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Abstract

This paper considers the American option pricing problem under regime-switching by using the method-of-lines (MOL) scheme. American option prices in each regime involve prices in all other regimes. We treat the prices from other regimes implicitly, thus guaranteeing consistency. Iterative procedures are required but very few iterative steps are needed in practice. Numerical tests demonstrate the robustness, accuracy and efficiency of the proposed numerical scheme. We compare our results with Buffington and Elliott (2002)'s analytical approximation under two regimes. Our MOL scheme provides improved results especially for out-of-the money options, possibly because they use a separation of variable approach to the PDEs which cannot hold around the early exercise region. We also compare our results with those of Khaliq and Liu (2009) and suggest that their implicit scheme can be improved.

1 Introduction

More than forty years since the seminal work of Black and Scholes (1973), their option pricing formula remains the key point of reference in derivatives markets. Nevertheless, it is now widely accepted that Black and Scholes' assumption of constant, or at least deterministic, volatility falls well short of adequately modelling the market. On the other hand, the high degree of tractability of the model clearly contributes to its undiminished popularity, and as option pricing models beyond Black/Scholes become more realistic,

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this tractability tends to be lost. Furthermore, this tractability applies to European options; however, options in practice very often are of the American type.

There is a recognition that volatility in reality evolves stochastically. A number of models have been developed, which treat volatility as a stochastic diffusion process (Ball and Roma (1994); Heston (1993); Stein and Stein (1991); Wiggins (1987), for example). An alternative approach is a regime–switching model, in which the key parameters of an asset depend on the market mode (or "regime") that randomly switches among a finite number of states. From an economic perspective, regime–switching behaviour captures the changing preferences and beliefs of investors concerning asset prices as the state of a financial market changes. A change of regime could be motivated in a number of ways: spontaneous changes in investors' degrees of optimism, shifts in economic policies, takeovers and so on. Since their introduction by Hamilton (1989), there has been a growing body of empirical evidence suggesting that the distributions of asset returns in some cases are better described by a regime–switching process (Bakshi et al. (1997); Bates (1996); Chernov et al. (1999, 2003); Eraker (2004), and Hamilton (1990)).

Various numerical methods have been proposed accordingly: Liu et al. (2006) develop a Fast Fourier Transform (FFT) method for regime–switching models and they propose a near–optimal FFT scheme to reduce computational complexity when the modulating Markov chain has a large state space. But they only study European options and it is difficult to apply the model to American options. Khaliq and Liu (2009) use a finite difference method, in which they consider an implicit approach by explicitly treating the linear terms from other regimes, resulting in computationally efficient algorithms. However, as we will see in Section 3.3.1, this scheme can be improved. Liu (2010) proposes a regime–switching recombining tree that grows only linearly as the number of time steps increases. This approach enables them to use a large number of time steps to obtain accurate approximations for both European and American option prices. Holmes et al. (2012) propose and implement a front–fixing finite element method for the free boundary American option problems, but they only study the two-regime case. Note that in most of the literature the Greeks and early–exercise boundaries are not reported, or not available for some schemes.

An American–style option allows the holder to exercise at any time up to and including the expiration date, and many traded options have this feature. Whilst there have been a lot of recent developments in pricing American options under stochastic volatility and jump–diffusion dynamics, much remains to be done when the dynamics of the underlying asset follow a regime–switching process. The general framework for American–style options under regime switching has been developed by Buffington and Elliott (2002). However, the coupled partial differential equation (PDE) system that describes the price remains to be fully explored and exploited under regime switching.

The Method–of–Lines (MOL henceforth) is a numerical scheme that has been successfully deployed to price American–style options with constant volatility, stochastic volatility and jumps (Chiarella et al., 2009; Meyer, 1998, 2015; Meyer and van der Hoek, 1997). A significant advantage of the MOL over lattice–based methods is that the option price, delta, gamma and optimal exercise boundary (OEB) are all computed in the solution process, whereas with lattice–based methods this is not the case (the calculation of the OEB can be particularly tedious with lattice–based methods). Furthermore, the MOL has been shown to offer greater computational efficiency over other numerical PDE methods for American–style options, such as the componentwise splitting method and the Crank–Nicolson scheme with projected successive over–relaxation (see Chiarella et al. (2009)).

In this paper, we implement an MOL numerical scheme for pricing American options with regime–switching models. Within each regime, the models assume constant volatility and constant interest rates as we aim to make a direct comparison with the implicit finite difference methods in Khaliq and Liu (2009) and the analytical approximation of Buffington and Elliott (2002). Moreover, the model could be extended to allow for stochastic volatility which is the topic of subsequent work.

We test the accuracy and speed of the proposed method and demonstrate its superiority over the above–mentioned alternatives. Buffington and Elliott (2002) use a separation of variable approach to the PDEs, resulting a pair of complicated nonlinear equations which still require substantial computational effort. In addition, their approach is difficult to extend to more regimes (more than 2) and it is very difficult to solve the coupled equations numerically, as Khaliq and Liu (2009) report in their numerical experiments. Khaliq and Liu (2009) explicitly treat the nonlinear terms and/or linear terms from other regimes to simplify the computation, but there is an upper bound on the time steps for their method, a limitation which does not apply to our approach.

The proposed method has several advantages in offering both numerical efficiency and modelling flexibility. Firstly, the proposed pricing model can be flexible enough to include multiple regimes (as opposed to some schemes which can only treat the two–regime case, see e.g. (Holmes et al., 2012)). Secondly, MOL computes the Greeks and the OEB in the solution process, thus reducing the burden of post-processing results.

The paper is organised as follows. Section 2 introduces the American option pricing problem under the regime switching model and the proposed MOL implicit scheme. Section 3 presents the numerical results and the comparison with other schemes. Section 4 concludes.

