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Empirical Pricing Performance in Long-Dated Crude Oil Derivatives:
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Benjamin Cheng, Christina Sklibosios Nikitopoulos and Erik Schlögl

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Empirical pricing performance on long-dated crude oil derivatives: Do models with stochastic interest rates matter?

Benjamin Cheng[a], Christina Sklibosios Nikitopoulos[a,*], Erik Schlögl[a]

[a] *University of Technology Sydney,*
Finance Discipline Group, UTS Business School,
PO Box 123 Broadway NSW 2007, Australia

Abstract

Does modelling stochastic interest rates beyond stochastic volatility improve pricing performance on long-dated crude oil derivatives? To answer this question, we examine the empirical pricing performance of two forward price models for commodity futures and options: a deterministic interest rate - stochastic volatility model and a stochastic interest rate - stochastic volatility model. Both models allow for a correlation structure between the futures price process, the futures volatility process and the interest rate process. By estimating the model parameters from historical crude oil futures prices and option prices, we find that stochastic interest rate models improve pricing performance on long-dated crude oil derivatives, with the effect being more pronounced when the interest rate volatility is relatively high. Several results relevant to practitioners have also emerged from our empirical investigations. With regards to balancing the trade-off between precision and computational effort, we find that two-factor models would provide good fit on long-dated derivative prices thus there is no need to add more factors. We also find empirical evidence for a negative correlation between crude oil futures prices and interest rates.

Keywords: Futures options pricing; Stochastic interest rates; Correlations; Long-dated crude oil derivatives;
JEL: C13, C60, G13, Q40

*Corresponding author

Email addresses: `Benjamin.T.Cheng@student.uts.edu.au` (Benjamin Cheng), `christina.nikitopoulos@uts.edu.au` (Christina Sklibosios Nikitopoulos), `erik.schlogl@uts.edu.au` (Erik Schlögl)

1. Introduction

The role of commodity markets in the financial sector has substantially increased over the last decade. A record high of $277 billion invested in commodity exchange-traded products was observed in 2009[1] (which was by 50 times larger than the decade earlier) with crude oil market being the most active commodity market. A variety of new products become available including exchange-traded products, managed futures funds, and hedge funds that boost activity in both short-term trading and long-term investment strategies. With the disappointing performance of the equity index market, commodities markets along with real estate become the new promising alternative investment vehicles with commodity index outperforming the S&P 500 index (see Figure 1) over the last decade. The crude oil futures and options are the world's most actively traded commodity derivatives forming a major part of these activities. The average daily open interest in crude oil futures contracts of all maturities has increased from 781,000 contracts in 2005 to 1,677,627 contracts in 2013. Even though the most active contracts are short-dated, the market for long-dated contracts has also substantially increased. The maturities of crude oil futures contracts and options on futures listed on CME Group have extended to 9 years in recent years. The average daily open interest in crude oil futures contracts with maturities of two years or more was at 118,000 contracts in December 2015 and reached a record high of 171,512 contracts in 2009.

It is well known that the impact of interest rate risk is more pronounced for long-dated derivative contracts than short-dated contracts.[2] Thus, in recent years stochastic volatility – stochastic interest rate derivative pricing models or the so-called hybrid pricing models have emerged for spot markets such as equities or foreign exchanges. Early models do not include correlations, or sufficient number of factors and many do not derive closed form derivative evaluations, see for instance, Amin and Jarrow (1992), Amin and Ng (1993), Bakshi, Cao, and Chen (1997), Grzelak and Oosterlee (2011). Bakshi et al. (2000) by using long-term equity anticipation securities with maturities up to three years, empirically investigate the pricing and hedging performances on four option pricing models, namely the Black and Scholes (1973) model, the stochastic volatility model, the stochastic volatility – stochastic interest rate model and the stochastic volatility jump model. Their results on pricing performance show that stochastic volatility jump model performs the best for short-term puts and the stochastic volatility model performs the best for long-term puts. They do not find evidence that stochastic volatility – stochastic interest rate models lead to consistent improvement in pricing performance. However in their hedging exercise, they find that the stochastic volatility – stochastic interest rate model helps improve empirical performance when it devises hedges of long-term options. van Haastrecht et al. (2009), van Haastrecht and Pelsser (2011) and Grzelak et al. (2012) combine the stochastic volatility model by Schöbel and Zhu (1999) as the spot price process and the Hull and White (1993) model as the stochastic interest rate process while allowing full correlations between the spot price process, its stochastic volatility process and the interest rate process. van Haastrecht et al. (2009) apply the hybrid model to the valuation of insurance options with long-term

[1]Source: www.barchart.com/articles/etf/commodityindex

[2]See for instance van Haastrecht, Lord, Pelsser, and Schrager (2009), Bakshi, Cao, and Chen (2000), and Grzelak, Oosterlee, and van Weeren (2012).

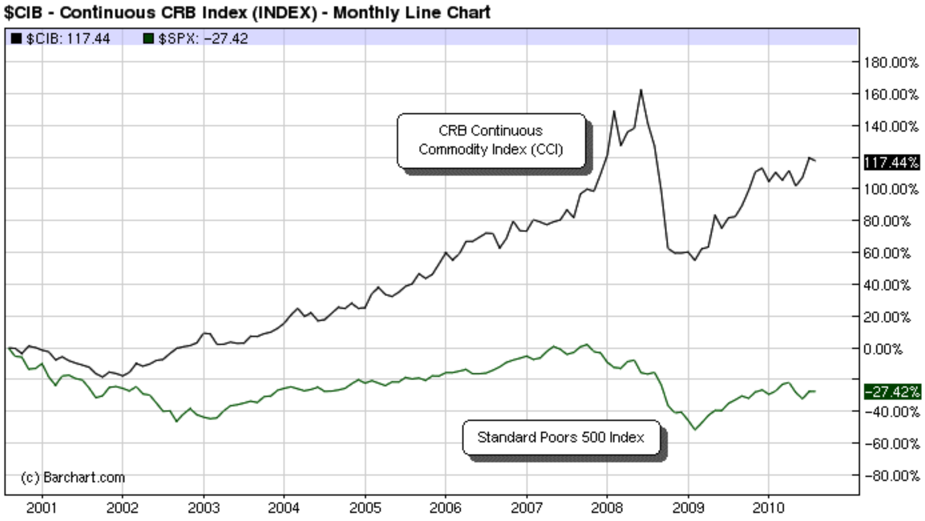

Figure 1: **CRB Continuous Commodity Index versus the S&P 500 Index.** Commodity index had outperformed the S&P 500 index over the last decade. Source: www.barchart.com/articles/etf/commodityindex

equity or foreign exchange (FX) exposure. Equity and FX markets have been extensively studied with hybrid pricing models, yet there is limited literature on commodity markets.

One of the earliest commodity derivative pricing models for crude oil is the Gibson and Schwartz (1990) model. They develop a two-factor joint diffusion process where the spot price follows a geometric Brownian motion and the net convenience yield follows an Ornstein-Uhlenbeck process, thereby the mean-reversion property of commodity prices is induced by the convenience yield process. This model does not feature stochastic volatility nor stochastic interest rates and only examines the pricing performance of relatively short-dated futures contracts. Schwartz (1997) confirms the importance of the stochastic interest rate process for longer maturity contracts. By employing Kalman filtering and fitting to futures contracts only, the pricing performance of three commodity derivative pricing models was empirically investigated. The first one-factor model could not adequately capture the market prices of futures contracts. Using crude oil futures contracts with a maturity of up to 17 months, the term structure of futures prices implied by the two- and three-factor models are indistinguishable. However, using proprietary crude oil forward prices with maturity of up to 9 years, the two- and three-factor models can imply quite different term structure of futures prices. The three-factor model with stochastic interest rates (6 percent infinite maturity discount bond yield is assumed) performs the best. Schwartz and Smith (1998) assume that the price changes from long-dated futures contracts represent fundamental modifications, which are expected to persist, whereas changes from the short-dated futures contracts represent ephemeral price shocks that are not expected to persist. Thus they propose a two-factor model where the first factor represents short-run deviations and the second process represents the equilibrium level. Their model provides better fitting to medium-term futures prices rather than to short-term and long-term futures prices. None of these models though feature stochastic volatility, an important feature for pricing long-dated contracts, see

Bakshi et al. (2000) and Cortazar, Gutierrez, and Ortega (2015). Furthermore, the above–mentioned models are spot price models. Note that a representative literature on pricing commodity contingent claims with spot price models includes Cortazar and Schwartz (2003), Casassus and Collin-Dufresne (2005), Cortazar and Naranjo (2006), and Fusai, Marena, and Roncoroni (2008).

Trolle and Schwartz (2009) introduce a commodity derivative pricing model that is based on the Heath, Jarrow, and Morton (1992) framework and features unspanned stochastic volatility. By using crude oil futures contracts of up to 5 years of maturity and futures options with maturity of up to about 1 year, they empirically demonstrate that because of the existence of unspanned stochastic volatility, options are not redundant contracts, in the sense that they cannot be fully replicated by portfolios consisting only the underlying futures contracts and some money in the saving accounts. The main reason they do not include options with longer maturities is that the option pricing formula developed in their paper only assumes deterministic interest rates. Assuming deterministic interest rates results in negligible pricing errors for short maturity contracts but for options with longer maturities, the pricing errors may be noticeable. By fitting their model to a longer dataset of crude oil futures and options, Chiarella, Kang, Nikitopoulos, and Tô (2013) consider a commodity pricing model within the Heath et al. (1992) framework and demonstrate that a hump-shaped crude oil futures volatility structure provides better fit to futures and option prices and improves hedging performance. However in this model, similar to the model proposed in Trolle and Schwartz (2009), deterministic interest rates are assumed. Pilz and Schlögl (2009) model commodity forward prices with stochastic interest rates driven by a multi-currency LIBOR Market Model and achieve a consistent cross-sectional calibration of the model to market data for interest rate options, commodity options and historically estimated correlations. Cortazar et al. (2015) investigate the pricing performance of different models to commodity prices, namely crude oil, gold and copper. The constant volatility model fits better futures prices but fitting to option prices improves significantly when a stochastic volatility model is considered on the expense of additional computational effort. Cortazar et al. (2015) do not consider stochastic interest rates.

Motivated by the limited literature dedicated to empirical research on pricing long-dated commodity derivatives by incorporating stochastic interest rates as well as the increasing importance of long-dated commodity derivative contracts to the financial markets, we consider an empirical investigation of the impact of stochastic interest rates to pricing models of long-dated crude oil derivative contracts. We employ two multi-factor stochastic volatility Heath et al. (1992) type models, one model with deterministic interest rate specifications and one model with stochastic interest rates as proposed in Cheng, Nikitopoulos, and Schlögl (2015). The models feature a full correlation structure and fit the entire initial forward curve by construction rather than generating it endogenously from the spot price as required in the spot price models. This class of models has an affine term structure representation that leads to quasi-analytical European vanilla futures option pricing equations. Furthermore, this class of forward price models allows us to model directly multiple futures prices simultaneously.[3] Spot price models require the modelling of the spot price and the interest rates,

[3]Note that it is the presence of convenience yields which permits us to specify directly the prices of

as well as making assumptions about the convenience yield to be able to specify the futures prices. The proposed approach directly models the full term structure of futures prices and provides tractable prices for futures options (something that it would be far more complicated to obtain with spot price models). Thus our model can be estimated by fitting to both futures prices and option prices.

By using extended Kalman filter maximum log-likelihood methodology, the model parameters are estimated from historical time series of both crude oil futures prices and crude oil futures option prices. Due to the large number of parameters required to be empirically estimated, the estimation process is treated in three stages. In stage one, we estimate the parameters of the interest rate models by fitting the implied yields to US Treasury yield rates. In stage two, we firstly run a sensitivity analysis on the correlations to determine how they impact the pricing of long-dated crude oil options. We find that the correlation between the stochastic volatility process and the stochastic interest rate process has negligible impact on prices of long-dated options, hence we assume this correlation to be zero and we do not estimate it. In stage three, we estimate the rest of the model parameters. Finally, we examine in-sample and out-of-sample pricing performance of the proposed models.

