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Pricing of Long-dated Commodity Derivatives with Stochastic Volatility and Stochastic Interest Rates

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Pricing of long-dated commodity derivatives with stochastic volatility and stochastic interest rates

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Abstract

Aiming to study pricing of long-dated commodity derivatives, this paper presents a class of models within the Heath, Jarrow, and Morton (1992) framework for commodity futures prices that incorporates stochastic volatility and stochastic interest rate and allows a correlation structure between the futures price process, the futures volatility process and the interest rate process. The functional form of the futures price volatility is specified so that the model admits finite dimensional realisations and retains affine representations, henceforth quasi-analytical European futures option pricing formulae can be obtained. A sensitivity analysis reveals that the correlation between the interest rate process and the futures price process has noticeable impact on the prices of long-dated futures options, while the correlation between the interest rate process and the futures price volatility process does not impact option prices. Furthermore, when interest rates are negatively correlated with futures prices then option prices are more sensitive to the volatility of interest rates, an effect that is more pronounced with longer maturity options.

Keywords: Futures options; Stochastic interest rates; Stochastic volatility; Correlations; Long-dated commodity derivatives

JEL: C60, G13, Q40

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1. Introduction

In recent years, markets for long-dated derivatives have become far more liquid and have attracted the attention of major market participants such as hedge funds and insurance companies[1]. The interest rate risk for short-term derivatives may be negligible, however it is not well studied whether ignoring the impact of interest rate risk could lead to significant mispricings of long-dated derivatives. Typically the sensitivity of an option's price with respect to the interest rate, measured by the option's rho (the partial derivative of the option price with respect to the interest rate), is an increasing function with time-to-maturity. One of the earliest empirical analysis of spot price models with stochastic interest rates is Rindell (1995). Using European stock index options with a maturity of up to two years from the Swedish option market, Rindell demonstrates that the option pricing model of Amin and Jarrow (1992) outperforms the original Black and Scholes (1973) model that assumes constant interest rates. Bakshi, Cao, and Chen (2000) perform empirical investigations on a variety of option pricing models with different levels of abstraction. One of their findings is that the hedging performance of delta hedging strategies of long-dated derivatives improves when the model incorporates stochastic interest rates.

Motivated by the new market for long-dated derivatives and the empirical findings, a new type of so-called hybrid pricing models is proposed in the literature. This type of models typically features a geometric Brownian motion for the spot price process with mean-reverting stochastic volatility and stochastic interest rate processes. Depending on the type of mean-reverting process and the correlation structure, the model may require some numerical approximations before leading to closed-form option pricing formulae. van Haastrecht, Lord, Pelsser, and Schrager (2009) and Grzelak, Oosterlee, and van Weeren (2012) introduce a hybrid spot price model by combining the stochastic volatility model by Schöbel and Zhu (1999) as the spot price process and the Hull and White (1993) model as the stochastic inter-

[1]For example, Long-term Equity Anticipation Securities (LEAPS) are long-dated (more than 1 year) put and call options on common stocks, equity indexes or American depositary receipts (ADRs), Power reverse dual-currency notes are long-dated FX hybrid products or option contracts on crude oil listed on NYMEX that extends to 9 years.

est rate process while allowing full correlations between the spot price process, its stochastic volatility process and the interest rate process. This model is dubbed as the Schöbel-Zhu Hull-White (hereafter SZHW) model. van Haastrecht et al. (2009) apply the model to the valuation of insurance options with long-term equity or foreign exchange (FX) exposure. The key advantage of this model is that it admits a closed-form pricing formula for European-style vanilla options, however there is a positive probability that the interest rate process or the stochastic volatility process becomes negative. Grzelak and Oosterlee (2011) propose two hybrid spot price models that specifically target the shortcoming of the SZHW model by replacing the mean-reverting Hull and White (1993) processes with the square-root processes (see Cox, Ingersoll, and Ross (1985)). While these models ensure that the stochastic volatility process is positive (Heston-Hull-White) or both the stochastic volatility process and stochastic interest rate process are positive (Heston-Cox-Ingersoll-Ross) the downside is that a closed-form option pricing formula cannot be obtained. However the authors propose approximations to obtain analytical characteristic functions. Grzelak and Oosterlee (2012) apply the (Heston-Hull-White) model to value FX options where both domestic and foreign interest rate processes are modelled by the Hull and White (1993) process.

The Black (1976) model is the earliest commodity pricing model. It is based on the Black and Scholes (1973) model and it is very popular among practitioners although it lacks many features of interest. This model assumes that the cost-of-carry formula holds and that net convenience yields are constant. Another drawback is that describing the futures price dynamics only by a geometric Brownian motion does not capture commodity price properties for instance changes in the shape of the futures curves and mean-reversion. In the spirit of Black and Scholes (1973), the earlier commodity pricing models specify exogenously a stochastic process for the spot price dynamics and then futures contracts are set to equal to the expected future spot price under the risk-neutral measure. Brennan and Schwartz (1985) consider a model where the spot price of the commodity follows a geometric Brownian motion and a convenient yield which is a deterministic function of the spot price. This model does not capture the mean-reversion behaviour of market observable commodity prices. The one-factor model of Schwartz (1997) overcomes this shortcoming by introducing a mean-reversion process for the log of the spot price. Another type of models is the spot price and convenience yield model. These two-factor models assume that the commodity prices and the convenience yield form a joint diffusion process with a non-zero correlation. Schwartz (1997) finds that two-factor models greatly improve the ability to describe the empirically observed price behaviour of copper, crude oil and gold.

The main drawback of spot commodity model is that the whole observed term structure of futures prices is implied. As a consequence it is difficult for a spot commodity model to capture certain features of the term structure of futures prices. Following the seminal paper by Heath et al. (1992) (hereafter HJM) which take the whole term structure of interest rate as an input to the model, Cortazar and Schwartz (1994) consider a commodity pricing model that uses all the information contained in the term structure of commodity futures prices. However, this model considers only deterministic volatility. Trolle and Schwartz (2009b) take a step further by considering unspanned stochastic volatility. This is critical because if there exists unspanned volatility in a given commodity, futures options are not redundant securities and they cannot be fully hedged using only the underlying futures contracts. Chiarella, Kang, Nikitopoulos, and Tô (2013) consider a commodity pricing model under the HJM framework. They demonstrate that the crude oil futures volatility structure is hump-shaped and thus allows increasing volatility at the short end of the implied volatility curve. However, in both papers from Trolle and Schwartz (2009b) and Chiarella et al. (2013), a deterministic function of interest rate is used to discount the payout of the options to present. For short-term and medium-term options, assuming deterministic interest rates results in negligible pricing errors however for long-term option contracts this error is not well known and it may not be negligible.

Pilz and Schlögl (2013) propose a joint model of commodity price and interest rate risk constructed analogously to the multi-currency LIBOR Market Model (see Schlögl (2002)). They also present a procedure to achieve a consistent fit of the model to market data for interest rates options, commodity options and historically estimated correlations. Although Pilz and Schlögl (2013) and the model proposed in this paper both feature stochastic interest rate and model commodity prices there are several differences. Pilz and Schlögl (2013) model forward commodity prices and they present a method to approximate the differences between futures and forwards as well as their implied volatilities. In this paper, we model directly prices of commodity futures contracts which are the most liquidly traded contracts (compared to forward contracts).

In this paper, we propose a commodity derivative pricing model featuring stochastic volatility and stochastic interest rates. The stochastic interest rate process is modelled by a Hull and White (1990) process and the volatility is modelled by an Ornstein-Uhlenbeck process. This model is within the HJM framework thus it fits the entire initial forward curve by construction rather than generating it endogenously from the spot price as in the spot pricing models. It is a continuous time multi-factor stochastic volatility model that allows for multiple volatility factors with flexible volatility structures. Empirical evidence in the crude oil market demonstrates that exponential decaying or hump-shaped are typical structures of its volatility factors, see Chiarella et al. (2013). In addition, the proposed model allows a full correlation structure between the underlying futures price process, the stochastic volatility process and the stochastic interest rate process. This feature is very important as empirical evidence has revealed that volatility is unspanned in commodity markets, see Trolle and Schwartz (2009b).

