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On Candlestick-based Trading Rules Profitability Analysis via Parametric Bootstraps and Multivariate Pair-Copula based Models

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Abstract

This study analyses the profitability of candlestick-based technical trading rules in currency futures markets. The main feature of this type of technical analysis is that it generates signals based on the relationships between several price series, i.e. open, high, low and close prices. Since the trading rules are not precise and mainly based on causal knowledge of traders, we use a fuzzy-system to mathematize and classify them. The profitability of the trading rules is then tested by means of a bootstrap technique using a multivariate parametric model, in order to cope with the multivariate premise of the candlesticks. The model employs different time series, a non-negative valued return (the absolute daily return), two non-negative valued ranges, one strictly positive range (the daily range) and a binary time series (the sign of the daily return). The multivariate distribution of the full model is constructed by means of a (Pair-)Copula approach, and we use a two-stage estimation method to identify its coefficients. The analysis is applied to data on British Pound, Swiss Francs and Japanese Yen futures prices from 1978 to 2013. The results show, that not all candlestick-based trading rules are profitable and only very few have a statistically significant predictive value in the analysed currency futures markets.

Keywords: Candlesticks, Technical Trading, Fuzzy-Systems, Parametric Bootstrap, Multiplicative Error Models, Pair-Copula, Futures Markets

JEL Classification: C10, C32, C51, F31, G1

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1. Introduction

Contrary to the efficient market hypothesis, technical analysis is based on the assumption that past prices and other trading information (such as trading volume) contain significant information for forecasting future price movements. If in an efficient market there are no potential economic profits (Jensen, 1978), i.e. risk-adjusted returns of acting on information net to all costs of doing so, short term positive returns are either an indication for market inefficiencies, or are due to time varying equilibrium returns, or to some factors based on the market microstructure (such as the impact of information on the behavior of investors and changes in regulatory conditions) (Brock et al., 1992). There are already a number of studies, which have analysed the statistical and economic significance of returns based on technical analysis, see for example Park and Irwin (2007) who survey 137 empirical studies from 1960 to 2004. Reported positive results in different financial markets, at least until the early 1990s, are nonetheless subject to various problems in their testing procedures, i.e. only some of them account for data snooping and many have difficulties in the estimation of risk and transaction costs.

More recent studies (after 1992) adopted the model-based bootstrap as a statistical test, introduced in conjunction with technical analysis in Brock et al. (1992). The basic idea of the test is to compare the conditional returns of the actual data with the ones of some artificial data, named hereafter *pseudo*-data, which is constructed by means of a postulated parametric model, the *null*-model. Generated by sampling from the distribution of the (i.i.d.) residuals of the model, the *pseudo*-data share some specific properties of the actual data, and, thus, the conditional returns based on it can be seen as realizations from the same, or at least from an asymptotically approximated distribution of conditional returns. By repeatedly generating *pseudo*-data for a larger number of iterations, usually 500-2000 iterations, one obtains an empirical distribution of the conditional returns, which can be used to analyse the conditional returns of the actual data by means of hypothesis testing. The survey conducted by Park and Irwin (2007) found 21 studies before 2004, which adopted the model-based bootstrap, where some of them found statistically significant positive returns even after considering transaction costs.

The model-based bootstrap applied as a statistical significance test after 1992 was nonetheless subject to criticism². First, regarding the profitability of technical trading rules, the statistical significance from the bootstrap tests has been interpreted in two different (conflicting) ways: use of bootstrapping trading rule profits for model selection and use of the bootstrap distribution, assuming an appropriate parametric model, to test if the trading rule profits from the actual data fall in the (say) 95% interval, meaning that they do not bring statistically significant returns. The

²See for example Maddala and Li (1996) and Ruiz and Pascual (2002).

first interpretation was used among others³ by Brock et al. (1992) and LeBaron (1991). Both analysed the moving average rules, which are based on daily (close price) returns. The second interpretation was used for example by Levich and Thomas (1993) (for moving average based trading rules) and also by Marshall et al. (2006), who analysed for the first time candlestick-based trading rules by means of bootstrap techniques. With this second interpretation however, the appropriateness of the chosen parametric model could not be tested and was thus only assumed. Second, as argued in Ruiz and Pascual (2002), should the serial dependence of the data be not correctly specified or highly complex (see Park and Irwin (2007)), the parametric bootstrap could be inconsistent, so alternative bootstrap methods (also nonparametric, such as the moving block bootstrap) could deal better with dependent time series data.

In turn, in the case of trading rules, which are based on more than one time series, as in the case of candlesticks, the *null*-model of the parametric bootstrap procedure must be also carefully chosen. We approach this problem by constructing a multivariate parametric *null*-model, that describes different volatility indicators, which are based on the close, open, high and low prices. Like the candlesticks, these volatility indicators are quantifying the relationships between the price series, as it will be shown throughout the paper.

Candlestick charting is a form of technical analysis, which dates back to 18th century Japan. It was introduced in the modern literature by Nison (1991). Candlestick charts require the opening, high, low and the closing prices of each analysed session, and represent them graphically on a time scale, as shown in Fig. 1. A session can be an arbitrary period of time, usually a trading day. The segment between the open and close prices of one session is called the body of the candlestick, whereas the segment between the high/low price and the body is called upper/lower shadow. Based on some specific forms of the candlesticks, one can analyse the evolution of the market. A typical example is a trend reversal signal, based on a candlestick called *hanging man*, shown on 29 January 1990 in Fig. 1. The candlestick has a small body and a long lower shadow and appears after a period of a sustained increase in prices. It signals a trend reversal, which in this context means that the trend is not likely to continue.

To test if the candlesticks have any statistically significant predictive value, one can employ bootstrap techniques. One can thus formulate a *null*-Hypothesis that the candlestick-based trading rules have no value, i.e. that the expected conditional h -day returns (usually for ten-day holding periods), given some candlestick signals, are zero. Then, by postulating a parametric model, one can generate *pseudo*-data, which has some common characteristics with the original data, and construct with these new realizations an empirical distribution of the conditional returns. However, because the candlesticks use more than one time series, the *null*-model must

³See also Tsay (1992) and Maddala and Li (1996) and Ruiz and Pascual (2002).

be carefully chosen. To account for the multivariate premise of the candlesticks, we construct a multivariate parametric model and generate, by sampling from the joint distribution of the error terms, the empirical distribution of the returns, conditional on the candlesticks signals. At this point one can see the candlestick signals as realizations of some daily returns and intra-daily ranges, which have been jointly modelled within the new model.

Our testing procedure is significantly different from the one presented in Marshall et al. (2006), which assumed a univariate *null*-model for the unconditional daily returns and resampled first its residuals, then some (raw) percentage differences between the underlying price of the modelled returns r_t^h and the remaining daily prices. The second resampling step of their procedure was motivated by the need to generate not only the underlying *pseudo*-prices, i.e. *pseudo*-close prices, but also the remaining ones, i.e. *pseudo*-open, high and low prices, in order to analyse the candlestick profitability. This could be however insufficient to ensure that the bootstrap method is correctly applied. In fact, the bootstrap technique is only valid if these raw vectors are i.i.d., a requirement, which was not tested. Furthermore, any correlation between different returns, which are being resampled in parallel, has to be taken into account, a requirement, which was also not analysed in Marshall et al. (2006).

The paper is organized as follows: In the next Section we construct a testing framework, to see if the candlesticks yield any profits on the analyzed time series and in Section 3 we describe the data and the candlestick recognition algorithm. In Section 4 we describe the multivariate parametric model. In Section 5 we discuss the empirical results and robustness of the bootstrap method, and Section 6 concludes. Some more technical details on the construction of the recognition algorithm can be found in Appendix A.