2 Regime Switching and Method of Lines

Without loss of generality, we assume in a risk–neutral world, $(\Omega,\ \mathcal{F},\ \mathbf{P})$ denotes a complete probability space, Ω is a non-empty set, $\mathcal{F}$ is a σ-algebra of subsets of Ω, and $\mathbf{P}$ is a probability measure. $\{\mathcal{F}_t\}_t$ is a filtration, under which all the processes below are defined. We denote as $S = \{S(t),\ t \in [0, T]\}$ the asset price and as $V(S, T)$ the price of American put option on this asset with maturity T and strike K.

2.1 American option pricing with regime switching

Let $\{\alpha_t\}$ be a continuous-time Markov chain taking values among m different states. Each state represents a particular regime and is labeled by i $(i = 1, 2, \dots, m)$. Then we have the regime generator

$$Q = \begin{Bmatrix} q_{11} & \cdots & q_{1m} \\ \cdots & \cdots & \cdots \\ q_{m1} & \cdots & q_{mm} \end{Bmatrix},$$

where $q_{i,j} \geq 0 \ \ (i \neq j)$ and $q_{i,i} = -\sum_{j \neq i} q_{i,j}$ (see Yin and Zhang (1998) for more details).

In each regime i we assume the classic Black-Scholes model, in which the asset price S follows a geometric Brownian motion:

$$dS = \mu S dt + \sigma_i S dB_t, \tag{1}$$

where μ is the drift, σ_i is the volatility in regime i and dB_t is the increment of a Wiener process. We assume B_t is uncorrelated with the Markov chain.

The corresponding American put option price $V_i(S,t)$ in each regime i satisfies the following free–boundary value problem (see Khaliq and Liu (2009), for example):

$$\begin{aligned} &\frac{\partial V_i}{\partial t} + \frac{1}{2}\sigma_i^2 S^2 \frac{\partial^2 V_i}{\partial S^2} + r_i S \frac{\partial V_i}{\partial S} - r_i V_i + \sum_{j \neq i} q_{ij}(V_j - V_i) = 0, \qquad && (S > S_i^*(t)) \\ &V_i(S,t) = K - S, && (0 < S < S_i^*(t)) \end{aligned}$$

where $i = 1, 2, \ldots, m$ and $S_i^*(t) = S_{\alpha\ i}^*(t)$ is the early exercise boundary in regime i. We also assume that σ_i and r_i are constants in each regime. The corresponding initial and boundary conditions are:

$$\begin{aligned} V_i(S,T) &= (K - S)^+, \\ \lim_{S \to \infty} V_i(S,t) &= 0, \\ \lim_{S \to S_i^*(t)} V_i(S,t) &= K - S_i^*(t), \\ \lim_{S \to S_i^*(t)} \frac{\partial V_i}{\partial S} &= -1, \\ S_i^*(T) &= K. \end{aligned}$$

2.2 Method of Lines

In this paper, the above free-boundary equations are solved by using a finite difference method, namely the MOL (see Meyer (2015)). We start working on the continuation equation only, which can be re-written as

$$\frac{\partial V_i}{\partial t} + \frac{1}{2}\sigma_i^2 S^2 \frac{\partial^2 V_i}{\partial S^2} + r_i S \frac{\partial V_i}{\partial S} - (r_i - q_{ii})V_i = -\sum_{j \neq i} q_{ij} V_j.$$

We introduce

$$\tau = T - t, \quad u = \frac{V}{K}, \quad x = \frac{S}{K},$$

and write the above equation in a normalised form:

$$-\frac{\partial u_i}{\partial \tau} + \frac{1}{2}\sigma_i^2 x^2 \frac{\partial^2 u_i}{\partial x^2} + r_i x \frac{\partial u_i}{\partial x} - (r_i - q_{ii})u_i = -\sum_{j \neq i} q_{ij} u_j.$$

There are several advantages to work with the normalised equation over the original one (Meyer, 1998; Chiarella et al., 2009; Meyer, 2015). It increases the computation accuracy, and the results are independent of specific parameters — the S and K values — thus making the code very flexible.

The key idea behind MOL is to replace a PDE with an equivalent system of one-dimensional Ordinary Differential Equations (ODEs) (Meyer, 2015).

In the finite difference method, the time difference term can be discretised using first and second order approximations:

$$\frac{\partial u}{\partial \tau} \approx \begin{cases} \frac{u-u^{n-1}}{\Delta\tau}, & n = 1,\ 2, \\ \frac{3(u-u^{n-1})-(u^{n-1}-u^{n-2})}{2\Delta\tau}, & n > 2. \end{cases} \tag{2}$$

The uniform time step is $\Delta\tau = \tau/(Nt)$ with Nt the number of time steps in the discretisation. The superscripts $n-1$ and $n-2$ represent the known values at $n-1$ and $n-2$ time steps, while we need to find the unknown value $u = u_i^n$. Here we have shortened notation by omitting the current regime i. (It will not be omitted where there is risk of confusion, e.g. if the regime is not i, as we will see soon).

We treat all other u terms implicitly (but we assume u_j is already known initially), and in regime i we get a second order ODE for u with respect to x in the following form:

$$u'' + d(x, \tau^n)u' - c(x, \tau^n)u = g(x, u_j, \tau^n).$$

Introducing the option delta and denoting it by v, we obtain two first order ODEs:

$$\begin{cases} u' = v, \\ v' = c(x, \tau^n)u - d(x, \tau^n)v + g(x, u_j, \tau^n). \end{cases} \tag{3}$$

Clearly, v' is the normalised gamma value. Thus the delta and gamma appear directly in the computation process, and do not have to be computed separately in a post-processing step.

