickly restored. However, during the crisis such large spreads persisted and apparent arbitrage opportunities do not seem to be taken.

1.2 One Discount Curve, Multiple Forward Curves

In addition to the presence of textbook arbitrage opportunities, the aforementioned anomalies also have implications for the pricing methodology for interest–rate derivative products, such as the ad–hoc modelling approach of "one discount curve, multiple forward curves" adopted by practitioners. The price of interest rate derivative products depends on the present value of future cash flows linked to interest rates. For the pricing purpose, we need forward curves to generate future cash flows and a yield curve to discount these cash flows.

Before the crisis, the standard market practice was to build a single curve to both generate and discount cash flows. A set of the most liquid interest–rate instruments based upon underlying rates of different tenors (e.g. deposits on 1M LIBOR, FRA or interest futures on 3M LIBOR and interest rate swaps on 6M LIBOR) are selected to construct the yield curve. Discount factors off the yield curve are used to calculate the forward rates (see e.g. Brigo and Mercurio 2006),

$$F(t; T_1, T_2) = \frac{1}{\tau(T_1, T_2)} \left(\frac{P(t, T_1)}{P(t, T_2)} - 1 \right), \qquad t \le T_1 \le T_2, \tag{1}$$

where $F(t;T_1,T_2)$ is the simple compounded forward rate contracted at t and applicable between the year fraction of the time interval $\tau(T_1,T_2)$. $P(t,T)$, also known as the discount factor, is the price at time t of a zero–coupon bond maturing at T with face value of unity. The pre–crisis single curve approach ensures the no–arbitrage relationship

$$P(t,T_2) = P(t,T_1)P(t;T_1,T_2), \qquad t \leq T_1 \leq T_2, \tag{2}$$

where $P(t;T_1,T_2)$ is the forward discount factor defined by $F(t,T_1,T_2)$ and $\tau(T_1,T_2)$ via,

$$P(t;T_1,T_2) = \frac{1}{1+F(t;T_1,T_2)\tau(T_1,T_2)}. \tag{3}$$

We employ an arbitrage strategy in Table 2 to prove Eqn. (2).

Table 2: Arbitrage Strategy

Strategy	t	T_1	T_2
Buy 1 bond maturing T_2	-$P(t,T_2)$		1
Short $\frac{P(t,T_2)}{P(t,T_1)}$ bonds maturing T_1	$P(t,T_2)$	- $\frac{P(t,T_2)}{P(t,T_1)}$	
Borrow $\frac{P(t,T_2)}{P(t,T_1)}$ cash at $F(t,T_1,T_2)$		$\frac{P(t,T_2)}{P(t,T_1)}$	- $\frac{P(t,T_2)}{P(t,T_1)}(1+F(t;T_1,T_2)\tau(T_1,T_2))$
Net Cash Flow	0	0	0

Because the net cash flow is zero both at t and T_1, to eliminate arbitrage opportunity we have to ensure

$$\frac{P(t,T_2)}{P(t,T_1)}(1+F(t;T_1,T_2)\tau(T_1,T_2)) = 1. \tag{4}$$

Eqn. (2) then is proved by putting together Eqns. (3) and (4). Eqn. (1) can also be proved from Eqn. (4). Eqn. (2) basically states that for a cash flow at T_2, its present value at t must be unique. We can either discount the cash flow by $P(t,T_2)$ in one step, or we can first discount from T_2 to T_1 by the forward discount factor $P(t;T_1,T_2)$, then discount from T_1 to t by the discount factor $P(t,T_1)$. From the way that the single yield curve is constructed before the crisis, we see that all discount factors and forward rates are calculated from a unique curve, hence the no–arbitrage relation is guaranteed.

Now if we consider a generic LIBOR $L(T_1,T_2)$ which is simply compounded between T_1 and T_2. $L(T_1,T_2)$ and the forward rate $F(t;T_1,T_2)$ is related by,

$$\lim_{T_1 \to t} F(t; T_1, T_2) = L(T_1, T_2). \tag{5}$$

It then follows from Eqn. (1) that

$$L(T_1, T_2) = \frac{1}{\tau(T_1, T_2)} \left(\frac{1}{P(T_1, T_2)} - 1 \right). \tag{6}$$

From the interest–rate derivative pricing perspective, forward rate $F(t; T_1, T_2)$ is the expectation of $L(T_1, T_2)$ at t under the T_2 forward measure (Geman et al. 1995), i.e.

$$E^{T_2}[L(T_1, T_2) \mid \mathcal{F}_t] = F(t; T_1, T_2), \qquad t \leq T_1 \leq T_2. \tag{7}$$

Eqn. (7) is an important tool to price LIBOR–linked derivatives, such as interest caps, floors and swaptions. It provides a link between LIBORs and forward rates, hence we can express the expected LIBOR under the associated forward measure by discount factors via Eqn. (1). Again, the internal consistency of the single curve framework is crucial in no-arbitrage pricing of such derivatives.

A yield curve is supposed to produce interest rates as a smooth function of any arbitrary time to maturity, hence a continuous function. However, in real markets we only have a set of instruments of discrete maturities quoted, including zero–coupon products such as deposits at LIBORs, and coupon–bearing products such as interest rate swaps. For the short–end of this discrete set of points on the yield curve, we compute the corresponding interest rates from the zero–coupon instruments. Given these yields, the longer–maturity zero–coupon yields can be recovered from the coupon–bond products by solving for them iteratively by forward substitution. This process is the so called bootstrap method in constructing yield curves. This discrete set of yields is calculated to eliminate arbitrage opportunities. For time points that fall between any two maturities in the discrete set, some interpolation scheme has to be employed because no instrument is quoted in the market corresponding to that maturity. Many arbitrary and different interpolation algorithms are used in practice (see Hagan and West 2006). Therefore, together with bootstrapping, any particular choice of interpolation completes the construction of yield curves.

As noted by Schlögl (2002) and subsequently Bianchetti (2010), such a yield curve is not strictly guaranteed to be free of arbitrage because discount factors through interpolation are not always consistent with those obtained by a stochastic interest rate model which belongs to the no–arbitrage framework developed by Heath et al. (1992). Researchers have extended arbitrage–free interpolation schemes from discrete to continuous settings (e.g. Schlögl 2002). In practice the transaction costs in general cancel such arbitrage opportunities. Therefore, this drawback of the single–currency–single–curve approach, as far as practitioners are concerned, was of second–order importance.

After the crisis, the single–curve approach described above is not valid. The reason is that the interest rate market is segmented and rates of different tenors display distinct dynamics, reflected in the large spreads in tenor swaps, as well in LIBOR vs. OIS of a given currency. Such "segmentation" reflects varying levels of risk premia driving rates of different tenors. The pre–crisis single–curve approach, which mixes instruments of different tenors of underlying rates characterised by significantly different risk premia, would result in inconsistencies across market segments. To consistently account for the market segmentation, as well as explain the reason that textbook arbitrage opportunities are not exploited, approaches based on explanatory factors are required. Recent studies generally attribute such market anomalies to default risk and liquidity risk, but acknowledge that a consistent framework incorporating these risks is not easy to construct (see, for example, Bianchetti (2010) and Mercurio (2010)) .

Bypassing a consistent framework, practitioners have tackled this issue by constructing multiple forward curves based on the length of the tenor to forecast future cash flows (i.e. 1M, 3M, 6M, 12M forward curves). Each forward curve is built with vanilla instruments homogeneous in the underlying rate tenor. For example, the 1M USD forward curve is bootstrapped with instruments on 1M USD LIBOR only. On the other hand, the curve for discounting has to be unique to preclude arbitrages. By the "Law of One Price", two identical future cash flows must have same present value. The unique discount curve is constructed with the pre–crisis approach, which mixes instruments on rates of different tenors.

The current practice of "one discount curve, multiple forward curves" contradicts the single curve approach which precludes arbitrage. Forward rates of a particular tenor are calculated from the corresponding forward curve, whereas the discount factors are from the discount curve. A natural consequence of this approach is that if we calculate the forward discount factor $P(t; T_1, T_2)$ from Eqn. (3), each curve would give us a different result. The present value of a particular cash flow is no longer unique. If we only use $P(t; T_1, T_2)$ off the discount curve, then the relationship defined by Eqn. (3) is immediately invalidated. Consequently, this created a clear need for a unified, consistent framework to reconcile inconsistencies and simplify the pricing methodologies of interest rate derivatives.

1.5. Motivation

We examine the issues existing in the tenor swap market. Based upon recent empirical studies, we propose a consistent framework to reconcile the differences between the classic "single curve" approach and practitioners' "multiple curve" approach. The remainder of this study is organised as follows. Section 2 reviews relevant literature. Section 3 sets up the model framework and Section 4 presents the empirical results. Based upon Section 4, Section 5 proposes a parametrically parsimonious model. Finally Section 6 concludes.

2. Literature Review

In this section we review studies which aim to explain and/or model the observed large spreads in the interest rate market. We separate these studies into two broad categories: the ad–hoc modelling approach and the fundamental approach, depending upon whether fundamental factors which drive market anomalies are explicitly examined.

2.1. Ad–Hoc Approach

The first approach is mainly adopted by quantitative practitioners to extend the existing interest rate derivative pricing models, such as the LIBOR Market Model (LMM) (e.g., Miltersen et al. (1997) and Brace et al. (1997)).

The classic LMM models the joint evolution of a set of consecutive forward LIBORs. Mercurio (2010) points out that two complications arise when we move to a multi–curve setting. The first is the co–existence of several yield curves. The second is that forward LIBORs are no longer equal to the corresponding ones defined by the discount curve. Mercurio addresses the first issue by adding extra dimensions to the vector of modelled rates and suitably modelling their instantaneous covariance structure. For the second issue, Mercurio models the joint evolution of forward rates calculated from the OIS discount curve[5] and the spread between OIS forward rates and forward LIBORs. For a given tenor, forward OIS rates are defined as

$$F_k(t) = F_D(t; T_{k-1}, T_k) = \frac{1}{\tau_k}\left(\frac{P_D(t, T_{k-1})}{P_D(t, T_k)} - 1\right), \qquad t \leq T_{k-1} \leq T_k, \tag{8}$$

where the subscript D refers to the discount curve built with the OIS rates, which are considered effective risk–free rates since the GFC. There are two reasons for directly modelling OIS forward rates. First, as in Kijima et al. (2009), which proposes a three yield–curve model (discount curve, LIBOR curve and government bond curve), the pricing measures in Mercurio (2010) (including the spot LIBOR measure Q_D^{τ} and the forward measure $Q_D^{T_k}$) are associated with the OIS discount

[5]Because OIS swap rates are perceived as entailing little default or liquidity risk, since the crisis market participants increasingly construct OIS–based discount curve to discount collateralised contracts.

curve. Secondly, forward swap rate depends on the OIS discount factors. The spread between forward LIBOR and forward OIS rate is defined as

$$S_k(t) = L_k(t) - F_k(t), \tag{9}$$

where $L_k(t)$ is the forward LIBOR for the given tenor. By construction, both $L_k(t)$ and $F_k(t)$ are martingales under the forward measure $Q_D^{T_k}$ with the zero–coupon bond $P_D^{T_k}$ as the numeraire. Therefore $S_k(t)$ is also a martingale under $Q_D^{T_k}$. $S_k(t)$ is modelled with a continuous and positive martingale which is independent of the OIS forward rate. The model is calibrated to the market caplet smile and model volatilities fit the market almost perfectly, though the sample size is small.

Bianchetti (2010) incorporates the forward basis to recover the no–arbitrage relationship between forward curves and the discount curve. The no–arbitrage relationship between two curves is expressed as

$$F_f(t;T_1,T_2)\tau_f(T_1,T_2) = F_d(t;T_1,T_2)\tau_d(T_1,T_2)BA_{fd}(t;T_1,T_2), \tag{10}$$

where the subscripts f and d denote forward curves and the discount curve from which forward rates (or discount factors) are extracted and obviously $\tau_f(T_1,T_2)$ $= \tau_d(T_1,T_2)$. The multiplicative forward basis $BA_{fd}(t;T_1,T_2)$ is the ratio between forward rates (or equivalently in terms of discount factors) from forward curves and from the discount curve

$$BA_{fd}(t;T_1,T_2) = \frac{F_f(t;T_1,T_2)\tau_f(T_1,T_2)}{F_d(t;T_1,T_2)\tau_d(T_1,T_2)} = \frac{P_d(t,T_2)}{P_f(t,T_2)}\frac{P_f(t,T_1)-P_f(t,T_2)}{P_d(t,T_1)-P_d(t,T_2)}. \tag{11}$$

Eqn. (11) can be easily derived from Eqn. (1). Hence the forward basis is a measure of the difference between the forward rates from the forward curve and forward rates from the discount curve. Alternatively, the additive forward basis $BA'_{fd}(t;T_1,T_2)$ is defined as

$$BA'_{fd}(t;T_1,T_2) = F_d(t;T_1,T_2)[BA_{fd}(t;T_1,T_2) - 1]. \tag{12}$$

In the single curve setting, the basis should be zero because there is only one curve, hence we expect $BA_{fd}(t;T_1,T_2) = 1$ and $BA'_{fd}(t;T_1,T_2) = 0$. Bianchetti (2010) then constructs the forward basis curve through bootstrapping. The finding is that the short–term forward basis is wide ranging, with the multiplicative forward basis ranging from 0.7 (12M tenor forward curve versus the discount curve) to 1.3 (1M tenor forward curve versus the discount curve). However, the longer term (up to 30 years maturity) forward basis tends to 1 (resp. 0) for the multiplicative case (resp. additive case). It is important to note that the term structure of the forward basis curve as constructed by Bianchetti (2010) oscillates. The oscillations are demonstrated especially in the longer term forward basis curve. This suggests that there may be some over–fitting in the bootstrap curve construction.

In Bianchetti (2010), the discount curve is built with the traditional "pre–crisis" approach. The instruments include liquid deposits, FRAs on 3M EURIBOR[6] and swaps on 6M EURIBOR. On the other hand, forward curves are constructed from instruments with homogeneous underlying tenor. For instance, 3M forward curve was based upon instruments linked to 3M EURIBOR. Hence the discount curve mixes rates of different underlying tenors with distinct dynamics, whereas a forward curve corresponds to one particular underlying tenor. Bianchetti (2010) therefore attributes oscillations in the forward basis curve to the amplification of small local differences between the two curves. The author also suggests to use the forward basis term structure as a tool to assess the distinct risk dynamics in the interest rate market because it provides a sensitive indicator of the tiny, but observable statical differences between different interest rate market sub–areas in the post GFC world.

As a sequel of Henrard (2007), Henrard (2010) proposes a framework to price interest rate derivatives based on different LIBOR tenors by introducing a deterministic, and maturity dependent, spread between the forward curve and the discount curve. In Henrard (2007) the spread is assumed to be constant across maturities. Hence this extension is a natural adaptation to the post–crisis market reality. Henrard (2010) assumes that the discount curve is given and proceeded to construct the forward curves based on the spreads. Simple vanilla instruments are selected to achieve this purpose, including FRA, futures and interest rate swaps. Henrard then proposes to extend this framework to cross–currency products and the object to be modelled is the cross–currency basis, which had also become substantially higher since the crisis.