2. Candlestick-based trading

Candlestick-based trading rules use in addition to the close prices also intra-daily information, i.e. the high, low and open prices, to generate a trade signal and are thus based on more than one (return) time series. Thereby, the relationship between the open (O_t), high (H_t), low (L_t) and close (C_t) prices at time t is classified as a candlestick with three main features, the body ($B_t = (|O_t - C_t|)/\mathcal{L}_t$), the upper shadow ($S_{u_t} = (H_t - \max(C_t, O_t))/\mathcal{L}_t$) and the lower shadow ($S_{l_t} = (\min(C_t, O_t) - L_t)/\mathcal{L}_t$). All three parts are normalized relative to the total length of the candlestick ($\mathcal{L}_t = H_t - L_t$). By means of the resulting relative lengths, different forms of candlesticks are then classified in two main classes, bullish or bearish forms (single lines for one day, or patterns for consecutive days).⁴ Fig. 1 shows a sample of candlesticks. The color of the candlestick (black or white) is determined by the relationship between the open and the close

⁴The analysed candlestick forms are described in Appendix A.

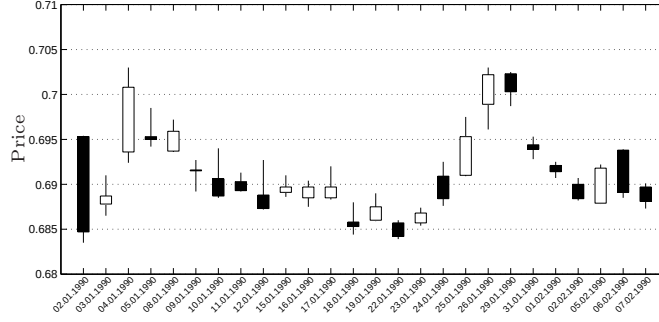


Figure 1: Candlestick sample of the exchange rate time series from the Japanese Yen futures.

price of the session, i.e. white if the close price is higher than the open price and black otherwise. To analyse the candlestick-based trading profits, each day is furthermore classified into a *buy* (b_t) signal, if the candlestick is bullish, a *sell* (s_t) signal, if the candlestick is bearish, or no signal, if otherwise. The candlestick-based returns are then generated as follows. For each *buy* (*sell*) signal purchase (sell) one dollar's worth futures contract at price P_t , hold it for h periods⁵, then sell (buy) it after $t+h$ periods, to obtain the return

$$r_t^h = \ln P_{t+h} - \ln P_t, \quad (1)$$

where the price P_t can be taken to be the close-price on the same day, i.e. C_t , the opening price on the next day, i.e. O_{t+1} , or the close-price on the next day, i.e. C_{t+1} . In this paper, the price P_t is taken to be the opening price on the next day, i.e. O_{t+1} .

The statistical significance of the conditional returns can be analysed by means of hypothesis testing. Thus the *null*-Hypothesis, that the candlestick-based trading rules have no value, i.e.

$$\mathcal{H}_{b0} : m_b = E(r_t^h | b_t) = 0, \quad (2)$$

$$\mathcal{H}_{s0} : m_s = E(r_t^h | s_t) = 0, \quad (3)$$

is analysed based on the distributions of the conditional returns, i.e. the distribution of r_t^h conditional on a buy-signal b_t respectively a sell-signal s_t , which are generally not known and have to be estimated. However, the buy and sell signals, which are generated based on candlesticks, are the realizations of some contemporaneous price differences between the open, high, low and close price. Thus, in constructing the conditional distribution of the analysed return, one has to take also these price differences into account. One of the main ideas of this paper, is that, these price differences can be also seen as the realizations of some daily returns and intra-daily ranges. Thus, the conditional distribution of the analysed return can be calculated from the joint

⁵Each candlestick is isolated from the others, i.e. no action is taken if during the h periods another candlestick gives a signal.

distribution of these variables. This can be done by postulating a multivariate *null*-model for the joint returns and ranges. Assuming that the *null*-model can accommodate some essential characteristics of these joint variables and by fitting it to the data, one obtains a number of residual time series, which should be independent and identically distributed (i.i.d.). By sampling from the distribution of the residuals, some *pseudo*-returns and ranges can be generated, which then share the same considered properties with the original data, but are independent and identically distributed. Based on these prices, one can calculate the empirical distribution of the analysed conditional returns.

As stated earlier, the candlesticks can be seen as the realizations of some daily and intra-daily returns and ranges. These should uniquely identify the three main features of the candlesticks, i.e. the body ($\mathcal{B}_t = (|O_t - C_t|)/\mathcal{L}_t$), the upper shadow ($S_{u_t} = (H_t - \max(C_t, O_t))/\mathcal{L}_t$) and the lower shadow ($S_{l_t} = (\min(C_t, O_t) - L_t)/\mathcal{L}_t$). For this purpose we choose the following daily and intra-daily returns and ranges as volatility estimators, based on the OLHC⁶ log-prices $o_t = \ln O_t$, $h_t = \ln H_t$, $l_t = \ln L_t$, and $c_t = \ln C_t$: the absolute returns, $|r_t| = |c_{t+1} - c_t|$, the indicator function for the sign of the daily return r_t , $\mathbb{I}[r_t > 0]$, and three OLHC (*range*-based) volatility indicators (Saichev et al., 2009): the daily range, $\sigma_{HL_t} = h_t - l_t$, the intradaily range $\sigma_{HC_t} = h_t - c_t$, and the *normalized high price*⁷, $\sigma_{HO_t} = h_t - o_t$. Our choice, more specifically, the decomposition of the daily returns in their volatility and direction-of-change, is motivated by the higher degree of dependence over time, that they exhibit, in comparison to that of the daily returns, as shown in Anatolyev and Gospodinov (2010).

Apart from the daily range, σ_{HL_t} , which takes strictly positive values, all other volatility indicators, σ_{HC_t} , σ_{HO_t} , and $|r_t|$, take non-negative values. They will be modelled by multiplicative error models (*MEM*), which were introduced in Engle and Gallo (2006). The indicator function $\mathbb{I}[r_t > 0]$ takes only binary values and will be modelled with a dynamic logit model (*direction model*), as in Anatolyev and Gospodinov (2010). The *direction model* will give the conditional *success probability*, i.e. the probability that $\mathbb{I}[r_t > 0] = 1$, given all available information to the time $t - 1$. Put together, the multivariate conditional distribution of $((\sigma_t, |r_t|, \mathbb{I}[r_t > 0])|\Omega_{t-1})$, where $\sigma_t = [\sigma_1 \ \sigma_2 \ \sigma_3]^T := [\sigma_{HL_t} \ \sigma_{HC_t} \ \sigma_{HO_t}]^T$, given the set of all available information to the time $t - 1$, Ω_{t-1} , will be constructed by means of a Copula approach. By this procedure, we are jointly estimating the conditional volatility based on high, low, open and close prices, as well as its influence on the direction of the daily returns.

⁶The letters stand for the open, low, high and close price.

⁷The term *normalized high* refers to the standard practice of setting, without loss of generality, the beginning of the time interval (τ_i, τ_e) , within which the observed realizations of the stochastic process $X(\tau)$, $\tau \in (\tau_i, \tau_e)$, are recorded, to zero, thus decreasing in practice the relevant quantities by the opening time, o_t . Consequentially, the *normalized high*, σ_{HO_t} , is equivalent to $\sup_{\tau \in (0, \mathcal{T})} X(\tau)$ and the *normalized close price*, $c_t - o_t$, to $X(\mathcal{T})$. We choose then the *normalized high price* as a volatility estimator, even though it is not the *best* estimator one can build based on the OLHC prices (see Saichev et al. (2009) for the *best* estimator).

3. Data and the Candlestick recognition algorithm

We use data for three currency futures contracts, traded at the Chicago Mercantile Exchange, i.e. British Pound, Swiss Francs and Japanese Yen futures contracts. The data for the time period from 3 January 1978 to 29 November 2013 are taken from Datastream. Furthermore, there has been a number of days, where the settlement price was outside the daily range. This problem is not unknown and results from the fact that the settlement price was determined after the market was closed, due to low market activity at the end of the trading session. In dealing with this matter, we have chosen the standard solution (see for example Pelletier (1996)), to update the high and low prices to include also the settlement price.