We use the so-called Riccati transformation method, and the solution can be expressed as (in each regime i)

$$u(x) = R(x)v(x) + w(x), \tag{4}$$

in which R and w satisfy the following equations:

$$\begin{cases} R' = 1 + d(x, \tau^n)R - c(x, \tau^n)R^2, & R(x_\infty) = 0, \\ w' = -c(x, \tau^n)Rw - Rg(x, u_j, \tau^n), & w(x_\infty) = h(\tau^n), \end{cases} \tag{5}$$

with $h(\tau^n)$ the appropriate boundary condition (see Meyer (2015) for details).

We check the c, d and g terms, and find that in each regime, the R term only needs to be calculated once, which makes the method very efficient.

For American options, it is important to find the free boundary (early exercise boundary) x^*, which satisfies the condition

$$1 - x^* = R(x^*) \cdot (-1) + w(x^*). \tag{6}$$

The x^* can be calculated from interpolation because we only have discrete values of R and w at the grid points. The Riccati transformation for the linear inhomogeneous system consists of three steps: the forward sweep to solve w; the determination of the boundary values x^*, $u(x^*)$ and $v(x^*)$; and the reverse sweep to solve u and v.

The above procedures finish one iteration step in all regimes, then we update the g term in Equation (3) using the latest u_j and repeat the whole process (note that the R term only needs to be solved once in each regime). We iterate until the price profile $u(x)$ converges to a desired level of accuracy for all regimes, i. e.,

$$\max_{l,i} |u_i^{k+1}(x_l) - u_i^k(x_l)| < 10^{-6}, \tag{7}$$

where i is the regime, l is the grid point in x dimension, and k is the iteration count number. One may argue that the iteration method is too slow, but in practice, only $4 \sim 5$ iterations are needed in each time step, making the code still an efficient one, as we will see in the next section.

The MOL procedures can be summarised by the following pseudo-code:

Algorithm 1 MOL algorithm

```
time=0
Initialize u and v
for n = 1 to Nt do
  time=dt*n
  calculate c and d terms
  if n == 1 or n == 3 then
    calculate R in equation (5)
  end if
  MAE=1000.     (initial guess of Maximum Absolute Error in equation (7))
  k=1
  uo = u, vo = v     (u and v at old iteration step k)
  while MAE ≥ 1.e-6 do
    for i = 1 to m do
      calculate g term
      forward sweep to solve w in equation (5)
      find the early exercise boundary x* in equation (6), u(x*) and v(x*)
      reverse sweep to solve v in equations (3) and (4), then u in equation (4)
      also get gamma value     (v' in equation (3))
    end for
    calculate MAE in equation (7)
    uo = u, vo = v
    k=k+1
  end while
end for
```

Table 1: American put option prices with no jump between regimes

S	$r_1 = 0.1,\ \sigma_1 = 0.8$	$r_2 = 0.05,\ \sigma_2 = 0.3$
6.00	3.66676242437	3.00000000000
9.00	2.37538560450	0.88831117801
12.00	1.60485395651	0.20354305568

3 Numerical Investigation

In this section, we present applications of the proposed numerical scheme, where we price an American put option with maturity $T = 1$ and strike $K = 9$. Firstly, we test the correctness and robustness with two-regime models that have no jumps and identical regimes. We also make comparison with the IMS1, IMS2 and Tree models studied by Khaliq and Liu (2009). In order to make a fair comparison, we will use the same parameters as in their paper. We also consider an extension to multiple regimes and study option sensitivities including delta and gamma.

3.1 No Jump between Regimes

Initially, to test the implementation of our numerical scheme, we consider the two-regime case of our model with regime generator

$$Q = \begin{Bmatrix} 0 & 0 \\ 0 & 0 \end{Bmatrix}.$$

This means that there is no jump between different regimes, so we should have two independent American option pricing problems. A direct comparison with standard MOL code would serve to test our code. So we compare with the results from the code developed in Meyer and van der Hoek (1997).

In this case, the infinite boundary $x(\infty) = 5.5$. The parameters (risk-free interest rate and volatility) in the two regimes are $r_1 = 0.1,\ \sigma_1 = 0.8$ and $r_2 = 0.05,\ \sigma_2 = 0.3$, respectively. One regime involves high interest rate and high volatility while the second regime assumes low interest rate and volatility. There are $Ns = 2500$ uniform grids in S dimension, and $Nt = 2000$ even steps in time dimension. We also choose three initial asset prices: $S = 6.00, 9.00$ and 12.00, corresponding to out-of-the money, at-the-money, and in-the-money options.

Table 1 shows the numerical results using our regime-switching MOL scheme with no jumps between the two regimes.

In Table 2, we list the American put option prices from the MOL code in Meyer and van der Hoek (1997) without regime-switching, and the parameters are the same as the ones assumed for the above mentioned regimes. Comparing Table 1 and Table 2, we could see that the error in each regime is smaller than 10^{-11}.

3.2 Identical Regimes

To further investigate the precision of our code, we use similar parameters as in Section 3.1, but use a non-zero Q matrix. Specifically, we assume the same interest rates and

Table 2: American put option prices from Meyer and van der Hoek (1997) (no regime-switching)

S	$r = 0.1,\ \sigma = 0.8$	$r = 0.05,\ \sigma = 0.3$
6.00	3.66676242437	3.00000000000
9.00	2.37538560450	0.88831117801
12.00	1.60485395651	0.20354305568

Table 3: American put option prices with identical regimes

S	$r_1 = 0.1,\ \sigma_1 = 0.8$	$r_2 = 0.1,\ \sigma_2 = 0.8$	no RS ($r = 0.1,\ \sigma = 0.8$)
6.00	3.66676242861	3.66676242869	3.66676242437
9.00	2.37538560691	2.37538560692	2.37538560450
12.00	1.60485395801	1.60485395802	1.60485395651

volatility in both regimes, namely, $r_1 = r_2 = 0.1$, and $\sigma_1 = \sigma_2 = 0.8$, with

$$Q = \begin{Bmatrix} -6 & 6 \\ 9 & -9 \end{Bmatrix}.$$

In this special case, there is jump between the two regimes, but they are actually the same regime, so we should expect the same prices in both these two regimes. In fact, we can see from Table 3 that the price differences in the two regimes are roughly 10^{-11}, which means we get identical results in the two regimes. The price errors from the standard case without regime switching (the rightmost column in Table 3) are roughly 10^{-9}, still much smaller than the tolerance level (10^{-6}). These results also illustrate the correctness and robustness of the proposed numerical scheme in these benchmark cases.