Fujii et al. (2009) proposes a Heath–Jarrow–Morton (HJM, see Heath et al. (1992)) model framework to adapt to new developments in the interest rate markets: large spreads in LIBOR vs. OIS and widespread use of collateral. The underlying quantities in the model are the instantaneous forward OIS rate and the spread, which measures the difference between the forward LIBOR under the collateralised forward measure and the OIS forward rate. The model is set up as follows,

$$dc(t,s) = \sigma_c(t,s) \cdot \left(\int_t^s \sigma_c(t,u)\, du \right) dt + \sigma_c(t,s) \cdot dW^Q(t), \tag{13}$$

$$\frac{dB(t,T;\tau)}{B(t,T;\tau)} = \sigma_B(t,T;\tau) \cdot \left(\int_t^s \sigma_c(t,s)\, ds \right) dt + \sigma_B(t,T;\tau) \cdot dW^Q(t), \tag{14}$$

[6]EURIBOR is the reference rate of unsecured borrowing of EUR between European prime banks within the euro zone.

where $c(t, T)$ is the instantaneous forward collateral rate and in Eqn. (13) the standard arbitrage–free HJM dynamics applies under the risk-neutral measure Q. $B(t, T; \tau)$ is the spread and by construction a martingale under the collateralised forward measure τ^c. τ stands for a particular LIBOR tenor. The stochastic differential equation is written as

$$\frac{dB(t, T; \tau)}{B(t, T; \tau)} = \sigma_B(t, T; \tau) \cdot dW^{\tau^c}(t). \tag{15}$$

The Brownian motion $W^{\tau^c}(t)$ under the measure τ^c is related to $W^Q(t)$ by the the Girsanov theorem (Girsanov 1960),

$$dW^{\tau^c}(t) = \left(\int_t^s \sigma_c(t, s)\, ds \right) dt + dW^Q(t). \tag{16}$$

The details of the volatility processes $\sigma_c(t, s)$ and $\sigma_B(t, T; \tau)$ are not specified in Fujii et al. (2009). It is also clear from Eqn. (14) that $\sigma_B(t, T; \tau)$ needs to be specified for all relevant LIBOR tenors (i.e. 1M, 3M, 6M and 12M), hence this is a high–dimensional approach.

These papers endeavour to reconcile inconsistencies caused by the multi–curve framework used by practitioners. They appear promising in fitting model prices to market prices by incorporating the spreads of LIBORs of different tenors. The drawback of this approach is that it does not relate the spreads to more fundamental risks, and thus does not attempt to explain why the textbook arbitrage opportunities seemingly created by the presence of these spreads are not exploited. Furthermore, one quickly ends with a multitude of basis spread dynamics, which should be related at a fundamental level. However, these relationships are not addressed by this ad–hoc approach.

2.2 Fundamental Approach

Different from the ad–hoc modelling approach, the fundamental approach aims to identify the risk factors causing market anomalies. Although market anomalies are commonly considered entailing default and liquidity risk premiums, empirical evidence shows that liquidity risk plays a more significant role.

2.2.1 Default Risk

Morini (2009) examines two particular instruments in interest rate markets: FRA and tenor swaps. Before the crisis, the market FRA rate was well approximated by the LIBOR–based replication. After the crisis, the LIBOR–based replication of the FRA rate has been persistently higher than the market quotes of FRA rates. Morini uses two different discount curves, the LIBOR–based curve and the OIS curve to bridge the gap between the market FRA and the replicated FRA by incorporating the basis spreads of LIBORs of different tenors. Therefore, two issues are reduced to one: why has the basis attached to the leg of the shorter–tenor LIBOR been persistently large and positive? Morini explicitly assumes an unexplained axiom proposed by Tuckman and Porfirio (2003) that lending at longer–tenor LIBOR involves higher counterparty default risk and liquidity risk than rolling lending at shorter–tenor LIBORs. Morini proposes that it is difficult to separate default risk and liquidity risk because the two risks are highly correlated. Hence Morini uses default risk only to approach the question. Morini conjectures that a LIBOR–panel bank today may not be a LIBOR bank in the future, due to its worsening credit rating. For example, the roll–over lender at 6M LIBOR can reassess the credit quality of the borrowing bank and may choose to replace with a counterparty that remains to be a LIBOR bank. There is a cap to how much the credit standing of a current LIBOR bank can worsen before it is excluded from the LIBOR Panel. This conjecture motivates Morini to model the spread of a generic LIBOR L^{X^0} over the market OIS rate E_M between time α and 2α as

$$S^{X^0}(\alpha, 2\alpha) = L^{X^0}(\alpha, 2\alpha) - E_M(\alpha, 2\alpha), \tag{17}$$

where $S^{X^0}(\alpha, 2\alpha)$ is the spread, X^0 denotes a generic LIBOR panel bank and the subscript M refers to market rate. The forward spread at time $t \leq \alpha$ is then the spread between the forward rate F_{Std} replicated by LIBORs and the forward rate E_{Std} replicated by OIS rates, i.e.

$$S^{X^0}(t; \alpha, 2\alpha) = F_{Std}(t; \alpha, 2\alpha) - E_{Std}(t; \alpha, 2\alpha). \tag{18}$$

A particular LIBOR counterparty is excluded from the LIBOR panel if

$$S^{X^0}(\alpha, 2\alpha) > S^{X^0}(t; \alpha, 2\alpha). \tag{19}$$

The interpretation of the inequality in (19) is that a current counterparty defaults if its LIBOR–OIS spread at α exceeds a pre–specified level. The spread thus is reduced to a call option with the strike $S^{X^0}(t; \alpha, 2\alpha)$. Morini further assumes that the spread evolves as a driftless geometric Brownian motion and prices the option with the standard Black–Scholes formula (Black and Scholes 1973). The formula is calibrated to market quotes of basis of EURIBORs and results closely track the shape of the traded 6M/12M basis from July 2008 to May 2009, though there are discrepancies in levels. Morini attributes level discrepancies to a lack of more appropriate volatility inputs during the sample period.

Taylor and Williams (2009) use a no–arbitrage model of term structure to examine the effect of default risk and liquidity risk on 3M LIBOR–OIS spread. They consider a range of possible measures of default risk, such as the credit default swap (CDS) premium, TIBOR–LIBOR spread[7] and asset–backed commercial paper spread. The effect of liquidity risk is measured by a dummy variable, Term Auction Facility (TAF). TAF was provided by the US Federal Reserve to inject liquidity into financial institutions. Results find that default risk measures explain most of the variations of LIBOR–OIS spread. The TAF dummy variable is either statistically insignificant or of the wrong sign[8].

2.2.2 Liquidity Risk

Brunnermeier and Pedersen (2009) develop a theoretical liquidity risk model in which market liquidity and funding liquidity reinforce each other. Market liquidity is defined as the ease of trading securities, including low bid–ask spread, market depth and market resilience. On the other hand, funding liquidity is the ease of raising funds, with own capital or loans. During the financial crisis, initial losses in the sub–prime mortgage market forced financial institutions to exit positions in other asset classes (e.g. stocks) to meet margin calls and other funding needs. Funding constraints prompted traders to sell securities at "fire sale" prices, which resulted in even larger losses. In such volatile market conditions, market liquidity also deteriorated and positions in illiquid assets (e.g. structured products due to highly customised nature and held–to–maturity investment strategy) were particularly difficult to unwind. Selling such assets meant even greater losses than selling in a liquid market. Both market liquidity and funding liquidity disappeared and banks faced a double jeopardy: they found it difficult to sell assets to raise funds exactly at a time it was difficult to borrow. The double "liquidity shock" forced them to hoard cash and other liquid instruments which they might otherwise have lent to others. They were reluctant to make lending to inter–bank counterparties for longer than three months (see Mollenkamp and Whitehouse (2008)). Brunnermeier (2009) identifies liquidity risk, lending channel, bank run and network effects as main amplification mechanisms through which a relatively small shock in the mortgage market transmitted to other asset classes and resulted in a full–blown financial crisis.

Ivashina and Scharfstein (2010) and Cornett et al. (2011) identify three factors which led banks to manage liquidity and reduce lending during the crisis. Firstly, the extent to which a bank is financed by short–term debt, as opposed to insured deposits. Short–term debts are subject to rollover risks[9]. On the other hand,

[7] TIBOR is the reference rate of unsecured lending of JPY to Japanese prime banks in the Tokyo interbank market. Taylor and Williams argue that because Japanese banks were less affected by the financial crisis than US banks, TIBOR–LIBOR spread reflected default risk differential between two markets.

[8] TAF announcements are supposed to decrease the level of LIBOR–OIS spread, hence the sign is expected to be negative.

[9] Rollover risk is associated with debt refinancing. It arises when existing debt is about to

insured deposits are a more stable source of capital. Before the crisis, financial institutions relied heavily on short–term funding, such as Asset–Backed Commercial Papers and Repurchase (Repo) Agreements, to finance their long–term assets. The average maturity of such instruments ranges from overnight to 90 days. After initial losses in mortgage securities, investors refused to roll over and banks had to refinance from other sources. The second factor is banks' exposure to credit–line draw downs. Ivashina and Scharfstein (2010) show that during the crisis firms drew on their credit lines primarily because of concerns about the ability of banks to fund these commitments, as well as due to firms' desire to enhance their own liquidity[10]. Lastly, on the asset side, banks holding illiquid loans and securities tended to increase holdings of liquid assets and decreased new lending.

In contrast to Taylor and Williams (2009), McAndrews et al. (2008) find that TAF announcements and operations significantly reduced the 3M LIBOR–OIS spread, which points to the importance of the liquidity risk premium. The authors argue that in order to test the effect of the TAF dummy variable, the dependent variable should be the change, not the level of the LIBOR–OIS spread. The use of the spread level as the dependent variable, as in Taylor and Williams (2009), is only valid under the assumption that the effect of TAF auction disappears immediately after the auction. If the liquidity risk premium stays low over days after the auction, the coefficient of the TAF dummy cannot be interpreted as the TAF effect.

Michaud and Upper (2008) aim to identify the drivers of the increase of the 3M LIBOR–OIS spread. Acknowledging that it is difficult to disentangle default risk and funding liquidity risk, as well as the measurement problem of bank–specific funding liquidity, Michaud and Upper examine only the effect of default risk and market liquidity risk. Funding liquidity is treated as an unobserved variable whose effects will appear as a residual once the impact of all other variables has been taken into account. The default risk is measured by the spread between the unsecured and secured interbank rate, as well as the CDS premia. The measures of market liquidity are number of trades, volume, bid–ask spreads and price impact of trades. The finding is that while default risk plays a role, the significance is stronger in market liquidity measures. Furthermore, due to potential positive correlation between default risk and funding liquidity risk, the effect of default risk may have been overestimated.

Acharya and Merrouche (2013) examine the UK interbank market during the crisis

mature and needs to be rolled over into new debt and interest rates increase. The debt issuer hence needs to refinance at a higher interest rate and incur more interest payments in the future. Recent studies on rollover risk during the GFC include Acharya et al. (2011) and He and Wei (2012).

[10]For example, FairPoint Communications drew down 200 million from the committed credit line supplied by Lehman Brothers as the lead bank on September 15th, 2008. In the SEC filing, the company "believes that these actions were necessary to preserve its access to capital due to Lehman Brothers' level of participation in the company's debt facilities and the uncertainties surrounding both that firm and the financial markets in general".

and empirical results are in favor of precautionary liquidity hoarding over default risk in explaining the increase of the 3M LIBOR–OIS spread. They find that liquidity hoarding substantially increased after structural breaks (e.g. BNP Paribas froze withdrawals on 08/09/2007, Bear Stearns in March 2008). Secondly, the hoarding of liquidity by banks was precautionary in nature, especially for banks with large losses in sub–prime mortgage securities. Thirdly, liquidity hoarding drove up interbank lending rates, both secured and unsecured. The effect of liquidity hoarding is to raise overnight inter–bank rates after the crisis. In contrast, before the crisis an increase in the overnight liquidity buffer was associated with a decline in overnight spreads. This confirms the authors' hypothesis that in stressed conditions banks only release liquidity at a premium that exceeds the direct cost of using the emergency lending facility offered by the central bank and the indirect stigma cost (e.g. bank run, credit line draw downs). The fact that the effects on rates are similar for secured and unsecured inter–bank rates implies that the market stresses were not *per se* due to default risk concerns. Instead, the stresses were most likely due to each bank engaging in liquidity hoarding as the precautionary response to its own heightened funding risk.

Schwarz (2010) is the first paper, to our best knowledge, to deliberately separate the effect of default risk and liquidity risk on LIBOR–OIS spread. Researchers commonly agree that it is difficult to disentangle default risk and funding liquidity risk, e.g. Michaud and Upper (2008) and Morini (2009). A bank with a funding shortage is more likely to default than a bank with ample funding. On the other hand, if a bank's credit rating worsens, it becomes more difficult to secure external funding. In fact, initial losses in the sub–prime mortgage market may have increased both default risk and funding liquidity risk. Hence, these two risk factors are highly interrelated. Schwarz measures market liquidity with the yield spread between German government bonds and KfW agency bonds. KfW bonds are fully guaranteed by the German government hence entail no default risk, but are less liquid in the bond market than the government bonds. The measure of default risk is the dispersion of borrowing rates of banks with different credit standings. Schwarz argues that a market–wide liquidity shock should have similar effect on banks' borrowing rates, hence the dispersion is relatively unchanged. On the other hand, a market-wide credit shock affects banks with bad credit rating more than banks with good credit, hence the dispersion increases. The correlation between the two risk measures is 0.07 and Schwarz claims that the regression results show the "clean" (i.e. independent) effect of each risk. The finding is that, though both risks are significant, nearly 70% of the increase of the 3M LIBOR–OIS spread and nearly 90% of the sovereign bond spread (Italy–Germany ten–year spread) increase can be explained by the market liquidity measure.

3. Model Set-up and Implementation

3.1 Liquidity risk, Basis Spreads and Limits to Arbitrage

We propose that liquidity risk is the fundamental factor that led to anomalies in the tenor swap market, as well as prevented arbitrage opportunities from being fully exploited. We choose a liquidity based model for observed basis spreads for two reasons. Firstly, liquidity is the factor which empirically seems to be driving the spreads in the market. Secondly, there is a need to reduce dimensionality from having a separate basis spread term structure for each tenor pair, which is the case in the "multiple curve" modelling approach. To illustrate the effect of liquidity risk, we revisit the arbitrage strategy in Table 1.

During the crisis, suppose a lender in the interbank (i.e. LIBOR) market rolls over two consecutive 3M lending. At the end of 3 months if there is funding shortage, the lender can choose not to make the second lending. In contrast, if the lender makes a 6M lending, there is no such flexibility. *Ceteris paribus*, because the 6M lending involves higher liquidity risk than the 3M roll–over lending, 6M LIBOR should entail a liquidity premium over the 3M LIBOR. As the crisis developed and intensified, liquidity risk was amplified, which led to large liquidity risk premium for longer term loans over the short term ones.