One of the issues of using HJM models is that in general they have infinite dimensional state space. By selecting suitable volatility structures,[2] the forward rate model can be reduced to a finite dimensional state space. Furthermore these volatility specifications are flexible enough to generate a wide range of shapes for the futures price volatility surface including exponentially decaying and hump-shaped. This class of models, however, by itself

[2]The HJM model in general is only Markovian in the entire forward curve thus requires an infinite number of state variables to determine its current state. However several papers (see for example Chiarella and Kwon (2001), Björk, Landén, and Svensson (2004) and Björk, Blix, and Landén (2006)) have proposed appropriate conditions on the volatility structure so that HJM models admit finite dimensional Markovian representations. This greatly improves its tractability to formulate closed-form option pricing formulae and it is important for its suitability in numerical evaluation techniques such as Monte Carlo simulations, finite difference or tree methods or Kalman filter.

does not conform to the general structure of affine term structure models. By introducing latent stochastic variables, this class of models has an affirm term structure representation hence quasi-analytical European vanilla futures option pricing formulae can be obtained. Therefore this class of models is well suited for estimation and calibration applications. Consequently these models can be fitted to both futures and option prices so that they have the potential to capture well the forward curves and the volatility smiles. Other mean-reverting processes for example the square-root process (see Cox et al. (1985)) can be used to model the stochastic interest rate or the stochastic volatility but some approximations are required in order to obtain closed-form solutions.

To understand better the impact of stochastic interest rates to the prices of futures options, a sensitivity analysis is performed to gauge the impact of the parameters of the interest rate process to futures option prices. Results show that the impact of the correlation between the stochastic interest rate process and the stochastic volatility process to option prices is negligible even when the time-to-maturity is twenty years. However, the impact of the correlation between stochastic interest rates and futures prices to the option prices is noticeable for longer maturity options. A fixed long-term level of interest rates applies the same discounting factor to the future payoff of the options with the same maturity and it is independent to the correlations between the stochastic interest rate process and other processes. The interest rate volatility impacts the option prices more severely than the long-term level of interest rates. When the volatility of interest rates is high, there is a nonlinear (convex) relationship between prices of long-dated options and the correlation coefficient between futures prices and interest rates. We find that, when the correlation between futures prices and interest rates is negative, option prices are more sensitive to interest rate volatility.

The remaining of the paper is structured as follows. Section 2 presents the proposed HJM term structure model with stochastic volatility and interest rates for pricing commodity futures prices and demonstrates that for certain volatility specifications, the model can be reduced to a finite dimensional state space. Section 3 shows that the model can be within the class of affine term structure models by introducing a latent stochastic variable, thus derives the formula for pricing European vanilla options on futures. Numerical investigations are presented in Section 4. Section 5 concludes.

2. Commodity Futures Prices

We consider a filtered probability space $(\Omega, \mathcal{F}_T, \mathbb{F}, \mathbb{P}), T \in [0, \infty)$ satisfying the usual conditions[3]. Here Ω is the state space, $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0,T]}$ is a set of σ-algebra representing measurable events and $\mathbb{P}$ is the historical (real-world) probability measure. We introduce $\sigma = \{\sigma_t; t \in [0, T]\}$ an n-dimensional generic stochastic volatility processes modelling the uncertainty in the commodity market. We let $F(t, T, \sigma_t)$ be the futures price of the commodity at time $t \geq 0$, for delivery at time $T \in [t, \infty)$ and the current state of the stochastic volatility process $\sigma_t \in \Re^n$. The spot price at time t of the underlying commodity, namely $S(t, \sigma_t)$ is obtained by taking the limit of the futures price as $T \to t$, i.e. $S(t, \sigma_t) = \lim_{T \to t} F(t, T, \sigma_t), t \in [0, T]$. We denote $r = \{r(t); t \in [0, T]\}$ the possibly multi-factor stochastic instantaneous short-rate process. Duffie (2001) by using no-arbitrage arguments demonstrates that the futures price process $F(t, T, \sigma_t)$ is a martingale under the equivalent risk-neutral probability measure with respect to the continuously compounded spot interest rate, denoted by $\mathbb{Q}$, namely,

$$F(t, T, \sigma_t) = \mathbb{E}^{\mathbb{Q}}\big[S(T, \sigma_T) | \mathcal{F}_t\big].$$

The commodity futures price process must follow a stochastic differential equation with zero drift. Thus, we assume that the commodity futures price process follows a driftless stochastic differential equation under the risk-neutral measure of the form:

$$\frac{dF(t, T, \sigma_t)}{F(t, T, \sigma_t)} = \sum_{i=1}^{n} \sigma_i^F(t, T, \sigma_t) dW_i^x(t), \tag{1}$$

where $\sigma_i^F(t, T, \sigma_t)$ are the $\mathbb{F}$-adapted futures price volatility processes for all $T > t$ and $W^x(t) = \{W_1^x(t), \ldots, W_n^x(t)\}$ is an n-dimensional Wiener process under the risk-neutral probability measure $\mathbb{Q}$ driving the commodity futures prices. The volatility process $\sigma_t =$

[3]The usual conditions satisfied by a filtered complete probability space are: (a) $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$ and (b) the filtration is right continuous

$\{\sigma_i(t), \ldots, \sigma_n(t)\}$ is an n-dimensional well-defined Markovian process following the dynamics:

$$d\sigma_i(t) = \mu_i^\sigma(t, \sigma_t)dt + \sigma_i^\sigma(t, \sigma_t)dW_i^\sigma(t), \tag{2}$$

for $i \in \{1, \ldots, n\}$, where $\mu_i^\sigma(t, \sigma_t)$ and $\sigma_i^\sigma(t, \sigma_t)$ are integrable and square-integrable real-valued functions respectively. We further specify the functional form of the drift and the volatility of the stochastic volatility process and we consider an extended version of the n-factor Schöbel-Zhu-Hull-White (ESZHW hereafter) model[4] as follows:

$$\frac{dF(t, T, \sigma_t)}{F(t, T, \sigma_t)} = \sum_{i=1}^{n} \sigma_i^F(t, T, \sigma_t)dW_i^x(t), \tag{3}$$

where, for $i = 1, 2, \ldots, n$,

$$d\sigma_i(t) = \kappa_i(\overline{\sigma}_i - \sigma_i(t))dt + \gamma_i dW_i^\sigma(t), \tag{4}$$

and

$$r(t) = \overline{r}(t) + \sum_{j=1}^{N} y_j(t), \tag{5}$$

$$dy_j(t) = -\lambda_j(t)y_j(t)dt + \theta_j dW_j^r(t), \quad \text{for} j = 1, 2, \ldots, N.$$

Note that $W_t^\sigma = \{W_1^\sigma(t), \ldots, W_n^\sigma(t)\}$ is an n-dimensional Wiener process under the risk-neutral probability measure driving the stochastic volatility process $\sigma_t = \{\sigma_1(t), \ldots, \sigma_n(t)\}$, $W^r(t) = \{W_1^r(t), \ldots, W_N^r(t)\}$ is an N-dimensional Wiener process under the risk-neutral probability measure driving the instantaneous short-rate process $r(t)$, for all $t \in [0, T]$, $\kappa_1, \ldots, \kappa_n$, $\overline{\sigma}_1, \ldots, \overline{\sigma}_n$ and $\theta_1, \ldots, \theta_N$ are constants, and $\{\lambda_i\}_{i=1,\ldots,N}$ and $\overline{r}$ are deterministic functions of time t. We further make the following assumptions on the correlation structure of the Wiener processes:

[4]see van Haastrecht et al. (2009)

$$
\begin{aligned}
dW_i^x(t)dW_j^x(t) &= \begin{cases} dt, & \text{if } i = j, \\ 0, & \text{otherwise} \end{cases} \\
dW_i^\sigma(t)dW_j^\sigma(t) &= \begin{cases} dt, & \text{if } i = j, \\ 0, & \text{otherwise} \end{cases} \\
dW_{\hat{\imath}}^r(t)dW_{\hat{\jmath}}^r(t) &= \begin{cases} dt, & \text{if } \hat{\imath} = \hat{\jmath}, \\ 0, & \text{otherwise} \end{cases} \\
dW_i^x(t)dW_j^\sigma(t) &= \begin{cases} \rho_i^{x\sigma} dt, & \text{if } i = j, \\ 0, & \text{otherwise} \end{cases} \\
dW_i^x(t)dW_{\hat{\jmath}}^r(t) &= \begin{cases} \rho_i^{xr} dt, & \text{if } \hat{\jmath} = 1, \\ 0, & \text{otherwise} \end{cases} \\
dW_i^\sigma(t)dW_{\hat{\jmath}}^r(t) &= \begin{cases} \rho_i^{r\sigma} dt, & \text{if } \hat{\jmath} = 1, \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
\tag{6}
$$

for $i \in \{1, \dots, n\}$, $j \in \{1, \dots, n\}$, $\hat{\imath} \in \{1, \dots, N\}$ and $\hat{\jmath} \in \{1, \dots, N\}$. The above-mentioned specifications entail the feature of unspanned stochastic volatility in the model. More specifically, when the Wiener processes $W_i^x(t)$ and $W_i^\sigma(t)$ are correlated, futures contracts can be used to partially hedge the volatility risk of the derivatives, while when the Wiener processes $W_i^x(t)$ and $W_i^\sigma(t)$ are uncorrelated, the volatility risk of the derivatives is unhedgeable by futures contracts. Note that for modelling convenience, we assume that only the first Wiener process of the interest rate process $W_1^r(t)$ can be correlated with the futures price process and the futures volatility process. If the Wiener processes of the interest rate process are uncorrelated, they can be disentangled from the expectation of the option's payoff function.