3.3 Comparing MOL with Other Numerical Schemes

In this section, we compare MOL with other numerical schemes in Khaliq and Liu (2009), who use the penalty method approach and implicit implementation of the θ-method. In the method IMS1 they replace the unknown V term with previously obtained value in the nonlinear terms, while in method IMS2 they further replace the unknown V terms in other regimes with known values. The parameters used here are also taken from Khaliq and Liu (2009), so that we can make direct comparison with their numerical results and Buffington and Elliott (2002)'s analytical approximation. The non-zero Q matrix is

$$Q = \begin{Bmatrix} -6 & 6 \\ 9 & -9 \end{Bmatrix}$$

and the volatility levels and risk-free interest rates in the two regimes are the same as Section 3.1.

Table 4 lists the American put option prices in regime 1 for 10 asset prices from different approaches. The second column are the results from Buffington and Elliott (2002)'s analytical approximation which still involves a lot of computations (see Khaliq and Liu (2009)), followed by IMS1, IMS2 and MTree results in Khaliq and Liu (2009)

Table 4: Comparison of American put option prices in Regime 1 for the two-regime case

S	BE [2002]	Khaliq and Liu (2009)			MOL ($r_1 = 0.1,\ \sigma_1 = 0.8$)		
		IMS1	IMS2	MTree	Nt, Ns	$4Nt, 4Ns$	$16Nt, 16Ns$
3.50	5.5000	5.5001	5.5001	5.5000	5.500000	5.500000	5.500000
4.00	5.0020	5.0067	5.0066	5.0031	5.003271	5.003266	5.003266
4.50	4.5359	4.5486	4.5482	4.5432	4.543307	4.543296	4.543296
6.00	3.4085	3.4198	3.4184	3.4144	3.414282	3.414282	3.414282
7.50	2.5870	2.5887	2.5867	2.5844	2.584181	2.584182	2.584183
8.50	2.1631	2.1598	2.1574	2.1560	2.155865	2.155869	2.155871
9.00	1.9810	1.9756	1.9731	1.9722	1.971988	1.971994	1.971995
9.50	1.8160	1.8090	1.8064	1.8058	1.805615	1.805621	1.805623
10.50	1.5309	1.5214	1.5187	1.5186	1.518486	1.518493	1.518495
12.00	1.1944	1.1827	1.1799	1.1803	1.180316	1.180325	1.180327

where the MTree results are taken as the benchmark. Firstly we compare with our model using the same parameters. We can see clearly that the prices from all other methods are not very close to the MTree results which is taken as the benchmark in Khaliq and Liu (2009), while our results are much closer to the benchmark values.

In order to find which method is more accurate and reliable, we further run convergence tests with higher resolution (more grid points in x and t dimensions). We specifically consider a $4Nt, 4Ns$ and a $16Nt, 16Ns$ resolutions. When the resolution is increased our MOL results are almost identical, thus suggesting fast convergence of the proposed numerical scheme. Compared with our numerical results, the values from the MTree model are still far from having converged and thus still need to be improved (e.g., using a larger number of steps). Later on, we will use the case with highest resolution in our results as the reference value.

The option prices in Regime 2 from all models are listed in Table 5. We perform the same comparison and confirm that the MOL results are more accurate and reliable.

Table 5: Comparison of American put option prices in Regime 2 for the two-regime case

S	BE (2002)	Khaliq and Liu (2009)			MOL ($r_2 = 0.05,\ \sigma_2 = 0.3$)		
		IMS1	IMS2	MTree	Nt, Ns	$4Nt, 4Ns$	$16Nt, 16Ns$
3.50	5.5000	5.5012	5.5012	5.5000	5.500000	5.500000	5.500000
4.00	5.0000	5.0016	5.0016	5.0000	5.000000	5.000000	5.000000
4.50	4.5083	4.5194	4.5190	4.5117	4.511912	4.511896	4.511896
6.00	3.3517	3.3565	3.3550	3.3503	3.350669	3.350669	3.350669
7.50	2.5125	2.5078	2.5056	2.5028	2.503292	2.503294	2.503296
8.50	2.0812	2.0722	2.0695	2.0678	2.068317	2.068322	2.068323
9.00	1.8967	1.8860	1.8832	1.8819	1.882445	1.882451	1.882453
9.50	1.7301	1.7181	1.7153	1.7143	1.714865	1.714871	1.714873
10.50	1.4440	1.4301	1.4272	1.4267	1.427336	1.427345	1.427346
12.00	1.1096	1.0945	1.0916	1.0916	1.092320	1.092329	1.092330