However, since the GFC tenor swaps are "almost" free of counterparty credit risk due to the widespread use of collateral. Johannes and Sundaresan (2007) find that due to collateral, market participants commonly view swaps as risk–free instruments and the cash flows should be discounted at the risk–free rate. Bianchetti (2010), Mercurio (2010) and Piterbarg (2010) also note that it makes sense to discount collateralised cash flows by the OIS rate, which is regarded as the best proxy for the risk–free rate. It is hence incorrect to compensate the party receiving 6M LIBOR with the liquidity premium in a tenor swap. To make the contract fair, a positive spread equal to the liquidity premium should be added to the 3M LIBOR. We propose that this is the reason the spread is always added to the shorter tenor rate. Large liquidity risk premium during the crisis hence also explains the large spreads quoted in tenor swaps.

In the arbitrage strategy proposed in Table 1, if a liquidity shock arrives between initiation and the end of 3 months, the lender may refuse to roll over the loan to the arbitrageur. Because the arbitrageur is committed to the 6M lending, he/she has to refinance in a stressed market. The potential loss due to refinancing (i.e. at a higher rate than $L^{3m}(0.25)$) could offset or even exceed the gain from the spreads in the swap. Therefore, although the market may appear rife with arbitrage opportunities, the strategy may break down.

3.2 Model and Implementation

We use an intensity model to describe the arrival time of liquidity shocks with a time-inhomogeneous Poisson process $N(t)$, with deterministic intensity $\lambda(t)$. The basic idea of intensity models is to describe the shock time τ as the first jump time of a Poisson process. Although shocks are not induced by observed market information or economic fundamentals, by formulation intensity models are suited to model credit spreads and calibrate to CDS data (see e.g. Brigo and Mercurio (2006)). In this study we adopt this technique and propose an intensity model for basis spreads in tenor swaps and calibrate to market data.

We consider a N-year maturity TS which exchanges the i–th tenor LIBOR plus a spread $B_{i,j,N}$ for the the j–th tenor LIBOR, where $i < j$ and $B_{i,j,N} > 0$. N, i and j are expressed in terms of year fractions. We assume that an arbitrageur follows the arbitrage strategy in Table 1. The arbitrageur gains $B_{i,j,N} * i$ at the end of each i–th tenor. We propose that the expected loss due to refinancing given a liquidity shock explains why the arbitrage strategy breaks down. Hence we impose the "fair pricing" condition that the expected loss offsets the expected gain. To gain more model tractability, we make several simplifying assumptions,

1) The tenor swap is perfectly collateralised with zero threshold, which means the posted collateral must be 100% of the contract's mark–to–market value. The amount of collateral is continuously adjusted with zero minimum transfer amount (MTA)[11]. Because daily margin call is quite common in the market, continuous adjustment should reasonably well approximate the actual practice (see Fujii et al. 2009).

2) The first jump of the Poisson process can occur within any shorter tenor of the swap and there can be at most one jump within each shorter tenor. Upon the first liquidity shock, the arbitrageur is unable to roll over the shorter tenor loan and has to refinance until the end of the associated longer tenor. The instantaneous loss rate due to refinancing is $\pi(t)$. The arbitrageur then shuts down the borrowing and lending in the arbitrage strategy at the end of the longer tenor within which the first jump occurs. To illustrate this assumption, suppose we have a 3M vs 12M tenor swap for 12 months, and in a purported arbitrage strategy against this swap we are borrowing at the shorter tenor and lending at the longer tenor. If the first liquidity shock occurs between initiation and 3 months, the borrowing and lending can only be shut down at the end of 12 months, due to the 12M lending.

3) We assume remaining risks are negligible for the arbitrageur, including the default risk of the longer tenor lending and the mark–to–market value of the tenor swap.

[11]MTA is the smallest amount of value that is allowable for transfer as collateral. This is the lower threshold below which the collateral transfer is more costly than the benefits.

Based on such simplifying assumptions, we calculate the present value (PV) of the expected gain and of the expected loss of the arbitrage strategy. We firstly examine the distribution of the first jump time τ. We assume that τ can occur within any shorter tenor. However, if τ arrives within the last shorter tenor for a given longer tenor, it is irrelevant because the arbitrageur can shut down the strategy at the end of the longer tenor without refinancing. Hence total number of relevant shorter tenors within which τ occurs is $\frac{N}{j} * (\frac{j}{i} - 1)$ and the PV of expected loss is expressed as

$$\begin{aligned} PV_{Loss} &= \sum_{k=1}^{K} \left(e^{\int_{T_k}^{T_{\eta(k)}} \pi(u)\,\mathrm{d}u} - 1 \right) P^Q(T_{k-1} < \tau \leq T_k) D^{OIS}(T_0, T_{\eta(k)}) \\ &= \sum_{k=1}^{K} \left(e^{\int_{T_k}^{T_{\eta(k)}} \pi(u)\,\mathrm{d}u} - 1 \right) \left(e^{-\int_0^{T_{k-1}} \lambda(u)\,\mathrm{d}u} - e^{-\int_0^{T_k} \lambda(u)\,\mathrm{d}u} \right) D^{OIS}(T_0, T_{\eta(k)}), \end{aligned} \tag{20}$$

where P^Q denotes the probability under the risk-neutral measure Q, $D^{OIS}(\cdot,\cdot)$ is the discount factor from the OIS curve. $K = \frac{N}{i}$ is the total number of shorter tenors until maturity and $T_k = k \cdot i$. η_k is expressed as

$$\eta_k = \min\,(m \,|\, T_k \leq T_{m \cdot n}) \cdot n, \tag{21}$$

where $n = \frac{j}{i}$. On the other hand, the PV of expected gain is

$$PV_{Gain} = \sum_{k=1}^{K} (B_{i,j,N} \cdot i) D^{OIS}(T_0, T_k). \tag{22}$$

The no–arbitrage condition is hence

$$\begin{aligned} & \sum_{k=1}^{K} \left(e^{\int_{T_k}^{T_{\eta(k)}} \pi(u)\,\mathrm{d}u} - 1 \right) \left(e^{-\int_0^{T_{k-1}} \lambda(u)\,\mathrm{d}u} - e^{-\int_0^{T_k} \lambda(u)\,\mathrm{d}u} \right) D^{OIS}(T_0, T_{\eta(k)}) \\ = & \sum_{k=1}^{K} (B_{i,j,N} * i) D^{OIS}(T_0, T_k). \end{aligned} \tag{23}$$

Given the OIS discount curve, we can use Eqn. (23) to calibrate the loss rate $\pi(t)$ and intensity function $\lambda(t)$ to the selected set of tenor swaps. In credit risk literature (e.g. Schönbucher (2003)) where the intensity model is used to calibrate credit spreads, joint calibration of the recovery rate and the deterministic intensity functions often produces unstable results. Hence the recovery rate, comparable to $\pi(t)$, is often made constant and estimated separately. We adopt this technique to estimate a constant loss rate π and $\lambda(t)$.

To obtain an estimate of π, as the first step we assume $\lambda(t) = \overline{\lambda}$, where $\overline{\lambda}$ is a constant and jointly calibrate π and $\overline{\lambda}$. To do this we use the mean–squared deviation function to obtain the optimal fit by minimizing the function by varying π and $\overline{\lambda}$:

$$\sum_{i=1}^{N} \left(\frac{PV^i_{Loss}(\pi, \overline{\lambda})}{PV^i_{Gain}} - 1 \right)^2, \tag{24}$$

where N is the number of tenor swaps used for the calibration. This measure uses relative deviations and hence is independent of the scale of individual present values.

In the second step, we use the estimated π from step 1 as the input and calibrate time–dependent and piecewise constant $\lambda(t)$ to the same set of selected swaps. To achieve a perfect fit and impose minimal structure on the intensity curve, we use the bootstrap method to strip $\lambda(t)$ from observed spreads. The bootstrap procedure is as follows,

1) Tenor swaps are ordered in the appropriate order[12].

2) $\lambda(t)$ is piecewise constant. We first find λ_1 such that

$$PV^1_{Loss}(\pi, \lambda_1) = PV^1_{Gain}. \tag{25}$$

We then work iteratively to evolve the intensity curve. Eventually, given λ_1, ..., λ_{N-1} we find λ_N such that

$$PV^N_{Loss}(\pi, \lambda_1, ..., \lambda_{N-1}; \lambda_N) = PV^N_{Gain}. \tag{26}$$

4. Data, Methodologies and Results

Because the set–up and implementation of the intensity model is currency independent, we collect USD data only. The calibration procedure is identical for currencies other than USD.

4.1 Construction of OIS Discount Factors

We use the standard bootstrap with interpolation method to construct the USD OIS discount factors required for both sides of Eqn. (23). To this end, we collect

[12]See details in Table 4 of Section 4.

USD OIS rates available from Bloomberg. Maturities include 1–week (1W), 2W, 1M, 2M, 3M, 4M, 5M, 6M, 7M, 8M, 9M, 10M, 11M, 1Y, 15M, 18M, 21M, 2Y, 3Y, 4Y, 5Y and 10Y. The sample period starts from July 28th, 2008, since when the 10Y OIS rates are available, and ends at April 2nd, 2013. OIS rates beyond the 10Y maturity are only quoted from September 27th, 2011.

In order to extend the OIS curve to 30Y maturity, we use the USD Fed Funds (FF) basis swap quotes to approximate OIS rates (see, for example, Bloomberg 2011). FF basis swaps exchange the non–compounded daily weighted average of the overnight FF effective rate[13] for a 90–day period plus a spread and 3M USD LIBOR flat, with quarterly payment frequency. On the other hand, two parties in an OIS agree to exchange the difference between interest accrued at the fixed rate and interest accrued at the daily compounded FF effective rate, with annual payment frequency. Although having different payment frequency and compounding conventions, both OIS and FF basis swaps are defined in terms of the daily reset FF effective rate, hence they are observables of the same underlying security.

By ignoring minor discrepancies such as compounding for weekends and holidays, Bloomberg (2011) proposes a quick approximation of OIS rates with IRS rates and FF basis swap spreads. Firstly, a fixed–floating FF swap can be set up by simultaneously entering an interest rate swap and an FF basis swap. In the interest rate swap, the fixed rate is received and the 3M LIBOR is paid. In the FF basis swap, the 3M LIBOR is received and FF rate plus the spread is paid. The net position is therefore interest rate swap fixed rate vs daily average FF rate plus the spread. Based upon this setup, let S_N and FF_N denote the N–year IRS fixed rate and FF basis swp spread, the OIS rate OIS_{t_N} can be approximated as

$$OIS_{t_N} = \left(\left(1 + \frac{\widehat{OIS_{t_N}}}{360} \right)^{90} - 1 \right) \times 4, \tag{27}$$

where

$$\widehat{OIS_{t_N}} = \left(1 + \frac{r_Q - FF_N}{4} \right)^{4} - 1, \tag{28}$$

and

$$r_Q = \left(\left(1 + \frac{S_N \times \frac{360}{365}}{2} \right)^{\frac{2}{4}} - 1 \right) \times 4, \tag{29}$$

[13]FF rate is the interest rate at which depository institutions trade funds held at the U.S. Federal Reserve with each other. The weighted average of FF rate across all transactions is the FF effective rate.

where Eqn. (29) converts the semiannually paid interest rate swap rate to quarterly paid (i.e. 3M LIBOR) rate and $\widehat{OIS_{t_N}}$ annualises the quarterly paid FF effective rate. OIS_{t_N} then is the OIS rate with the daily compounding adjustment.

We then collect the interest rate swap rates and FF basis spreads with corresponding maturities to approximate 12Y, 15Y, 20Y, 25Y and 30Y OIS rates. These maturities are chosen because they started to be quoted from September 27th, 2011. Because FF basis swaps are quoted from September 22nd, 2008, we approximate OIS rates from September 22nd, 2008 to September 26th, 2011. To evaluate how well this method performs, we compare actual quotes of OIS rates and approximated OIS rates from September 27th, 2011 to April 2nd, 2013. Table 3 shows that the approximated rates track the actual rates reasonably well.

Table 3: Average OIS Rate Approximation Errors

Maturity	12-year	15-year	20-year	25-year	30-year
Basis Points	0.71	0.94	1.17	0.98	0.70
Percentage	0.37%	0.43%	0.51%	0.40%	0.28%

We therefore have OIS rates with maturities from 1 week up to 30 years. Since OIS swaps have annual payment frequency, there is only one exchange of payments up to 1 year. Therefore to bootstrap the OIS curve up to 1 year, OIS rates are treated as deposit rates. With the day count convention of *Actual*/365, the OIS discount factors are calculated as

$$D^{OIS}(t_i) = \frac{1}{1 + \tau_i \cdot OIS(t_i)}, \tag{30}$$

where τ_i is the year fraction of maturity t_i. Similar to using interest rate swap rates to construct the LIBOR discount curve, OIS discount curves from 1Y to 30Y are extracted from par OIS rates with the standard bootstrap method. Eqns. (31), (32) and (33) summarise this method:

$$OIS(t_N) \cdot \sum_{i=1}^{N} D_{t_i}^{OIS} + D_{t_N}^{OIS} = 1, \tag{31}$$

where N is the total number of payments. Discount factors $D_{t_N}^{OIS}$ are iteratively obtained with

$$D_{t_N}^{OIS} = \frac{1 - OIS(t_N) \cdot \sum_{i=1}^{N-1} D_{t_i}^{OIS}}{1 + OIS(t_N)}, \tag{32}$$

and for maturities not quoted from Bloomberg, OIS rates are linearly interpolated with available quotes:

$$OIS_t = OIS_{t_i} + \left(\frac{t - t_i}{t_{i+1} - t_i}\right) \times (OIS_{t_{i+1}} - OIS_{t_i}), \quad t_i < t < t_{i+1}. \tag{33}$$

4.2 Bootstrap Liquidity Spreads

We bootstrap piecewise constant intensity λ_t to achieve perfect fits without imposing any functional form. We collect available tenor swap data from Bloomberg, including 1M vs 3M, 3M vs 6M and 3M vs 12M swaps. We aim to include all tenor swap instruments in order to capture as much market information as possible and evolve maturities up to 30 years. To illustrate our bootstrap approach, consider the 1M vs 3M swap with 3M maturity. Based upon the model assumptions, the first liquidity shock can arrive between 0 and 1 month, 1 and 2 months or 2 and 3 months. However, if the shock is between 2 and 3 months, it is irrelevant because the arbitrageur shuts down the strategy at the end of 3 months and does not need to refinance. Therefore, we assume a constant intensity between 0 and 2 months and use the 1M vs 3M swap with 3M maturity to calculate the intensity λ_1 with Eqn. (23). With λ_1, we are then able to calculate the constant intensity λ_2 between 2 and 3 months, by using the 3M vs 6M swap with 6M maturity. With this approach, we establish 36 piecewise constant intensities, which are summarised in Table 4.

In the bootstrap procedure, we exclude 3M v 12M tenor swaps for two reasons. Firstly, 3M v 12M quotes are only available from August 6th, 2009. Secondly, in our approach to extending bootstrap intervals, 3M v 12M tenor swaps are redundant once we have used 3M v 6M swaps. Because 3M v 6M swaps have been quoted for a much longer period, we propose the quotes should be more consistent and reliable. Having established the bootstrap procedure, we use Eqn. (23) to calculate piecewise constant intensities. We start from the 1M v 3M with 3M maturity swap to calculate λ_1, then work iteratively to find λ_2, λ_3, ..., λ_{36}.