3.1. Summary statistics for the analysed time series

Table 1 shows the summary statistics of the annualized log return series and ranges for the analysed futures contracts in the time period 3 January 1978 to 29 November 2013. All returns and ranges are positively skewed and strongly leptokurtic, with a kurtosis value much higher than that of a normal distribution, which is 3. The sample autocorrelations, although modest, are all positive and at least up to lag 30 outside the 95% confidence interval for a independently and identically distributed (i.i.d.) process. This indicates that the absolute return and ranges are not i.i.d.. In addition, all correlation coefficients are significantly different from zero, which indicates that they are not statistically independent. The point biserial correlation coefficients between the sign of the return and all other variables are all positive, indicating that negative daily returns tend to coincide with higher volatilities. Aside from the daily range, σ_{HL_t} , all volatility proxies exhibit up to almost 3% share of zeros, quantified as n_z/N , i.e. as the proportion of zeros in the total number of observations. Finally, the Engle test for heteroscedasticity is applied to the residuals, i.e. returns-mean(returns). The test rejects the null hypothesis of no conditional heteroscedasticity in favor of the alternative hypothesis that an ARCH(k) model describes the series of squared residuals.

3.2. Candlestick statistics

Since the candlestick form can be defined by means of linguistic rules, we employ a static system, based on fuzzy-logic, to mathematize this rules and thus automatically identify different classes of candlestick single lines and patterns. The fuzzy-system has a fixed structure and uses in general min / max operators to process the signals. It is also known as a Mamdani-type fuzzy-system, after E.H. Mamdani, its developer⁸. In this paper, the fuzzy-system is composed of five parts. The first three parts are used for classifying the single line forms, the fourth for the pattern

⁸See Mamdani (1974) for the first application of fuzzy logic in the field of automatic control and Dourra and Siy (2002) for an application in the technical analysis field.

Table 1: Summary statistics of the returns and ranges for the GBP, CHF and JPY futures (in annualized percentage terms) for the total period, i.e. 8915 observations for the GBP futures, 8891 observations for the CHF futures and 8937 observations for the JPY futures. Numbers marked with * are significant at the 1% level.

	σ_{HL_t}	σ_{HC_t}	σ_{HO_t}	$ r_t $	σ_{HL_t}	σ_{HC_t}	σ_{HO_t}	$ r_t $	σ_{HL_t}	σ_{HC_t}	σ_{HO_t}	$ r_t $
GBP					CHF				JPY			
Min	0.052	0	0	0	0.046	0	0	0	0.074	0	0	0
Max	23.60	21.84	18.08	12.98	24.79	24.42	16.78	27.05	24.29	13.70	24.16	20.77
Mean	2.06	1.00	1.03	1.21	2.51	1.24	1.26	1.45	2.16	1.10	1.12	1.30
SD	1.21	0.99	0.99	1.15	1.33	1.12	1.19	1.36	1.41	1.03	1.21	1.26
Skewness	3.46	3.99	2.95	2.18	2.59	3.10	2.32	2.87	3.03	2.62	3.74	2.61
Kurtosis	35.20	46.55	27.55	11.32	21.38	32.45	14.17	28.69	24.18	16.51	35.91	18.34
Autocorrelation $\rho(i)$ at lag i												
$\rho(1)$	0.42*	0.18*	0.11*	0.13*	0.31*	0.08*	0.10*	0.09*	0.45*	0.17*	0.20*	0.14*
$\rho(10)$	0.34*	0.13*	0.09*	0.14*	0.23*	0.08*	0.06*	0.09*	0.30*	0.18*	0.13*	0.08*
$\rho(20)$	0.30*	0.14*	0.08*	0.10*	0.20*	0.10*	0.05*	0.07*	0.28*	0.15*	0.11*	0.09*
$\rho(30)$	0.27*	0.10*	0.08*	0.09*	0.17*	0.05*	0.03*	0.05*	0.25*	0.13*	0.09*	0.05*
Kendall correlations												
σ_{HL_t}	0.36*	0.30*	0.41*		0.33*	0.32*	0.37*		0.42*	0.36*	0.35*	
σ_{HC_t}		-0.12*	0.15*			-0.13*	0.14*			-0.03*	0.16*	
σ_{HO_t}			0.15*				0.12*				0.11*	
Point biserial correlation coefficient												
$\mathbb{I}[r_t > 0]$	0.08*	0.05*	0.05*	0.15*	0.06*	0.04*	0.04*	0.14*	0.06*	0.04*	0.04*	0.14*
Share of zero observations												
n_z/N	0	0.0048	0.029	0.021	0	0.0055	0.03	0.016	0	0.0044	0.025	0.018
Engle test for residual heteroscedasticity (p-values)												
$k = 1$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.00	0.00
$k = 2$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

forms and the fifth for identifying the buy/sell signal based on the candlestick single line/pattern form, prior trend and, if necessary, the next day confirmation. For example, the candlestick named *hammer*, which has a single line form, can be defined using the following linguistic rule:

If the candlestick has a small body, almost no upper shadow and a very long lower shadow, which is at least twice the body length, then it has a hammer form.

The identification of a buy signal is done using the following linguistic rule:

If the candlestick has a hammer form, the prior trend is negative, and the next day opens higher than the body of the hammer, then set a buy signal.

Other forms and trend definitions can be also formulated by linguistic rules. These and the construction of the fuzzy-systems are described in Appendix A. Tables 2 and 3 summarize the results of the candlestick single line/pattern classification algorithm. As indicated in Nison (1991), a long-term exponential moving average of 65 days of the close prices, which gives more weight to the most recent observations, is used to determine the prior trend. Alternative choices are given in Appendix A. As in Marshall et al. (2006), the profitability is measured over a period of ten days following a buy/sell signal, after entering the market on the opening price of the next day, as the proportion of times when the conditional returns are greater/smaller than zero. Hence, a

profitable candlestick should give a correct signal in more than 50% of the time.⁹ Furthermore, the buy/sell signals are considered individually when the profits are calculated. The results show that almost all single lines are profitable in more than 50% of the time. However, the bullish and bearish patterns are generally not profitable on either of the analysed future price series. In addition, it can be seen that the strength of most trading rules is very low. The strength is given as a result of the fuzzy system and is dependent on the individual strengths of the form of the pattern, their prior trends and next day confirmations, see Appendix A for more details. To test if the candlestick single lines and patterns have statistically significant predictive value, we employ a bootstrap technique based on a multivariate parametric model. The test will be described in the following sections.

4. A (Pair-)Copula-based vector multiplicative error model (Copula-vMEM) for jointly modelling the volatility and the direction-of-change of daily returns

The purpose of this Section is to build a model, which includes the volatility indicators chosen in Section 2 as well as the sign of the daily returns. This is motivated by the idea that the candlesticks can be seen also as realizations of these variables, and, thus, the distribution of the returns r_t^h (from eq. (1)), conditional on a buy/sell signal, can be constructed from the joint distribution of these variables. The distribution of the conditional returns can then be used to test whether the hypotheses from eq. (2) and (3) can be rejected. This approach is different from the one used by Marshall et al. (2006), firstly because, knowing the joint distribution of all indicators, the parametric bootstrapping method is applied more accurately, by sampling not only from the distribution of the residuals of the daily returns but also from the ones of the other residuals, accounting also for the correlations between them.

Furthermore, the new model has also an economic interpretation of its own. The model makes a connection between Engle and Gallo (2006), who increase the forecasting performance of financial asset volatility by jointly modelling different volatility indicators and Anatolyev and Gospodinov (2010), who study the dependence over time between the volatility of absolute returns and the direction-of-change of daily returns. By connecting these two approaches, one can analyse the dependence and, hence, the forecastability of return signs based on an improved volatility measure, which incorporates not only daily but also intra-daily information.

4.1. Model specification for the volatility indicators

As indicated in the summary statistics, shown in Table 1, the time series σ_{HCt} , σ_{DRt} , σ_{Ht} and $|r_t|$ have non-negative values (or are strictly positive, as in the case of σ_{DRt}) and, hence, the

⁹As stated in Marshall et al. (2006), this profitability measure is only indicative of the candlestick performance, since it does not take the size of the profits into account. For example, a candlestick-based rule could be correct in less than 50% of the time, but still be profitable overall, if it yields substantially bigger profits than losses.