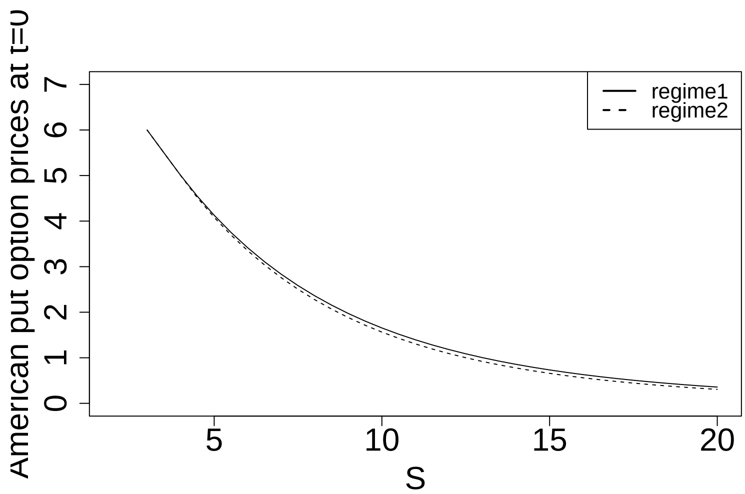

Figure 1: American put option price curves at $t = 0$ with two regimes. The maturity $T = 1$ and strike $K = 9$. The upper curve is in Regime 1 and the lower curve is in Regime 2.

We plot the option prices from our model as a function of asset price S at $t = 0$ in Figure 1. The upper curve corresponds to Regime 1 and the lower curve to Regime 2. The surface plots of the option prices are displayed in Figure 2 as a function of S and t. Both figures produce very smooth curves and surfaces, confirming that the MOL method produces prices which are reliable in this sense.

3.3.1 Improving the Implicit Scheme IMS2

To avoid the complications of implicit schemes, Khaliq and Liu (2009) explicitly treat the nonlinear terms and linear terms from other regimes in their second implicit scheme IMS2. Here we point out a problem with this scheme.

From Equation (3.6) in Khaliq and Liu (2009), we have that

$$\begin{aligned}\frac{V_i^{j,n+1} - V_i^{j,n}}{\Delta t} &+ \frac{1}{2}\sigma_i^2 S_j^2 \left[\theta \frac{\delta_S^2 V_i^{j,n+1}}{(\Delta S)^2} + (1-\theta)\frac{\delta_S^2 V_i^{j,n}}{(\Delta S)^2}\right] \\ &+ r_i S_j \left[\theta \frac{\Delta_S V_i^{j,n+1}}{\Delta S} + (1-\theta)\frac{\Delta_S V_i^{j,n}}{\Delta S}\right] - (r_i - q_{ii})\left[\theta V_i^{j,n+1} + (1-\theta) V_i^{j,n}\right] \\ &+ \sum_{l \neq i} q_{il} V_l^{j,n+1} + \frac{\varepsilon C}{V_i^{j,n+1} + \varepsilon - q(S_j)} = 0, \quad (8)\end{aligned}$$

where $V_i^{j,n} = V(\alpha_i, S_j, n\Delta t)$ is the unknown price at regime i, asset price j and time step n.

To illustrate the problem, we only consider the simplest case where all regimes are the same, i.e., $\alpha_1 = \alpha_2 = ... = \alpha_m, V_l^{j,n} = V_i^{j,n}$ (j is another regime indicator). Using

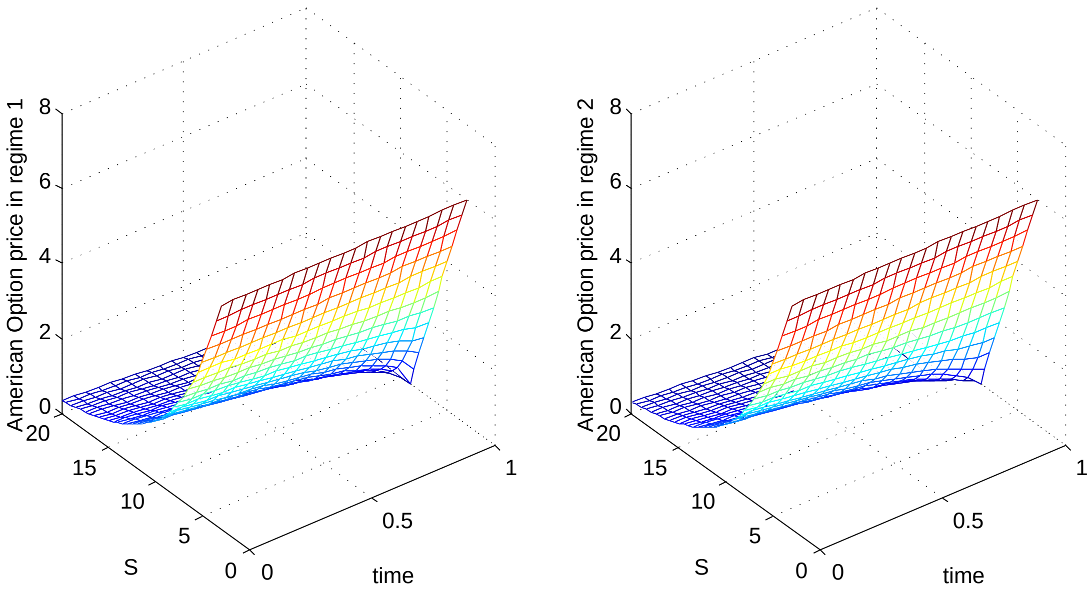

Figure 2: American put option price surface with two regimes. The maturity $T = 1$ and strike $K = 9$. The left panel is in Regime 1 and the right panel is in Regime 2.

the relationship $q_{ii} = -\sum_{l \neq i} q_{il}$ the equation can be written as

$$...-r_i \left[\theta V_i^{j,n+1} + (1-\theta)V_i^{j,n}\right] + q_{ii}\left[(\theta-1)V_i^{j,n+1} + (1-\theta)V_i^{j,n}\right] + \frac{\varepsilon C}{V_i^{j,n+1} + \varepsilon - q(S_j)} = 0,$$

which is

$$... - r_i \left[\theta V_i^{j,n+1} + (1-\theta)V_i^{j,n}\right] + \frac{\varepsilon C}{V_i^{j,n+1} + \varepsilon - q(S_j)} + q_{ii}\left[(\theta-1)(V_i^{j,n+1} - V_i^{j,n})\right] = 0.$$

The last term on the left hand side usually is not equal to zero unless $\theta = 1$, which is introduced by the explicit treatment of V_l in all other regimes. This extra term can not be eliminated even if finer grids or high order time difference schemes are adapted, thus this scheme needs to be further improved.