Let $X(t,T) = \log F(t,T,\sigma_t)$ be the natural logarithm of the futures price process then by an application of Itô's lemma, it follows that:

$$
dX(t,T) = -\frac{1}{2}\sum_{i=1}^{n}\left(\sigma_i^F(t,T,\sigma_t)\right)^2 dt + \sum_{i=1}^{n}\sigma_i^F(t,T,\sigma_t)dW_i^x(t). \tag{7}
$$

In Appendix A, we show that the spot price follows the SDE:

$$\frac{dS(t,\sigma_t)}{S(t,\sigma_t)} = \zeta(t,\sigma_t)dt + \sum_{i=1}^{n} \sigma_i^F(t,t,\sigma_t)dW_i^x(t), \tag{8}$$

with the instantaneous spot cost of carry $\zeta(t,\sigma_t)$ satisfying the relationship

$$\begin{aligned}\zeta(t,\sigma_t) =& \frac{\partial}{\partial t}\log F(0,t,\sigma_0) - \sum_{i=1}^{n}\int_0^t \sigma_i^F(u,t,\sigma(u))\frac{\partial}{\partial t}\sigma_i^F(u,t,\sigma(u))\,du \\ &+ \sum_{i=1}^{n}\int_0^t \frac{\partial}{\partial t}\sigma_i^F(u,t,\sigma(u))\,dW_i^x(u).\end{aligned}$$

2.1. Finite Dimensional Realisations

The commodity HJM model (3) is Markovian in an infinite dimensional state space due to the fact that the futures price curve is an infinite dimensional object. For the system to admit finite dimensional realisations, it is necessary to impose certain functional forms to the volatility terms $\sigma_i^F(t,T,\sigma_t)$, see Chiarella and Kwon (2003). By employing methods of Lie algebra, Björk et al. (2004) and Björk et al. (2006) show that the volatility terms $\sigma_i^F(t,T,\sigma_t)$ admit finite dimensional realisations, if and only if, $\sigma_i^F(t,T,\sigma_t)$ can be expressed in the following form:

$$\sigma_i^F(t,T,\sigma_t) = \varpi_i(T-t)\alpha_i(t,\sigma_t),$$

where $\alpha_i : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ are square-integrable real-valued functions and $\varpi_i : \mathbb{R} \to \mathbb{R}$ are quasi-exponential functions. A quasi-exponential function $f : \mathbb{R} \to \mathbb{R}$ has the general form

$$f(x) = \sum_i e^{m_i x} + \sum_j e^{n_j x}[p_j(x)\cos(k_j x) + q_j(x)\sin(k_j x)],$$

where m_i, n_i and k_i are real numbers and p_j and q_j are real polynomials. To reduce the system (3) to a finite dimensional state space, we assume a simplified version of this form where the commodity futures price volatility process is assumed to be:

$$\begin{aligned}\sigma_i^F(t,T,\sigma_t) &= \varpi_i(T-t)\alpha_i(t,\sigma_t) \\ &= (\xi_{0i} + \xi_i(T-t))e^{-\eta_i(T-t)}\sigma_i(t),\end{aligned} \tag{9}$$

with ξ_{0i}, ξ_i and $\eta_i \in \mathbb{R}$ for all $i \in \{1, \cdots, n\}$. This volatility specification allows for a variety of volatility structures such as exponentially decaying and hump-shaped which are typical volatility structures in commodity market, see for example Chiarella et al. (2013) or Trolle and Schwartz (2009a).

Proposition 1. *Using the volatility specifications* (9)*, the dynamics* (7) *can derive the logarithm of the instantaneous futures prices at time* t *with maturity* T *in terms of* $6n$ *state variables, namely* $x_i(t), y_i(t), z_i(t), \phi_i(t), \psi_i(t)$ *and* $\sigma_i(t)$*:*

$$\begin{aligned}\log F(t,T,\sigma_t) = \log F(0,T,\sigma_0)&\\ -\frac{1}{2}\sum_{i=1}^{n}\Big(\gamma_{1i}(T-t)x_i(t)+\gamma_{2i}(T-t)y_i(t)+\gamma_{3i}(T-t)z_i(t)\Big)&\\ +\sum_{i=1}^{n}\Big(\beta_{1i}(T-t)\phi_i(t)+\beta_{2i}(T-t)\psi_i(t)\Big),&\end{aligned} \tag{10}$$

where for $i=1,\ldots,n$*, the deterministic functions are defined as:*

$$\begin{aligned}\beta_{1i}(T-t) &= \varpi_i(T-t) = \big(\xi_{i0}+\xi_i(T-t)\big)e^{-\eta_i(T-t)},\\ \beta_{2i}(T-t) &= \xi_i e^{-\eta_i(T-t)},\\ \gamma_{1i}(T-t) &= \beta_{1i}(T-t)^2,\\ \gamma_{2i}(T-t) &= 2\beta_{1i}(T-t)\beta_{2i}(T-t),\\ \gamma_{3i}(T-t) &= \beta_{2i}(T-t)^2,\end{aligned}$$

and the state variables $x_i, y_i, z_i, \phi_i, \psi_i$ *satisfy the following SDEs:*

$$\begin{aligned}dx_i(t) &= \big(-2\eta_i x_i(t)+\sigma_i^2(t)\big)\,dt,\\ dy_i(t) &= \big(-2\eta_i y_i(t)+x_i(t)\big)\,dt,\\ dz_i(t) &= \big(-2\eta_i z_i(t)+2y_i(t)\big)\,dt,\\ d\phi_i(t) &= -\eta_i\phi_i(t)dt+\sigma_i(t)dW_i^x(t),\\ d\psi_i(t) &= \big(-\eta_i\psi_i(t)+\phi_i(t)\big)dt,\end{aligned} \tag{11}$$

subject to the initial condition $x_i(0) = y_i(0) = z_i(0) = \phi_i(0) = \psi_i(0) = 0$*. The abovementioned* $5n$ *state variables are associated with the* n *stochastic volatility processes* $\sigma_i(t)$*,* $i \in \{1,\ldots,n\}$ *see equations* (4) *so that the whole system includes* $6n$ *state variables.*

Proof. See Appendix B. □

The stochastic interest rate process $r(t)$ does not affect the futures prices process per se, but it matters when we consider pricing options on futures contracts.

3. Affine Class Transformation

The initially infinite-dimensional Markovian model is now reduced to a model with $6n+N$ state variables . The additional N state variables are from the stochastic interest rate process specified by the equation (5). Note that this system is not affine. When the volatility specifications (9) are applied to the dynamics (7), the $\sigma_i^2(t)$ term is not an affine transformation of $\sigma_i(t)$, which is required for the system to admit a closed-form characteristic function of $X(t,T) = \log F(t,T,\sigma_t)$, see Duffie, Pan, and Singleton (2000) section 2.2 for the details. By introducing a latent stochastic variable $\nu_i(t) \triangleq \sigma_i^2(t)$ with $\nu_t = \{\nu_1(t), \ldots, \nu_n(t)\}$, then this system can be transformed to the following affine system:

$$dX(t,T) = -\frac{1}{2}\sum_{i=1}^{n} \beta_{1i}^2(T-t)\nu_i(t)dt + \sum_{i=1}^{n} \beta_{1i}(T-t)\sqrt{\nu_i(t)}dW_i^x(t),$$

where, for $i = 1, 2, \ldots, n$,

$$\begin{aligned} d\sigma_i(t) &= \kappa_i(\overline{\sigma}_i - \sigma_i(t))dt + \gamma_i dW_i^\sigma(t), \\ d\nu_i(t) &= 2\kappa_i\big(\overline{\sigma}_i\sigma_i(t) + \frac{\gamma_i^2}{2\kappa_i} - \nu_i(t)\big)dt + 2\gamma_i\sqrt{\nu_i(t)}dW_i^\sigma(t), \end{aligned}$$