To assess the impact of stochastic interest rates under different market conditions, we consider two subsamples; the period $2005-2007$ that was characterised by relatively volatile interest rates and the period $2011-2012$ that exhibited very low interest rates (consequently a very stable interest rate market). According to the numerical analysis performed in Cheng et al. (2015), the volatility of the interest rates itself, rather than the level of the interest rates plays important role to the pricing of long-dated commodity derivatives. By comparing the pricing errors of the two models (deterministic interest rates against stochastic interest rates), we find that the RMSEs from the stochastic interest rates counterpart are lower than the deterministic interest rates counterpart, an effect that holds as the maturity of the crude oil futures options increases. These results are more pronounced during the period of relatively high volatility of the interest rates, consistent with the numerical findings in Cheng et al. (2015). During the period of very low interest rates and interest rate volatility, there are no noticeable differences between the stochastic and deterministic interest rate models. We also investigated the number of model factors required to provide better pricing performance on long-dated crude oil derivatives contracts. We conclude that adding more factors does not improve the pricing performance (see also Schwartz and Smith (1998) and Cortazar et al. (2015)). Finally, there is empirical evidence for a negative correlation between crude oil futures prices and interest rates.

The remaining paper is structured as follows. Section 2 presents a brief description of the proposed term structure models for pricing commodity derivatives and gives a description of the crude oil derivatives data used in our empirical analysis. Section 3 provides the details of the estimation methodology. Section 4 presents the estimation results and discusses the empirical findings of pricing long-dated crude oil derivatives. Section 5 concludes.

futures to several maturities, simultaneously on the same underlying, without introducing inconsistency to the model.

2. Model and data description

2.1. Modelling Commodity Derivatives

This section presents a brief description of the HJM term structure models used in the empirical analysis. The proposed models are both stochastic volatility models but one model features stochastic interest rates (see Cheng et al. (2015) for more details) and the other model is restricted to deterministic interest rate specifications.

2.1.1. The stochastic interest rate model

We consider $\sigma = \{\sigma_t; t \in [0, T]\}$ an n-dimensional stochastic volatility process and we denote as $F(t, T, \sigma_t)$ the futures price of the commodity at time $t \geq 0$, for delivery at time $T \in [t, \infty)$ and as $r = \{r(t); t \in [0, T]\}$ the instantaneous short-rate process. We further consider the extended version of the n-factor Schöbel-Zhu-Hull-White model[4], under the risk-neutral measure, as shown below:

$$\frac{dF(t, T, \sigma_t)}{F(t, T, \sigma_t)} = \sum_{i=1}^{n} \sigma_i^F(t, T, \sigma_t) dW_i^x(t), \tag{1}$$

where, for $i = 1, 2, \ldots, n$,

$$d\sigma_i(t) = \kappa_i(\overline{\sigma}_i - \sigma_i(t))dt + \gamma_i dW_i^\sigma(t), \tag{2}$$

and for $j = 1, 2, \ldots, N$,

$$r(t) = \overline{r}(t) + \sum_{j=1}^{N} r_j(t), \tag{3}$$
$$dr_j(t) = -\lambda_j(t) r_j(t) dt + \theta_j dW_j^r(t).$$

The functional form of the volatility term structure $\sigma_i^F(t, T, \sigma_t)$ is specified as follows:

$$\sigma_i^F(t, T, \sigma_t) = (\xi_{0i} + \xi_i(T - t))e^{-\eta_i(T-t)}\sigma_i(t) \tag{4}$$

with ξ_{0i}, ξ_i, and $\eta_i \in \mathbb{R}$ for all $i \in \{1, \ldots, n\}$. This volatility specification allows for a variety of volatility structures such as exponentially decaying and hump-shaped which are typical volatility structures in commodity market and admits finite dimensional realisations. We denote with $\text{Call}(t, F(t, T, \sigma_t); T_o)$ the price of the European style call option with maturity T_o and strike K on the futures price $F(t, T, \sigma_t)$ maturing at time T. The price of a call option can be expressed as the discounted expected payoff under the risk-neutral measure of the form:

$$\text{Call}(t, F(t, T, \sigma_t); T_o) = \mathbb{E}_t^{\mathbb{Q}}\left(e^{-\int_t^{T_o} r(s)\, ds}\left(e^{X(T_o, T)} - K\right)^+\right) \tag{5}$$

where $X(t, T) = \log F(t, T, \sigma_t)$. By using Fourier inversion technique, Duffie, Pan, and Singleton (2000) provide a semi-analytical formula for the price of European-style vanilla

[4] see van Haastrecht et al. (2009)

options under the class of affine term structure. With a slight modification of the pricing equation in Duffie et al. (2000), the equation (5) can be expressed as[5]:

$$\begin{aligned}\text{Call}(t, F(t,T,\sigma_t); T_o) = e^{-\int_t^{T_o} \overline{r}(s)\,ds} \prod_{i=2}^{N} \mathbb{E}_t^{\mathbb{Q}}\big[e^{-\int_t^{T_o} r_i(s)\,ds}\big] \times \\ \big[G_{1,-1}(-\log K) - K G_{0,-1}(-\log K)\big],\end{aligned} \tag{6}$$

where

$$G_{a,b}(y) = \frac{\phi(t; a, T_o, T)}{2} - \frac{1}{\pi}\int_0^\infty \frac{\Im[\phi(t; a + \mathbf{i}bu, T_o, T)e^{-iuy}]}{u}\,du. \tag{7}$$

The characteristic function ϕ can be expressed as:

$$\begin{aligned}\phi(t; \upsilon, T_o, T) = \exp\big\{&A(t;\upsilon,T_o) + B(t;\upsilon,T_o)X(t,T) + C(t;\upsilon,T_o)r_1(t) \\ &+ \sum_{i=1}^{n} D_i(t;\upsilon,T_o)\nu_i(t) + \sum_{i=1}^{n} E_i(t;\upsilon,T_o)\sigma_i(t)\}.\end{aligned} \tag{8}$$

where the functions $A(t;\upsilon,T_o), B(t;\upsilon,T_o), C(t;\upsilon,T_o), D_i(t;\upsilon,T_o)$ and $E_i(t;\upsilon,T_o)$ in equation (8) satisfy the complex-valued Ricatti ordinary differential equations. Note that $\mathbf{i}^2 = -1$ and $\Im(x + \mathbf{i}y) = y$. The logarithm of the futures prices at time t with maturity T can be expressed in terms of $6n$ state variables, namely $x_i(t)$, $y_i(t)$, $z_i(t)$, $\phi_i(t)$, $\psi_i(t)$ and $\sigma_i(t)$ (see Proposition 1.2 in Cheng et al. (2015)):

$$\begin{aligned}\log F(t,T,\sigma_t) = \log F(0,T,\sigma_0) - \frac{1}{2}\sum_{i=1}^{n}\Big(\gamma_{1i}(T-t)x_i(t) + \gamma_{2i}(T-t)y_i(t) + \gamma_{3i}(T-t)z_i(t)\Big) \\ + \sum_{i=1}^{n}\Big(\beta_{1i}(T-t)\phi_i(t) + \beta_{2i}(T-t)\psi_i(t)\Big),\end{aligned} \tag{9}$$

where for $i = 1, \ldots, n$ the deterministic functions are defined as:

$$\begin{aligned}&\beta_{1i}(T-t) = \big(\xi_{i0} + \xi_i(T-t)\big)e^{-\eta_i(T-t)}, \quad \beta_{2i}(T-t) = \xi_i e^{-\eta_i(T-t)}, \\ &\gamma_{1i}(T-t) = \beta_{1i}(T-t)^2, \quad \gamma_{2i}(T-t) = 2\beta_{1i}(T-t)\beta_{2i}(T-t), \quad \gamma_{3i}(T-t) = \beta_{2i}(T-t)^2,\end{aligned}$$

and the state variables $x_i, y_i, z_i, \phi_i, \psi_i$ satisfy the following SDE:

$$\begin{aligned}dx_i(t) &= \big(-2\eta_i x_i(t) + \sigma_i^2(t)\big)\,dt, \\ dy_i(t) &= \big(-2\eta_i y_i(t) + x_i(t)\big)\,dt, \\ dz_i(t) &= \big(-2\eta_i z_i(t) + 2y_i(t)\big)\,dt, \\ d\phi_i(t) &= -\eta_i\phi_i(t)dt + \sigma_i(t)dW_i^x(t), \\ d\psi_i(t) &= \big(-\eta_i\psi_i(t) + \phi_i(t)\big)dt,\end{aligned} \tag{10}$$

subject to the initial condition $x_i(0) = y_i(0) = z_i(0) = \phi_i(0) = \psi_i(0) = 0$.

[5] The product starts at $i = 2$ because the state variables $r_2, \ldots, r_N$ are uncorrelated to other state variables. However r_1 is correlated.

2.1.2. The deterministic interest rate model

The dynamics of the futures price process and the stochastic volatility process remain the same as specified in equations (1) and (2) but the stochastic interest rate process is replaced by a deterministic discount function. This discount function $P_{NS}(t,T)$ is obtained by fitting a Nelson and Siegel (1987) curve each trading day to US treasure yields with maturities of $1,2,3$ and 5 years similarly to Chiarella et al. (2013). Nelson and Siegel propose a functional form of the time-t instantaneous forward rate as follows:

$$f_{NS}(t,T) = b_0 + b_1 e^{-a(T-t)} + b_2 a(T-t) e^{-a(T-t)}$$

where a, b_0, b_1 and b_2 are parameters to be determined. The yield to maturity on a US treasure bill is:

$$\begin{aligned} y_{NS}(t,T) &= \frac{\int_t^T f_{NS}(t,u)\,du}{T-t} \\ &= b_0 + \frac{(b_1+b_2)(1-e^{-a(T-t)})}{a(T-t)} - b_2 e^{-a(T-t)}. \end{aligned} \tag{11}$$

In each trading day we determine the parameters by minimising the square of the errors between the observable treasury yields and the implied yields $y_{NS}(t,T)$. The discount function for the option with maturity T is simply $P_{NS}(t,T) = e^{-y_{NS}(t,T)(T-t)}$.

2.2. Data description

To better capture the impact of interest rates to the crude oil derivative prices, we select two two-year periods to estimate the model parameters. The first period is from 1st August 2005 to 31st July 2007 as it represents a period of relatively high interest rates (over 4.6% , see Table 1) and high interest rate volatility. The second period is from 1st January 2011 to 31st December 2012 and it represents a period of extremely low interest rates (below 0.5% for maturities under three years, see Table 1). The rationale here is that higher interest rate volatility has more impact on option prices with long maturities. [6]

2.2.1. Interest rate data

We use the US treasury yield rates[7] as the proxy to estimate the parameters in the interest rate process in our model as well as to convert the prices of American options to European options. The reason for this choice over other rates such as LIBOR is that the options in the dataset are exchange-traded options, hence there is no credit risk involved. There are over ten different maturities in the dataset and we choose only four sets of yield rates (the 1-year, 2-year, 3-year and 5-year yields) that best match the maturities of our options. Summary statistics of the yields are presented in Table 1. Note that the standard deviation of the 5-year bond yields estimated in the period between January 2011 and December 2012 (0.539%) is almost two-fold that (0.288%) of the corresponding yield between the period August 2005 and July 2007. The reason for this seemingly higher volatility is due to the fact that the

[6]See Cheng et al. (2015) for a detailed analysis of the impact of different interest rate parameters have on commodity derivative prices.

[7]Data were obtained from www.treasury.gov

5-year yield at the beginning of 2011 is around 2% and it increases to 2.4% in early February 2011 and then it quickly plummets to less than 1% in the beginning of 3rd, 2011. To have a better understand of the volatility of the interest rates during those two periods, we also present the linearly as well as nonlinearly detrended standard deviation of the interest rates. The linearity and nonlinearity are removed by subtracting a least-squares polynomial fit of degree 1 and degree 2 respectively. We observe that the nonlinearly detrended standard deviation of the 5-year yield is lower during 2011 period.