Table 2: Bullish single lines/patterns statistics for the JPY, CHF and GBP futures contracts: Column 2 shows the number of identified candlestick forms, Column 3 shows the largest output value, Column 4 shows the smallest output value, Column 5 and 6 show the mean and median output values of the identified candlesticks, Column 7 shows the proportion of times that profits for a ten-day period following a buy signal are greater than zero.

Bullish Single Lines	$N(y \neq 0)$	$\max y$	$\min y$	$\text{mean } y$	$\text{median } y$	$\text{profit} > 0$
GBP futures						
<i>Long White Candle</i> (LWC)	2195	1.000	0.200	0.503	0.600	0.507
<i>White Marubozu</i> (WM)	10	1.000	1.000	1.000	1.000	0.700
<i>Closing White Marubozu</i> (CWM)	33	1.000	0.515	0.755	0.750	0.424
<i>Opening White Marubozu</i> (OWM)	230	1.000	0.525	0.751	0.735	0.535
<i>Dragonfly Doji</i> (DD)	1	1.000	1.000	1.000	1.000	0.000
<i>White Paper Umbrella</i> (WPU)	815	1.000	0.200	0.362	0.200	0.523
<i>Black Paper Umbrella</i> (BPU)	901	0.985	0.191	0.399	0.400	0.509
CHF futures						
<i>Long White Candle</i> (LWC)	2093	1.000	0.200	0.503	0.600	0.519
<i>White Marubozu</i> (WM)	10	1.000	1.000	1.000	1.000	0.800
<i>Closing White Marubozu</i> (CWM)	30	0.935	0.510	0.713	0.665	0.633
<i>Opening White Marubozu</i> (OWM)	213	1.000	0.525	0.755	0.740	0.516
<i>Dragonfly Doji</i> (DD)	9	1.000	1.000	1.000	1.000	0.556
<i>White Paper Umbrella</i> (WPU)	753	0.985	0.200	0.384	0.400	0.509
<i>Black Paper Umbrella</i> (BPU)	823	0.995	0.191	0.392	0.400	0.485
JPY futures						
<i>Long White Candle</i> (LWC)	2016	1.000	0.200	0.488	0.600	0.509
<i>White Marubozu</i> (WM)	2	1.000	1.000	1.000	1.000	0.500
<i>Closing White Marubozu</i> (CWM)	35	1.000	0.510	0.741	0.730	0.571
<i>Opening White Marubozu</i> (OWM)	202	1.000	0.515	0.755	0.750	0.475
<i>Dragonfly Doji</i> (DD)	2	1.000	1.000	1.000	1.000	1.000
<i>White Paper Umbrella</i> (WPU)	683	0.985	0.200	0.362	0.200	0.483
<i>Black Paper Umbrella</i> (BPU)	732	0.985	0.200	0.399	0.400	0.496
Bullish Patterns	$N(y \neq 0)$	$\max y$	$\min y$	$\text{mean } y$	$\text{median } y$	$\text{Profit} > 0$
GBP futures						
<i>Hammer</i> (HA)	2	0.005	0.005	0.005	0.005	0.000
<i>Bullish Engulfing</i> (BuE)	78	0.035	0.005	0.010	0.005	0.487
<i>Piercing Line</i> (PL)	120	0.055	0.005	0.011	0.005	0.492
<i>Bullish Harami</i> (BH)	174	0.040	0.005	0.011	0.010	0.511
<i>Three Inside Up</i> (TIU)	78	0.040	0.005	0.010	0.010	0.449
<i>Three Outside Up</i> (TOU)	33	0.035	0.005	0.010	0.005	0.485
<i>Tweezer Bottom</i> (TB)	7	0.025	0.005	0.013	0.015	0.143
CHF futures						
<i>Hammer</i> (HA)	6	0.015	0.005	0.008	0.007	0.333
<i>Bullish Engulfing</i> (BuE)	97	0.035	0.005	0.010	0.010	0.526
<i>Piercing Line</i> (PL)	168	0.030	0.005	0.010	0.010	0.482
<i>Bullish Harami</i> (BH)	183	0.040	0.005	0.011	0.010	0.366
<i>Three Inside Up</i> (TIU)	84	0.035	0.005	0.012	0.010	0.321
<i>Three Outside Up</i> (TOU)	43	0.035	0.005	0.010	0.010	0.512
<i>Tweezer Bottom</i> (TB)	12	0.030	0.005	0.013	0.010	0.583
JPY futures						
<i>Hammer</i> (HA)	7	0.020	0.005	0.011	0.010	0.286
<i>Bullish Engulfing</i> (BuE)	82	0.030	0.005	0.010	0.010	0.549
<i>Piercing Line</i> (PL)	133	0.045	0.005	0.010	0.010	0.436
<i>Bullish Harami</i> (BH)	156	0.025	0.005	0.010	0.010	0.455
<i>Three Inside Up</i> (TIU)	71	0.025	0.005	0.009	0.010	0.408
<i>Three Outside Up</i> (TOU)	30	0.030	0.005	0.010	0.007	0.633
<i>Tweezer Bottom</i> (TB)	9	0.015	0.005	0.008	0.005	0.111

Table 3: Bearish single lines/patterns statistics for the JPY, CHF and GBP futures contracts: Column 2 shows the number of identified candlestick forms, Column 3 shows the largest output value, Column 4 shows the smallest output value, Column 5 and 6 show the mean and median output values of the identified candlesticks, Column 7 shows the proportion of times that profits for a ten-day period following a sell signal are greater than zero.

Bearish Single Lines	$N(y \neq 0)$	$\max y$	$\min y$	$\text{mean } y$	$\text{median } y$	profit > 0
GBP futures						
<i>Long Black Candle</i> (LBC)	1900	1.000	0.200	0.489	0.600	0.491
<i>Black Marubozu</i> (BM)	8	1.000	1.000	1.000	1.000	0.625
<i>Opening Black Marubozu</i> (OBM)	247	1.000	0.515	0.765	0.750	0.502
<i>Closing Black Marubozu</i> (CBM)	16	1.000	0.520	0.700	0.660	0.562
<i>Gravestone Doji</i> (GD)	1	1.000	1.000	1.000	1.000	0.000
<i>White Shooting Star</i> (WSS)	749	0.995	0.191	0.377	0.300	0.537
<i>Black Shooting Star</i> (BSS)	708	0.995	0.200	0.355	0.200	0.510
CHF futures						
<i>Long Black Candle</i> (LBC)	2113	1.000	0.200	0.492	0.600	0.506
<i>Black Marubozu</i> (BM)	10	1.000	1.000	1.000	1.000	0.700
<i>Opening Black Marubozu</i> (OBM)	248	1.000	0.510	0.746	0.740	0.548
<i>Closing Black Marubozu</i> (CBM)	21	0.910	0.520	0.655	0.630	0.571
<i>Gravestone Doji</i> (GD)	4	1.000	1.000	1.000	1.000	1.000
<i>White Shooting Star</i> (WSS)	765	0.985	0.191	0.386	0.400	0.506
<i>Black Shooting Star</i> (BSS)	732	0.985	0.200	0.371	0.400	0.510
JPY futures						
<i>Long Black Candle</i> (LBC)	2107	1.000	0.200	0.494	0.600	0.480
<i>Black Marubozu</i> (BM)	6	1.000	1.000	1.000	1.000	0.500
<i>Opening Black Marubozu</i> (OBM)	211	1.000	0.525	0.740	0.730	0.517
<i>Closing Black Marubozu</i> (CBM)	30	0.945	0.565	0.762	0.792	0.600
<i>Gravestone Doji</i> (GD)	55	1.000	1.000	1.000	1.000	0.545
<i>White Shooting Star</i> (WSS)	942	0.990	0.200	0.397	0.400	0.486
<i>Black Shooting Star</i> (BSS)	870	0.980	0.200	0.391	0.400	0.515
Bearish Patterns	$N(y \neq 0)$	$\max y$	$\min y$	$\text{mean } y$	$\text{median } y$	Profit > 0
GBP futures						
<i>Hanging Man</i> (HM)	5	0.015	0.005	0.008	0.005	0.400
<i>Bearish Engulfing</i> (BeE)	77	0.040	0.005	0.009	0.005	0.519
<i>Dark Cloud Cover</i> (DCC)	142	0.025	0.005	0.009	0.005	0.437
<i>Bearish Harami</i> (BH)	196	0.045	0.005	0.010	0.005	0.459
<i>Three Inside Down</i> (TID)	89	0.045	0.005	0.009	0.005	0.483
<i>Three Outside Down</i> (TOD)	38	0.030	0.005	0.010	0.005	0.553
<i>Tweezer Top</i> (TT)	12	0.020	0.005	0.010	0.007	0.917
CHF futures						
<i>Hanging Man</i> (HM)	4	0.035	0.005	0.016	0.013	0.500
<i>Bearish Engulfing</i> (BeE)	100	0.040	0.005	0.011	0.007	0.430
<i>Dark Cloud Cover</i> (DCC)	158	0.065	0.005	0.012	0.010	0.462
<i>Bearish Harami</i> (BH)	211	0.055	0.005	0.011	0.010	0.474
<i>Three Inside Down</i> (TID)	104	0.040	0.005	0.011	0.010	0.471
<i>Three Outside Down</i> (TOD)	51	0.040	0.005	0.011	0.010	0.490
<i>Tweezer Top</i> (TT)	11	0.030	0.005	0.011	0.010	0.273
JPY futures						
<i>Hanging Man</i> (HM)	4	0.025	0.005	0.017	0.020	0.500
<i>Bearish Engulfing</i> (BeE)	100	0.045	0.005	0.011	0.010	0.540
<i>Dark Cloud Cover</i> (DCC)	122	0.045	0.005	0.012	0.010	0.434
<i>Bearish Harami</i> (BH)	232	0.050	0.005	0.013	0.010	0.483
<i>Three Inside Down</i> (TID)	124	0.050	0.005	0.012	0.010	0.484
<i>Three Outside Down</i> (TOD)	48	0.040	0.005	0.011	0.010	0.583
<i>Tweezer Top</i> (TT)	9	0.030	0.005	0.012	0.010	0.222