According to the above analysis, the prices in the regime–related terms should be treated the same way, either explicitly or implicitly. An explicit treatment of those terms could impose severe time-step constraints, while in the implicit treatment, the model becomes very complicated, and one would use iterative methods, or solve a large matrix inversion.

In contrast, in the MOL implementation, we avoid this problem by treating all u_j terms implicitly. Because of the implicit method, the scheme should be unconditionally stable, which means that there is no upper limit on the size of the time steps. We use limited $\Delta\tau$ in the code only because of a high accuracy requirement. However, Khaliq and Liu (2009) find an upper bound on the size of the time steps in order for their schemes to satisfy a discrete version of the positivity constraint for American option values in their penalty method approach.

Because we use an iteration method, one might worry about the efficiency of the code, so we perform several tests to check the efficiency of our scheme. Table 6 lists the option prices from our model and 4 cases using smaller numbers of time steps (larger time step sizes $\Delta\tau$) $Nt/2,\ Nt/4,\ Nt/10,\ Nt/20$. In each case, only $4 \sim 5$ iterations are needed in each time step. It is clearly seen that when the time step is increased to 20 times larger, the results are still reasonably accurate up to 3 decimal places, and they are usually closer to the true solution than Khaliq and Liu (2009)'s two implicit methods. We further find the CPU time[1] for each case is almost perfectly proportional to the number of time steps Nt.

Table 6: Comparison of American put option prices in Regime 1 for the two-regime case with different time steps

S	$Ns = 2500, Nt = 2000, T = 1$				
	Nt	$Nt/2$	$Nt/4$	$Nt/10$	$Nt/20$
3.50	5.500000	5.500000	5.500000	5.500000	5.500000
4.00	5.003271	5.003269	5.003266	5.003259	5.003244
4.50	4.543307	4.543303	4.543295	4.543267	4.543208
6.00	3.414282	3.414272	3.414248	3.414160	3.413971
7.50	2.584181	2.584166	2.584131	2.584003	2.583732
8.50	2.155865	2.155848	2.155810	2.155667	2.155369
9.00	1.971988	1.971971	1.971931	1.971785	1.971479
9.50	1.805615	1.805597	1.805557	1.805408	1.805099
10.50	1.518486	1.518468	1.518427	1.518278	1.517972
12.00	1.180316	1.180299	1.180260	1.180118	1.179830
CPU time (seconds)	14.17	7.08	3.58	1.43	0.73

In Table 7, we compare the errors in Regime 1 in different models, using the highest resolution case with $16Nt, 16Ns$ as the "true results". IMS1 and IMS2 data are taken from Table 4, while the worst MOL data with $Nt/20$ in Table 6 is used for comparison. Whether in the maximum or in the average values, the MOL errors are always one order smaller than the other two methods. Thus using the MOL model we could get rather accurate results in a very efficient manner by using relatively large time steps.

Table 7: Comparison of American put option prices in regime 1 ($r_1 = 0.1,\ \sigma_1 = 0.8$) for the two-regime cases

absolute error	IMS1	IMS2	$Nt/20$
max	0.00552	0.00490	0.00052
ave	0.00351	0.00190	0.00034

[1]The code was compiled by the GNU Fortran 95 compiler with double precision without any optimisation option, and run on a personal computer with i5-2400 CPU @3.10GHz and 3.0GB memory. The Operating System was Windows XP Professional with SP3.

3.4 Allowing for Multiple Regimes

The proposed numerical scheme can be easily extend to allow for multiple regimes. Here we report the numerical results when there are four regimes with the generator

$$Q = \begin{Bmatrix} -1 & 1/3 & 1/3 & 1/3 \\ 1/3 & -1 & 1/3 & 1/3 \\ 1/3 & 1/3 & -1 & 1/3 \\ 1/3 & 1/3 & 1/3 & -1 \end{Bmatrix},$$

which means that the market could stay in one regime and/or jump to any other regime with equal probability.

Table 8 lists the volatilities and risk-free interest rates in the four regimes, and all other parameters are the same as in previous cases. Note that Regime 1 has lowest r and highest σ, Regime 4 has highest r and lowest σ, while the parameters in Regimes 2 and 3 are between the previous two regimes. Thus we expect highest option price in Regime 1, lowest price in Regime 4, and intermediate prices in the other two regimes.

Table 8: The volatilities and risk-free interest rates used in the four-regime cases

regime i	r	σ
1	0.02	0.9
2	0.1	0.5
3	0.06	0.7
4	0.15	0.2

In Figure 3 we plot the option prices from the four-regime model described above as a function of asset price S at $t = 0$. Different regimes are identified by different colors. It's obvious that all 4 curves are very smooth. The surface plots of the option prices are displayed in Figure 4 as a function of S and t. Both figures confirm that the MOL model produces smooth and reliable prices in all different regimes.

As we have mentioned before, one feature of the MOL is that the early exercise boundary (x^* in Equation (6)) is computed in the solution process. Figure 5 displays the evolution of the early exercise boundaries ($S^* = x^* \cdot K$) in the four regimes. The plots are all smooth and similar to the results obtained from non-regime–switching models.