Period: August 2005 - July 2007				
Maturity in years	1	2	3	5
Mean	4.773%	4.682%	4.648%	4.640%
Standard Deviation	0.389%	0.309%	0.297%	0.288%
Detrended S.D.	0.262%	0.250%	0.251%	0.247%
Nonlinear Detrended S.D.	0.115%	0.177%	0.198%	0.210%
Kurtosis	0.411	0.268	0.012	-0.252
Skewness	-1.225	-0.797	-0.558	-0.334
Period: January 2011 - December 2012				
Maturity in years	1	2	3	5
Mean	0.178%	0.363%	0.565%	1.140%
Standard Deviation	0.052%	0.173%	0.310%	0.539%
Detrended S.D.	0.049%	0.122%	0.194%	0.264%
Nonlinear Detrended S.D.	0.036%	0.086%	0.131%	0.172%
Kurtosis	-0.026	0.783	0.343	-0.520
Skewness	0.450	1.415	1.324	0.991

Table 1: **Descriptive Statistics of US Treasury yields.** The table displays the descriptive statistics for the 1-, 2-, 3- and 5-year US treasury yields from August 2, 2005 to July 31, 2007 and from January 1, 2011 to December 31, 2012. The first period represents a period with high volatility of interest rates and the second represents a period with low levels and volatility of interest rates.

2.2.2. Crude oil derivatives data

We use Light Sweet Crude Oil (WTI) futures and options traded on the NYMEX[8] which is one of the richest dataset available on commodity derivatives. The number of available futures contracts per day has increased from 24 on the 3rd January 2005 and reached over 40 in 2013. The futures dataset has 145,805 lines of data and the options dataset has close to 5 million lines of data. Due to the enormous amount of data, for estimation purposes we make a selection of contracts based on their liquidity and we use the open interest of the futures as the proxy of its liquidity. Liquidity is generally very low for contracts with less than 14 days to expiration while for contracts with more than 14 days to expiration

[8]The database has been provided by CME.

liquidity increases significantly. Consequently we use only contracts with more than 14 days to expiration. Figure 2 shows the open interest of the futures contracts by the time-to-maturity for the first nine months and then the open interest of the futures contracts with maturities more than one year by calendar month. It is clear that liquidity is mainly concentrated on short-maturity contracts and on the December contracts. Thus we select the first seven monthly contracts, then the next three contracts with maturity either on March, June, September and December and then all contracts with maturity on December. We also filter out abnormalities such as futures price of zero and zero open interest. Thus on daily basis, we use around $10-15$ futures contracts for the period 2005 to 2007 and $15-17$ futures contracts for the period 2011 to 2012 extending our dataset of futures maturities to eight years.

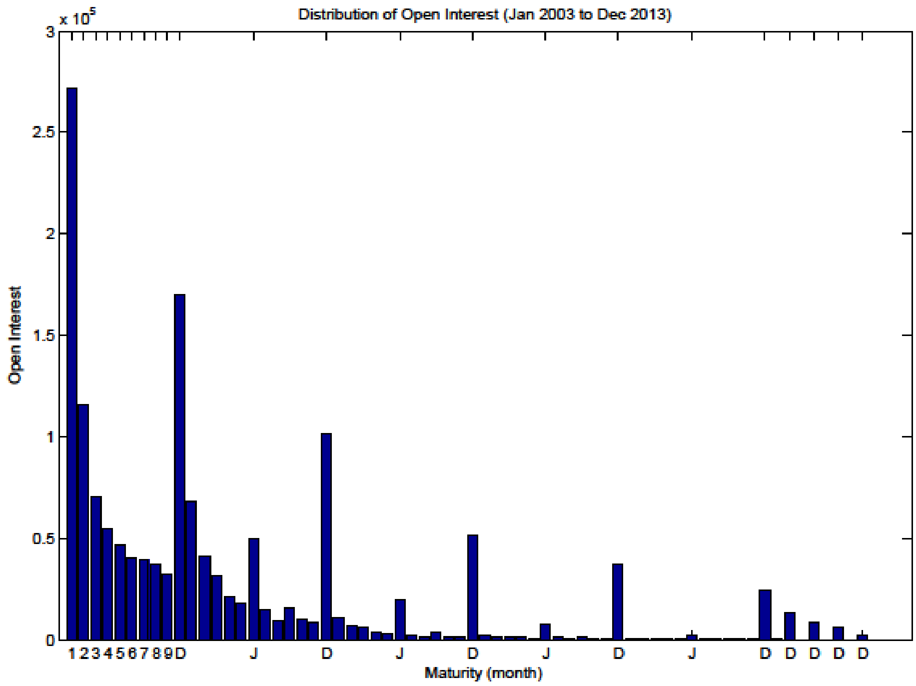

Figure 2: **Liquidity of crude oil futures contracts by calendar month.** The plot shows the liquidity of the first nine months from the trading day as well as December and June contracts in the following years. It shows that futures contracts with next-month delivery date are the most liquid and liquidity gradually decreases over the following months. Futures contracts with maturities in December are very liquid even after a few years and June contracts are moderately liquid. Contracts with maturities more than two years are very illiquid in other months.

For the crude oil futures option dataset, we select the underlying futures contracts used in the crude oil futures dataset. That is, we take options on the first seven monthly futures contracts and the next three contracts with maturity on either March, June, September and all December contracts with maturities of up to 5 years. For each option maturity, we consider six moneyness intervals, 0.86–0.905, 0.905–0.955, 0.955–1.005, 1.005–1.055, 1.055–1.105 and 1.105–1.15. We define moneyness as the option strike divided by the price of the underlying futures contract. In each of the moneyness interval we use only the out-of-the-money (OTM) and at-the-money (ATM) options that are closest to the interval mean. OTM options are generally more liquid. Beside that, the OTM options have lower early exercise approximation errors. On daily basis we use around $50-77$ options for the period $2005-2007$ and $77-95$ options for the period $2011-2012$. After sorting the data, we are left with $74,073$ future contracts and $297,878$ option contracts over a period of $5,103$ trading days.

2.2.3. American to European options conversion

The prices of the options in the dataset are American-style options whereas our pricing model is for European-style options. For the conversion of American prices for European prices, we first back out the implied volatility from the prices of American options using the approximation method provided by Barone-Adesi and Whaley (1987) and then we use the Black (1976) formula together with this implied volatility to compute the prices of the corresponding European options. The only parameter that is not immediately available is the constant interest rate yield covering the corresponding maturity of the options. To find out the constant interest rate yield corresponding to the maturity of the options to be converted, we use the yield of the one-month treasury bill as the instantaneous short-rate together with the interest rate parameters that we estimated previously to calculate the no arbitrage zero coupon bond price corresponding to its maturity. Then we can obtain the constant interest yield from the bond price. Further details are provided in Appendix D.

3. Estimation method

Several methodologies have been proposed in the literature to estimate the parameters of stochastic models such as efficient method of moments, see Gallant, Hsieh, and Tauchen (1997), maximum likelihood estimation, see Chen and Scott (1993) and the Kalman filter method. Duffee and Stanton (2012) perform an extensive analysis and comparison on these methods and conclude that the Kalman filter is the best method among these three. In this paper, we adopt the Kalman filter methodology to estimate the parameters of our model. For the purpose of parameter estimation, we let $\overline{r}(t), \lambda_i(t)$ and $\theta_i(t)$ to be constants for all i, i.e. $\overline{r}(t) = \overline{r}, \lambda_i(t) = \lambda_i$ and $\theta_i(t) = \theta_i, \forall i$. With these time-varying functions taken as constants the option pricing formula (6) is reduced to:

$$\mathrm{Call}(t, F(t,T,\sigma_t); T_o) = e^{-\overline{r}(T_o-t)} \prod_{i=2}^{N} \exp(-A_i(T_o-t)r_i(t) + D_i(T_o-t)) \times \\ \big[G_{1,-1}(-\log K) - K G_{0,-1}(-\log K)\big], \tag{12}$$

where

$$A_i(\tau) = \frac{1-e^{-\lambda_i \tau}}{\lambda_i},$$

$$D_i(\tau) = \Big(-\frac{\theta_i^2}{2\lambda_i^2}\Big)\Big(A_i(\tau) - \tau\Big) - \frac{\theta_i^2 A_i(\tau)^2}{4\lambda_i}.$$

There are $34 + 2N$ [9] parameters that need to be estimated in our 3-factor model and $24 + 2N$ for the 2-factor model. Due to the large number of parameters, it is computationally expensive to estimate all the parameters at once. Thus, we subdivide the task into three steps as follow.

1. We firstly estimate the stochastic interest rate process by using US Treasury yield rates.
2. We perform a sensitivity analysis over the correlations of the model to determine their impact on futures and option prices and see if there are any correlations with negligible impact on the prices.
3. We estimate the remaining of the model parameters.

We estimate the parameters of the interest rate dynamics by using Kalman filter and maximum likelihood estimates (see Appendix A for details). The estimation method for the futures price volatility processes σ_i^F and the volatility processes $\sigma_i(t)$ involves using the extended Kalman filter and maximum likelihood (see Appendix C for details), where the model is re-expressed in a state-space form which consists of the system equations and the observation equation and then maximum likelihood method is employed to estimate the state variables.
The system equations describe the evolution of the underlying state variables. In this model, the state vector is [10] $X_t = \{X_t^i, i = 1, 2, \ldots, n\}$ where X_t^i consists seven state variables $x_i(t), y_i(t), z_i(t), \phi_i(t), \psi_i(t), \sigma_t^i$ and ν_t^i, see (10). This system can be put in a state-space discrete evolution form as follow:

$$\mathbf{X}_{t+1} = \mathbf{\Phi}_0 + \mathbf{\Phi}_X \mathbf{X}_t + \omega_{t+1}. \tag{13}$$

The ω_t with $t = 0, 1, 2, \ldots$ are independent to each other, with zero means and with the covariance matrices conditional on time t a deterministic function of the state variable $\sigma(t)$.
The observation equation is:

$$\mathbf{Z}_t = h(\mathbf{X}_t, \mathbf{u}_t),\ \mathbf{u}_t \sim \text{i.i.d. } N(\mathbf{1}, \mathbf{\Omega}), \tag{14}$$

where $\mathbf{u}_t$ is a vector of i.i.d. multiplicative Gaussian measurement errors with covariance matrix Ω. The observation equation can be constructed by (9) which relates log futures prices linearly to the state variables $x_i(t), y_i(t), z_i(t), \phi_i(t)$ and $\psi_i(t)$. However equation (6) relates option prices to the state variables through nonlinear expressions. The application of the extended Kalman filter for parameter estimation involves linearising the observation function h and making the assumption that the disturbance terms $\{\omega_t\}_{t=0,1,\ldots}$ in equation (13) follow the multivariate normal distributions. From the Kalman filter recursions, we can compute the likelihood function.

[9] Three of each $\xi_0, \xi, \eta, \kappa, \gamma, \rho_{xr}, \rho_{x\sigma}, \rho r\sigma, \Lambda, \Lambda^\sigma$, N of each λ_i, θ_i and one of each $\overline{r}, f_0, \sigma_f, \sigma_o$. $\Lambda, \Lambda^\sigma, f_0, \sigma_f, \sigma_o$ are the market price of risk, market price of volatility risk, initial time-homogenous futures curve, measurement noise of the futures and options respectively. See Appendix Appendix B and Appendix C for details

[10] Note that we use $X(t, T)$ as the logarithm of the futures prices in (5). In this section on Kalman filter, we re-define X_t to be a vector of state variables.

When interest rates are assumed to be stochastic, the interest rates needed for discounting at each future date is calculated from the estimated values of the state variables of the interest rate model. In particular, we have the estimated parameters of the interest rate model $\bar{r}$, λ_j, θ_j, where $j = 1, 2, 3$ (for the 3-factor model specifications). On each date, Kalman filter updates $r_j(t)$. With known parameters and state variables, we use equation (6) to calculate the theoretical option prices. Note that r_2 and r_3 are independent so they are not in the main option pricing function $G_{a,b}(y)$. When interest rates are assumed to be deterministic, the rates used for discounting at each future date are specified from equation (11).