and

$$\begin{aligned} r(t) &= \overline{r}(t) + \sum_{j=1}^{N} y_j(t), \qquad (12) \\ dy_j(t) &= -\lambda_j(t)y_j(t)dt + \theta_j dW_j^r(t), \quad \text{for } j = 1, 2, \ldots, N, \end{aligned}$$

with the correlations of the Wiener processes the same as those in (6). For $t \le T_o \le T$ and $\upsilon \in \mathbb{C}$, the $y_1(t)$-discounted characteristic functions of the logarithm of the futures prices $\phi(t) \triangleq \phi(t, X(t,T), r(t), \nu_t, \sigma_t; \upsilon, T_o, T)$:

$$\begin{aligned} \phi(t;\upsilon,T_o,T) &\triangleq \mathbb{E}_t^{\mathbb{Q}}\big[e^{-\int_t^{T_o} y_1(u)\,du} \exp\big\{\upsilon \log F(T_o,T,\sigma_{T_o})\big\}\big] \\ &= \mathbb{E}_t^{\mathbb{Q}}\big[e^{-\int_t^{T_o} y_1(u)\,du} \exp\big\{\upsilon X(T_o,T)\big\}\big] \end{aligned} \qquad (13)$$

can be expressed as:

$$\begin{aligned} \phi(t;\upsilon,T_o,T) = \exp\big\{&A(t;\upsilon,T_o) + B(t;\upsilon,T_o)X(t,T) + C(t;\upsilon,T_o)r(t) \\ &+ \sum_{i=1}^{n} D_i(t;\upsilon,T_o)\nu_i(t) + \sum_{i=1}^{n} E_i(t;\upsilon,T_o)\sigma_i(t)\big\}. \end{aligned} \qquad (14)$$

Lemma 1. *The functions $A(t;\upsilon,T_o), B(t;\upsilon,T_o), C(t;\upsilon,T_o), D_i(t;\upsilon,T_o)$ and $E_i(t;\upsilon,T_o)$ in equation* (14) *satisfy the following complex-valued Ricatti ordinary differential equations:*

$$\begin{aligned}
\frac{\partial B}{\partial t} =& 0,\\
\frac{\partial C}{\partial t} =& \lambda_1 C + 1,\\
\frac{\partial D_i}{\partial t} =& -\frac{1}{2}\beta_{1i}^2(T-t)(B-1)B - 2(\rho_i^{x\sigma}\beta_{1i}(T-t)\gamma_i B - \kappa_i)D_i - 2\gamma_i^2 D_i^2,\\
\frac{\partial E_i}{\partial t} =& -2\bar{\sigma}_i\kappa_i D_i - \rho_i^{xr}\theta_1\beta_{1i}(T-t)BC - 2\rho_i^{r\sigma}\theta_1\gamma_i CD\\
& - (2\gamma_i^2 D_i - \kappa_i + \rho_i^{x\sigma}\beta_{1i}(T-t)\gamma_i B)E_i,\\
\frac{\partial A}{\partial t} =& -\frac{1}{2}\theta_1^2 C^2 - \sum_{i=1}^{n}\gamma_i^2 D_i - \sum_{i=1}^{n}(\kappa_i\bar{\sigma}_i + \frac{1}{2}\gamma_i^2 E_i + \rho_i^{r\sigma}\theta_1\gamma_i C)E_i,
\end{aligned} \tag{15}$$

where $i \in \{1,\dots,n\}$, subject to the terminal condition $\phi(T_0) = e^{\upsilon X(T_0,T)}$.

Proof. See Appendix C. □

In the next section we present the quasi-analytical pricing formulae for European options on futures that this affine transformation leads to.

Pricing of European Options on Futures

We denote with $\mathrm{Call}(t, F(t,T,\sigma_t); T_o)$ and $\mathrm{Put}(t, F(t,T,\sigma_t); T_o)$ the price of the European call and put option, respectively, with maturity T_o and strike K on the futures price $F(t,T,\sigma_t)$ maturing at time T. The price of a call option can be expressed as the discounted expected payoff under the risk-neutral measure of the form:

$$\mathrm{Call}(t, F(t,T,\sigma_t); T_o) = \mathbb{E}_t^{\mathbb{Q}}\Big(e^{-\int_t^{T_o} r(s)\,ds}\big(e^{X(T_o,T)} - K\big)^+\Big). \tag{16}$$

By using Fourier inversion technique Duffie et al. (2000) provide a semi-analytical formulae for the price of European-style vanilla options under the class of affine term structure. With a slight modification of the pricing equation in Duffie et al. (2000), the equation (16) can be expressed as:

$$\begin{aligned}
\mathrm{Call}(t, F(t,T,\sigma_t); T_o) = e^{-\int_t^{T_o}\bar{r}(s)\,ds}\prod_{i=2}^{N}\mathbb{E}_t^{\mathbb{Q}}\big[e^{-\int_t^{T_o} y_i(s)\,ds}\big]\times\\
\big[G_{1,-1}(-\log K) - K G_{0,-1}(-\log K)\big],
\end{aligned} \tag{17}$$

where

$$G_{a,b}(y) = \frac{\phi(t; a, T_o, T)}{2} - \frac{1}{\pi} \int_0^\infty \frac{\Im[\phi(t; a + \mathbf{i}bu, T_o, T)e^{-iuy}]}{u} \, du. \tag{18}$$

Note that $\mathbf{i}^2 = -1$ and $\Im(x + \mathbf{i}y) = y$. Note that, the product starts at $i = 2$ because the equation (17) is conditional to the specifications of the correlation structure (6) that allows only the first factor of the interest rate process to be correlated with the futures price process. For European put options, the discounted expected payoff is:

$$\begin{aligned} \text{Put}(t, F(t, T, \sigma_t); T_o) &= \mathbb{E}_t^{\mathbb{Q}}\Big(e^{-\int_t^{T_o} r(s)\, ds}\big(K - e^{X(T_o,T)}\big)^+\Big) \\ &= e^{-\int_t^{T_o} \overline{r}(s)\, ds} \prod_{i=2}^{N} \mathbb{E}_t^{\mathbb{Q}}\big[e^{-\int_t^{T_o} y_i(s)\, ds}\big] \times \\ &\quad \big[K G_{0,1}(\log K) - G_{1,1}(\log K)\big]. \end{aligned} \tag{19}$$

With finite dimensional realisations and affine class transformation the commodity pricing model (3) admits a closed-form option pricing formula. Closed-form option pricing formulae greatly facilitate model estimation and calibration as we shall see in a subsequent paper where the model is fitted to the most active commodity derivatives market, namely the crude oil market. Note that the proposed model is not limited to pricing only commodity futures options. The model can be easily adjusted to price any type of options on futures, for instance options on index futures. The model characteristics are examined next by performing a sensitivity analysis.

4. Numerical investigations

In this section we numerically investigate the implications of allowing the interest rate processes to be stochastic and possibly correlated to both of the future price process and the future stochastic volatility process. To be specific we want to assess the effect of the following parameters of the interest rate model to commodity futures option prices:

- ρ^{xr} — the correlation between the future price process and the interest rate process.
- $\rho^{r\sigma}$ — the correlation between the future price volatility process and the interest rate process.

- $\overline{r}$ — the long-term level of interest rates.

- θ — the volatility of the interest rate process.

The correlation $\rho^{x\sigma}$ between the futures price process and the volatility process has been well studied (see for example Schöbel and Zhu (1999)) so we omit the discussion here. In this numerical exercise, aiming to amplify and concentrate solely on the impact of the stochastic interest rates, we set $\rho^{x\sigma} = 0$. For simplicity also we use the one-dimensional version of the ESZHW model (that is equations (3) and (4) with $n = 1$ and equation (5) with $N = 1$) with $\overline{\sigma}(t) = \overline{\sigma}$. Furthermore we simplify the volatility structure of the futures price process by assuming $\xi_{01} = 1, \xi_1 = 0$ and $\eta_1 = 0$ (see equation (9)) thus reducing it to a constant volatility, namely $\sigma^F(t, T, \sigma_t) = \sigma_t$. This allows us to target our investigations specifically on the impact of the parameters of the stochastic interest rate to option prices. The one-dimensional ESZHW model used in this analysis is summarised below:

$$\frac{dF(t,T,\sigma_t)}{F(t,T,\sigma_t)} = \sigma_t \, dW^x(t), \tag{20}$$

$$d\sigma_t = \kappa(\overline{\sigma} - \sigma_t)dt + \gamma dW^\sigma(t), \tag{21}$$

$$dr(t) = \lambda(\overline{r} - r(t))dt + \theta dW^r(t), \tag{22}$$

with,

$$\begin{aligned} dW^x(t)dW^r(t) &= \rho^{xr} dt, \\ dW^x(t)dW^\sigma(t) &= 0, \\ dW^r(t)dW^\sigma(t) &= \rho^{r\sigma} dt. \end{aligned}$$

Table 1 displays the parameter values used in our numerical investigations, where F_0, r_0, σ_0 is the initial futures price, the initial interest rate and the initial stochastic volatility respectively.