3.5 Option Greeks

The Greeks, delta and gamma, can be found as part of the computation in our MOL scheme. Delta is the sensitivity of the option price to variations in the asset prices, which is v in Equation (3), and gamma is the sensitivity of delta to variations in the asset prices, which is v' in Equation (3). Figure 7 shows the delta of the option at $t = 0$ in the four regimes. The green color is the result from v in Equation (3) which computes the option's delta, and the black color is the calculated delta $\partial V_i/\partial S$. The curves from the two methods agree very well, which shows that the numerical inaccuracy is very tiny. The deltas in each regime are very smooth in the continuation regions, and they are zero in the early–exercise regions. Such structures are typical in the American option

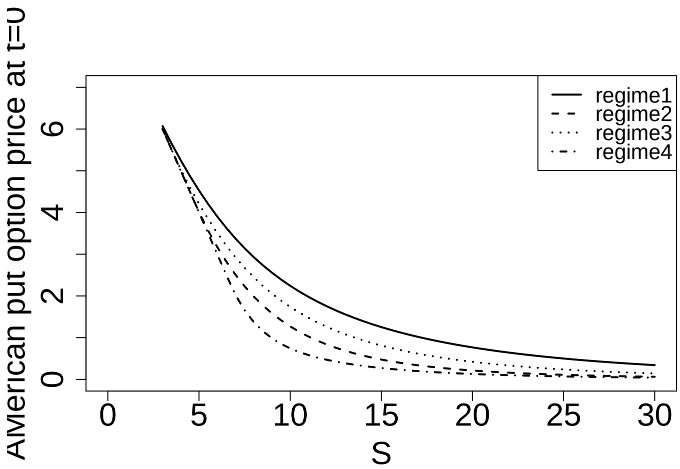

Figure 3: American put option price curves at $t = 0$ in the four regimes for the four-regime case with $T = 1$ and $K = 9$. From top to bottom, the curves are in regime 1, 3, 2 and 4, respectively. Parameters for this case are listed in Table 8.

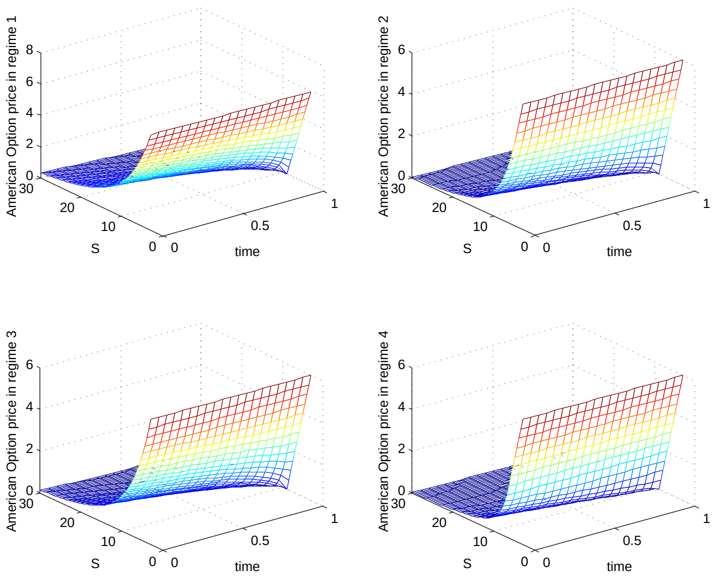

Figure 4: American put option price surface in the four regimes for the four-regime case with $T = 1$ and $K = 9$. From top to bottom, from left to right, the panels are in regime 1, 2, 3 and 4, respectively.. Parameters for this case are listed in Table 8.

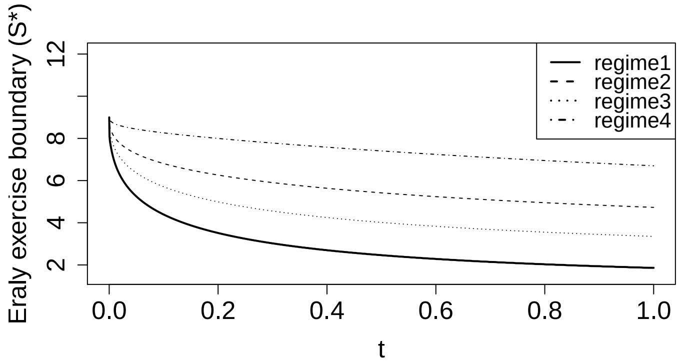

Figure 5: The early exercise boundaries (S^*) in the four regimes for the four-regime case with $T = 1$ and $K = 9$. From top to bottom, the curves are in regime 4, 2, 3 and 1, respectively. Parameters for this case are listed in Table 8.

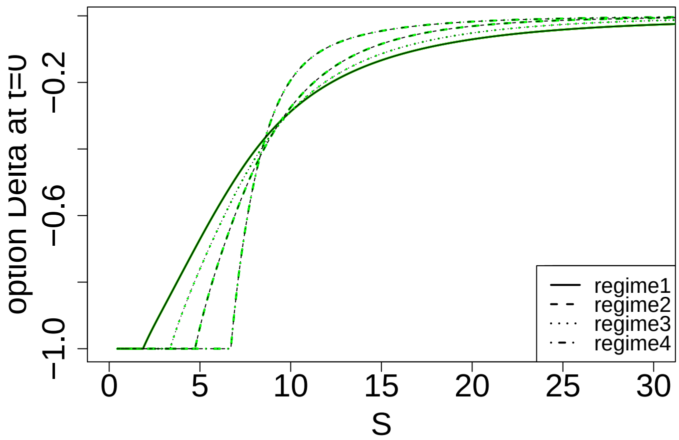

Figure 6: American put option Delta at $t = 0$ in the four regimes for the four-regime case with $T = 1$ and $K = 9$. Parameters for this case are listed in Table 8.

problems without regime–switching (de Frutos, 2006), but the slopes are all lower than the corresponding non-regime–switching results.