3.1. Computational details

The program is written in Matlab. The log-likelihood function is maximised by using Matlab's "fminsearch" routine to search for the minimal point of the negative of the log-likelihood function. "fminsearch" is an unconstrained nonlinear optimisation routine and it is derivative-free. The Ricatti ordinary differential equations of the characteristic function ϕ in equation (8) is solved by Matlab's "ode23" which is an automatic step-size Runge-Kutta-Fehlberg integration method. This method uses lower order formulae comparing to other ODE-methods which can be less accurate, but the advantage is that this method is fast comparing to other methods. The integral in (6) is computed by the Gauss-Legendre quadrature formula with 19 integration points and truncating the integral at 33 and we find that these numbers of integration points and truncation of the integral provide a good trade-off between computational time and accuracy.

3.1.1. Methods to reduce computational time

One of the biggest challenges is the formidable amount of computational time required for the estimation of parameters. Although this model admits quasi-analytic solutions for option pricing, however complex-valued numerical ODE approximation together with complex-valued numerical integral are needed to be performed for each option price. Further for each day of the crude oil data, numerical Jacobian is needed to be calculated for the linearisation of option prices in the Kalman filter update. To complete one day of the data, which typically involves the calculation of around 70 options and its Jacobian for Kalman filter update may take 10 to 15 mins on a desktop running a second generation quad-core i7 processer. So that is about $5,000$ to $7,500$ minutes for the program to process two years (about 500 trading days) of data in order to calculate one log-likelihood. Matlab's "fminsearch" routine may take several hundreds of iterations for it to converge to a local maximum. The key observation to massively reduce the computational time is that given a set of parameters of our model the characteristic function $\phi(t; a + \mathbf{i}bu, T_o, T)$ (see (7))is a function of a, b, u, T_o, T. T_o and T are set to be the same because all the crude oil options traded in CME expire only a few days before the underlying futures contracts. So for each iteration we precalculate six tables for the characteristic functions. These are $\phi(t; 0, T, T)$, $\phi(t; 1, T, T)$, $\phi(t; 1 - \mathbf{i}u, T, T)$, $\phi(t; -\mathbf{i}u, T, T)$, $\phi(t; 1 + \mathbf{i}u, T, T)$ and $\phi(t; \mathbf{i}u, T, T)$ where values of the variable u are determined by the 19 integration points on the interval from 0 to 33 calculated by using Gauss-Legendre quadrature and the values of the maturity T are $14, 15, 16, \ldots, 1850$ because the shortest maturity is only 14-day and the longest maturity is five-year. Observing the fact that each calculation of the characteristic function is independent of each other, we also take advantage of the parallel toolbox available in Matlab by using Matlab's "parfor" loop. This reduces the time required to process one iteration (2 years of data) from $5,000$ to $7,500$ minutes to around 2 minutes. The total time required for the parameters to converge to a local maximum could still take a few days.

4. Estimation results

This section provides estimation results of the parameters of our model and also the in-sample and out-of-sample pricing performance of the model.

4.1. Interest rate process

To determine the most suitable number of factors, we first use Kalman filter to estimate the parameters of a 1-, 2- and 3-factor affine term structure models for the interest rate process. The details of the estimation process can be found in Appendix A and the results are summarised in Table 2. We choose the 3-factor affine term structure model for our interest rate process because the three-factor model provides the best fit for all maturities and it aligns with the literature, see for example Litterman and Scheinkman (1991). From the results, the estimated long-term mean level $\overline{r}$ in the first period is 4.96% which is a lot higher than in the second period -0.20%. The parameter estimates are consistent with the statistical properties of the interest rates over these two periods; one period with high levels and volatility of interest rates and the other period with much lower interest rates.

4.2. Sensitivity analysis of the correlations

A sensitivity analysis on the correlations ρ^{xr} and $\rho^{r\sigma}$ is performed to determine their impact on the futures and option prices. To perform the analysis we firstly estimate the models assuming zero correlations between the futures price process and the interest rate process (i.e. $\rho_i^{xr} = 0$) for both periods ($2005-2007$ and $2011-2012$). The estimation results of fitting the two-factor model and the three-factor model to crude oil futures and options are shown on Table 3 and Table 4, respectively. Then we use the parameters estimated in these two models to price call options with different correlation structure. In our example, we choose an ITM option with a maturity of 4,000 days and a strike of \$100. These numbers are chosen for illustration purposes only and the results are presented in Table 5.

We make several observations from this analysis. The first is that the impact of the correlation between the stochastic interest rate process and the stochastic volatility ($\rho^{r\sigma}$) to the option prices is insignificant even for maturity as long as 4000 days. For $\rho^{xr} = 0$[11], we see that the percentage difference of the option price (comparing $\rho^{r\sigma} = 0.50$ and $\rho^{r\sigma} = 0$) is $(10.51 - 10.416)/10.51 = 0.894\%$ for the period $2005 - 2007$ and the percentage difference of the option price (comparing $\rho^{r\sigma} = 0.50$ and $\rho^{r\sigma} = 0$) is $(28.201 - 28.187)/28.201 = 0.05\%$ for the period $2011 - 2012$. The second observation is that the correlation between stochastic interest rate process and the underlying futures price process (ρ_{xr}) has some effect to the option prices. More specifically, the impact to the option prices in the period $2005-2007$ is more than twice compared to the impact in the period from $2011-2012$. For

[11] Throughout this section, we use the shorthanded notation $\rho^{xr} = 0$ to refer to $\rho_1^{xr} = \rho_2^{xr} = \rho_3^{xr} = 0$, and the same interpretation for $\rho^{r\sigma}$.

Period 1: Aug 05 – Jul 07

	1-factor	2-factor		3-factor		
	$i=1$	$i=1$	$i=2$	$i=1$	$i=2$	$i=3$
λ_{0i}	0.0492	-0.0401	3.9081	0.0752	0.2766	1.5860
θ_i	0.0058	0.0058	0.0164	0.0152	0.0080	0.0130
$\overline{r}$	4.1592%	3.9112%		4.9611%		
$\log \mathrm{L}$	11255	12803		13240		
rmse 1yr	4.1033%	1.0425%		0.4885%		
rmse 2yr	1.1798%	0.4110%		0.2823%		
rmse 3yr	0.6045%	0.3392%		0.1559%		
rmse 5yr	0.5367%	0.1346%		0.0476%		

Period 2: Jan 11 – Dec 12

	1-factor	2-factor		3-factor		
	$i=1$	$i=1$	$i=2$	$i=1$	$i=2$	$i=3$
λ_{0i}	-0.6491	-0.3913	0.5144	0.1223	-0.4987	-1.2297
θ_i	0.0010	0.0027	0.0025	0.0021	0.0019	3.98E-05
$\overline{r}$	0.0700%	-0.2828%		-0.2018%		
$\log \mathrm{L}$	11602	13057		13595		
rmse 1yr	94.7269%	22.5815%		15.3528%		
rmse 2yr	18.4326%	5.6910%		3.8036%		
rmse 3yr	7.6132%	2.8091%		1.2320%		
rmse 5yr	1.0640%	0.3006%		0.0661%		

Table 2: **Parameter estimates of the interest rate process.** The table displays the parameter estimates of the interest rate process using a one-, two- and three-factor model over two periods. The first period represents a period with high volatility of interest rates and the second represents a period with low levels and volatility of interest rates. Three-factor models provide the best fit for all maturities.

	Period 1: Aug 2005 – Jul 2007		Period 2: Jan 2011 – Dec 2012	
	$i = 1$	$i = 2$	$i = 1$	$i = 2$
ξ_{0i}	0.0564	1.2693	0.9683	0.1988
ξ_i	0.1034	0.0297	0.0015	0.0046
η_i	0.0872	0.1892	0.0049	0.2704
κ_i	0.0859	0.0193	0.0053	0.1222
γ_i	0.0181	0.0488	0.0205	0.2114
ρ_i^{xr}	0	0	0	0
$\rho_i^{x\sigma}$	-0.4126	-0.4416	-0.4376	0.0127
Λ_i	0.1145	0.4785	-0.0130	0.2238
Λ_i^{σ}	-0.0090	-0.0917	-2.1178	0.7654
f_0	7.305		3.675	
σ_f	2.00%		2.00%	
σ_O	6.87%		6.95%	
$\log \mathrm{L}$	-60139		-62160	
RMSE Futures	2.8635%		1.6565%	
RMSE Imp.Vol. 4mth	1.2075%		1.7917%	
RMSE Imp.Vol. 12mth	1.4787%		1.7302%	
RMSE Imp.Vol. 2yr	1.9997%		1.4997%	
RMSE Imp.Vol. 3yr	2.3210%		0.9885%	
RMSE Imp.Vol. 4yr	2.4634%		1.0235%	
RMSE Imp.Vol. 5yr	2.6908%		1.3064%	

Table 3: **Parameter estimates of a two-factor model for crude oil futures and options with $\rho^{xr} = 0$.** The table displays the maximum-likelihood estimates for the two-factor model specifications over the two two-year periods, namely, August, 2005 to July, 2007 and January 2011 to December 2012. The model assumes zero correlations between the futures price process and the interest rate process. Note that f_0 is the time-homogenous futures price at time 0, namely $F(0, t) = f_0, \forall t$. The quantities σ_f and σ_o are the standard deviations of the log futures prices measurements errors and the option price measurement errors, respectively. We normalised the long run mean of the volatility process, $\overline{\sigma}_i$, to one to achieve identification.

	Period 1: Aug 2005 – Jul 2007			Period 2: Jan 2011 – Dec 2012		
	i=1	i=2	i=3	i=1	i=2	i=3
ξ_{0i}	0.0295	0.3164	2.0226	0.6734	0.3183	0.0143
ξ_i	0.2710	0.0334	0.0068	0.0037	0.0104	0.0519
η_i	0.1748	0.3526	0.0209	0.0053	0.2972	0.0597
κ_i	0.0287	0.0670	-0.0319	0.0105	0.1606	0.0025
γ_i	-0.0311	1.0054	0.0064	-0.0516	-0.2799	0.0750
ρ_i^{xr}	0	0	0	0	0	0
$\rho_i^{x\sigma}$	-0.0894	-0.0962	-0.1767	0.4008	0.0015	0.0159
Λ_i	-0.3211	1.6172	-16.4285	0.2504	0.0037	-0.0586
Λ_i^σ	1.1087	0.1652	-1.8788	-0.2761	0.8139	-1.0503
f_0		9.0851			5.5110	
σ_F		1.27%			1.66%	
σ_O		3.15%			11.34%	
log L		-38670			-31096	
RMSE Futures		1.2995%			1.2535%	
RMSE Imp.Vol. 4mth		1.1827%			1.7927%	
RMSE Imp.Vol. 12mth		1.6173%			1.7064%	
RMSE Imp.Vol. 2yr		1.9285%			1.5372%	
RMSE Imp.Vol. 3yr		2.1934%			1.0613%	
RMSE Imp.Vol. 4yr		2.5390%			0.8927%	
RMSE Imp.Vol. 5yr		2.9145%			0.9096%	

Table 4: **Parameter estimates of a three-factor model for crude oil futures and options with $\rho^{xr} = 0$.** The table displays the maximum-likelihood estimates for the three-factor model specifications over the two two-year periods, August, 2005 to July, 2007 and January 2011 to December 2012. The model assumes zero correlations between the futures price process and the interest rate process. Note that f_0 is the time-homogenous futures price at time 0, namely $F(0, t) = f_0, \forall t$. The quantities σ_f and σ_o are the standard deviations of the log futures prices measurements errors and the option price measurement errors, respectively. We normalised the long run mean of the volatility process, $\overline{\sigma}_i$, to one to achieve identification.