We investigate next the following questions. What is the impact of the correlation coefficients ρ^{xr} and $\rho^{r\sigma}$ to the model option prices? How does the impact of these correlation coefficients to option prices charge as the maturity of the options increases? What is the impact of the stochastic interest rate parameters, long-term level $\overline{r}$ and volatility θ?

Table 1: **Parameter Values**

λ	$\overline{r}$	θ	κ	$\overline{\sigma}$	γ	F_0	K	r_0	σ_0
4	0.05	0.05	4	0.5	0.6	100	100	0.05	0.05

The table displays the parameter values use in the sensitivity analysis.

4.1. The correlation coefficients ρ^{xr} and $\rho^{r\sigma}$

To gauge the impact of ρ^{xr} and $\rho^{r\sigma}$ to the option prices we vary the correlations from -0.75 to 0.75 with 0.25 as the incremental value and compare these prices with the option prices of the restricted models with $\rho^{xr} = \rho^{r\sigma} = 0$. We are thus considering the following ratio:

$$\text{Ratio}(\rho^{xr}, \rho^{r\sigma}; T) = \frac{\text{OptionPrice}(\rho^{xr}, \rho^{r\sigma}; T)}{\text{OptionPrice}(0, 0; T)}. \tag{23}$$

This ratio as a function of the two correlations is plotted in Figure 1 for four call options maturities, namely $0.5, 3, 10$ and 20 years. From the surface plots of the ratio as a function of ρ^{xr} and $\rho^{r\sigma}$ two observations can be made. The first is that the ratio is virtually unchanged as $\rho^{r\sigma}$ varies from -0.75 to 0.75 in all four plots. This implies that the correlation between the interest rate process and the future price volatility process has minimal impact to call option prices. The second observation is that the correlation between the interest rate process and the future price process has minimal impact for short maturities (T=0.5 years) but the impact gradually becomes more distinct as the maturity increases. At $T = 20$, the price difference between taking correlation (ρ^{xr}) into account and assuming zero correlation ($\rho^{xr} = 0$) can be more than 10%.

Figure 2 depicts the cross-section of Figure 1 with $\rho^{r\sigma} = 0$. From this plot we see that, when the futures price process and the interest rate process are negatively (positively) correlated, the call option price is more expensive (less expensive) than the option prices computed when ignoring this correlation. The effect is more pronounced for options with longer maturities. We can intuitively understand the impact of the correlation coefficient ρ^{xr} to option prices by considering the behaviour of the futures price process and the interest rate process. Consider paths that lead to increasing futures prices such that a call option at

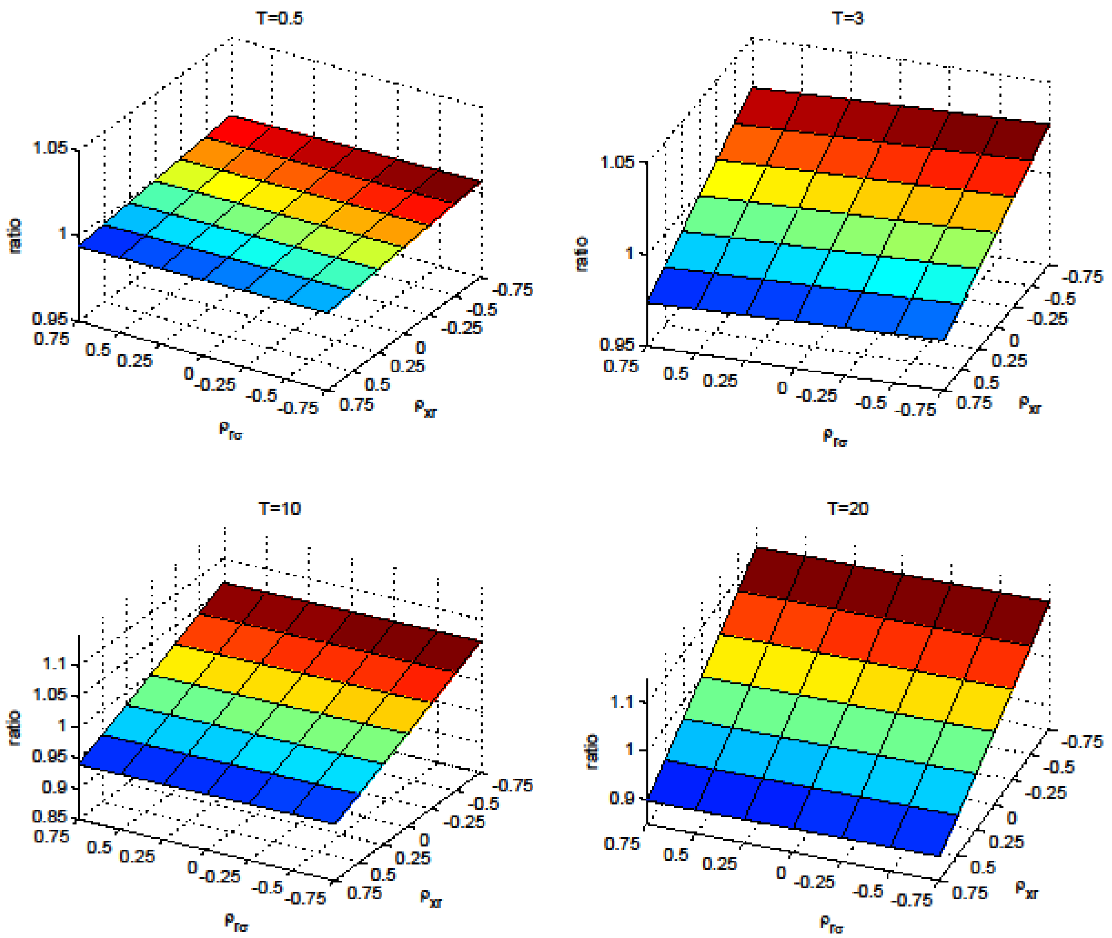

Figure 1: **Impact of the correlation coefficients ρ^{xr} and $\rho^{r\sigma}$ to call option prices** The plots display the ratio of option prices (between prices from a model with full correlation structure and prices from a restricted model with zero correlation) for a range of correlation coefficient parameter values and for four maturities; 0.5, 3, 10 and 20 years.

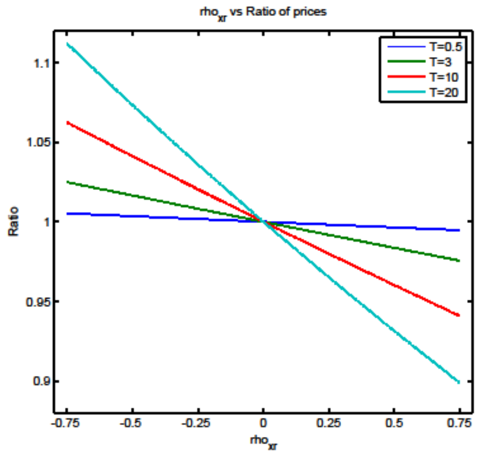

Figure 2: **Impact of the correlation between futures prices and interest rates to call option prices**

The plot shows ratio of option prices (between prices from a model with full correlation structure and prices from a restricted model with $\rho^{r\sigma} = 0$) for a range of correlation coefficient parameter values and for four different maturities; 0.5, 3, 10 and 20 years.

maturity is in-the-money.[5] In these paths, a negative correlation between futures prices and interest rates implies decreasing interest rates, hence the discounted payoffs at maturity of these particular paths are relatively expensive. Conversely, when the correlation is positive, the discounted payoffs at maturity are relatively less expensive. To sum up, a negative (positive) correlation means that we are discounting the payoffs of the in-the-money paths less (more) severely, hence the price of the option is more (less) expensive.