The bootstrap results demonstrate two problems. Firstly, the term structure of calibrated intensities severely oscillates. Secondly, many of the intensities are negative. Severe oscillations are an undesirable property for the term structure of intensities. Even worse, negative intensities invalidate the fundamental model assumption because $\lambda(t)$ is a positively valued function. To illustrate, Figure 2 shows the bootstrap results on Oct 10th, 2008, with $\pi = 0.1$ which minimises the deviation function in Eqn. (24). The order of intensities in Figure 2 follows the sequence of intensities constructed in Table 4.

Table 4: Bootstrap Piecewise Constant Intensities

Piecewise Constant λ_t	Bootstrap Interval	Tenor Swap
λ_1	0-2 months	1v3 3-month
λ_2	2-3 months	3v6 6-month
λ_3	3-5 months	1v3 6-month
λ_4	5-8 months	1v3 9-month
λ_5	8-9 months	3v6 1-year
λ_6	9-11 months	1v3 1-year
λ_7	11-15 months	3v6 18-month
λ_8	15-17 months	1v3 18-month
λ_9	17-21 months	3v6 2-year
λ_{10}	21-23 months	1v3 2-year
λ_{11}	23-33 months	3v6 3-year
λ_{12}	33-35 months	1v3 3-year
λ_{13}	35-45 months	3v6 4-year
λ_{14}	45-47 months	1v3 4-year
λ_{15}	47-57 months	3v6 5-year
λ_{16}	57-59 months	1v3 5-year
λ_{17}	59-69 months	3v6 6-year
λ_{18}	69-71 months	1v3 6-year
λ_{19}	71-81 months	3v6 7-year
λ_{20}	81-83 months	1v3 7-year
λ_{21}	83-93 months	3v6 8-year
λ_{22}	93-95 months	1v3 8-year
λ_{23}	95-105 months	3v6 9-year
λ_{24}	105-107 months	1v3 9-year
λ_{25}	107-117 months	3v6 10-year
λ_{26}	117-119 months	1v3 10-year
λ_{27}	119-141 months	3v6 12-year
λ_{28}	141-143 months	1v3 12-year
λ_{29}	143-177 months	3v6 15-year
λ_{30}	177-179 months	1v3 15-year
λ_{31}	179-237 months	3v6 20-year
λ_{32}	237-239 months	1v3 20-year
λ_{33}	239-297 months	3v6 25-year
λ_{34}	297-299 months	1v3 25-year
λ_{35}	299-357 months	3v6 30-year
λ_{36}	357-359 months	1v3 30-year

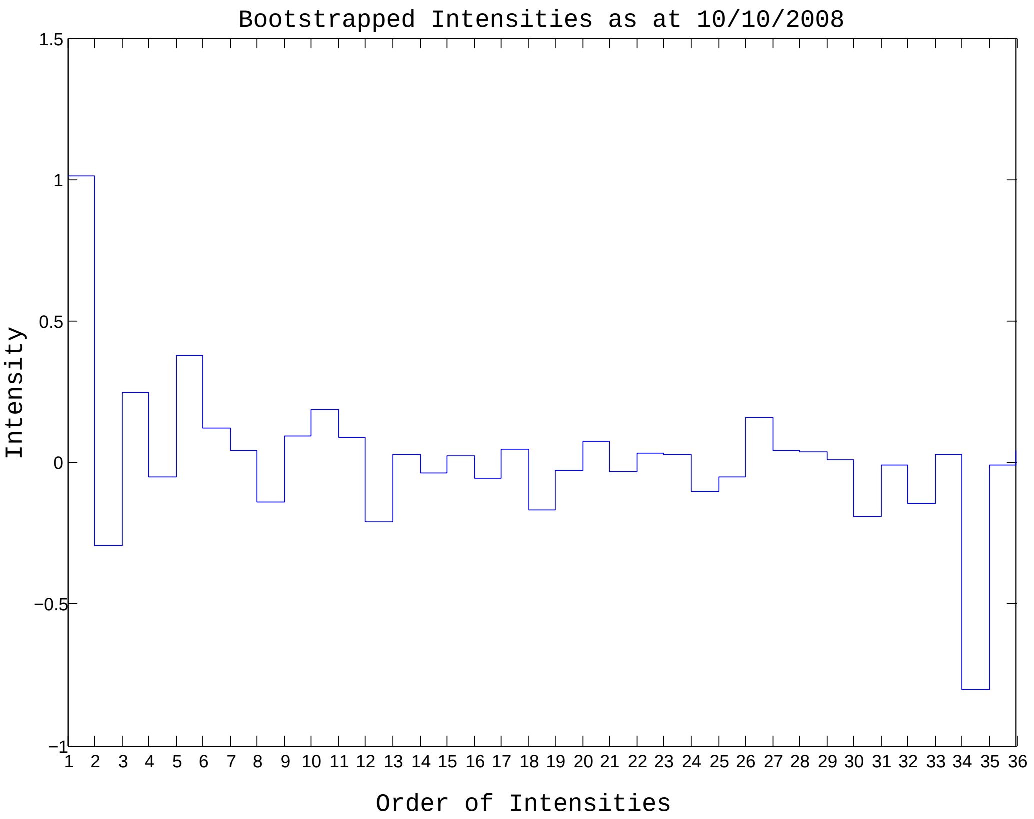

Figure 2: **Bootstrapped Piecewise Constant Intensities as at 10/10/2008.**

4.3 Analytical Analyses

To understand the cause of the problems shown in the bootstrap results, we perform an analytical analysis of the sample data. We conjecture that the oscillations and negative intensities result from the data. In the standard bootstrap of interest rate term structure with LIBORs, in general the shape of the curve is monotonic or humped, without oscillations. The positivity of the curve is also almost always guaranteed. Compared to interest rate swaps, tenor swaps, especially those of longer maturities, are only recently quoted. Table 5 summarises the starting year of interest rate swaps (IRS) and tenor swaps of various maturities,

Table 5: Starting Year of Swap Quotes from Bloomberg

Maturity	1-yr	5-yr	10-yr	15-yr	20-yr	25-yr	30-yr
IRS	1996	1988	1988	1994	1994	1999	1994
1M v 3M TS	1997	1997	1997	2008	2007	2008	2007
3M v 6M TS	1997	1997	1997	2008	2008	2008	2008

We find from Table 5 that tenor swaps beyond 10Y maturity have been quoted for a much shorter period of time, compared to corresponding interest rate swaps. In addition, since these instruments were introduced, financial markets have experienced turmoils such as the GFC and European sovereign–debt crisis. We hence suspect that the tenor swap market is much less mature and consistent than the interest rate swap market and the quotes may have caused the problems in the bootstrap results. To find out whether this is the case, we analyze respectively the shape of the term structure of the quoted spreads of 1M v 3M swaps and 3M v 6M swaps. To control for transaction costs, we also consider the bid–ask spread of the quotes. The analysis takes the following steps,

1) For each sample date, we extract the bid rate and the ask rate of the basis spread for each swap and list them in two rows. The ask rates are in the upper row and bid rates in the lower row. For each row, we order the rates in the ascending order of swap maturity. Hence a matrix S of 2 rows and 19 columns is formed for 1M v 3M swaps (17 columns for 3M v 6M swaps). Matrix elements are denoted as $S_{i,j}$, where i is the row index and j is the column index.

3) For 1M v 3M swaps, a matrix R of 2 rows and 19 columns (17 columns for 3M v 6M swaps) is used to record the results of the analysis. We initialise R by setting $R_{1,1} = S_{1,1}$ and $R_{2,1} = S_{2,1}$. For $i = 1$, we evolve the matrix R along the columns as follows,

If $R_{i,j} \geq S_{i,j+1} \geq S_{i+1,j+1}$, *then* $R_{i,j+1} = S_{i,j+1}$;

If $S_{i+1,j+1} < R_{i,j} < S_{i,j+1}$, *then* $R_{i,j+1} = R_{i,j}$;

Else, $R_{i,j+1} = S_{i+1,j+1}$.

For $i = 2$, the algorithm is,

If $R_{i,j} \geq S_{i-1,j+1} \geq S_{i,j+1}$, *then* $R_{i,j+1} = S_{i-1,j+1}$;

If $S_{i,j+1} < R_{i,j} < S_{i-1,j+1}$, *then* $R_{i,j+1} = R_{i,j}$;

Else, $R_{i,j+1} = S_{i,j+1}$.

The above algorithm is designed such that potential oscillations in the term structure of spreads are minimised. The rationale is that the actual transacted spread should always be bounded by the bid and the ask rate. Hence, in evolving the spread curve, we set the spread that spans a particular interval at the rate which minimises the change from the previous spread, with the constraint that the rate must be within the bid-ask bounds. As a result, for 1M v 3M swaps, we have two term structures of spreads on each sample date, one consists of $R_{1,1}, R_{1,2}, ..., R_{1,19}$ and the other consists of $R_{2,1}, R_{2,2}, ..., R_{2,19}$.

We then examine the shape of the spread curves resulting from the algorithm. If both term structures oscillate, we conclude that even after considering the bid-ask spread, the oscillations still persist on that sample date. By this criteria, we identify 90 days which shows oscillations for the 1M v 3M swaps and 68 days for the 3M v 6M swaps. After counting for overlapping days, there are 142 distinct days of oscillations. Table 6 shows a breakdown of these days by years.

Table 6: Days of Oscillations by Year

Year	2008	2009	2010	2011	2012	2013
Days	33	11	22	62	12	2

It is worth noting that the sample period is from September 22nd, 2008 to April 2nd, 2013. Hence for 2008 and 2013 we do not have a full year. However we can observe that a large number of oscillation days occurred in the last three months of 2008. The number of such days dropped significantly during 2009, but started to pick up in 2010 and intensified in 2011. Since 2012 such anomalies have stabilised and only occurred infrequently. Such an observation broadly corresponds to major financial market developments during the sample period. The late 2008 marked the peak the GFC. The European sovereign debt–crisis emerged in early 2010, intensified during 2011 and started to stabilise since mid–2012.

This finding lends support to our conjecture that during market turbulence, the quotes of tenor swaps are inconsistent and the shape of the spread curve is not well–behaved. This could potentially be explained by the observation that tenor swap market is less mature and developed, a problem that could be amplified during stressed market conditions.

The other analysis we perform on the sample data is related to the negative quotes. Because we propose that the transacted spread should be bounded by the bid and the ask rate, if the ask rate is negative, a negative transacted spread is necessarily implied. In principle, the arbitrage strategy in Table 1 can be reversed[14] to exploit the negative basis spread attached to the shorter tenor LIBOR. However, based on the model setup, a negative spread would imply a negative intensity or a negative loss rate or both. Hence model assumptions are violated. There are 74 days in our sample on which negative ask rates are quoted. The distribution of these days are as follows,

Table 7: Days of Negative Spreads

Year	2008	2009	2010	2011	2012	2013
Days	39	29	6	0	0	0

From Table 7 we observe that the days with negative spreads are concentrated around the peak of the GFC. We surmise that inconsistent and less meaningful quotes may have resulted from large volatility and uncertainty associated with the market stresses. Taking into account overlapping days, we identify 192 distinct days with oscillations and/or negative spreads. In our subsequent analysis, we decide to exclude these days from the sample because these data lack meaningful behaviour and/or conflict with the proposed model, and we justify this choice by the above argument that these days represent anomalies due to an immature market, which seem to be disappearing as the tenor swap market matures. As a result, the final sample period includes 787 trading days.

4.4 Global Optimisation

Recognizing the problems with the bootstrap results and data issues, we instead calibrate model parameters with global optimisation. By optimisation, structures and constraints can be imposed to avoid oscillations and negative intensities. To achieve best fits, both intensities and loss rates are made time–dependent and piecewise constant.

[14]By reversing the strategy, the arbitrageur should borrow at 6M LIBOR and lend at 3M LIBOR. In the tenor swap, the arbitrageur receives 6M LIBOR and pays the 3M LIBOR plus the negative spread.

The global optimisation is implemented as follows. We denote $PV_{GainBid}$ and $PV_{GainAsk}$ respectively as the PV of the gains based on the bid rate and the ask rate of the tenor swaps spread. To fully take account of transaction costs in the arbitrage strategy proposed in Table 1, we subtract the LIBOR–LIBID[15] spread (assumed to be fixed at 12.5 bps, see Coyle (2001)) from the bid rate of basis spread to calculate $PV_{GainBid}$. This is appropriate because when the arbitrageur lends funds, the deposit rate is the LIBID rate. Therefore the lower bound of the arbitrage profit is $PV_{GainBid}$. We then minimise the loss function G:

$$G = \sum_{i=1}^{N} [\max(PV^i_{Loss} - PV^i_{GainAsk}, 0) + \max(PV^i_{GainBid} - PV^i_{Loss}, 0)]^2, \tag{34}$$

where $N = 36$ is the number of swaps used in the global optimisation. The loss function is chosen such that for a given set of parameters π_i and λ_i, the optimisation error is zero if the following condition is satisfied,

$$PV^i_{GainBid} \leq PV^i_{Loss} \leq PV^i_{GainAsk}. \tag{35}$$

In (35) we set $PV^i_{GainBid}$ as the lower bound and $PV^i_{GainAsk}$ as the upper bound for PV^i_{Loss}. If PV^i_{Loss} based upon the calibrated parameters falls within the bounds, we assume that the PV_{Gain} based on the actual transacted rate is matched and the error is set to zero. Therefore we only have positive error terms if PV^i_{Loss} is below the lower bound or above the upper bound.

In order to obtain sensible fits and avoid severe oscillations, we also impose a measure of smoothness on the optimisation. We use the following smooth measure to penalise large oscillations of the intensities,

$$\begin{aligned} Smooth_\lambda &= \sum_{i=1}^{N-2} [(\lambda_{i+2} - \lambda_{i+1}) - (\lambda_{i+1} - \lambda_i)]^2 \\ &= \sum_{i=1}^{N-2} (\lambda_{i+2} + \lambda_i - 2\lambda_{i+1})^2. \end{aligned} \tag{36}$$

We therefore minimise the objective function, i.e. the weighted sum of the loss function and the smoothness measure:

$$Scale_{loss} \cdot G + Scale_{smooth} \cdot \ Smooth_\lambda, \tag{37}$$

[15]London Interbank Bid Rate is a bid rate at which a bank is willing to borrow from other banks, while LIBOR is the ask rate.

where $Scale_{loss}$ is the damping factor on the loss function and $Scale_{smooth}$ is the damping factor on the smoothness measure. The damping factors can be adjusted, depending on the main objective of the optimisation. If the dominating objective is to minimise the fitting errors, $Scale_{loss}$ should be assigned a higher weight than $Scale_{\lambda}$. On the other hand, if the main objective is to have a smooth term structure of intensities without large fluctuations, $Scale_{\lambda}$ should be assigned a higher weight than $Scale_{loss}$.