Period 2005-2007

		$\rho^{r\sigma}$						
		-0.50	-0.30	-0.15	0	0.15	0.30	0.50
ρ^{xr}	-0.50	10.967	10.922	10.889	10.857	10.825	10.794	10.753
	-0.30	10.823	10.78	10.748	10.717	10.686	10.656	10.617
	-0.15	10.717	10.675	10.644	10.613	10.583	10.554	10.516
	0	10.611	10.57	10.54	10.51	10.481	10.453	10.416
	0.15	10.506	10.466	10.437	10.408	10.38	10.353	10.317
	0.30	10.403	10.364	10.335	10.307	10.28	10.253	10.218
	0.50	10.266	10.228	10.201	10.174	10.148	10.122	10.089

Period 2011-2012

		$\rho^{r\sigma}$						
		-0.50	-0.30	-0.15	0	0.15	0.30	0.50
ρ^{xr}	-0.50	28.605	28.6	28.595	28.591	28.587	28.583	28.577
	-0.30	28.449	28.443	28.439	28.435	28.43	28.426	28.421
	-0.15	28.332	28.326	28.322	28.318	28.314	28.309	28.304
	0	28.215	28.209	28.205	28.201	28.197	28.193	28.187
	0.15	28.098	28.093	28.089	28.085	28.081	28.077	28.071
	0.30	27.982	27.977	27.973	27.969	27.965	27.961	27.955
	0.50	27.828	27.823	27.819	27.815	27.811	27.807	27.802

Table 5: **Call Option prices for varying correlation coefficients.** Futures=100, strike=100, maturity=4000 days. We denote for example $\rho^{xr} = -0.50$ to mean $\rho_1^{xr} = \rho_2^{xr} = \rho_3^{xr} = -0.50$ and similarly for $\rho^{r\sigma}$.

instance, the percentage difference of the option price (comparing $\rho^{xr} = 0$ and $\rho^{xr} = -0.50$) is $(10.51 - 10.857)/10.51 = -3.30\%$ for the period $2005 - 2007$ and the percentage difference of the option price (comparing $\rho^{xr} = 0$ and $\rho^{xr} = -0.50$) is $(28.201 - 28.591)/28.201 = -1.38\%$ for the period $2011 - 2012$. Since the impact of the correlation between the stochastic interest rate process and the stochastic volatility to the option price is negligible, we set $\rho_1^{r\sigma} = \rho_2^{r\sigma} = \rho_3^{r\sigma} = 0$ (see also Cheng et al. (2015) for similar observations) and we only estimate ρ_i^{xr}.

4.3. The correlation ρ_i^{xr}

Table 6 and Table 7 present the estimates of a two-factor and a three-factor model, respectively, when the correlation coefficient ρ_i^{xr} between the futures price process and interest rate process is non-zero. We find that these correlations are quite high ranging from -0.64 to 0.59 underscoring the important relationship between interest rates and futures prices. We observe that, especially over the high interest rate volatility period $2005 - 2007$, these correlations are always negative. Studies such as Akram (2009), Arora and Tanner (2013) and Frankel (2014) provide empirical evidence for a negative relationship between between oil prices and interest rates. Akram (2009) conducts an empirical analysis based on structural VAR models estimated on quarterly data over the period $1990 - 2007$. One of his results suggests that there is a negative relationship between the real oil prices and real interest rates. The empirical results from Arora and Tanner (2013) suggest that oil price consistently falls with unexpected rises in short-term real interest rates through the whole sample period from 1975 to 2012. Another result their paper suggests is that oil prices have become more responsive to long-term real interest rates over time. Frankel (2014) presents and estimates a "carry trade" model of crude oil prices and other storable commodities. Their empirical results support the hypothesis that low interest rates contribute to the upward pressure on real commodity prices via a high demand for inventories. Even though our empirical analysis does not refer to the correlations between the actual financial observables as the above studies do, it reveals a negative correlation between innovations of the crude oil futures prices and innovations in the interest rates process. This implies that crude oil futures prices and crude oil spot prices have similar response to changes in the interest rates.

We compare the results with the models ignoring the correlation coefficient between the futures price process and interest rate process, thus assume $\rho_i^{xr} = 0$, see Table 3 and Table 4. In the low interest rate period $2011 - 2012$ there is no improvement in the fit of both futures prices and implied volatility by incorporating the correlation ρ_i^{xr}. This is mainly because during that period interest rate and its volatility are very low and the interest rate process has very little impact in the option prices. However, we observe some improvement in the root-mean-square error of the implied volatility in the period $2005 - 2007$. For instance, the root-mean-square error of the implied volatility of options with five years to maturity improves from 2.6908% to 2.4606%. For periods where the interest rates would be more volatile, we would expect a more substantial improvement as it has been shown in Cheng et al. (2015).

4.4. Pricing performance of long-dated derivatives

From Table 6 and Table 7, we note that the three-factor model outperforms the two-factor model in the root-mean-square error of futures prices for the period $2005 - 2007$ and

	Period 1: Aug 2005 – Jul 2007		Period 2: Jan 2011 – Dec 2012	
	$i=1$	$i=2$	$i=1$	$i=2$
ξ_{0i}	0.0564	1.2693	0.9683	0.1988
ξ_i	0.1034	0.0297	0.0015	0.0046
η_i	0.0872	0.1892	0.0049	0.2704
κ_i	0.0859	0.0193	0.0053	0.1222
γ_i	0.0181	0.0488	0.0205	0.2114
ρ_i^{xr}	-0.6166	-0.3014	0.5420	-0.6409
$\rho_i^{x\sigma}$	-0.4126	-0.4416	-0.4376	0.0127
Λ_i	0.1145	0.4785	-0.0130	0.2238
Λ_i^{σ}	-0.0090	-0.0917	-2.1178	0.7654
f_0	7.305		3.675	
σ_f	2.00%		2.00%	
σ_O	6.87%		6.95%	
$\log \mathrm{L}$	-59977		-61299	
RMSE Futures	2.8408%		1.6563%	
RMSE Imp.Vol. 4mth	1.2174%		1.7916%	
RMSE Imp.Vol. 12mth	1.4736%		1.7233%	
RMSE Imp.Vol. 2yr	1.9605%		1.4987%	
RMSE Imp.Vol. 3yr	2.2144%		0.9987%	
RMSE Imp.Vol. 4yr	2.2947%		1.0082%	
RMSE Imp.Vol. 5yr	2.4606%		1.2727%	

Table 6: **Parameter estimates of a two-factor model for crude oil futures and options.** The table displays the maximum-likelihood estimates for the two-factor model specifications for the two two-year periods, namely, August, 2005 to July, 2007 and January 2011 to December 2012. Here f_0 is the time-homogenous futures price at time 0, namely $F(0,t) = f_0, \forall t$. The quantities σ_f and σ_o are the standard deviations of the log futures prices measurements errors and the option price measurement errors, respectively. We normalise the long run mean of the volatility process, $\overline{\sigma}_i$, to one to achieve identification.

	Period 1: Aug 2005 – Jul 2007			Period 2: Jan 2011 – Dec 2012		
	i=1	i=2	i=3	i=1	i=2	i=3
ξ_{0i}	0.0295	0.3164	2.0226	0.6734	0.3183	0.0143
ξ_i	0.2710	0.0334	0.0068	0.0037	0.0104	0.0519
η_i	0.1748	0.3526	0.0209	0.0053	0.2972	0.0597
κ_i	0.0287	0.0670	-0.0319	0.0105	0.1606	0.0025
γ_i	-0.0311	1.0054	0.0064	-0.0516	-0.2799	0.0750
ρ_i^{xr}	-0.6222	-0.3822	-0.5444	-0.3167	0.59167	-0.4583
$\rho_i^{x\sigma}$	-0.0894	-0.0962	-0.1767	0.4008	0.0015	0.0159
Λ_i	-0.3211	1.6172	-16.4285	0.2504	0.0037	-0.0586
Λ_i^σ	1.1087	0.1652	-1.8788	-0.2761	0.8139	-1.0503
f_0		9.0851			5.5110	
σ_F		1.27%			1.66%	
σ_O		3.15%			11.34%	
$\log \mathrm{L}$		-35893			-31372	
RMSE Futures		1.3067%			1.2452%	
RMSE Imp.Vol. 4mth		1.1792%			1.7925%	
RMSE Imp.Vol. 12mth		1.6091%			1.7078%	
RMSE Imp.Vol. 2yr		1.8815%			1.5345%	
RMSE Imp.Vol. 3yr		2.0380%			1.0497%	
RMSE Imp.Vol. 4yr		2.3487%			0.8897%	
RMSE Imp.Vol. 5yr		2.6539%			0.9139%	

Table 7: **Parameter estimates of a three-factor model for crude oil futures and options.** The table displays the maximum-likelihood estimates for the three-factor model specifications for the two two-year periods, August, 2005 to July, 2007 and January 2011 to December 2012. Here f_0 is the time-homogenous futures price at time 0, namely $F(0,t) = f_0, \forall t$. The quantities σ_f and σ_o are the standard deviations of the log futures prices measurements errors and the option price measurement errors, respectively. We normalise the long run mean of the volatility process, $\overline{\sigma}_i$, to one to achieve identification.

the period 2011 − 2012. No improvement can be observed in the root-mean-square error of the implied volatility between the two-factor model and the three-factor model in both periods for shorter maturity options. For the period 2005 − 2007, the two-factor model seems to perform slightly better for the very long maturities of the 4-year and 5-year implied volatility. Thus, as it has been also shown in Schwartz and Smith (1998) and Cortazar et al. (2015), adding more factors does not improve pricing on long-dated commodity contracts.

We also examine the impact of including stochastic interest rate in a commodity pricing model. We compare the pricing performance of our proposed three-factor models that include stochastic volatility / stochastic interest rates with equivalent models (same parameter estimates as the proposed models) but with deterministic interest rate (i.e. discounted by the corresponding treasury yields). We also use the model parameters estimated in the previous section to assess out-of-sample performance by re-run the scenarios with extended data. The data extends from previously 31st July 2007 to 31st December 2007 for the first period and from 31st December 2012 to 31st December 2013 for the second period. Figure 4 displays the average of the daily root-mean-square error of the implied volatility over all maturities and for the two sample periods used in our analysis. Figure 4 and Figure 5 display the average of the daily root-mean-square error of the implied volatility for 2, 3, 4 and 5 years, respectively. Since the liquidity of the crude oil long-dated contracts is concentrated in the December contracts, we assess the model fit to December contracts which are in the second, third, forth and fifth year to maturity. The results of the in-sample and out-of-sample analysis are also included. From the graphs two observations can be made. The first observation is that models that incorporate stochastic interest rate have a noticeable impact to pricing performance as the maturity increases. The second observation is that considering stochastic interest rate is relevant when interest rate volatility is at least moderately high; during the low interest rate volatility period, minimal improvement in pricing performance can be observed. The results are tabulated in Table 8.

	Period 1: Aug 05 – Jul 07				Period 2: Jan 11 – Dec 12			
	IS		OFS		IS		OFS	
	Sto	Det	Sto	Det	Sto	Det	Sto	Det
4mth	1.18%	1.18%	1.30%	1.34%	1.75%	1.75%	1.88%	1.88%
12mth	1.61%	1.63%	2.43%	2.59%	1.72%	1.72%	1.64%	1.67%
2year	1.90%	1.98%	2.38%	2.58%	1.53%	1.51%	1.59%	1.65%
3year	2.39%	2.68%	2.50%	3.15%	1.27%	1.18%	1.43%	1.60%
4year	2.75%	3.13%	2.87%	3.77%	1.02%	0.89%	1.54%	1.81%
5year	3.11%	3.58%	3.23%	4.06%	1.04%	0.89%	1.91%	2.26%

Table 8: **RMSE of implied volatility.** The table displays the RMSE between the observed implied volatility and the implied volatility from the estimated model for different maturities. In-sample and out-of-sample analysis is included.

5. Conclusion

We empirically assess, on long-dated crude oil derivatives, the pricing performance of modelling stochastic interest rates in addition to stochastic volatility. We consider the stochastic volatility – stochastic interest rate, forward price model developed in Cheng et al. (2015) and the nested version of deterministic interest rate specifications. The model parameters were estimated from historical time series of crude oil futures prices and option prices over two periods characterised by different interest rate market conditions; one period with high interest rate volatility and one period with very low interest rate volatility.