4.2. The long-term level of interest rates $\overline{r}$

We now turn our attention to the parameter $\overline{r}$ — the long-term level of interest rates. Figure 3 shows the call option prices and their ratios when the long-term interest rate $\overline{r}$ varies from 1% to 5% to 10% for two maturities; 0.5 and 20 years. We set $\rho^{r\sigma} = 0$ thus

[5]Paths leading to out-the-money options have no impact to the option price as the payoff in these paths is zero.

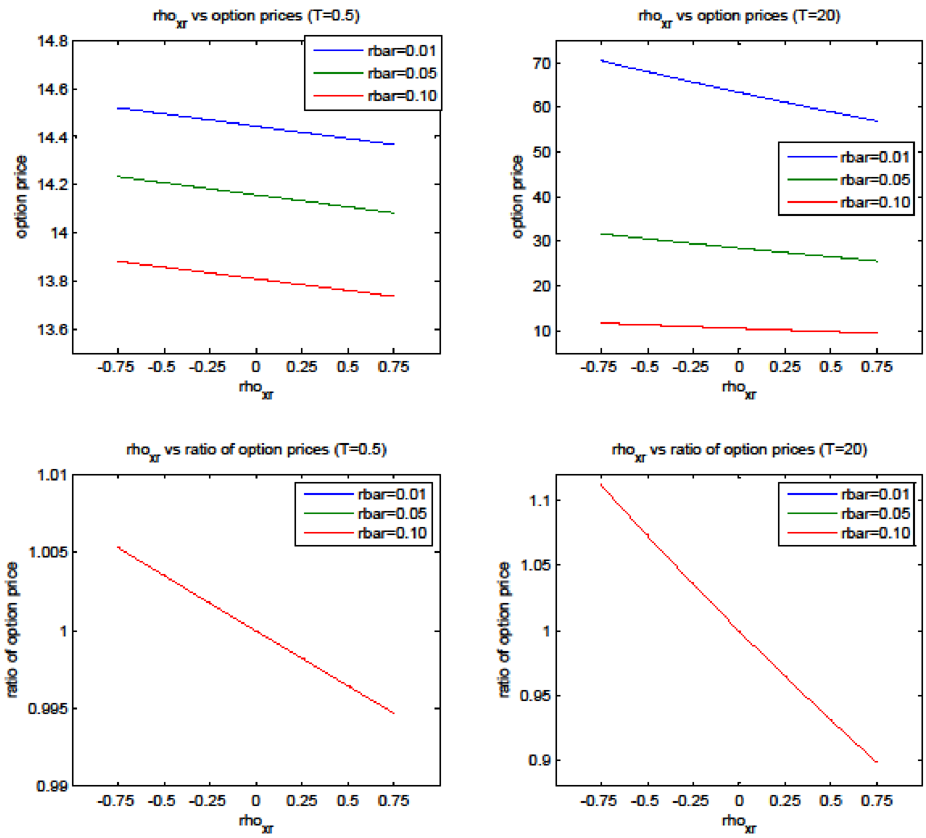

Figure 3: **Impact of the long term level of interest rates to call option prices** This plot displays option prices (top two graphs) and the ratios of option prices (bottom two graphs) for two different maturities, $T = 0.5$ and $T = 20$, and for $\{\overline{r}, r_0\} = \{0.01, 0.01\}, \{0.05, 0.05\}, \{0.10, 0.10\}$.

consider the ratio of the option prices given $\rho^{xr}, T, \overline{r}$ as:

$$\mathrm{Ratio}(\rho^{xr}; T, \overline{r}) = \frac{\mathrm{OptionPrice}(\rho^{xr}; T, \overline{r})}{\mathrm{OptionPrice}(0; T, \overline{r})}. \tag{24}$$

One interesting observation from Figure 3 is that the level of $\overline{r}$ has no impact on the ratio of the option prices. This can be easily seen from the term $e^{-\int_t^{T_o} \overline{r}(s)\,ds}$ in (17). In this analysis we set $\overline{r}(t) = \overline{r}$ so that the integral reduces to $e^{-\overline{r}(T_o - t)}$ and it cancels out in (24). The implication on option pricing is that by assuming that the interest rate volatility θ does not vary, the correlation ρ^{xr} impacts the option price equally during a period of higher interest rates and a period of lower interest rates. Higher long-term interest rates merely discount the future payout more severely. However the interest rate volatility θ has a significant impact on option prices varies and this is outlined in the next subsection.

4.3. The interest rate volatility θ

Figure 4 plots call option prices and the options ratios for varying ρ^{xr} with four different parameter values of interest rate volatility θ and two maturities; $T = 0.5$ and $T = 20$. Comparing Figure 4 with Figure 3, we conclude that the interest rate volatility has a substantial impact to long-dated futures option prices. For long-dated options and when θ is very high, there is a nonlinear (convex) relationship between ρ^{xr} and the option prices. More specifically, the higher the volatility of interest rates, the higher (lower) the options prices when interest rates are negatively (positively) correlated with the futures prices. In addition, option prices from models with interest rates being negatively correlated to the futures prices are more sensitive to the volatility of interest rate.

5. Conclusion

In this paper we propose a commodity derivatives pricing model featuring stochastic volatility and stochastic interest rate. The functional form of the volatility function is chosen such that the model admits finite dimensional realisations and leads to affine representations, hence quasi-analytical European vanilla option pricing formulae can be obtained. Numerical investigations to gauge the impact of parameters of the interest rate model to option prices are performed.

Several conclusions can be drawn from these investigations. Firstly, the correlation between the stochastic interest rates and the stochastic futures price volatility process has negligible impact on the prices even for very long-dated options. Secondly, the correlation between the stochastic interest rates and the stochastic futures price process has noticeable impact on prices of long-dated options but remains negligible for short-dated options. Thirdly, the impact of the long-term level of interest rate to option prices does not depend on the correlation between stochastic interest rates and the stochastic futures price process. Lastly, as the volatility of interest rates increases, the value of the option increases with the impact be more pronounced for longer-maturity options and when the correlation between

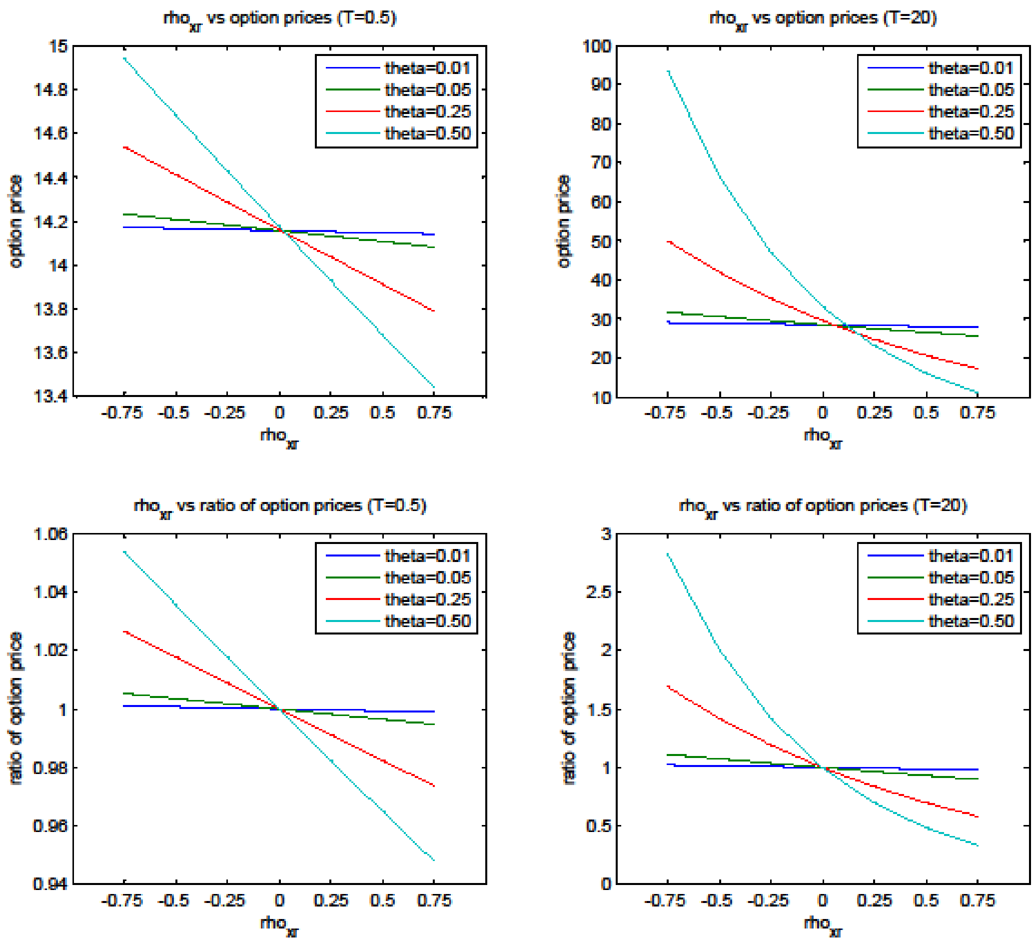

Figure 4: **Impact of the volatility of interest rates to call option prices**

The plot displays option prices (top two graphs) and ratios of option prices (bottom two graphs) for two different maturities $T = 0.5$ and $T = 20$ and for $\theta = 0.01, 0.05, 0.25, 0.50$.

futures prices and interest rate is negative.