We also plot the gamma profile at $t = 0$ in Figure 8. The green color is for the v'/K in the computation, and the black color is the calculated $\partial^2 V_i/\partial S^2$ after the run. The matching of the two results demonstrates the high accuracy and low dissipation of the code. We also notice the non-oscillation feature in the plot, which is challenging in the numerical simulation of American option problems due to the discontinuity in the second–order derivative at the early exercise boundary S^* (de Frutos, 2006; Forsyth and Vetzal, 2014). We also observe that the jump points are the early exercise boundaries in the four regimes.

In Regime 1 where the market has highest volatility σ and lowest risk-free rate r,

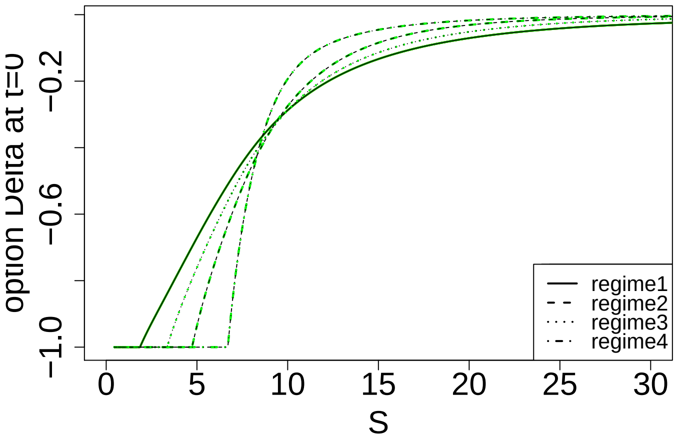

Figure 7: American put option Delta at $t = 0$ in the four regimes for the four-regime case with $T = 1$ and $K = 9$. Parameters for this case are listed in Table 8.

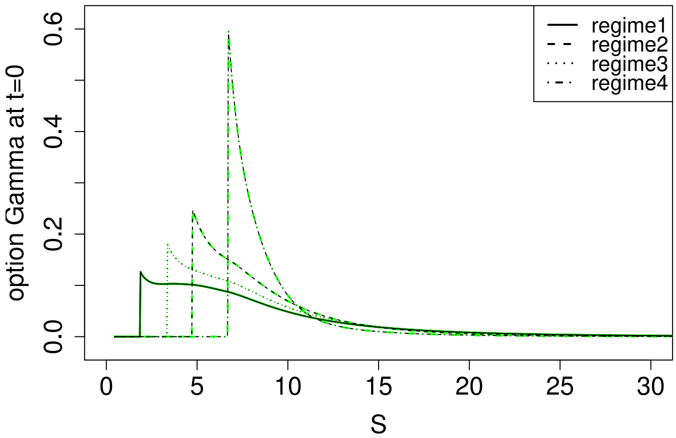

Figure 8: American put option Gamma at $t = 0$ in the four regimes for the four-regime case with $T = 1$ and $K = 9$. Parameters for this case are listed in Table 8.

the option has highest price (see Figure 3), and it is exercised earlier at very low asset price S (also see Figure 5) and the hedging delta is the lowest. In contrast, in Regime 4 where the market has lowest volatility σ and highest risk-free rate r, the option has the lowest price, and it is exercised much later at very high asset price S and the hedging delta is the highest. In Regimes 2 and 3 the volatility levels and risk-free rates have intermediate values, so the option prices and delta values are between the results in the other two regimes.

For the four-regime case, we perform several tests with different resolutions, and list the results in Regime 1 in Table 9. Columns 3 and 4 are high–resolution tests with a larger number of grid points in the time and x dimensions: $4Nt$, $4Ns$ and $16Nt$, $16Ns$, respectively. Such tests illustrate the convergence of the simulation code, so our MOL code is reliable and accurate. In the last column we use a very large time step (number of time steps is $Nt/20$) and still get very close approximation results.

Table 9: Comparison of American put option prices in regime 1 for the four-regime case

S	$Ns = 2500,\ Nt = 2000,\ T = 1,\ r_1 = 0.02,\ \sigma_1 = 0.9$			
	$Nt,\ Ns$	$4Nt,\ 4Ns$	$16Nt,\ 16Ns$	$Nt/20,\ Ns$
3.50	5.647802	5.647747	5.647745	5.647657
4.00	5.248409	5.248360	5.248359	5.248230
4.50	4.874721	4.874679	4.874677	4.874509
6.00	3.904401	3.904360	3.904359	3.904099
7.50	3.143172	3.143146	3.143145	3.142807
8.50	2.735863	2.735841	2.735840	2.735473
9.00	2.557578	2.557567	2.557567	2.557180
9.50	2.394160	2.394143	2.394144	2.393758
10.50	2.106300	2.106290	2.106290	2.105895
12.00	1.754401	1.754398	1.754398	1.754007

4 Conclusion

In this paper, we have developed an implementation of the Method-of-Lines numerical approximation scheme that allows for multiple regime settings, and solve the American option pricing problem using this method. Numerical results show that this approach is superior to other numerical methods in both accuracy and efficiency. The superior performance of this method stems from the implicit treatment of variables in all regimes and the second order approximation in the time difference term. In addition, the Greeks (delta and gamma) and the optimal exercise boundary are all computed in the solution process, thus reducing the burden of postprocessing needing to obtain these additional numerical results.

The current numerical scheme is very simple because the volatilities and interest rates are constant in each regime. Extensions to the method are possible to include stochastic volatility and/or stochastic interest rates in each regime. This will be the topic of future research.

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