To implement the global optimisation scheme, we set initial conditions for intensities as

$$\lambda_i = e^{(-0.1T_i)} \cdot \frac{Spread_1}{100}, \qquad \forall i \in [1, 2, ..., N], \tag{38}$$

and apply the constraint

$$0.00001 < \lambda_i < 0.99999, \qquad \forall i \in [1, 2, ..., N]. \tag{39}$$

The time-decay function in Eqn. (38) is chosen because based upon the analysis of the sample data, the shape of the spread curve is monotonically decreasing on most of the trading days. T_i stands for the maturity of each bootstrap interval end. The weight factor of the decay function $\frac{Spread_1}{100}$ is used to assign different sets of initial intensities for each sample date, based upon the spread level of 1M v 3M tenor swaps with 3M matutiry on that day. Eqn. (38) is used to both avoid oscillations and ensure smoothness of the calibrated intensities. The constraint in Eqn. (39) is imposed to guarantee positivity of intensities, as well as prevent unusually high values.

We also set initial conditions for the loss rates. As we have no view on the shape of the loss rates, a constant is chosen as initial inputs for the optimisation:

$$\pi_i = 0.01, \qquad \forall i \in [1, 2, ..., N], \tag{40}$$

Similarly, positive bounds are imposed:

$$0.0001 < \pi_i < 0.1, \qquad \forall i \in [1, 2, ..., N]. \tag{41}$$

4.5 Optimisation Results

We have three key results from the proposed global optimisation scheme, in relation to the fitting errors, intensities and loss rates.

1) For all sample dates the fitting error is zero, i.e. the loss function G in Eqn. (34) is zero and the condition in (35) is satisfied. The perfect fits are not surprising given that we make both intensities and loss rates deterministic, step–wise constant functions of time, hence 72 degrees of freedom. The bounds we set for the intensities and the loss rates are satisfied for all sample days.

2) The term structure of the calibrated intensities displays well–behaved shape on most of the sample days. Table 8 summarises the various shapes of the intensity curves, where Inc & Hump (resp. Dec & Hump) refers to the humped shape that firstly increases (resp. decreases) then decreases (resp. increases). Given the initial condition we set for intensities in Eqn. (38), it is expected that most intensity curves monotonically decrease. Humped shapes, though not many, are also produced by the optimisation scheme. While monotonic and humped shapes are common in observed term structures, oscillations are not desirable. However, for all these 15 days, the curve only oscillates once. On the other hand, the bootstrapped intensity curve in Figure 2 oscillates repeatedly.

Table 8: Shapes of Intensity Curve

Shape	Decrease	Inc & Hump	Dec & Hump	Oscillate
Days	734	26	12	15

3) The loss rate curve oscillates on 522 sample days.

Figures 3, 4 and 5 are used to illustrate the optimisation results. We show respectively the fits, intensity curve and loss rate curve produced by global optimisation on Oct 10th, 2008. In Figure 3 we observe that for all tenor swaps included in the sample, the PV of loss based on the calibrated intensities and loss rates is bounded by the PV of gains based on the bid rate of the spread (lower bound) and the ask rate of the spread (upper bound). Hence the fitting error is zero. In Figure 4 we see the calibrated piecewise constant intensities are monotonically decreasing. However, the loss rate curve in Figure 5 repeatedly oscillates.

5. Parsimonious Modelling of Intensity and Loss Rate

5.1 Introduction

Our optimisation results in Subsection 4.5 show symptoms of overparameterisation. Firstly, the perfect fits are expected because both intensities and loss rates

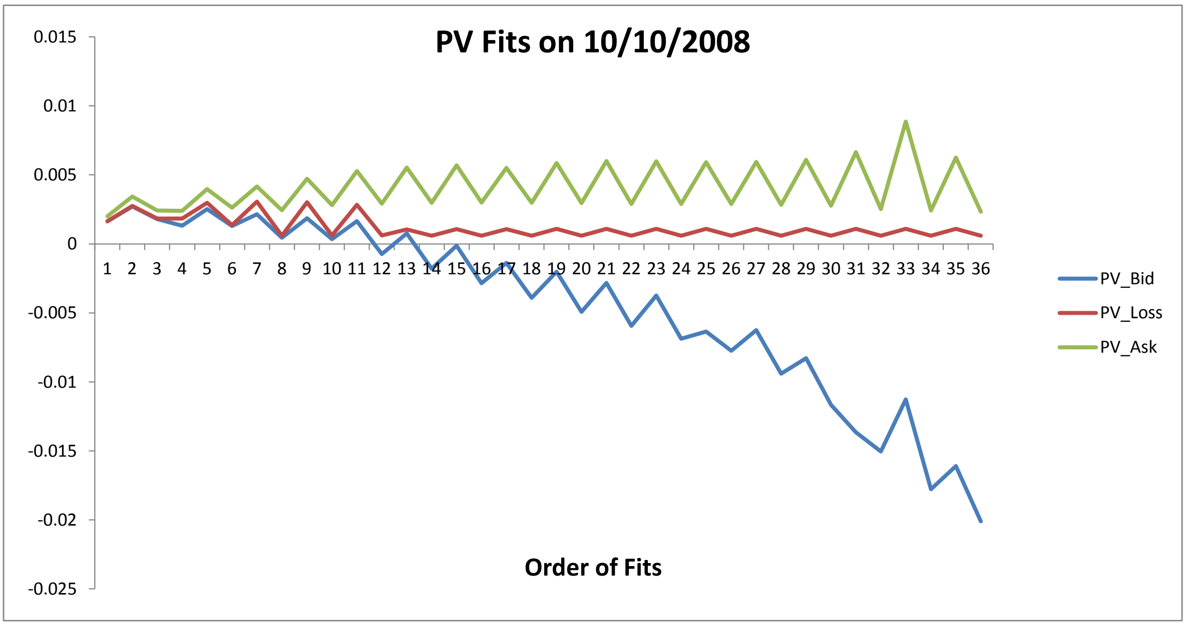

Figure 3: **Global optimisation fits as at 10/10/2008.**

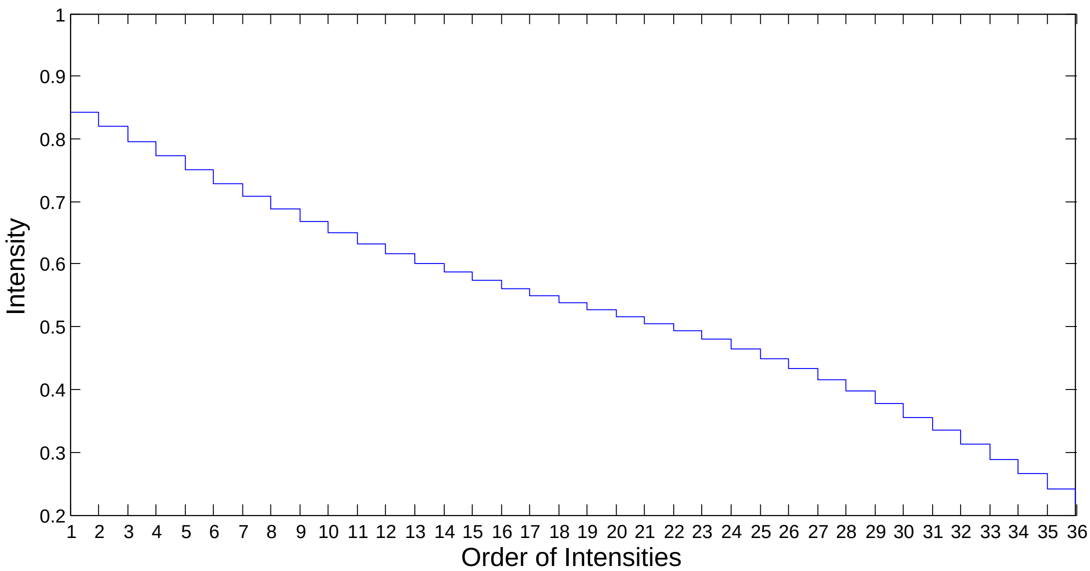

Figure 4: **Global optimisation Intensity Curve as at 10/10/2008.**

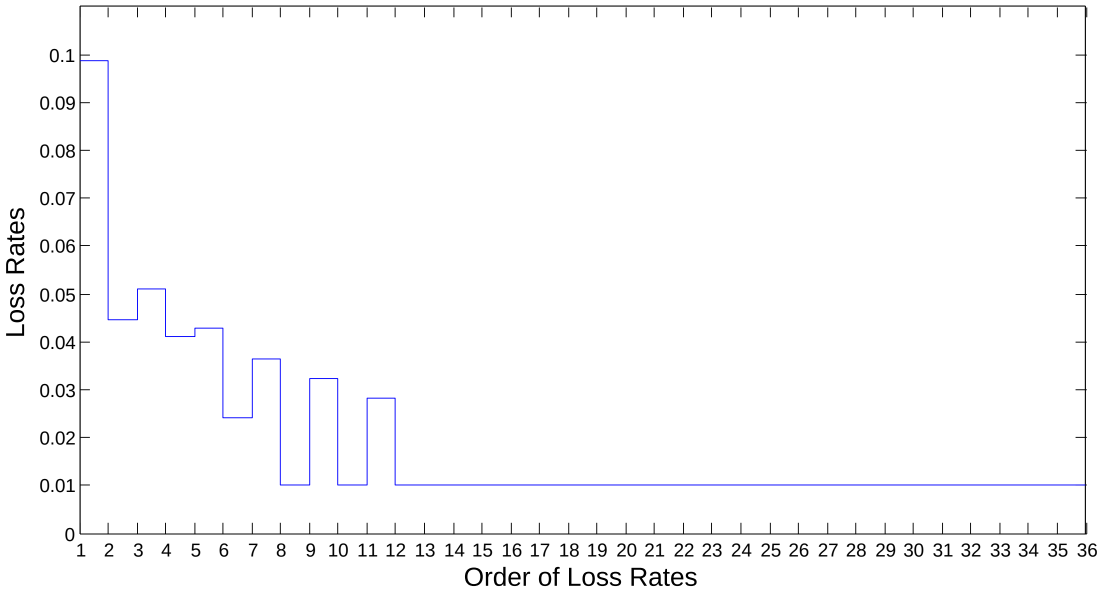

Figure 5: **Global optimisation Loss Rate Curve as at 10/10/2008.**

are made piecewise constant, deterministic functions of time, resulting in 72 degrees of freedom. Secondly, though we impose the initial condition in Eqn. (38) which decays exponentially, humped shapes and oscillations are generated by the optimisation. We therefore suspect that the monotonically decreasing initial condition is too restrictive and not flexible enough to capture a richer dynamic of the intensities. Lastly, the loss rate curves show a lack of structure through repeated oscillations.

A model with a large number of parameters, such as the one used in our optimisation, is able to perfectly fit the observed data. However, it is less likely to explain well than a parsimonious model which assumes more smoothness. Furthermore, the fitting errors of the parsimonious model may represent an opportunity to study the systematic and idiosyncratic features of the data that the model fails to capture (see, for example, Nelson and Siegel (1987)).

We are thus motivated to propose a parametrically parsimonious model for both intensities and loss rates, which allows us to capture a family of curve shapes. Nelson and Siegel (1987) proposed a model to fit the term structure of interest rates. The Nelson–Siegel model is consistent with a level–slope–curvature factor interpretation of the term structure (see e.g. Litterman and Scheinkman (1991)) and widely used in academia and practice. Nelson and Siegel (1987) models the instantaneous forward rate $f(\tau)$ as

$$f(\tau) = \beta_0 + \beta_1 e^{(-\tau/s)} + \beta_2 (\tau/s) e^{(-\tau/s)}, \tag{42}$$

where τ is time to maturity and β_0, β_1, β_2 and s are constants to be estimated.

The model hence consists of three factors: the constant β_0 represents the long term interest level, the exponential decay function $\beta_1 e^{(-\tau/s)}$ and a Laguerre function in the form of xe^{-x}. The role of the factors can be seen by examining the limiting behavior of time to maturity. If we let $\tau \to \infty$, the second and the third factor vanish and the long-term forward rate converges to β_0. As $\tau \to 0$, the Laguerre function vanishes and the forward rate converges to $\beta_0 + \beta_1$. Hence $-\beta_1$ measures the slope of the yield curve, where a positive (negative) β_1 represents a downward (upward) slope. Lastly, the Laguerre function represents the curvature of the yield curve and the shape parameter $s > 0$ determines the rate at which the slope and the curvature decay to zero. The location of the maximum (minimum) value of the curvature is determined by the value of s. Small (large) values of s correspond to rapid (slow) decay and therefore suitable for fitting curvatures at low (longer) maturities.

By pre–specifying a grid of shape parameters, Nelson and Siegel (1987) transformed the non–linear model in Eqn. (42) to a linear model and performed ordinary least squares (OLS) regressions for 37 data sets. The regression results explained a large fraction of the variations in the yields of treasury bills, with a median R^2 of 96%. Although the best–fitting shape parameter s varies for different data sets, by fixing s at its median value for all data sets only resulted in little loss of explanatory power. An important observation in Nelson and Siegel (1987) is that by plotting the time series of the estimated parameters, a breaking point October 1982 was identified, after which the importance of the curvature factor was evidently less. Since October 1982, both the magnitudes and variations of estimated β_1 and β_2 became much smaller and the yield curve shapes became simpler and more stable. The authors attributed such a structural break to the change of Federal Reserve monetary policy in October 1982, a switch from stabilising the monetary aggregates to stabilising interest rates. Nelson and Siegel argued that the market quotes may have become more accurate with more certainty in interest rates and resulted in simpler, lower–order yield curves. Such a structural break effect is particularly relevant with our study. It would be interesting to see if the intensity and loss rate curves become more stable during more recent periods than during the crises.

Therefore, in this section we employ a Nelson–Siegel type model, which allows us to parsimoniously describe intensities and loss rates of the liquidity shock. To account for randomness, we also propose a preliminary stochastic model for these two parameters.

5.2 Model Set-up

We propose a Nelson–Siegel type model for both the intensity $\lambda(t)$ and the instantaneous loss rates $\pi(t)$:

$$\lambda(\tau) = \lambda_0 + \lambda_1 e^{(-\tau/S_1)} + \lambda_2(\tau/S_1)e^{(-\tau/S_1)}, \tag{43}$$

$$\pi(\tau) = \pi_0 + \pi_1 e^{(-\tau/S_2)} + \pi_2(\tau/S_2)e^{(-\tau/S_2)}, \tag{44}$$

where τ is time to maturity. λ_0, λ_1, λ_2 (π_0, π_1, π_2) are respectively coefficient for the level, slope and curvature factor of the intensity (loss rate) and S_1 (S_2) is the shape parameter for the intensity (loss rate). Hence for each sample trading day there are 8 parameters to estimate, a drastic reduction compared to the global optimisation. In general researchers fix the shape parameter and estimate the linearised version of the Nelson–Siegel model parameters. However, the linear regression method has been reported to behave erratically over time and have large variances. Annaert et al. (2012) showed that these problems result from multi–collinearity. Alternatively, nonlinear optimisation techniques can be used to estimate model parameters. The drawback of this approach is that the estimators are sensitive to the initial values used in the optimisation (see Cairns and Pritchard (2001)). In the global optimisation scheme of Section 4 we deliberately account for transaction costs and fit the PV of losses with respect to two bounds; and in the present context we will also follow the optimisation approach and the associated conditions of Eqn. (34) and (35).