The implications of the results from our sensitivity analysis may be relevant to practitioners who trade derivative contracts with long maturities, especially at time when volatility of interest rates is high and there exists a strong negative correlation between the interest rates and futures prices. A company who underwrites options in this situation using models with deterministic interest rate could be selling them at prices lower than from models that incorporate stochastic interest rates, therefore exposing the company to interest rate risk. However at a time when the volatility of interest rates is low or when there is low correlation between the interest rates and futures prices, a model with deterministic interest rate may be sufficient. In a forthcoming paper, we present an empirical study of the proposed model using the most liquid commodity market for long-dated futures options contracts, namely the crude oil market.

Appendix A. Instantaneous Spot Cost of Carry

Starting with the SDE of the logarithm of the futures price process $X(t,T) = \log F(t,T,\sigma_t)$, an application of the Itô's lemma derives

$$dX(t,T) = -\frac{1}{2}\sum_{i=1}^{n}\big(\sigma_i^F(t,T,\sigma_t)\big)^2 dt + \sum_{i=1}^{n}\sigma_i^F(t,T,\sigma_t)dW_i^x(t). \tag{A.1}$$

By integrating (A.1) we get:

$$\begin{aligned} &F(t,T,\sigma_t) = \\ &\qquad F(0,T,\sigma_0)\exp\Big(-\frac{1}{2}\sum_{i=1}^{n}\int_0^t \big(\sigma_i^F(u,T,\sigma_u)\big)^2 du + \sum_{i=1}^{n}\int_0^t \sigma_i^F(u,T,\sigma_u)\, dW_i^x(u)\Big). \end{aligned} \tag{A.2}$$

By letting the maturity T approaches time t, (A.2) derives the dynamics of the commodity spot price as:

$$\begin{aligned} S(t,\sigma_t) &= \lim_{T\to t} F(t,T,\sigma_t) \\ &= F(0,t,\sigma_0)\exp\Big(-\frac{1}{2}\sum_{i=1}^{n}\int_0^t \big(\sigma_i^F(u,t,\sigma_u)\big)^2 du + \sum_{i=1}^{n}\int_0^t \sigma_i^F(u,t,\sigma_u)\, dW_i^x(u)\Big), \end{aligned} \tag{A.3}$$

equivalently,

$$\begin{aligned} &\log S(t,\sigma_t) = \log F(0,t) \\ &\qquad -\frac{1}{2}\sum_{i=1}^{n}\int_0^t \big(\sigma_i^F(u,t,\sigma_u)\big)^2 du + \sum_{i=1}^{n}\int_0^t \sigma_i^F(u,t,\sigma_u)\, dW_i^x(u)\Big). \end{aligned} \tag{A.4}$$

By applying the stochastic Leibniz differential rule, we obtain (8).

Appendix B. Finite Dimensional Realisation for the Futures Process

Starting from (A.2) we need to calculate two integrals:

$$I = \int_0^t \sigma_i^F(u,T,\sigma_u)\, dW_x(u) \tag{B.1}$$

$$II = \int_0^t \big(\sigma_i^F(u,T,\sigma_u)\big)^2 du \tag{B.2}$$

We substitute the volatility specifications (9) to get:

$$
\begin{aligned}
I = & \int_0^t \big(\xi_{0i} + \xi_i(T-u)\big) e^{-\eta_i(T-u)} \sigma_i(u)\, dW_i^x(u) \\
= & \int_0^t \big(\xi_{0i} + \xi_i(T-t) + \xi_i(t-u)\big) e^{-\eta_i(T-t)} e^{-\eta_i(t-u)} \sigma_i(u)\, dW_i^x(u) \\
= & \big(\xi_{0i} + \xi_i(T-t)\big) e^{-\eta_i(T-t)} \int_0^t e^{-\eta_i(t-u)} \sigma_i(u)\, dW_i^x(u) \\
& + \xi_i e^{-\eta_i(T-t)} \int_0^t (t-u) e^{-\eta_i(t-u)} \sigma_i(u)\, dW_i^x(u) \\
= & \beta_{1i}(T-t)\phi_i(t) + \beta_{2i}(T-t)\psi_i(t),
\end{aligned}
$$

where the deterministic functions are defined as:

$$
\begin{aligned}
\beta_{1i}(T-t) &= \big(\xi_{0i} + \xi_i(T-t)\big) e^{-\eta_i(T-t)}, \\
\beta_{2i}(T-t) &= \xi_i e^{-\eta_i(T-t)},
\end{aligned}
$$

and the state variables are defined by:

$$
\begin{aligned}
\phi_i(t) &= \int_0^t e^{-\eta_i(t-u)} \sigma_i(u)\, dW_i^x(u), \\
\psi_i(t) &= \int_0^t (t-u) e^{-\eta_i(t-u)} \sigma_i(u)\, dW_i^x(u).
\end{aligned}
$$

Next

$$
\begin{aligned}
II = & \int_0^t \big(\sigma_i^F(u, T, \sigma_u)\big)^2 \, du \\
= & \int_0^t \big(\xi_{0i} + \xi_i(T-u)\big)^2 e^{-2\eta_i(T-u)} \sigma_i(u)^2 \, du \\
= & \int_0^t \Big(\beta_{1i}(T-t) + \beta_{2i}(T-t)(t-u)\Big)^2 e^{-2\eta_i(t-u)} \sigma_i(u)^2 \, du \\
= & \gamma_{1i}(T-t)x_i(t) + \gamma_{2i}(T-t)y_i(t) + \gamma_{3i}(T-t)z_i(t),
\end{aligned}
$$

where the deterministic functions are defined as:

$$
\begin{aligned}
\gamma_{1i}(T-t) = & \beta_{1i}(T-t)^2, \\
\gamma_{2i}(T-t) = & 2\beta_{1i}(T-t)\beta_{2i}(T-t), \\
\gamma_{3i}(T-t) = & \beta_{2i}(T-t)^2,
\end{aligned}
$$

and the state variables are defined as:

$$
\begin{aligned}
x_i(t) &= \int_0^t e^{-2\eta_i(t-u)}\sigma_i(u)^2\,du \\
y_i(t) &= \int_0^t (t-u)e^{-2\eta_i(t-u)}\sigma_i(u)^2\,du \\
z_i(t) &= \int_0^t (t-u)^2 e^{-2\eta_i(t-u)}\sigma_i(u)^2\,du.
\end{aligned}
$$

Alternatively, the $5n$ state variables $\phi_i(t), \psi_i(t), x_i(t), y_i(t)$ and $z_i(t)$ can be expressed in differential form as (11).

Appendix C. Derivation of the Riccati ODE

Starting from (13), first observation is that this equation it not a martingale, but multiplying it by a discount factor $e^{-\int_0^t y_1(u)\,du}$, it can be turned into a martingale. Defining $\phi_t = \phi(t, x(t), y_1(t), \nu_t, \sigma_t)$ and by applying Itô's lemma we have

$$
d\big(e^{-\int_0^t y_1(u)\,du}\phi(t)\big) = e^{-\int_0^t y_1(u)\,du}\big(-y_1(t)\phi(t)dt + d\phi(t)\big)
$$