sample distribution would not have a support in $\mathbb{R}$. One possibility is to take logs, however this will not work if there are exact zeros in the data. To cope with the problem, and thereby avoiding the addition of some (small) constant, which heavily influences the finite sample estimates, we employ the multiplicative error model introduced by Engle (2002). For each point in time, the realization x_{i_t} , with $i = 1, \dots, 4$, corresponding to the random variable σ_{DR_t} ($i = 1$), σ_{HC_t} ($i = 2$), σ_{H_t} ($i = 3$), resp. $|r_t|$ ($i = 4$), is described as the product of a deterministic scale factor μ_{i_t} and a multiplicative error, ε_{i_t} , i.e.

$$x_{i_t} = \mu_{i_t} \cdot \varepsilon_{i_t}, \quad \varepsilon_{i_t} \sim \mathcal{D}(1, \varphi), \quad (4)$$

where the distribution $\mathcal{D}(1, \varphi)$ has a positive or a non-negative support. The unit-mean assumption ensures that the scale factor μ_{i_t} acts as the conditional mean, given past information. The conditional mean is specified by a dynamic equation with a base specification which includes for each series Q lagged innovations $\boldsymbol{\alpha}_i^T \mathbf{x}_{t-q}$, P lagged conditional means $\boldsymbol{\beta}_i^T \boldsymbol{\mu}_{t-p}$, and a constant ω_i , as

$$\mu_{i_t} = \omega_i + \sum_{q=1}^Q \boldsymbol{\alpha}_i^T \cdot \mathbf{x}_{t-q} + \sum_{p=1}^P \boldsymbol{\beta}_i^T \cdot \boldsymbol{\mu}_{t-p}, \quad (5)$$

where $\boldsymbol{\theta}_i = [\boldsymbol{\alpha}_i^T \quad \boldsymbol{\beta}_i^T]$ gather the coefficients of the variables $\mathbf{x} = [x_{1t-1} \quad x_{2t-1} \quad x_{3t-1} \quad x_{4t-1}]^T$ resp. $\boldsymbol{\mu}_{t-1} = [\mu_{1t-1} \quad \mu_{2t-1} \quad \mu_{3t-1} \quad \mu_{4t-1}]^T$ and account for the terms usually encountered in a GARCH(p, q) - type specification for the conditional variance of the underlying realization. In fact, as it was shown in Engle (2002), this model specification with unit-mean Gamma distributions for the error terms is equivalent, at least regarding the first order optimality conditions, to a GARCH(p, q) specification for the square root of the variable of interest, hereby assuming Gaussian error distributions. This fact will be used in Section 5 for the coefficient estimation. Thus, we choose as marginal density functions of the non-negative error terms the family of unit-mean Gamma distributions, with the density function

$$f_{\varepsilon_{i_t}}(x, k_i, 1/k_i) = \frac{k_i^{k_i}}{\Gamma(k_i)} \cdot x^{k_i-1} \cdot \exp(-k_i x),$$

and with $k_i > 0$ as shape parameter. Using Gamma distributions for the error terms, the variables $x_{i_t} | \Omega_{t-1}$ will be $\text{Gamma}(k_i, \mu_{i_t}/k_i)$ distributed. Other choices could be Inverse-Gamma, Weibull, Lognormal or mixtures of them. Since all time series except the daily range, contain exact zeros, one must restrict the shape parameter k_i to one, yielding a special case of the Gamma distribution, the unit-mean exponential distribution, or to 0.5, yielding the chi-squared distribution. This potentially conservative choice is due to the presence of zeros in the volatility proxies, which however does not exceed a 3% threshold. An alternative specification could be the mixture distribution of Malec (2013) and Hautsch et al. (2013), which is a new type of discrete-continuous

mixture distribution with a point-mass at zero and a flexible continuous distribution for strictly positive values.

4.2. The direction model

Based on the findings in Anatolyev and Gospodinov (2010), we assume that the conditional distribution of the sign of the return, $\mathbb{I}[r_t > 0]$, given the past information set, Ω_{t-1} is of Bernoulli type. Its probability mass function will thus be $f_{\mathbb{I}[r_t > 0]}(y) = s_t^y(1 - s_t)^{1-y}$, where s_t is the conditional *success probability*, i.e. the probability $\Pr_{t-1}[\mathbb{I}[r_t > 0] = 1]$.¹⁰ This probability will be modelled using a dynamic logit model, i.e.

$$s_t = \frac{\exp(\lambda_t)}{1 + \exp(\lambda_t)} \quad (6)$$

with

$$\lambda_t = \omega_d + \sum_{k=1}^{Q_d} \alpha_{d_k} y_{t-k} + \sum_{l=1}^{P_d} \beta_{d_l} \lambda_{t-l}. \quad (7)$$

The parameters $\phi_d = [\omega_d \ \alpha_{d_k} \ \beta_{d_l}]$ will be estimated via the Maximum Likelihood (ML) method. It can be seen from eq. (5) and (7), that there are no contemporaneous cross-effects between the volatility indicators and the direction-of-change of the daily returns. This choice is motivated by the advantage of conducting the parameter estimation for each model independently and, subsequently, estimate the joint density function by means of Copulas.

4.3. Analysis of the joint density function of the full model $f_{\sigma_t, r_t}(\mathbf{x}, y)$

Following the decomposition procedure for the returns r_t , as $r_t = |r_t| \text{sgn}(r_t)$, Anatolyev and Gospodinov (2010) study the forecastability of the direction-of-change of the returns from the volatility of absolute returns, by jointly modelling both variables and employing a Copula approach to generate the joint distribution. Based on their analysis, we construct an extension to include a multivariate case of volatility indicators, i.e. the case where the predictability of the signs of the daily returns is inferred not only from the volatility of the daily returns but also from intra-daily data, the range-based volatility indicators from Section 2. The following theorem states that the joint density function of all variables in the full model is the product between the density functions of the volatility indicators, and the density function of a Bernoulli distributed variable with a *success probability* parameter $\rho_t(F_{\sigma_{1_t}}(x_1|\mu_{1_t}), \dots, F_{\sigma_{3_t}}(x_3|\mu_{3_t}), F_{|r_t|}(x_4|\mu_{4_t}))$.