We integrate $\lambda(\tau)$ and $\pi(\tau)$ specified by Eqn. (43) and (44). The "fair value" condition in Eqn. (23) then becomes

$$\sum_{k=1}^{K}(e^{A}-1)(e^{-B}-e^{-C})D^{OIS}(T_0,T_{\eta(k)}) = \sum_{k=1}^{K}(B_{i,j,N}\cdot i)D^{OIS}(T_0,T_k), \tag{45}$$

where A, B and C are respectively:

$$\begin{aligned} A &= \int_{T_k}^{T_{\eta(k)}} \pi(u)\,\mathrm{d}u \\ &= (T_{\eta(k)} - T_k)\pi_0 - \pi_1 S_2\left(e^{(-T_{\eta(k)}/S_2)} - e^{(-T_k/S_2)}\right) \\ &\quad - \pi_2\left(e^{(-T_{\eta(k)}/S_2)}(T_{\eta(k)} + S_2) - e^{(-T_k/S_2)}(T_k + S_2)\right), \end{aligned} \tag{46}$$

$$\begin{aligned} B &= \int_{0}^{T_{k-1}} \lambda(u)\,\mathrm{d}u \\ &= T_{k-1}\lambda_0 + (\lambda_1+\lambda_2)S_1(1-e^{(-T_{k-1}/S_1)}) - \lambda_2 T_{k-1}e^{(-T_{k-1}/S_1)}, \end{aligned} \tag{47}$$

$$\begin{aligned} C &= \int_{0}^{T_k} \lambda(u)\,\mathrm{d}u \\ &= T_k\lambda_0 + (\lambda_1+\lambda_2)S_1(1-e^{(-T_k/S_1)}) - \lambda_2 T_k e^{(-T_k/S_1)}. \end{aligned} \tag{48}$$

Eqn. (45) is then used in the optimisation to calibrate the parameters for $\lambda(\tau)$ and $\pi(\tau)$.

5.3 Optimisation Scheme

We establish the constraints and initial values for the optimisation, which is used to calibrate parameters of λ and π.

5.3.1 Optimisation Constraints

We establish parameter constraints for the nonlinear optimisation scheme. For the intensity parameters, because λ_0 is the long–term level of intensity, we require that

$$0.00001 \leq \lambda_0 \leq 0.22872, \tag{49}$$

where the lower bound (resp. upper bound) is the minimum (resp. maximum) λ_0 calculated from the global optimisation results in Section 4. The shape parameter S_1 is bounded by the maturities of tenor swaps data and as such

$$0 \leq S_1 \leq 30. \tag{50}$$

The constraints for λ_1 and λ_2 are derived from the positivity of the model in Eqn. (43). Take the first derivative of Eqn. (43) with respect to τ we have

$$\lambda_\tau = \frac{-\lambda_1}{S_1} e^{(-\tau/S_1)} - \frac{\lambda_2}{S_1}(\tau/S_1)e^{(-\tau/S_1)} + \frac{\lambda_2}{S_1} e^{(\tau/S_1)}. \tag{51}$$

Letting (51) equal to 0, we obtain

$$\tau = \left(\frac{\lambda_2 - \lambda_1}{\lambda_2}\right) S_1. \tag{52}$$

The second derivative of (51) with respect to τ is

$$\lambda_{\tau\tau} = \frac{1}{S_1^2}\left(\lambda_1 - 2\lambda_2 + \frac{\lambda_2 \tau}{S_1}\right) e^{(-\tau/S_1)}. \tag{53}$$

Substitute (52) into (53), the second derivative becomes

$$\lambda_{\tau\tau} = -\frac{\lambda_2}{S_1^2} e^{\left(\frac{\lambda_1 - \lambda_2}{\lambda_2}\right)}. \tag{54}$$

Therefore the second derivative is positive if $\lambda_2 < 0$. It follows that the function $\lambda(\tau)$ has the local minimum at $\tau = (\frac{\lambda_2 - \lambda_1}{\lambda_2})S_1$ and the function value is

$$\lambda(\tau) = \lambda_0 + \lambda_2 e^{\left(\frac{\lambda_1 - \lambda_2}{\lambda_2}\right)}. \tag{55}$$

To ensure the positivity of the minimum, we examine the location of critical value of τ in (52). If $\tau = (\frac{\lambda_2 - \lambda_1}{\lambda_2})S_1 < 0$, because $\lambda_2 < 0$, we must have $\lambda_2 > \lambda_1$. Then all is required is that $\lambda(0) = \lambda_0 + \lambda_1 > 0$, which ensures that $\lambda(\tau) > 0$ for all positive maturities. Therefore, in this case the constraints are

$$\lambda_2 < 0, \quad \lambda_2 > \lambda_1, \quad \lambda_0 + \lambda_1 > 0. \tag{56}$$

On the other hand, if $\tau = (\frac{\lambda_2 - \lambda_1}{\lambda_2})S_1 \geq 0$, then $\lambda_1 \geq \lambda_2$. We require that $\lambda_0 + \lambda_2 e^{\left(\frac{\lambda_1 - \lambda_2}{\lambda_2}\right)} > 0$, which leads to $\lambda_2 > \frac{-\lambda_0}{e^{\left(\frac{\lambda_1 - \lambda_2}{\lambda_2}\right)}}$. Therefore the constraint for λ_2 is

$$\lambda_2 < \lambda_1, \quad \frac{-\lambda_0}{e^{\left(\frac{\lambda_1 - \lambda_2}{\lambda_2}\right)}} < \lambda_2 < 0. \tag{57}$$

In a parallel fashion, the constraints of the loss rate parameters are set as follows:

$$0.0001 \leq \pi_0 \leq 0.02, \tag{58}$$

and

$$0 \leq S_2 \leq 30, \tag{59}$$

The constraints for π_1 and π_2 are

$$\pi_2 < 0, \quad \pi_2 > \pi_1, \quad \pi_0 + \pi_1 > 0, \tag{60}$$

or

$$\pi_2 < \pi_1, \quad \frac{-\pi_0}{e^{\left(\frac{\pi_1 - \pi_2}{\pi_2}\right)}} < \pi_2 < 0. \tag{61}$$

5.3.2 Initial Values

To promote stability and smoothness in the estimated parameters, we use the estimated parameters on one day as the initial values for the next day. For the first sample date, 03/03/2009, the initial values are chosen and listed in Table 9.

Table 9: Initial Values for Optimisation as at 03/03/2009

Parameter	λ_0	λ_1	λ_2	S_1	π_0	π_1	π_2	S_2
Value	0.1789	0.6406	-0.35	2	0.01	0.0702	-0.06	2

Based upon the global optimisation results in Section 4, we fix he value of λ_0 by the 30Y intensity. We then approximate $\lambda(0) = \lambda_0 + \lambda_1$ by the 2M intensity. Taking the difference of $\lambda(0)$ and λ_0, we obtain the initial value of λ_1. Initial values of π_0 and π_1 are chosen with the same procedure. We set the shape parameters S_1 and S_2 to be 2, which means at such initial values the location of the hump or trough of the Laguerre function $\beta_2(\frac{\tau}{s})e^{(-\frac{\tau}{s})}$ is at $\tau = 2$. Finally, we search over a grid of values for λ_2 and π_2 and choose the set of values, which in conjunction with other initial values, produces the least optimisation error.

5.4 Results

The calibration results of the model parameters are presented in Figures 6 and 7. In Figure 6 we observe that, except for the initial sample period when the market was still experiencing turmoils, the intensity parameters, λ_0, λ_1, λ_2 and S_1, show little time variations. In Figure 7 we have similar findings for the loss rate parameters π_2 and S_2. On the other hand, π_0 and π_1 exhibit significant time variations. Different from the estimation of default risk, where the standard calibration to market instruments (e.g. CDS spreads) normally assumes a constant recovery rate and calibrates time–varying default intensities, our optimisation jointly calibrates loss rates and intensities. We therefore propose that the time variations of liquidity risk in our model are mainly captured by the loss rate parameters.

The calibrated model parameters fit 504 sample days perfectly, or a proportion of 64% of the whole sample period. It needs to be pointed out that, it is not the objective of the Nelson–Siegel type model to achieve perfect fits. Instead, we aim to identify the underlying structure of the model fitted to the data. Fitting errors may present an opportunity to examine the systematic and idiosyncratic features of the sample data. To this end, we study the distribution of the fitting errors across the sample period. Table 10 presents the distribution of the largest 20% fitting errors. The sample mean and standard deviation of fitting errors are summarised in Table 11.

Table 10: Distribution of the Largest 20% Fitting Errors

Year	2009	2010	2011	2012	2013
Days	114	19	14	8	3

Table 11: Sample Mean and Standard Deviations of Fitting Errors

Year	2009	2010	2011	2012	2013
mean	9.12E-08	3.24E-10	1.73E-09	3.90E-11	6.51E-11
standard deviation	3.44E-07	1.89E-09	9.81E-09	2.40E-10	1.23E-10

We see from Table 10 that the large fitting errors are highly concentrated around the early sample period. Table 11 shows that the fitting errors of the more recent sample period (2012 and 2013) are characterised by both lower level and volatility of the fitting errors, which lends support for our conjecture that the consistency of tenor swap market has improved since the crisis and liquidity risk priced into the market for basis swaps has gradually stabilised.

We also examine shapes of the intensity curve and the loss rate curve. As expected, both curves are well behaved without oscillations. On all sample days, the intensity curve firstly decreases then increases. The loss curves either monotonically increase or initially decrease then increase. Figures 8 illustrates possible shapes of these curves.

Lastly, in Figure 7 we find that the upper bound (2%) of π_0 imposed for the optimisation is binding and hit on 24 sample days. We therefore increase the upper bound (to 3% and 4% respectively) and re–optimise. The results are summarised in Figures 9 and 10 (for 3%) and Figures 11 and 12 (for 4%). We see that the upper bound is only hit on 5 days (for 3%) and 1 day (for 4%) and such days all fall at the very beginning the sample period. We conclude that the boundary hitting is due to the heightened market stress and there is no need to further increase the upper bound of the long–term loss rate. The distribution of the fitting errors and curve shapes of intensity and loss rate are virtually unchanged after increasing the upper bound for π_0.

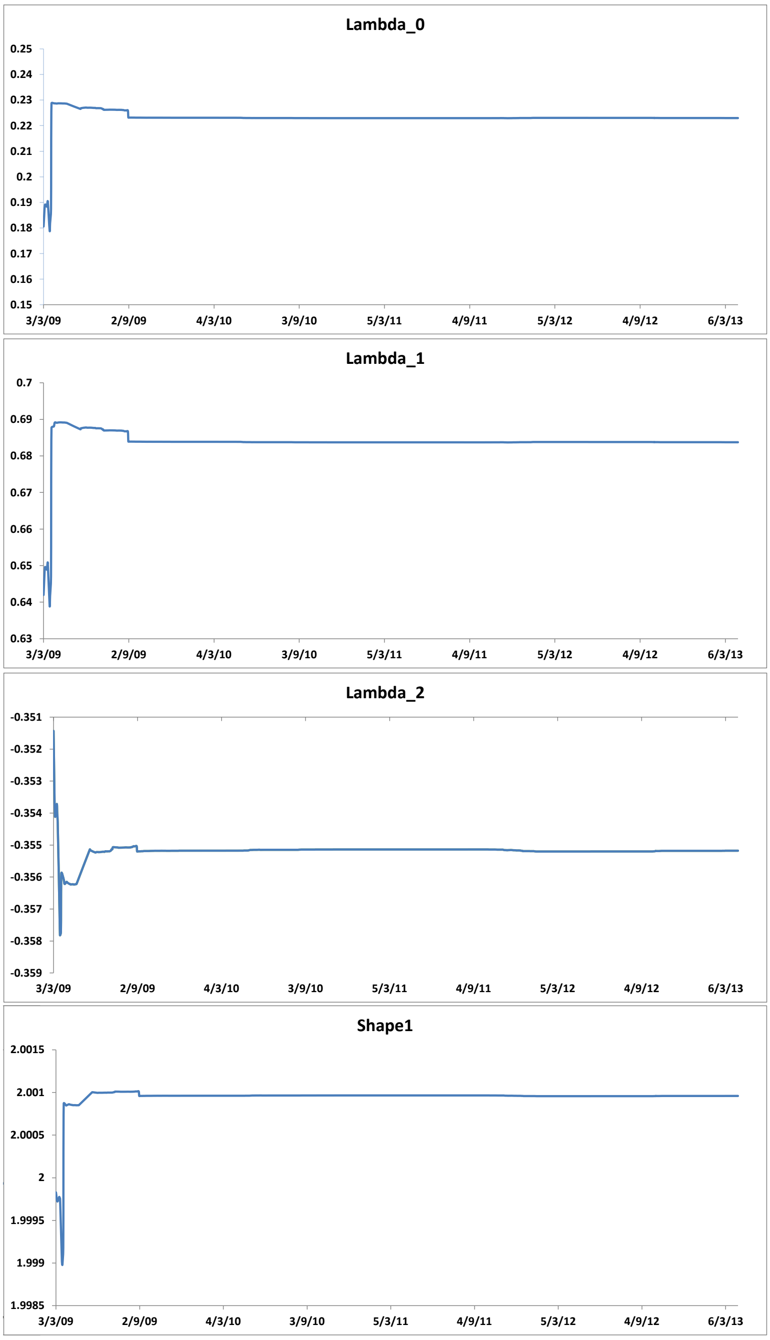

Figure 6: **Calibrated parameters of the Nelson-Siegel type model in Eqn. (43). Parameters include λ_0, λ_1, λ_2 and S_1. The upper bound of π_0 is 0.02.**