Since $e^{-\int_0^t y_1(u)\,du}\phi(t)$ is a martingale, its drift term must be zero and as a result we have:

$$
-y_1(t)\phi(t) + \text{drift term of } d\phi(t) = 0. \tag{C.1}
$$

Using the correlation structure in (6) we have:

$$
\begin{aligned}
d\phi(t) &= \frac{\partial \phi}{\partial t}dt + \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y_1}dy_1 + \sum_{i=1}^{n}\frac{\partial \phi}{\partial \nu_i}d\nu_i + \sum_{i=1}^{n}\frac{\partial \phi}{\partial \sigma_i}d\sigma_i + \frac{1}{2}\frac{\partial^2 \phi}{\partial x^2}dx^2 \\
&+ \frac{1}{2}\frac{\partial^2 \phi}{\partial y_1^2}dy_1^2 + \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 \phi}{\partial \nu_i^2}d\nu^2 + \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 \phi}{\partial \sigma_i^2}d\sigma^2 + \frac{\partial^2 \phi}{\partial x \partial y_1}dxdy_1 + \sum_{i=1}^{n}\frac{\partial^2 \phi}{\partial x \partial \nu_i}dxd\nu_i \\
&+ \sum_{i=1}^{n}\frac{\partial^2 \phi}{\partial x \partial \sigma_i}dxd\sigma_i + \sum_{i=1}^{n}\frac{\partial^2 \phi}{\partial y_1 \partial \nu_i}dy_1d\nu_i + \sum_{i=1}^{n}\frac{\partial^2 \phi}{\partial y_1 \partial \sigma_i}dy_1d\sigma_i \\
&+ \underbrace{\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 \phi}{\partial \nu_i \partial \sigma_j}d\nu_j d\sigma_j}_{=0 \text{ if } i\neq j} + \underbrace{\sum_{i=1}^{n}\sum_{j>i}^{n}\frac{\partial^2 \phi}{\partial \nu_i \partial \nu_j}d\nu_j d\nu_j}_{=0} + \underbrace{\sum_{i=1}^{n}\sum_{j>i}^{n}\frac{\partial^2 \phi}{\partial \sigma_i \partial \sigma_j}d\sigma_j d\sigma_j}_{=0} \\
&= \phi(t)\left(\frac{\partial A}{\partial t} + x\frac{\partial B}{\partial t} + y_1\frac{\partial C}{\partial t} + \sum_{i=1}^{n}\left(\nu_i\frac{\partial D_i}{\partial t} + \sigma_i\frac{\partial E_i}{\partial t}\right)\right)dt \\
&+ \phi(t)\left(B\Big(-\frac{1}{2}\sum_{i=1}^{n}\beta_{1i}^2(T-t)\nu_i(t)dt + \sum_{i=1}^{n}\beta_{1i}(T-t)\sqrt{\nu_i(t)}dW_i^x(t)\Big)\right) \\
&+ \phi(t)\left(C\Big(-\lambda y_1(t))dt + \theta dW_r(t)\Big)\right) \\
&+ \phi(t)\left(\sum_{i=1}^{n}D_i\Big((-2\nu_i(t)\kappa_i + 2\kappa_i\overline{\sigma}_i\sigma_i(t) + \gamma_i^2)dt + 2\gamma_i\sqrt{\nu_i(t)}dW_i^\sigma(t)\Big)\right) \\
&+ \phi(t)\left(\sum_{i=1}^{n}E_i\Big(\kappa_i(\overline{\sigma}_i - \sigma_i(t))dt + \gamma_i dW_i^\sigma(t)\Big)\right) \\
&+ \frac{1}{2}\phi(t)B^2\Big(\sum_{i=1}^{n}\beta_{1i}^2(T-t)\nu_i(t)dt\Big) + \frac{1}{2}\phi(t)C^2\theta^2 dt \\
&+ \frac{1}{2}\phi(t)\left(4\sum_{i=1}^{n}D_i^2\gamma_i^2\nu_i dt\right) + \frac{1}{2}\phi(t)\sum_{i=1}^{n}E_i^2\gamma_i^2 dt \\
&+ \phi(t)BC\sum_{i=1}^{n}\theta\beta_{1i}(T-t)\sigma_i(t)\rho_i^{xr}dt + 2\phi(t)B\sum_{i=1}^{n}D_i\gamma_i\rho_i^{x\sigma}\nu_i(t)\beta_{1i}(T-t)dt \\
&+ B\phi(t)\sum_{i=1}^{n}E_i\rho_i^{x\sigma}\gamma_i\beta_{1i}(T-t)\sigma_j dt + 2\phi(t)C\sum_{i=1}^{n}D_i\theta\gamma_i\rho_i^{\sigma r}\sigma_i(t)dt \\
&+ C\phi(t)\sum_{i=1}^{n}E_i\rho_i^{\sigma r}\theta\gamma_i dt + 2\phi(t)\sum_{i=1}^{n}D_jE_i\gamma_i^2\sigma_i(t)dt.
\end{aligned}
$$

Combining this result with (C.1) we get:

$$
\begin{aligned}
0 &= \frac{\partial A}{\partial t} + x(t)\frac{\partial B}{\partial t} + y_1(t)\frac{\partial C}{\partial t} + \sum_{i=1}^{n}\nu_i(t)\frac{\partial D_i}{\partial t} + \sum_{i=1}^{n}\sigma_i(t)\frac{\partial E_i}{\partial t} - \frac{1}{2}B\sum_{i=1}^{n}\beta_{1i}^2(T-t)\nu_i(t)\\
&- C\lambda y_1(t) + \sum_{i=1}^{n} D_i(-2\nu_i(t)\kappa_i + 2\kappa_i\overline{\sigma}_i\sigma_i(t) + \gamma_i^2)\\
&+ \sum_{i=1}^{n} E_i\kappa_i(\overline{\sigma}_i - \sigma_i(t)) + \frac{1}{2}B^2\sum_{i=1}^{n}\beta_{1i}^2(T-t)\nu_i(t) + \frac{1}{2}C^2\theta^2\\
&+ 2\sum_{i=1}^{n} D_i^2\gamma_i^2\nu_i(t) + \frac{1}{2}\sum_{i=1}^{n}E_i^2\gamma_i^2 + BC\sum_{i=1}^{n}\theta\beta_{1i}(T-t)\sigma_i(t)\rho_i^{xr}\\
&+ 2B\sum_{i=1}^{n} D_i\gamma_i\rho_i^{x\sigma}\nu_i(t)\beta_{1i}(T-t) + B\sum_{i=1}^{n}E_i\rho_i^{x\sigma}\gamma_i\beta_{1i}(T-t)\sigma_i(t)\\
&+ 2C\sum_{i=1}^{n} D_i\theta\gamma_i\rho_i^{\sigma r}\sigma_i(t) + C\sum_{i=1}^{n}E_i\rho_i^{\sigma r}\theta\gamma_i + 2\sum_{i=1}^{n}D_iE_i\gamma_i^2\sigma_i(t) - y_1(t).
\end{aligned}
$$

Rearranging the above equation, we derive:

$$
\begin{aligned}
0 &= x(t)\frac{\partial B}{\partial t},\\
0 &= y_1(t)\frac{\partial C}{\partial t} - \lambda y_1(t)C - y_1(t),\\
0 &= \nu_i(t)\frac{\partial D_i}{\partial t} - \frac{1}{2}\beta_{1i}^2(T-t)\nu_i(t)B - 2\kappa_i\nu_i(t)D_i + \frac{1}{2}\beta_{1i}^2(T-t)\nu_i(t)B^2,\\
&+ 2\nu_i(t)\gamma_i^2D_i^2 + 2\rho_i^{x\sigma}\beta_{1i}(T-t)\gamma_i\nu_i(t)BD_i,\\
0 &= \sigma_i(t)\frac{\partial E_i}{\partial t} + 2\bar{\sigma}_i\kappa_i\sigma_i(t)D_i - \kappa_i\sigma_i(t)E_i + \rho_i^{xr}\theta\beta_{1i}(T-t)\sigma_i(t)BC,\\
&+ \rho_i^{x\sigma}\beta_{1i}(T-t)\gamma_i\sigma_i(t)BE_i + 2\rho_i^{r\sigma}\theta\gamma_i\sigma_i(t)CD_i + 2\gamma_i^2\sigma_i(t)D_jE_i,\\
0 &= \frac{\partial A}{\partial t} + \frac{1}{2}\theta^2C^2 + \sum_{i=1}^{n}\gamma_i^2D_i + \sum_{i=1}^{n}\kappa_i\bar{\sigma}_iE_i + \frac{1}{2}\sum_{i=1}^{n}\gamma_i^2E_i^2\\
&+ \sum_{i=1}^{n}\rho_i^{r\sigma}\theta\gamma_iCE_i,
\end{aligned}
$$

where $i \in \{1,\ldots,n\}$, subject to the terminal condition $\phi(T_0) = e^{\upsilon X(T_o,T_o)}$. This implies that $A(T_0;\upsilon,T_0) = C(T_0;\upsilon,T_0) = D_i(T_0;\upsilon,T_0) = E_i(T_0;\upsilon,T_0) = 0, \forall i \in \{1,\ldots,n\}$ and $B(T_0;\upsilon,T_0) = \upsilon$.

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