Theorem 1. *The joint density/mass function $f_{\sigma_t, r_t}(\mathbf{x}, y)$ can be represented as*

$$f_{\sigma_t, r_t}(\mathbf{x}, y) = \prod_{i=1}^3 f_{\sigma_{i_t}}(x_i|\mu_{i_t}) \cdot f_{|r_t|}(x_4|\mu_{4_t}) \cdot \rho_t(F_{\sigma_{1_t}}(x_1|\mu_{1_t}), \dots, F_{\sigma_{3_t}}(x_3|\mu_{3_t}), F_{|r_t|}(x_4|\mu_{4_t}))^y \cdot [1 - \rho_t(F_{\sigma_{1_t}}(x_1|\mu_{1_t}), \dots, F_{\sigma_{3_t}}(x_3|\mu_{3_t}), F_{|r_t|}(x_4|\mu_{4_t}))]^{1-y}, \quad (8)$$

¹⁰See for example Gujarati and Porter (2002).

with

$$\rho_t(\mathbf{u}) = 1 - \partial_1 \partial_2 \partial_3 \partial_4 C_{\sigma_t, r_t}(\mathbf{u}, 1 - s_t), \quad (9)$$

where $u_i = F_{\sigma_{i_t}}(x_i | \mu_{i_t})$, $i = 1, \dots, 3$ and $u_4 = F_{|r_t|}(x_4 | \mu_{4_t})$ are the transformed realizations of the volatility indicators using the respective cumulative distribution functions, i.e. the probability integral transforms; $s_t = \Pr_{t-1}[\mathbb{I}[r_t > 0] = 1]$ represents the conditional probability that the sign of the daily return equals one, given the set of all available information to the time $t - 1$, Ω_{t-1} , $C_{\sigma_t, r_t}(\mathbf{u}, v)$ is a five-dimensional Copula function associated with the joint cumulative distribution function of the continuous random vector $[\sigma_t \mid r_t]$ and the discrete random variable $\mathbb{I}[r_t > 0]$ and $\partial_i C_{\sigma_t, r_t}(\mathbf{u}, v)$ means the partial derivative of the Copula function with respect to the i -th argument.

Proof. The conditional density function of the full model, $f_{\sigma_t, r_t}(\mathbf{x}, y)$, can be computed from the multivariate cumulative distribution function of the volatility indicators and the sign of the return. This distribution can be constructed from a Copula function, i.e.

$$F_{\sigma_t, r_t}(\mathbf{x}, y) = C_{\sigma_t, r_t}(F_{\sigma_{1_t}}(x_1 | \mu_{1_t}), \dots, F_{\sigma_{3_t}}(x_3 | \mu_{3_t}), F_{|r_t|}(x_4 | \mu_{4_t}), F_{\mathbb{I}[r_t > 0]}(y)), \quad (10)$$

where $C_{\sigma_t, r_t}(\mathbf{u}, v)$ is a five-dimensional Copula, $[\sigma_t \mid r_t]$ is a continuous random vector and $\mathbb{I}[r_t > 0]$ is a discrete random variable. Noting that y takes only binary values and, thus, using finite differences, it follows

$$f_{\sigma_t, r_t}(\mathbf{x}, y) = \partial_1 \partial_2 \partial_3 \partial_4 F_{\sigma_t, r_t}(\mathbf{x}, y) - \partial_1 \partial_2 \partial_3 \partial_4 F_{\sigma_t, r_t}(\mathbf{x}, y - 1). \quad (11)$$

Using the Copula description from eq. (10) and the facts that

$$f_{|r_t|}(x_4) = \frac{\partial F_{|r_t|}(x_4)}{\partial x_4}, \text{ and } f_{\sigma_i}(x_i) = \frac{\partial F_{\sigma_i}(x_i)}{\partial x_i}, \quad \forall i = 1, \dots, 3,$$

eq. (11) becomes

$$f_{\sigma_t, r_t}(\mathbf{x}, y) = \prod_{i=1}^3 f_{\sigma_i}(x_i) \cdot f_{|r_t|}(x_4) \cdot [\partial_1 \partial_2 \partial_3 \partial_4 C_{\sigma_t, r_t}(\mathbf{u}, F_{\mathbb{I}[r_t > 0]}(y)) - \partial_1 \partial_2 \partial_3 \partial_4 C_{\sigma_t, r_t}(\mathbf{u}, F_{\mathbb{I}[r_t > 0]}(y - 1))]. \quad (12)$$

As in Anatolyev and Gospodinov (2010), we denote the expression in square brackets with $\Delta_t(y)$. We know also that the Bernoulli type cumulative distribution function, $F_{\mathbb{I}[r_t > 0]}(y)$, which can be generated by recursion from its cumulative distribution function, is

$$F_{\mathbb{I}[r_t > 0]}(y) = \begin{cases} 0, & y < 0, \\ 1 - s_t, & 0 \leq y < 1, \\ 1, & y \geq 1, \end{cases} \quad (13)$$

where $s_t = E_{t-1}[\mathbb{I}[r_t > 0]]$ is the conditional *probability of success* $\Pr_{t-1}[\mathbb{I}[r_t > 0] = 1]$. For $y = 0$ follows then from eq. (12) and (13) that

$$\Delta_t(0) = \partial_1 \partial_2 \partial_3 \partial_4 C_{\sigma_t, r_t}(\mathbf{u}, 1 - s_t) - \partial_1 \partial_2 \partial_3 \partial_4 C_{\sigma_t, r_t}(\mathbf{u}, 0)$$

and for $y = 1$ that

$$\Delta_t(1) = \partial_1 \partial_2 \partial_3 \partial_4 C_{\sigma_t, r_t}(\mathbf{u}, 1) - \partial_1 \partial_2 \partial_3 \partial_4 C_{\sigma_t, r_t}(\mathbf{u}, 1 - s_t).$$

From the properties of the Copula function, we know that $C_{\sigma_t, r_t}(\mathbf{u}, 0) = 0$ and that $C_{\sigma_t, r_t}(\mathbf{u}, 1) = C_{\sigma_t, |r_t|}(\mathbf{u})$ (a four-dimensional marginal of the Copula). Furthermore, because the random vector $[F_{\sigma_{1_t}}(x_1) \cdots F_{\sigma_{3_t}}(x_3) \mid F_{|r_t|}(x_4)]$ is uniformly distributed on $[0, 1]$, the marginal density function $c_{\sigma_t, |r_t|}(\mathbf{u})$ is

$$\begin{aligned} \partial_1 \partial_2 \partial_3 \partial_4 C_{\sigma_t, r_t}(\mathbf{u}, 1) &= \partial_1 \partial_2 \partial_3 \partial_4 C_{\sigma_t, |r_t|}(\mathbf{u}) \\ &= c_{\sigma_t, |r_t|}(\mathbf{u}) \Big|_{\mathbf{u}=[F_{\sigma_{1_t}}(x_1 | \mu_{1_t}) \cdots F_{\sigma_{3_t}}(x_3 | \mu_{3_t}) \mid F_{|r_t|}(x_4 | \mu_{4_t})]} = 1, \end{aligned}$$

where $c_{\sigma_t, |r_t|}(\mathbf{u})$ is the associated density function of the marginal Copula $C_{\sigma_t, |r_t|}(\mathbf{u})$. Thus, it follows that

$$\begin{aligned}\Delta_t(0) &= \partial_1 \partial_2 \partial_3 \partial_4 C_{\sigma_t, r_t}(\mathbf{u}, 1 - s_t) = 1 - \rho_t(\mathbf{u}) \\ &= 1 - \rho_t(F_{\sigma_{1_t}}(x_1 | \mu_{1_t}), \dots, F_{\sigma_{3_t}}(x_3 | \mu_{3_t}), F_{|r_t|}(x_4 | \mu_{4_t})), \\ \Delta_t(1) &= 1 - \partial_1 \partial_2 \partial_3 \partial_4 C_{\sigma_t, r_t}(\mathbf{u}, 1 - s_t) = \rho_t(\mathbf{u}) \\ &= \rho_t(F_{\sigma_{1_t}}(x_1 | \mu_{1_t}), \dots, F_{\sigma_{3_t}}(x_3 | \mu_{3_t}), F_{|r_t|}(x_4 | \mu_{4_t})).\end{aligned}$$