Several observations can be drawn from our empirical analysis. First, stochastic volatility forward price models that incorporate stochastic interest rates improve pricing performance on long-dated crude oil derivatives. Second, this improvement to the pricing performance is more pronounced when the interest rate volatility is high. Third, three-factor models provide better fit to futures prices of all maturities compared to two-factor models. However, for long-dated crude oil option prices, more factors do not tend to improve the pricing performance. Fourth, there is empirical evidence for a negative correlation between the futures price process and the interest rate process especially over periods of high interest rate volatility. Fifth, the correlation between futures price volatility process and the interest rate process has no impact to the pricing of long-dated crude oil contracts.

The empirical results presented in this paper can provide useful insights to practitioners. Adding more factors in a pricing model may provide better fit to historical market data, but it comes with additional computational effort. Our results show that for long-dated crude oil commodities adding more factors does not improve the pricing performance thus models with at least two-factors will provide a good fit. Another point that would be of interest to practitioners is that stochastic interest rates matter for crude oil derivatives with maturities of two years and above and only if the volatility of interest rates is high. Otherwise using deterministic interest rates will be sufficient. From a historical perspective, interest rates experience high volatility from time to time, for instance short-interest rate volatility has reached the highest levels in early eighties with volatilities reaching 56.4 basis, see Roley and Troll (1983). While it is quite rare to encounter cases with such a high interest rate volatility especially in current financial conditions, modelers should recognize the impact of stochastic interest rates when pricing long-dated commodity derivative contracts.

Appendix A. Estimation of n-dimensional affine term structure models

We use Kalman filter to estimate the parameters for the N-factor affine term structure models. Then we estimate and compare their corresponding log-likelihoods and root-mean-square errors. The n-factor affine term structure models have the form:

$$
\begin{aligned}
r(t) &= \overline{r} + \sum_{i=1}^{N} r_i(t), \\
dr_i(t) &= -\lambda_i r_i(t)dt + \theta_i \, dW_i(t), \; i \in 1, 2, \ldots, N \\
dW_i \, dW_j &= \begin{cases} dt & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

Solving the SDE of $r_i(t)$ we get:

$$
r_i(t) = r_i(s) e^{-\lambda_i (t-s)} + \theta_i \int_s^t e^{-\lambda_i (t-u)} \, dW_i(u).
$$

Redefining the notations $r_t^i \triangleq r_i(t)$ and $r_{t-1}^i \triangleq r_i(t - \Delta t)$ where $\Delta t = t - s$, it can be re-expressed into a state-space form as follow:

$$
r_t^i = \Phi_i r_{t-1}^i + \omega_{it}, \; \omega_{it} \sim \text{iid} \, N(0, Q_i),
$$

where

$$
\begin{aligned}
\Phi_i &= e^{-\lambda_i \Delta t}, \\
Q_i &= \frac{\theta_i^2}{2\lambda_i} \big(1 - e^{-2\lambda_i \Delta t}\big) = \frac{\theta_i^2}{2\lambda_i} \big(1 - \Phi_i^2\big).
\end{aligned}
$$

In the real world, the instantaneous risk-free rates do not exist. However, the government bonds with time-to-maturities $\tau = \{\tau_1, \tau_2, \ldots, \tau_n\}$ are observable and are traded frequently in exchanges. They can be used as proxies to estimate the parameters in our model. We choose the bonds with maturities of one year, two years, three years and five years to be our proxies. Now, let the time-to-maturity of a j-year bond be τ_j then the arbitrage-free bond pricing formula can be expressed as:

$$
\begin{aligned}
B(r_t, T_j) &= \mathbb{E}^{\mathbb{Q}} \big[e^{-\int_t^{t+\tau_j} r(u) \, du} | \mathcal{F}_t \big], \text{with } T_j = t + \tau_j \\
&= e^{-\overline{r}\tau_j} \prod_{i=1}^{N} \mathbb{E}^{\mathbb{Q}} \big[e^{-\int_t^{t+\tau_j} r_i(u) \, du} | \mathcal{F}_t \big], \\
&= e^{-\overline{r}\tau_j} \prod_{i=1}^{N} \exp(-A_i(\tau_j) r_t^i + D_i(\tau_j)), \qquad \text{(A.1)} \\
A_i(\tau_j) &= \frac{1 - e^{-\lambda_i \tau_j}}{\lambda_i}, \\
D_i(\tau_j) &= \Big(-\frac{\theta_i^2}{2\lambda_i^2} \Big) \Big(A_i(\tau_j) - \tau_j \Big) - \frac{\theta_i^2 A_i(\tau_j)^2}{4\lambda_i}.
\end{aligned}
$$

The objective now is that given the observable bond prices, Kalman filter is used together with maximum likelihood method to estimate the parameters $\overline{r}, \lambda_i$ and θ_i.
Let the observable bond price at time t with time-to-maturities τ_j be $z(t, T_j)$ where $T_j = t + \tau_j$ and define the logarithm of this bond price to be $Z(t, T_j) = \log z(t, T_j)$. So that the logarithm of bond price is a linear function of the hidden state variable r_t. We further let $Y(t, T_j) = Z(t, T_j) + \overline{r}\tau_j - \sum_{i=1}^{N} D_i(\tau_j)$. The Kalman filter equation becomes:

$$\begin{bmatrix} r_t^1 \\ r_t^2 \\ \vdots \\ r_t^N \end{bmatrix} = \begin{bmatrix} \Phi_1 & 0 & \dots & 0 \\ 0 & \Phi_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Phi_N \end{bmatrix} \begin{bmatrix} r_{t-1}^1 \\ r_{t-1}^2 \\ \vdots \\ r_{t-1}^N \end{bmatrix} + \omega_t, \ \omega_t \sim \text{iid}\, N(0, Q),$$

$$\begin{bmatrix} Y(\tau_1) \\ Y(\tau_2) \\ Y(\tau_3) \\ Y(\tau_5) \end{bmatrix} = - \begin{bmatrix} A_1(\tau_1) & A_2(\tau_1) & \dots & A_N(\tau_1) \\ A_1(\tau_2) & A_2(\tau_2) & \dots & A_N(\tau_2) \\ A_1(\tau_3) & A_2(\tau_3) & \dots & A_N(\tau_3) \\ A_1(\tau_5) & A_2(\tau_5) & \dots & A_N(\tau_5) \end{bmatrix} \begin{bmatrix} r_t^1 \\ r_t^2 \\ \vdots \\ r_t^N \end{bmatrix} + u_t,$$

$$u_t \sim \text{iid}\, N(0, \Omega),$$

where

$$Q = \begin{bmatrix} Q_1 & 0 & \dots & 0 \\ 0 & Q_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & Q_N \end{bmatrix}$$

$$\Omega = \begin{bmatrix} \omega^2 & 0 & \dots & 0 \\ 0 & \omega^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \omega^2 \end{bmatrix}.$$

The above Kalman filter equation can be re-expressed in matrix notation as:

$$r_t = \Phi r_{t-1} + \omega_t, \ \omega_t \sim \text{iid}\, N(0, Q),$$
$$Y(\tau) = -A r_t + u_t, u_t \sim \text{iid}\, N(0, \Omega).$$

To apply the maximum likelihood method, we would need to find a set of parameters which maximises the log likelihood function. Let the log likelihood function be a function of $\xi \triangleq \{\overline{r}, \lambda_1, \dots, \lambda_N, \theta_1, \dots, \theta_N, \Omega\}$, ie:

$$\log L = \log L(\xi).$$

Now, given a set of parameters ξ we calculate its corresponding likelihood $\log L$ as follows: Initialisation:

$$\hat{r}_{0|0} = \mathbb{E}(r_0) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$
$$P_{0|0} = \text{var}(r_0) = \mathbf{0}_{N\times N},$$

The Kalman filter yields,

$$\hat{r}_{t|t-1} = \Phi \hat{r}_{t-1|t-1},$$
$$P_{t|t-1} = \Phi P_{t-1|t-1}\Phi' + Q,$$

and

$$\hat{r}_{t|t} = \hat{r}_{t|t-1} + P_{t|t-1}(-A)' F_t^{-1} \varepsilon_t,$$
$$P_{t|t} = P_{t|t-1} - P_{t|t-1}(-A)' F_t^{-1}(-A)\, P_{t|t-1},$$

where

$$\varepsilon_t = Y(t,T) + A(\tau) r_t$$
$$\mathfrak{F}_t = -A(\tau)\, P_{t|t-1}(-A(\tau))' + \Omega.$$

The log-likelihood function $\log L(\xi)$ is constructed as:

$$\log L = -\frac{1}{2} N \log(2\pi) - \frac{1}{2}\sum_{t=1}^{T} \log |\mathfrak{F}_t| - \frac{1}{2}\sum_{t=1}^{T} \epsilon_t' \mathfrak{F}_t^{-1} \epsilon_t.$$

Because the $-\frac{1}{2} N \log(2\pi)$ part is a constant for any choice of ξ, it therefore has no contribution to the maximisation problem, and can be dropped for estimation proposes.

Appendix B. The system equation

From equations (9) we get the dynamics of the model as,

$$\log F(t,T,\sigma_t) = \log F(0,T) - \frac{1}{2}\sum_{i=1}^{n} \Big(\gamma_{1i}(T-t)x_i(t) + \gamma_{2i}(T-t)y_i(t) + \gamma_{3i}(T-t)z_i(t)\Big) + \sum_{i=1}^{n} \Big(\beta_{1i}(T-t)\phi_i(t) + \beta_{2i}(T-t)\psi_i(t)\Big),$$

where the deterministic functions are defined as:

$$\begin{aligned}
\beta_{1i}(T-t) &= \big(\xi_{i0} + \xi_i(T-t)\big)e^{-\eta_i(T-t)},\\
\beta_{2i}(T-t) &= \xi_i e^{-\eta_i(T-t)},\\
\gamma_{1i}(T-t) &= \beta_{1i}(T-t)^2,\\
\gamma_{2i}(T-t) &= 2\beta_{1i}(T-t)\beta_{2i}(T-t),\\
\gamma_{3i}(T-t) &= \beta_{2i}(T-t)^2,
\end{aligned}$$

and the state variables satisfy the following SDE under the risk-neutral measure:

$$\begin{aligned}
dx_i(t) &= \big(-2\eta_i x_i(t) + \nu_i(t)\big)\,dt,\\
dy_i(t) &= \big(-2\eta_i y_i(t) + x_i(t)\big)\,dt,\\
dz_i(t) &= \big(-2\eta_i z_i(t) + 2y_i(t)\big)\,dt,\\
d\phi_i(t) &= -\eta_i\phi_i(t)dt + \sqrt{\nu_i(t)}dW_i^x(t),\\
d\psi_i(t) &= \big(-\eta_i\psi_i(t) + \phi_i(t)\big)dt,\\
d\sigma_i(t) &= \kappa_i(\overline{\sigma}_i - \sigma_i(t))dt + \gamma_i dW_i^\sigma(t),\\
d\nu_i(t) &= (-2\nu_i(t)\kappa_i + 2\kappa_i\overline{\sigma}_i\sigma_i(t) + \gamma_i^2)dt + 2\gamma_i\sqrt{\nu_i(t)}dW_i^\sigma(t).
\end{aligned}$$