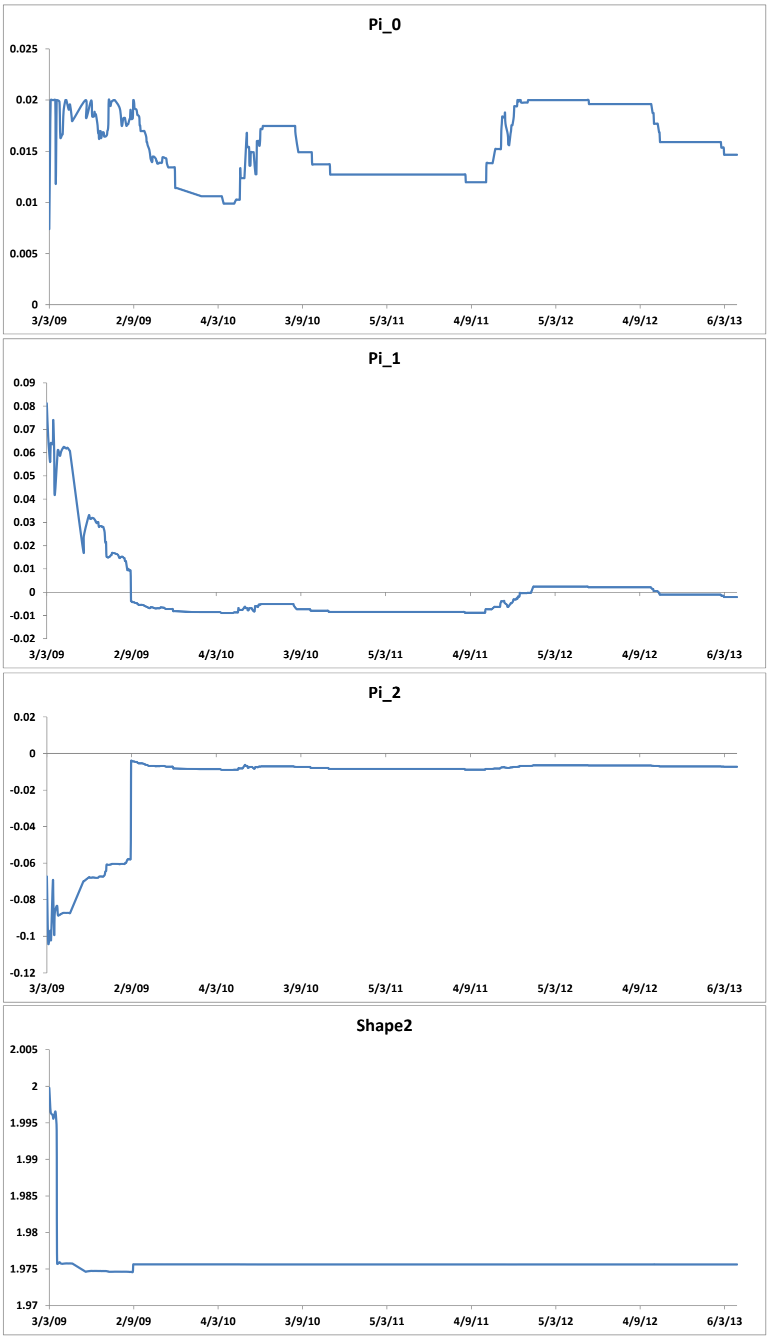

Figure 7: **Calibrated parameters of the Nelson-Siegel type model in Eqn. (44). Parameters include π_0, π_1, π_2 and S_2. The upper bound of π_0 is 0.02.**

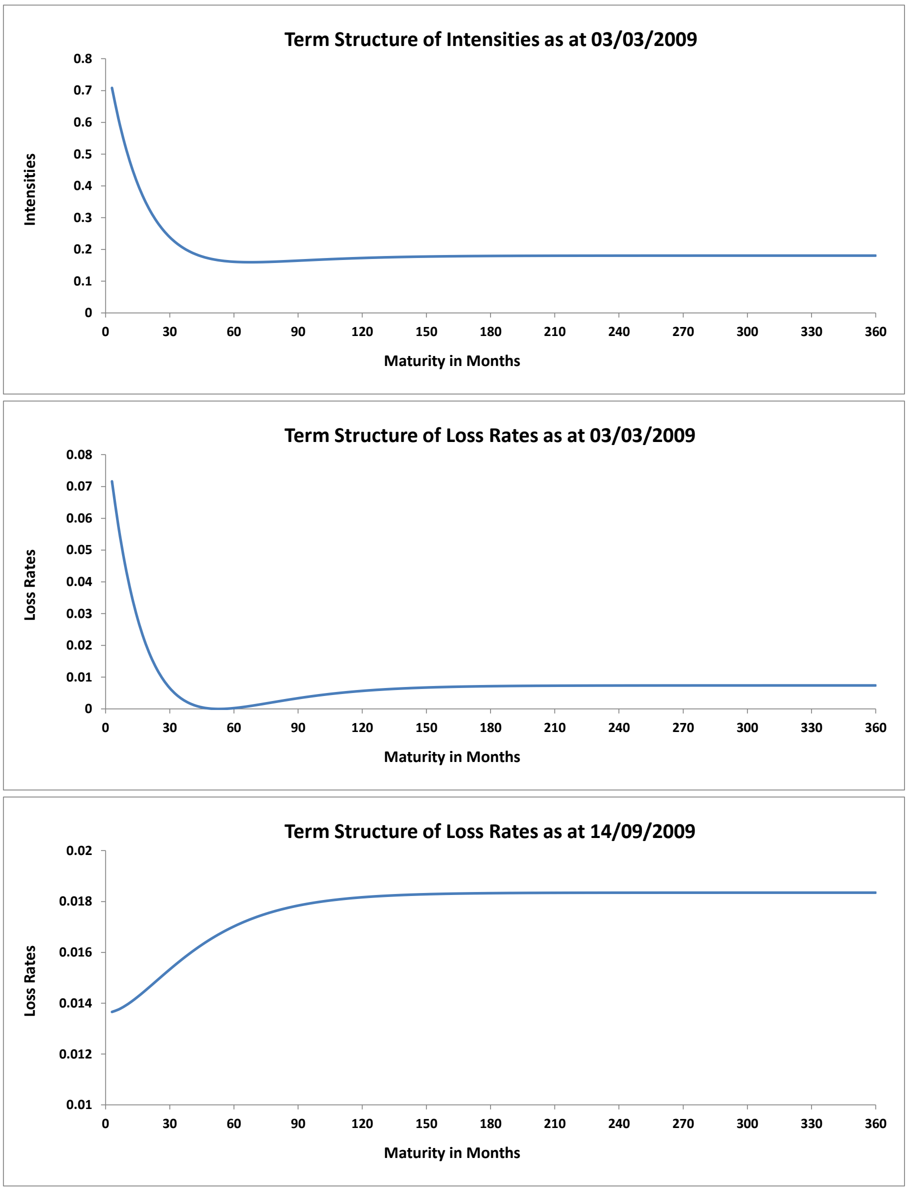

Figure 8: **Shapes of the calibrated intensity curve and loss rate curves.**

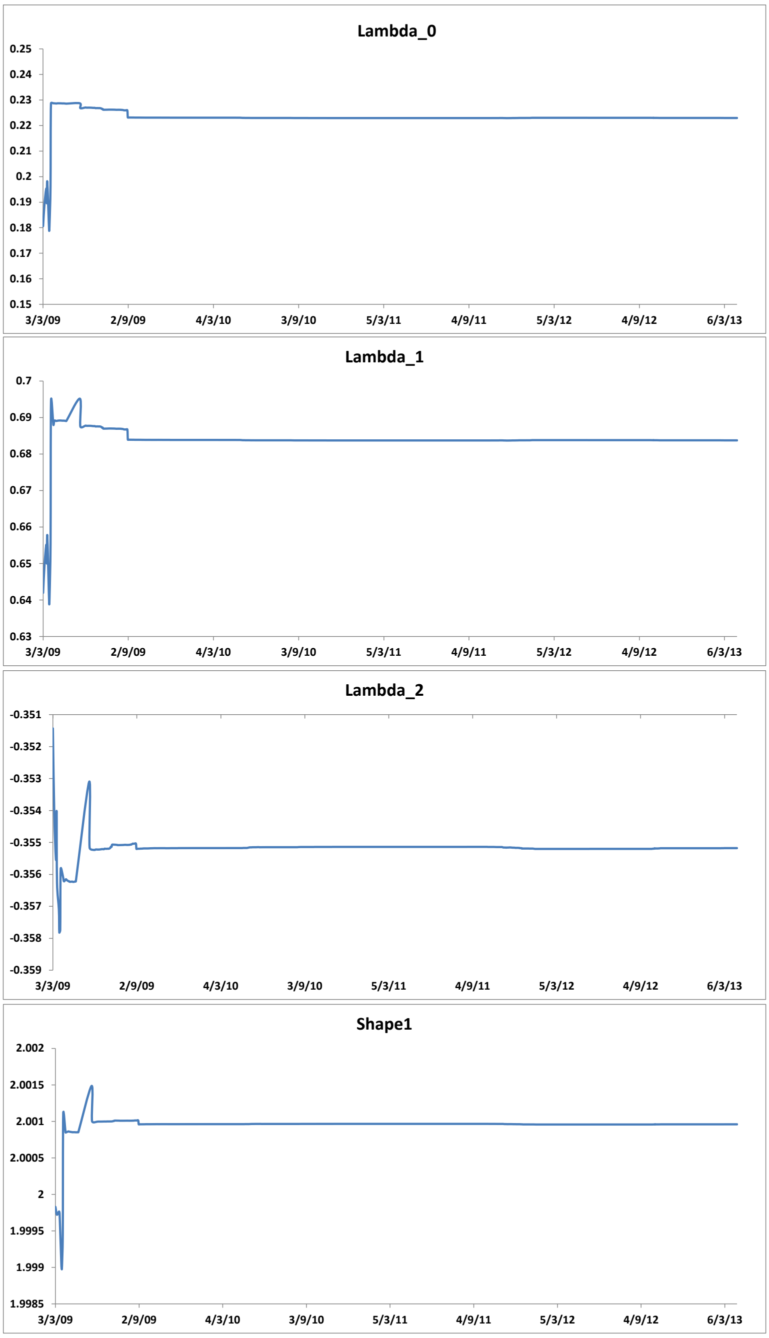

Figure 9: **Calibrated parameters of the Nelson-Siegel type model in Eqn. (43). Parameters include λ_0, λ_1, λ_2 and S_1. The upper bound of π_0 is 0.03.**

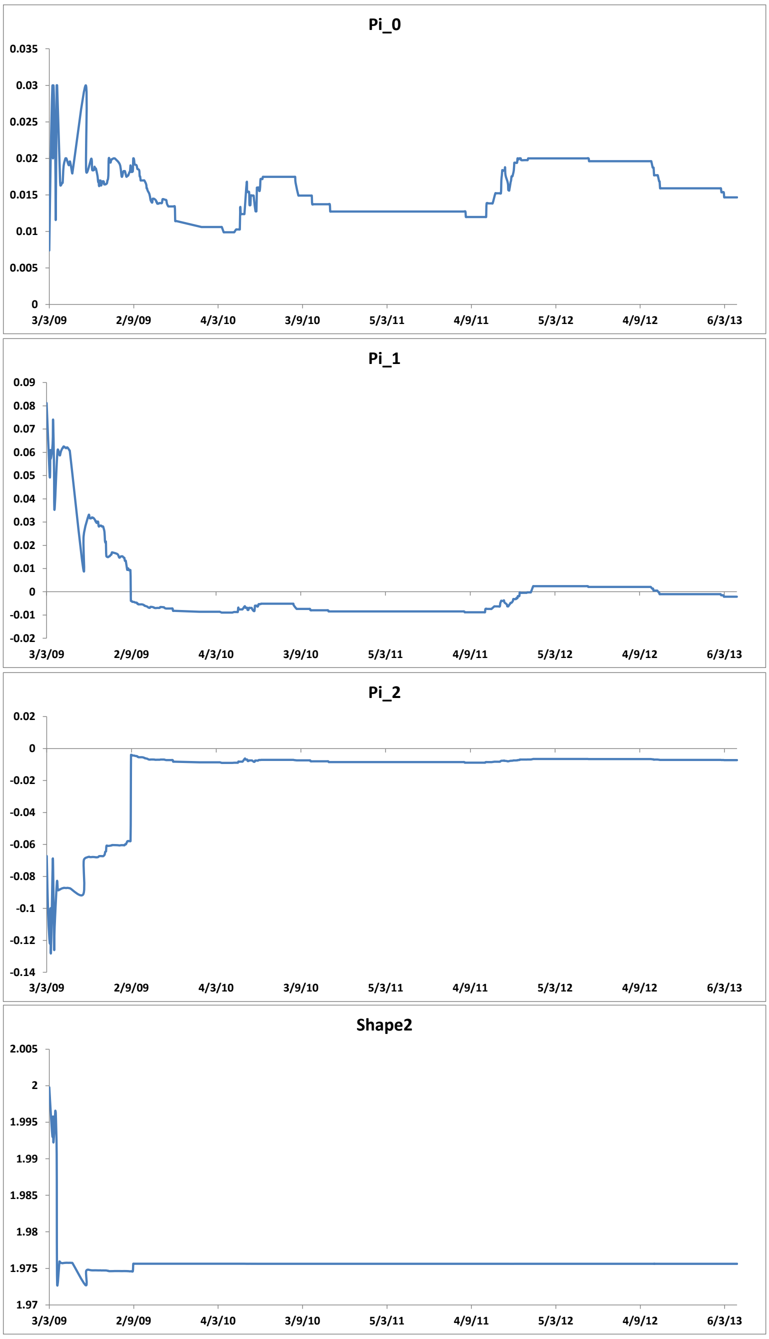

Figure 10: **Calibrated parameters of the Nelson-Siegel type model in Eqn. (44). Parameters include π_0, π_1, π_2 and S_2. The upper bound of π_0 is 0.03.**

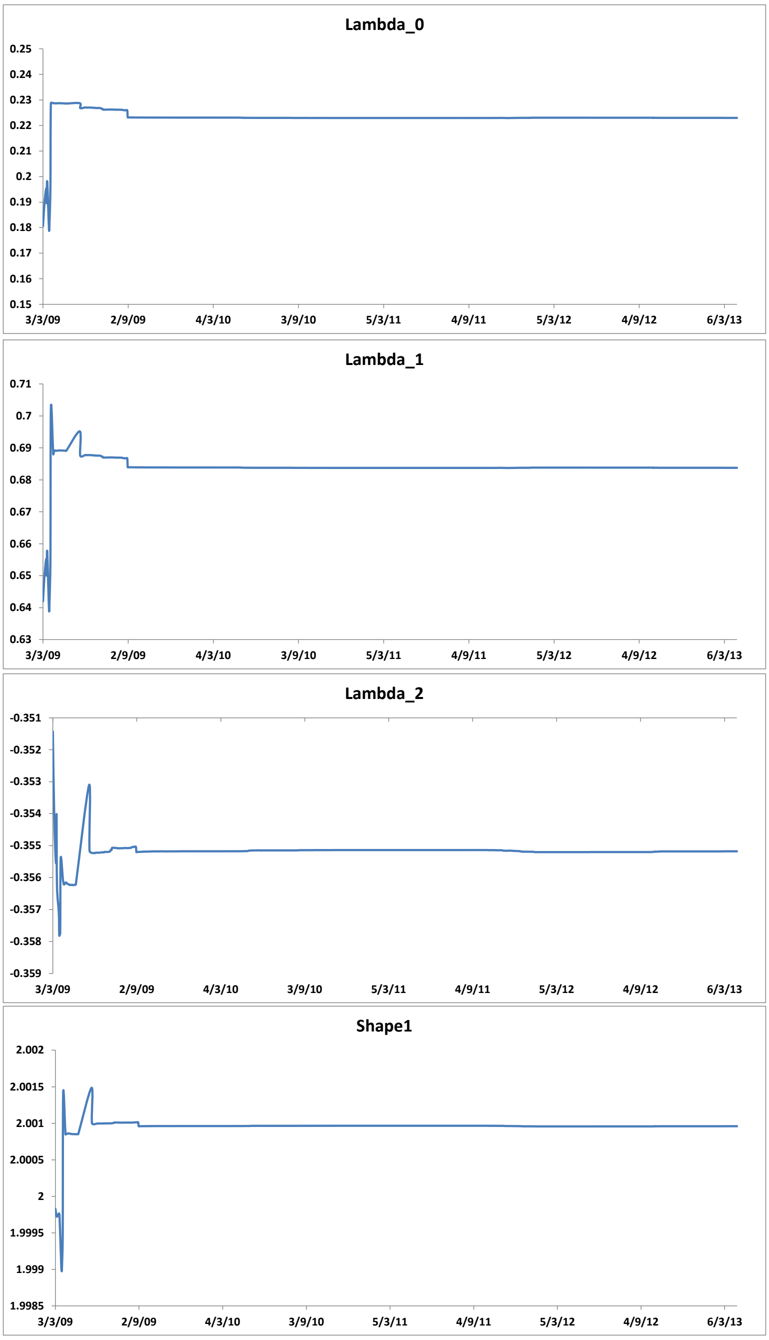

Figure 11: **Calibrated parameters of the Nelson-Siegel type model in Eqn. (43). Parameters include λ_0, λ_1, λ_2 and S_1. The upper bound of π_0 is 0.04.**