Furthermore, it holds that $\Delta_t(1) + \Delta_t(0) = 1$, and that $\Delta_t(y) = \Delta_t(1)^y \Delta_t(0)^{1-y}$, with $y \in \{0, 1\}$. Thus, $\Delta_t(y)$ can be seen as the density function of a Bernoulli distributed random variable with the realization y and *success probability parameter* $\rho_t(F_{\sigma_{1_t}}(x_1 | \mu_{1_t}), \dots, F_{\sigma_{3_t}}(x_3 | \mu_{3_t}), F_{|r_t|}(x_4 | \mu_{4_t}))$. It follows that

$$\begin{aligned}\Delta_t(y) &= \rho_t(F_{\sigma_{1_t}}(x_1 | \mu_{1_t}), \dots, F_{\sigma_{3_t}}(x_3 | \mu_{3_t}), F_{|r_t|}(x_4 | \mu_{4_t}))^y \\ &\quad \cdot [1 - \rho_t(F_{\sigma_{1_t}}(x_1 | \mu_{1_t}), \dots, F_{\sigma_{3_t}}(x_3 | \mu_{3_t}), F_{|r_t|}(x_4 | \mu_{4_t}))]^{1-y}.\end{aligned}\tag{14}$$

Introducing eq. (14) in eq. (12), it follows that

$$\begin{aligned}f_{\sigma_t, r_t}(\mathbf{x}, y) &= \prod_{i=1}^3 f_{\sigma_{i_t}}(x_i | \mu_{i_t}) \cdot f_{|r_t|}(x_4 | \mu_{4_t}) \cdot \rho_t(F_{\sigma_{1_t}}(x_1 | \mu_{1_t}), \dots, F_{\sigma_{3_t}}(x_3 | \mu_{3_t}), F_{|r_t|}(x_4 | \mu_{4_t}))^y \\ &\quad \cdot [1 - \rho_t(F_{\sigma_{1_t}}(x_1 | \mu_{1_t}), \dots, F_{\sigma_{3_t}}(x_3 | \mu_{3_t}), F_{|r_t|}(x_4 | \mu_{4_t}))]^{1-y},\end{aligned}$$

which is the conclusion of the theorem. $\square$

Some details must be given on the choice of the multivariate Copula function, encompassed in eq. (9). This will be done in the next section.

4.4. Copula specification and Pair-Copula decomposition

Choosing the multivariate Copula function in eq. (9) may prove a difficult task. First, one has to give the analytical form of the mixed fourth order partial derivative of the chosen Copula. Second, the set of higher-dimensional Copula types is smaller than the one of bivariate Copulas. Some possible choices with a simple closed form are the multivariate Archimedean Copulas, see Nelsen (2006). For example, the multivariate version of the Clayton Copula is

$$C_\alpha(\mathbf{u}) = \left[1 + \sum_{i=1}^n (u_i^{-\alpha} - 1) \right]^{-1/\alpha}, \quad \alpha \geq 0.\tag{15}$$

The Clayton Copula leads to independence between the marginals for $\alpha \rightarrow 0$. Furthermore, it allows only for positive dependence between the marginals. Another choice could be the multivariate version of the Gumbel-Copula

$$C_\alpha(\mathbf{u}) = \exp \left[- \left(\sum_{i=1}^n \tilde{u}_i^\alpha \right)^{1/\alpha} \right],\tag{16}$$

with $\tilde{u}_i = \log u_i$, which leads to independence for $\alpha \rightarrow 1$. A third example is the Frank Copula, i.e.

$$C_\alpha(\mathbf{u}) = -\frac{1}{\alpha} \ln \left(1 + \frac{(\exp(-\alpha u_1) - 1) \cdot \dots \cdot (\exp(-\alpha u_n) - 1)}{(\exp(-\alpha) - 1)^{n-1}} \right),\tag{17}$$

with $\alpha > 0$, which leads to independence for $\alpha \rightarrow 0$.

To increase the flexibility of the joint density function, $f_{\sigma_t, r_t}(\mathbf{x}, y)$, we employ also a Pair-Copula decomposition, based on the work of Aas et al. (2009) and Stöber (2013). With this decomposition, the multivariate density function will be specified as a product of marginal density functions and some bivariate Copulas. By this approach, the Copulas do not have to belong to the same family and can be chosen based on a goodness-of-fit measure.¹¹ The Pair-Copula decomposition builds the remaining topic of this section.

A further method of constructing the joint density function is to decompose it into¹²

$$f_{\sigma_t, r_t}(\mathbf{x}, y) = f_{1|2,3,4,5}(x_1|x_2, x_3, x_4, y) f_{2|3,4,5}(x_2|x_3, x_4, y) f_{3|4,5}(x_3|x_4, y) f_{4|5}(x_4|y) f_5(y). \quad (18)$$

The decomposition from eq. (18) is also applicable with other variable orderings. Here, the conditional density function of the absolute returns, given the sign of the returns, $f_{4|5}(x_4|y)$, was isolated due to its special meaning, i.e. the density function of the absolute returns conditional on the sign of the returns. Thus, the conditional density function $f_{4|5}(x_4|y)$ can be derived from its cumulative distribution function, as

$$\begin{aligned} f_{4|5}(x_4|y) &= \frac{\partial}{\partial x_4} F_{4|5}(x_4|y) = \frac{\partial}{\partial x_4} \left(\frac{F_{4,5}(x_4, y) - F_{4,5}(x_4, y-1)}{F_5(y) - F_5(y-1)} \right) \\ &= \frac{\partial}{\partial x_4} \left(\frac{C_{4,5}(u_4, F_5(y)) - C_{4,5}(u_4, F_5(y-1))}{F_5(y) - F_5(y-1)} \right) \\ &= \frac{\partial_1 [C_{4,5}(u_4, F_5(y)) - C_{4,5}(u_4, F_5(y-1))]}{f_5(y)} f_4(x_4), \end{aligned} \quad (19)$$

where $\partial_1 C(u, v)$ denotes the partial derivative of the bivariate Copula with respect to the first argument, i.e. $\partial_1 C(u, v) = \frac{\partial C(u, v)}{\partial u}$. In eq. (19) we derived the conditional cumulative distribution function $F_{4|5}(x_4|y)$ from the unconditional joint distribution function $F_{4,5}(x_4, y)$, using the facts that the variable y is discrete and that, for a pair of random variables, it holds (see Stöber (2013))

$$F_{X|W}(x|w) = f_W(w)^{-1} \cdot \frac{\partial F_{X,W}(x, w)}{\partial w} \quad \text{and} \quad F_{W|Z}(w|z) = (D_z[F_Z(z)])^{-1} D_z[F_{W,Z}(w, z)], \quad (20)$$

where X, W are continuous random variables, Z is a discrete random variable, and $D_z[\cdot]$ is the difference operator with respect to the variable z .

The third term of eq. (18) can be described also by means of a bivariate Copula, as (see Patton (2006))

$$\begin{aligned} f_{3|4,5}(x_3|x_4, y) &= \frac{f_{3,5|4}(x_3, y|x_4)}{f_{5|4}(y|x_4)} = \frac{c_{3,5;4}(u_{3|4}, u_{5|4}) f_{3|4}(x_3|x_4) f_{5|4}(y|x_4)}{f_{5|4}(y|x_4)} \\ &= c_{3,5;4}(u_{3|4}, u_{5|4}) f_{3|4}(x_3|x_4), \end{aligned} \quad (21)$$

¹¹See Aas et al. (2009) for details about goodness-of-fit measures.