However, in the physical world we need to account for the market price of risk and the market price of volatility risk namely λ_i and λ_i^V. They are specified as:

$$\begin{aligned}
dB_i^x(t) &= dW_i(t) - \lambda_i\sqrt{\nu_i(t)}\,dt,\\
dB_i^\sigma(t) &= dW_i^\sigma(t) - \lambda_i^\sigma\sqrt{\nu_i(t)}\,dt,
\end{aligned}$$

where, $B_i^x(t)$ and $B_i^\sigma(t)$ are Brownian motions under the physical measure. The dynamics of the state variables under the physical measure is:

$$\begin{aligned}
dx_i(t) &= \big(-2\eta_i x_i(t) + \nu_i(t)\big)\,dt,\\
dy_i(t) &= \big(-2\eta_i y_i(t) + x_i(t)\big)\,dt,\\
dz_i(t) &= \big(-2\eta_i z_i(t) + 2y_i(t)\big)\,dt,\\
d\phi_i(t) &= \big(-\eta_i\phi_i(t) + \lambda_i\nu_i(t)\big)dt + \sqrt{\nu_i(t)}dB_i^x(t),\\
d\psi_i(t) &= \big(-\eta_i\psi_i(t) + \phi_i(t)\big)dt,\\
d\sigma_i(t) &= \big(\kappa_i\overline{\sigma}_i - (\kappa_i - \gamma_i\lambda_i^\sigma)\sigma_i(t)\big)dt + \gamma_i\,dB_i^\sigma(t),\\
d\nu_i(t) &= \big((2\gamma_i\lambda_i^\sigma - 2\kappa_i)\nu_i(t) + 2\kappa_i\overline{\sigma}_i\sigma_i(t) + \gamma_i^2\big)dt + 2\gamma_i\sqrt{\nu_i(t)}dB_i^\sigma(t).
\end{aligned}$$

This equation can be more succinctly written in matrix form as:

$$dX_t^i = (\Psi_i - \mathcal{K}_i X_t^i)dt + \Sigma_i d\overline{B}_i(t)$$

where $X_t^i = (x_i(t), y_i(t), z_i(t), \phi_i(t), \psi_i(t), \sigma_i(t), \nu_i(t))'$, $B_i(t) = (B_i^x(t), B_i^\sigma(t))'$ and

$$\Psi_i = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \kappa_i\overline{\sigma}_i \\ \gamma_i^2 \end{pmatrix} \mathcal{K}_i = \begin{pmatrix} 2\eta_i & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 2\eta_i & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 2\eta_i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta_i & 0 & 0 & -\lambda_i \\ 0 & 0 & 0 & -1 & \eta_i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \kappa_i - \gamma_i\lambda_i^\sigma & 0 \\ 0 & 0 & 0 & 0 & 0 & -2\kappa_i\overline{\sigma}_i & 2\kappa_i - 2\gamma_i\lambda_i^\sigma \end{pmatrix},$$

$$\Sigma_i = (\Sigma_i^1 + \sqrt{\nu_t^i}\Sigma_i^2) \cdot R_i^{\frac{1}{2}},\ \Sigma_i^1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \gamma_i \\ 0 & 0 \end{pmatrix},\ \Sigma_i^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 2\gamma_i \end{pmatrix}$$

where $R_i^{\frac{1}{2}}$ is the 2×2 Cholesky decomposition of the correlation matrix for the Wiener Processese:

$$R_i^{\frac{1}{2}} = \begin{pmatrix} 1 & 0 \\ \rho_i^{x\sigma} & \sqrt{1 - (\rho_i^{x\sigma})^2} \end{pmatrix}.$$

Note that $R_i^{\frac{1}{2}}$ is just a notation and it means:

$$R_i \triangleq R_i^{\frac{1}{2}}(R_i^{\frac{1}{2}})' = \begin{pmatrix} 1 & \rho_i^{x\sigma} \\ \rho_i^{x\sigma} & 1 \end{pmatrix}.$$

Applying the stochastic chain rule to $e^{\mathcal{K}_i t}X_t^i$, we have

$$\begin{aligned} d\big(e^{\mathcal{K}_i t}X_t^i\big) &= e^{\mathcal{K}_i t}\mathcal{K}_i X_t^i dt + e^{\mathcal{K}_i t} dX_t^i \\ &= e^{\mathcal{K}_i t}\Psi_i dt + e^{\mathcal{K}_i t}\Sigma_i dB_i(t). \end{aligned}$$

It follows that $X_s^i, s > t$ is given by

$$X_s^i = e^{-\mathcal{K}_i(s-t)}X_t^i + \int_t^s e^{-\mathcal{K}_i(s-u)}\Psi_i\, du + \int_t^s e^{-\mathcal{K}_i(s-u)}\Sigma_i\, dB_i(u). \tag{B.1}$$

When we discretise equation (B.1) by letting $t \in \{0, \Delta t, 2\Delta t, \ldots\}$ we get:

$$X_{t+1}^i = \underbrace{e^{-\mathcal{K}_i\Delta t}}_{=\Phi_X^i} X_t^i + \underbrace{\int_t^{t+\Delta t} e^{-\mathcal{K}_i(t+\Delta t-u)}\Psi_i\, du}_{=\Phi_0^i} + \underbrace{\int_t^{t+\Delta t} e^{-\mathcal{K}_i(t+\Delta t-u)}\Sigma_i\, dB_i(u)}_{\omega_{t+1}^i}.$$

The mean of X_s^i conditional on the information at time t is given by:

$$\mathbb{E}_t\big[X_s^i\big] = e^{-\mathcal{K}_i(s-t)}X_t^i + \int_t^s e^{-\mathcal{K}_i(s-u)}\Psi_i\,du,$$

and covariance matrix of X_s^i conditional on the information at time t is given by:

$$\begin{aligned}
\mathrm{cov}_t\left(X_s^i,(X_s^i)'\right) =& \mathbb{E}_t\Big[\Big(\int_t^s e^{-\mathcal{K}_i(s-u)}\Sigma_i\,dB_i(u)\Big)\Big(\int_t^s e^{-\mathcal{K}_i(s-u)}\Sigma_i\,dB_i(u)\Big)'\Big] \\
=& \int_t^s e^{-\mathcal{K}_i(s-u)}\Sigma_i^1 R_i(\Sigma_i^1)'(e^{-\mathcal{K}_i(s-u)})'du \\
&+ \int_t^s \mathbb{E}_t(\sigma^i(u))e^{-\mathcal{K}_i(s-u)}\Sigma_i^1 R_i(\Sigma_i^2)'(e^{-\mathcal{K}_i(s-u)})'du \\
&+ \int_t^s \mathbb{E}_t(\sigma^i(u))e^{-\mathcal{K}_i(s-u)}\Sigma_i^2 R_i(\Sigma_i^1)'(e^{-\mathcal{K}_i(s-u)})'du \\
&+ \int_t^s \mathbb{E}_t(\nu^i(u))e^{-\mathcal{K}_i(s-u)}\Sigma_i^2 R_i(\Sigma_i^2)'(e^{-\mathcal{K}_i(s-u)})'du.
\end{aligned}$$

We know that the conditional distribution of $\sigma_i(u)$ given information at time t is normally distributed with mean and variance as follow:

$$\begin{aligned}
\mathbb{E}_t\big(\sigma_i(u)\big) &= \sigma_i(t)e^{-\kappa_i(u-t)} + \overline{\sigma}(1-e^{-\kappa_i(u-t)}), \\
\mathrm{var}_t\left(\sigma_i(u)\right) &= \frac{\gamma_i^2}{2\kappa_i}(1-e^{-2\kappa_i(u-t)}).
\end{aligned}$$

So,

$$\begin{aligned}
\mathbb{E}_t\big(\nu^i(u)\big) &= \mathbb{E}_t\big(\sigma_i^2(u)\big) \\
&= \mathrm{var}_t\left(\sigma_i(u)\right) + \Big(\mathbb{E}_t\big(\sigma_i(u)\big)\Big)^2.
\end{aligned}$$

Putting the three factors together, we obtain:

$$X_t = \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix}, B(t) = \begin{pmatrix} B^1(t) \\ B^2(t) \\ B^3(t) \end{pmatrix},$$

$$\Psi = \begin{pmatrix} \Psi^1 \\ \Psi^2 \\ \Psi^3 \end{pmatrix}, \mathcal{K} = \begin{pmatrix} \mathcal{K}_1 & 0 & 0 \\ 0 & \mathcal{K}_2 & 0 \\ 0 & 0 & \mathcal{K}_3 \end{pmatrix}, \Sigma = \begin{pmatrix} \Sigma_1 & 0 & 0 \\ 0 & \Sigma_2 & 0 \\ 0 & 0 & \Sigma_3 \end{pmatrix},$$

$$\mathrm{cov}_t(X_s,(X_s)') = \begin{pmatrix} \mathrm{cov}_t(X_s^1,(X_s^1)') & 0 & 0 \\ 0 & \mathrm{cov}_t(X_s^2,(X_s^2)') & 0 \\ 0 & 0 & \mathrm{cov}_t(X_s^3,(X_s^3)') \end{pmatrix}.$$

The system equation, therefore can be written in discrete form as:

$$X_t = \Phi_0 + \Phi_X X_{t-1} + \omega_t,\ \omega_t \sim\ N(0, Q_t), \tag{B.2}$$
$$(\omega_t \text{ and } \omega_\tau \text{ are independent for } t \neq \tau) \tag{B.3}$$

where

$$\Phi_0 = \int_t^{t+\Delta t} e^{-\mathcal{K}(t+\Delta t-u)}\Psi\, du,$$
$$\Phi_X = e^{-\mathcal{K}\Delta t}.$$

Appendix C. The extended Kalman filter and the maximum likelihood method

There are two sets of equations in this model. One is the system equation as shown in equation (B.2). This equation describes the evolution of the unobservable states

$$X_t = (X_t^1, X_t^2, \ldots)',$$

where $X_t^i = (x_i(t), y_i(t), z_i(t), \phi_i(t), \psi_i(t), \sigma_i(t), \nu_i(t))'$. The other equation is the observation equation that links the unobservable state variables with the market-observable variables (i.e. futures and option prices) and is of the form:

$$Z_t = h(X_t, u_t),\ u_t \sim \text{i.i.d.}\, N(\mathbf{1}, \Omega). \tag{C.1}$$

Here we assume that u_t and u_τ are independent for $u \neq \tau$. We model the noise u_t to be multiplicative with the cross-correlation of the noise to be zero (i.e. Ω is a diagonal matrix). For estimation purposes, we assume that $\Omega = \begin{bmatrix} \Omega_f & 0 \\ 0 & \Omega_O \end{bmatrix}$ where Ω_f is a diagonal matrix with constant noise σ_f and Ω_O with σ_O. σ_f and σ_O are the measurement noise and they are to be estimated from Kalman filter. Note that for options, the function h is not a linear function, see equation (6). It is however a linear function for futures, see equation (9).
Let the number of futures observed in the market at time t be m, i.e. $F_{t,1}, \ldots, F_{t,m}$. Let the number of options on futures observed in the market at time t be n, i.e. $O_{t,1}, \ldots, O_{t,n}$ then we have:

$$Z_t = (\log F_{t,1}, \ldots, \log F_{t,m}, O_{t,1}, \ldots, O_{t,n})'.$$

Let $\hat{X}_t = \mathbb{E}_t[X_t]$ and $\hat{X}_{t|t-1} = \mathbb{E}_{t-1}[X_t]$ denote the expectation of X_t given information at time t and time $t-1$ respectively and let P_t and $P_{t|t-1}$ denote the corresponding estimation error covariance matrices. That is

$$\text{var}(X_t - \hat{X}_{t|t-1}) = P_{t|t-1},$$
$$\text{var}(X_t - \hat{X}_t) = P_t.$$

We linearise the h-function around $\hat{X}_{t|t-1}$ by Taylor series expansion and we ignore higher order terms:

$$h(X_t) = h(\hat{X}_{t|t-1}) + H(\hat{X}_{t|t-1})(X_t - \hat{X}_{t|t-1}) + \text{Higher Order Terms}.$$

The function H is the Jacobian of $h(\cdot)$, and it is given by

$$H \triangleq \begin{pmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial h_n}{\partial x_1} & \frac{\partial h_n}{\partial x_2} & \cdots & \frac{\partial h_n}{\partial x_n} \end{pmatrix},$$

where $h(X) = (h_1(X), h_2(X), \ldots, h_n(X))'$ and $X = (X_1, X_2, \ldots, X_n)'$. For simplicity, we shall let H_t to represent $H(\hat{X}_{t|t-1})$. The Kalman filter yields

$$\hat{X}_{t|t-1} = \Phi_0 + \Phi_X \hat{X}_{t-1},$$
$$P_{t|t-1} = \Phi_X P_{t-1} \Phi_X' + Q_t,$$

and

$$\hat{X}_t = \hat{X}_{t|t-1} + P_{t|t-1} H_t' F_t^{-1} \epsilon_t,$$
$$P_t = P_{t|t-1} - P_{t|t-1} H_t' F_t^{-1} H_t P_{t|t-1},$$

where

$$\epsilon_t = Z_t - h(\hat{X}_{t|t-1}),$$
$$F_t = H_t P_{t|t-1} H_t' + D_k \Omega D_k',$$

where

$$D_k \triangleq \frac{\partial h(X,u)}{\partial u}|_u.$$

The log-likelihood function is constructed as:

$$\log L = -\frac{1}{2}\log(2\pi)\sum_{t=1}^{T} N_t - \frac{1}{2}\sum_{t=1}^{T}\log|F_t| - \frac{1}{2}\sum_{t=1}^{T}\epsilon_t' F_t^{-1}\epsilon_t,$$

where N_t here represents the sum of the number of future and option contracts of day t.