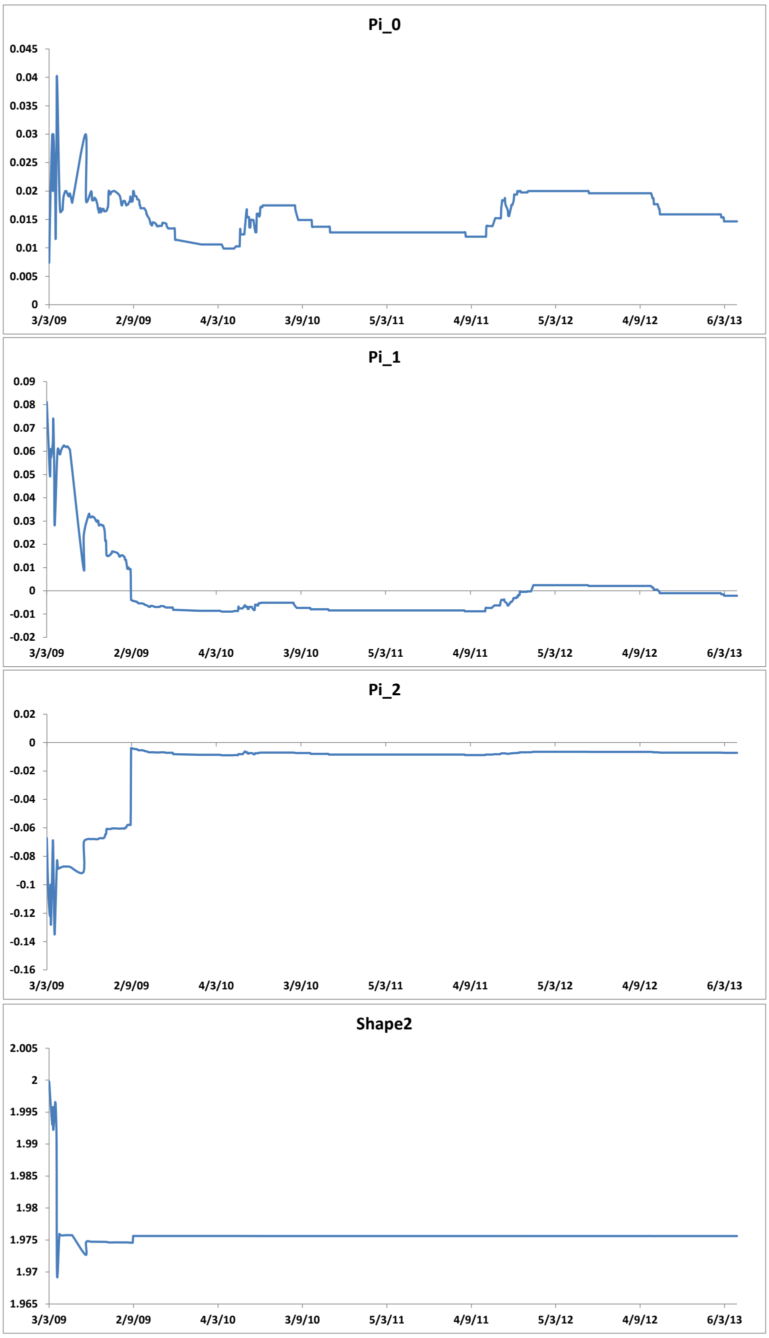

Figure 12: **Calibrated parameters of the Nelson-Siegel type model in Eqn. (44). Parameters include π_0, π_1, π_2 and S_2. The upper bound of π_0 is 0.04.**

5.5 A Stochastic Model

We have thus far modelled time–dependent, deterministic intensity and loss rate. To account for potential randomness, we set up a preliminary stochastic model for these two parameters.

As in Section 3, we assume that when a liquidity shock occurs, the arbitrageur is unable to roll over the shorter tenor loan and has to refinance until the end of the associated longer tenor, shutting down the borrowing and lending in the arbitrage strategy at the end of the longer tenor within which the first jump occurs. However suppose now that the liquidity shock is triggered by a Cox process (Cox 1955) with stochastic intensity λ. Assume that this process is independent of any interest rate dynamics and is given by a sum of d independent factors y_i, i.e.

$$\lambda(t) = \sum_{i=1}^{d} y_i(t), \tag{62}$$

where the y_i follows the Cox–Ingersoll–Ross (CIR) dynamics (Cox et al. 1985) under the pricing measure, i.e.

$$dy_i(t) = (\theta_i - a_i y_i(t))dt + \sigma_i \sqrt{y_i(t)} dW_i^{\lambda}(t), \tag{63}$$

where $dW_i^{\lambda}(t) \quad (i = 1, \cdots, d)$ are independent Wiener processes. The CIR–type model is chosen for its analytical tractability, as well as the guaranteed positivity of the modelled object, with the condition which ensures that the origin is inaccessible. In order to keep the model analytically tractable we do not allow for time–dependent coefficients at this stage. Since each of the factors follow independent CIR–type dynamics, the sufficient condition for each factor to remain positive is $2\theta_i > \sigma_i^2, \forall i$, as discussed for the one–factor case in Cox et al. (1985).

For stochastic intensity λ, the LHS of the "fair value" condition proposed in Eqn. (23) becomes

$$\sum_{k=1}^{K} \left(e^{\int_{T_k}^{T_{\eta(k)}} \pi(u)\,\mathrm{d}u} - 1 \right) \left(E\left[e^{-\int_0^{T_{k-1}} \lambda(u)\,\mathrm{d}u} \right] - E\left[e^{-\int_0^{T_k} \lambda(u)\,\mathrm{d}u} \right] \right) D^{OIS}(T_0, T_{\eta(k)}). \tag{64}$$

The expectations under the pricing measure can be evaluated in the same manner as the multi–factor CIR zero coupon bond price given in Chen and Scott (1995), i.e.

$$E\left[e^{-\int_t^T \lambda(u)\,\mathrm{d}u} \,\middle|\, \mathcal{F}_t \right] = A(t,T) \cdot e^{-B(t,T)y(t)}, \tag{65}$$

with

$$y(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_d(t) \end{pmatrix},$$

and $B(t,T)$ similarly a vector with components

$$B_i(t,T) = 2w_i \left(e^{c_i(T-t)} - 1\right). \tag{66}$$

Furthermore,

$$A(t,T) = \prod_{i=1}^{d} A_i(t,T), \tag{67}$$

with

$$A_i(t,T) = \left(2c_i w_i e^{\frac{1}{2}(c_i+a_i)(T-t)}\right)^{(2\theta_i/\sigma_i^2)}. \tag{68}$$

The coefficients c_i and w_i are given by

$$w_i = \left((c_i + a_i)e^{c_i(T-t)} + c_i - a_i\right)^{-1}, \tag{69}$$

$$c_i = \sqrt{a_i^2 + 2\sigma_i^2}. \tag{70}$$

Similarly, in addition we may assume stochastic dynamics for the refinancing loss rate as well by setting π as a sum of $\tilde{d}$ independent factors z_i:

$$\pi(t) = \sum_{i=1}^{\tilde{d}} z_i(t), \tag{71}$$

with the stochastic process independent of interest rates and λ and is given by

$$dz_i(t) = (\xi_i - b_i z_i(t))dt + \gamma_i \sqrt{z_i(t)} dW_i^{\pi}(t), \tag{72}$$

where $dW_i^{\pi}(t) \quad (i = 1, \cdots, \tilde{d})$ are independent Wiener processes. We interpret the stochastic π as the instantaneous spread representing the cost of refinancing after a liquidity shock, meaning that if a liquidity shock occurs at time τ between T_{k-1}

and T_k, the actual refinancing cost is represented by an implicit term structure of refinancing spreads. Denote this actual refinancing spread cost for the period from T_k to $T_{\eta(k)}$ by $\tilde{\pi}(\tau)$, which is a continuously compounded, per annum rate. Then we model this by,

$$e^{\tilde{\pi}(\tau)(T_{\eta(k)}-T_k)} = \frac{E\left[e^{-\int_\tau^{T_k}\pi(s)\,\mathrm{d}s}\,\middle|\,\mathcal{F}_\tau\right]}{E\left[e^{-\int_\tau^{T_{\eta(k)}}\pi(s)\,\mathrm{d}s}\,\middle|\,\mathcal{F}_\tau\right]}. \tag{73}$$

Thus quantity with a direct economic interpretation is $\tilde{\pi}(\tau)$, observable at the time τ of the liquidity shock, while the modelling of π as a multi–factor CIR–type process serves to endow the $\tilde{\pi}(\tau)$ with a tractable stochastic dynamic; i.e. the term structure of incremental refinancing costs a the τ is given by the "discount factors":

$$E\left[e^{-\int_\tau^{T}\pi(s)\,\mathrm{d}s}\,\middle|\,\mathcal{F}_\tau\right] = \tilde{A}(\tau,T)\cdot e^{-\tilde{B}(\tau,T)z(\tau)}, \tag{74}$$

with

$$z(\tau) = \begin{pmatrix} z_1(\tau) \\ \vdots \\ z_{\tilde{d}}(\tau) \end{pmatrix},$$

and $\tilde{B}(\tau,T)$ similarly a vector with components:

$$\tilde{B}(\tau,T) = 2\tilde{w}_i\left(e^{\tilde{c}_i(T-\tau)}-1\right). \tag{75}$$

Furthermore,

$$\tilde{A}(\tau,T) = \prod_{i=1}^{\tilde{d}} \tilde{A}_i(\tau,T), \tag{76}$$

with

$$\tilde{A}_i(\tau,T) = \left(2\tilde{c}_i\tilde{w}_i e^{\frac{1}{2}(\tilde{c}_i+b_i)(T-\tau)}\right)^{(2\xi_i/\gamma_i^2)}. \tag{77}$$

The coefficients $\tilde{c}_i$ and $\tilde{w}_i$ are given by,

$$\tilde{w}_i = \left((\tilde{c}_i+b_i)e^{\tilde{c}_i(T-\tau)}+\tilde{c}_i-b_i\right)^{-1}, \tag{78}$$

$$\tilde{c}_i = \sqrt{b_i^2 + 2\gamma_i^2}. \tag{79}$$

Under the present independence assumptions, it therefore remains to calculate

$$\begin{aligned} E\left[e^{\tilde{\pi}(\tau)(T_{\eta(k)}-T_k)}\right] &= \frac{\tilde{A}(\tau,T_k)}{\tilde{A}(\tau,T_{\eta(k)})} E\left[e^{(\tilde{B}(\tau,T_{\eta(k)})-\tilde{B}(\tau,T_k))z(\tau)}\right] \\ &= \frac{\tilde{A}(\tau,T_k)}{\tilde{A}(\tau,T_{\eta(k)})} \prod_{i=1}^{\tilde{d}} E\left[e^{(\tilde{B}_i(\tau,T_{\eta(k)})-\tilde{B}_i(\tau,T_k))z_i(\tau)}\right]. \end{aligned} \tag{80}$$

If $\tilde{B}_i(\tau,T_{\eta(k)}) - \tilde{B}_i(\tau,T_k) < \frac{1}{2}$, we can apply Lemma 2.5 in Schlögl and Schlögl (2000) to obtain

$$E\left[e^{(\tilde{B}_i(\tau,T_{\eta(k)})-\tilde{B}_i(\tau,T_k))z_i(\tau)}\right] = \frac{e^{(\zeta L/(1-2L))}}{(1-2L)^{\delta/2-1}}, \tag{81}$$

with

$$L = \tilde{B}_i(\tau,T_{\eta(k)}) - \tilde{B}_i(\tau,T_k), \tag{82}$$

$$\zeta = \frac{4b_i e^{-b_i(\tau-T_0)}}{\gamma_i^2(1-e^{-b_i(\tau-T_0)})} z_i(T_0), \tag{83}$$

and

$$\delta = \frac{4\xi_i}{\gamma_i^2}, \tag{84}$$

where δ and ζ are respectively the degrees of freedom and the non–centrality parameter for the non–central chi–squared distribution function $\chi^2(\cdot,\delta,\zeta)$. The density function of $\chi^2(\cdot,\delta,\zeta)$ has the following representation (see, for example, Johnson and Kotz (1970)):

$$p_{\chi^2(\delta,\zeta)}(x) = \frac{e^{(-\frac{1}{2}(\zeta+x))}}{2^{\delta/2}} \sum_{j=0}^{\infty} \frac{\zeta^j}{2^{2j} j! \Gamma(\frac{\delta}{2}+j)} x^{(\delta/2+j-1)}. \tag{85}$$

The density function of z is

$$p_z(x) = c p_{\chi^2(\delta,\zeta)}(cx), \tag{86}$$

where

$$c = \frac{4b_i}{\gamma_i^2 \big(1 - e^{-b_i(\tau - T_0)}\big)}. \tag{87}$$

If $\tilde{B}_i(\tau, T_{\eta(k)}) - \tilde{B}_i(\tau, T_k) \geq \frac{1}{2}$, the calculation of this expectation is less tractable and we resort to numerical integration techniques.

We therefore propose that the stochastic models of λ and π can be used to estimate the parameters in Eqn. (63) an (72), with observed spreads in the tenor swap market.

6. Conclusion

In this study we focus on the high–dimensional modelling problem existing in the single–currency tenor swap market. Based on recent empirical studies, we propose an intensity–based model to describe the arrival time of liquidity shocks in the interbank market. With the no–arbitrage argument and non–linear constrained optimisations, we calibrate the model parameters to quoted basis spreads in tenor swaps. Our model reduces the dimensionality of the problem down to two factors: the intensity and the loss rate characterising the driving liquidity risk. In contrast to the approach prevalent in the credit risk literature, the intensities and loss rates are calibrated simultaneously and results show that loss rates display more variations than intensities. Another advantage of our modelling approach, compared to the ad–hoc modelling approach typically adopted by practitioners, is that our model is motivated by the driving risk of market anomalies. It is hence more explanatory and consistent with market fundamentals. The results also demonstrate that since the turmoil of the GFC, the tenor swap market is in the process of maturing and stabilising.

In order to account for potential randomness, as a preliminary step, we also set up stochastic CIR–type models for the intensity and the loss rate. We show that under certain conditions closed form solutions exist, which can be used to tractably estimate the model parameters. The parameters of the stochastic model developed in this study can be estimated, with either closed form solutions or numerical techniques, in order to examine its ability to fit observed basis spreads — this is a topic of ongoing research.

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