The log-futures price dynamics that we develop earlier in equation (9) is time-inhomogeneous and fits the initial futures curve by construction. For the purpose of estimation, it is more convenient to work with a model that the initial futures curve is time-homogeneous. We achieve this by assuming that the initial futures curve $F(0,T) = f_0$ for all time T, where f_0 is a constant representing the long-term futures price (futures with infinite maturity) which is then estimated in the Kalman filter. In the estimation process, the long-run mean of the volatility process $\overline{\sigma}_i$ is normalised to one to achieve identification, see Dai and Singleton (2000).

Appendix D. Conversion of American to European Option Prices

Because the option pricing equation (6) provides the price for European options, not American options that are the options of our database, American option prices would have to be conversed to their associated European option prices before parameter estimation. This conversion, including the approximation of the early exercise premium, we follow the same approach proposed by Broadie, Chernov, and Johannes (2007) for equity options and applied by Trolle and Schwartz (2009) for commodity options. Basically, we apply the method outlined in Barone-Adesi and Whaley (1987) to get the implied volatility of the American options and then use Black (1976) to calculate the associated price of the European options. There is an unobservable parameter which is needed for this calculate namely r, the constant risk-free interest rate. We use government bonds (such as US treasury bills and bonds) as the observable data together with Kalman filter and maximum likelihood method to estimate the parameters in our multi-factor stochastic volatility model (3), then calculate the price of a fair zero-coupon bond. Then use this bond price to calculate the constant risk-free rate.

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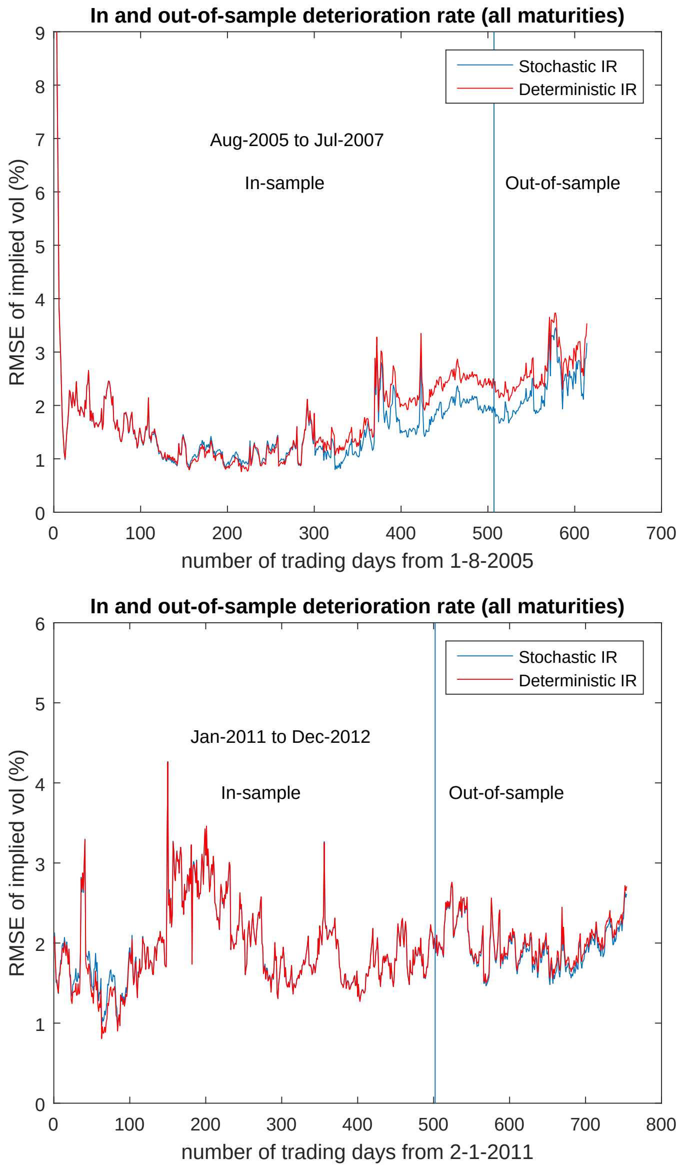

Figure 3: **RMSE of implied volatility for all maturities.** The graph displays the average of the daily root-mean-square error of the implied volatility over all maturities. The top panel displays the fitting starting from August 2005 and the bottom panel displays the fitting starting from January 2011. The root-mean-square error is defined to be between the observed implied volatility and the implied volatility from the estimated model.

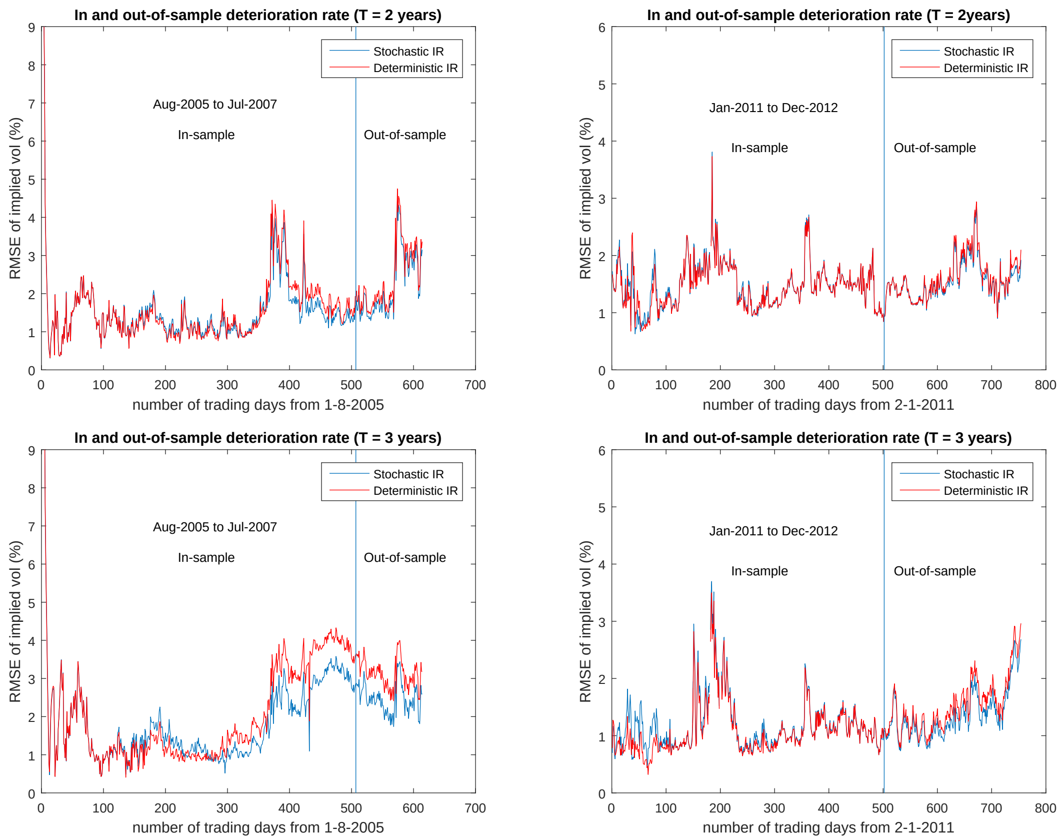

Figure 4: **RMSE of implied volatility with 2-year and 3-year maturities.** The four graphs show the daily root-mean-square error of the implied volatility (the RMSE between the observed implied volatility and the implied volatility from the estimated model. The two graphs on the left show the RMSE of options fitted to December contracts which are in the second and third year starting from August 2005. The two graphs on the right have a starting day at January 2011 during the period when the interest rate volatility is low.

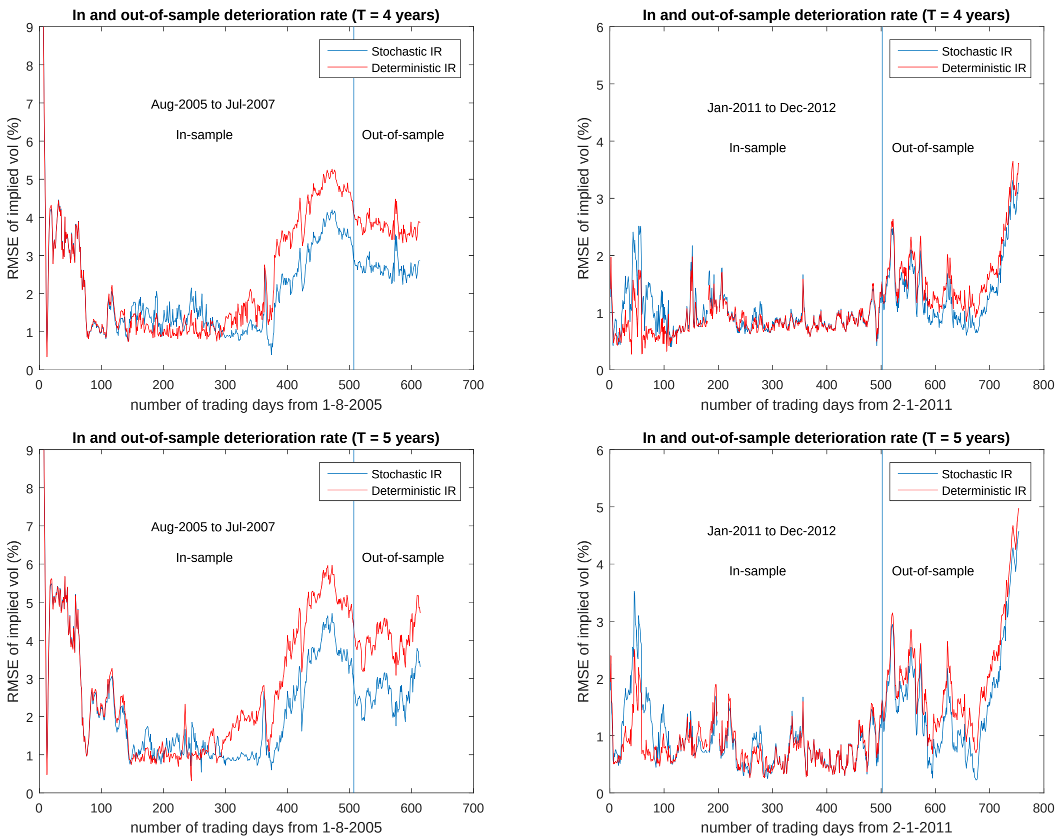

Figure 5: **RMSE of implied volatility with 4-year and 5-year maturities.** The four graphs show the daily root-mean-square error of the implied volatility (the RMSE between the observed implied volatility and the implied volatility from the estimated model). The two graphs on the left show the RMSE of options fitted to December contracts which are in the forth and fifth year starting from August 2005. The two graphs on the right have a starting day at January 2011 during the period when the interest rate volatility is low.