¹²For simplicity, we used in this decomposition a slightly different notation for the density functions and their arguments. For example, the density function $f_i(x_i)$ is equivalent to $f_{\sigma_i}(x_i|\mu_i)$.

where $c_{3,5;4}(u_{3|4}, u_{5|4})$ is the density function of a bivariate Copula¹³ with the arguments $u_{3|4} = F_{3|4}(x_3|x_4)$ and $u_{5|4} = F_{5|4}(y|x_4)$. These arguments can be determined further from bivariate Copulas using eq. (20), as

$$u_{3|4} = F_{3|4}(x_3|x_4) = \frac{\partial F_{3,4}(x_3, x_4)}{\partial x_4} \bigg/ \frac{\partial F_4(x_4)}{\partial x_4} = \frac{\partial C_{3,4}(u_3, u_4)}{\partial x_4} / f_4(x_4) = \partial_2 C_{3,4}(u_3, u_4). \quad (22)$$

Analogously, it follows that

$$u_{5|4} = F_{5|4}(y|x_4) = \partial_1 C_{4,5}(u_4, F_5(y)). \quad (23)$$

Using a similar decomposition as in eq. (19), we can write the density function $f_{3|4}(x_3|x_4)$ as $f_{3|4}(x_3|x_4) = c_{3,4}(u_3, u_4)f_3(x_3)$.

The second term of eq. (18) can be decomposed further as

$$\begin{aligned} f_{2|3,4,5}(x_2|x_3, x_4, x_5) &= \frac{f_{2,3|4,5}(x_2, x_3|x_4, y)}{f_{3|4,5}(x_3|x_4, y)} = \frac{c_{2,3;4,5}(u_{2|4,5}, u_{3|4,5})f_{2|4,5}(x_2|x_4, y)f_{3|4,5}(x_3|x_4, y)}{f_{3|4,5}(x_3|x_4, y)} \\ &= c_{2,3;4,5}(u_{2|4,5}, u_{3|4,5})f_{2|4,5}(x_2|x_4, y), \end{aligned} \quad (24)$$

with¹⁴

$$\begin{aligned} u_{2|4,5} &= F_{2|4,5}(x_2|x_4, y) = \partial_2 C_{2,5;4}(u_{2|4}, u_{5|4}), \\ u_{3|4,5} &= F_{3|4,5}(x_3|x_4, y) = \partial_2 C_{3,5;4}(u_{3|4}, u_{5|4}), \end{aligned}$$

and, using a similar transformation as in eq. (22), it follows that $u_{2|4} = F_{2|4}(x_2|x_4) = \partial_2 C_{2,4}(u_2, u_4)$. The conditional density function $f_{2|4,5}(x_2|x_4, y)$ from eq. (24) can be written as in eq. (21) as $f_{2|4,5}(x_2|x_4, y) = c_{2,5;4}(u_{2|4}, u_{5|4})f_{2|4}(x_2|x_4)$, with $f_{2|4}(x_2|x_4) = c_{2,4}(u_2, u_4)f_2(x_2)$.

Finally, the first term $f_{1|2,3,4,5}(x_1|x_2, x_3, x_4, y)$ can be decomposed as

$$\begin{aligned} f_{1|2,3,4,5}(x_1|x_2, x_3, x_4, y) &= \frac{f_{1,2|3,4,5}(x_1, x_2|x_3, x_4, y)}{f_{2|3,4,5}(x_2|x_3, x_4, y)} \\ &= c_{1,2;3,4,5}(u_{1|3,4,5}, u_{2|3,4,5})f_{1|3,4,5}(x_1|x_3, x_4, y), \end{aligned} \quad (25)$$

with¹⁵

$$\begin{aligned} u_{1|3,4,5} &= F_{1|3,4,5}(x_1|x_3, x_4, y) = \partial_2 C_{1,3;4,5}(u_{1|4,5}, u_{3|4,5}), \\ u_{2|3,4,5} &= F_{2|3,4,5}(x_2|x_3, x_4, y) = \partial_2 C_{2,3;4,5}(u_{2|4,5}, u_{3|4,5}), \end{aligned}$$

where $u_{1|4,5} = F_{1|4,5}(x_1|x_4, x_5) = \partial_2 C_{1,5;4}(u_{1|4}, u_{5|4})$, $u_{2|4,5} = F_{2|4,5}(x_2|x_4, x_5) = \partial_2 C_{2,5;4}(u_{2|4}, u_{5|4})$, and $u_{3|4,5} = F_{3|4,5}(x_3|x_4, y) = \partial_2 C_{3,5;4}(u_{3|4}, u_{5|4})$. The last component of eq. (25) is

$$f_{1|3,4,5}(x_1|x_3, x_4, y) = \frac{f_{1,3|4,5}(x_1, x_3|x_4, y)}{f_{3|4,5}(x_3|x_4, y)} = c_{1,3;4,5}(u_{1|4,5}, u_{3|4,5})f_{1|4,5}(x_1|x_4, y),$$

¹³We use the notation $c_{3,5;4}$ instead of $c_{3,5|4}$ to denote a bivariate Copula, whose marginals are some conditional cumulative density functions, and to distinguish it from a conditional Copula density function.

¹⁴See Joe (1996), Property 2.b.

¹⁵See Joe (1996), Property 2.c.

with $f_{1|4,5}(x_1|x_4, y) = c_{1,5;4}(u_{1|4}, u_{5|4})f_{1|4}(x_1|x_4)$, $u_{1|4} = F_{1|4}(x_1|x_4) = \partial_2 C_{1,4}(u_1, u_4)$ and finally $f_{1|4}(x_1|x_4) = c_{1,4}(u_1, u_4)f_1(x_1)$.

Summing up, the joint density function of the full model can be decomposed as

$$\begin{aligned} f_{\sigma_t, r_t}(\mathbf{x}, y) &= c_{1,2;3,4,5}(u_{1|3,4,5}, u_{2|3,4,5}) \cdot c_{1,3;4,5}(u_{1|4,5}, u_{3|4,5}) \cdot c_{1,5;4}(u_{1|4}, u_{5|4}) \cdot c_{1,4}(u_1, u_4) \\ &\quad \cdot c_{2,3;4,5}(u_{2|4,5}, u_{3|4,5}) \cdot c_{2,5;4}(u_{2|4}, u_{5|4}) \cdot c_{2,4}(u_2, u_4) \\ &\quad \cdot c_{3,5;4}(u_{3|4}, u_{5|4}) \cdot c_{3,4}(u_3, u_4) \\ &\quad \cdot \partial_1[C_{4,5}(u_4, F_5(y)) - C_{4,5}(u_4, F_5(y-1))] \\ &\quad \cdot f_1(x_1) \cdot f_2(x_2) \cdot f_3(x_3) \cdot f_4(x_4). \end{aligned} \quad (26)$$

This decomposition is known as the canonical Pair-Copula decomposition, see Aas et al. (2009) for details. As shown in Anatolyev and Gospodinov (2010), the term $\partial_1[C_{4,5}(u_4, F_5(y)) - C_{4,5}(u_4, F_5(y-1))]$ can be described as a Bernoulli-type density function, in this case with a *success probability parameter* $\eta_t(F_4(x_4))$, i.e.

$$\partial_1[C_{4,5}(u_4, F_5(y)) - C_{4,5}(u_4, F_5(y-1))] = \eta_t(F_4(x_4))^y [1 - \eta_t(F_4(x_4))]^{1-y},$$

with

$$\eta_t(u) = 1 - \partial_1 C_{4,5}(u, 1 - s_t). \quad (27)$$

In the case of conditional independence between the absolute returns $|r_t|$ and the indicator function for the sign of daily returns $\mathbb{I}[r_t > 0]$, e.g. $C_{4,5}(u_4, v)$ is the product Copula, then the *success probability parameter* $\eta_t(F_4(x_4))$ is equal to the *success probability parameter* s_t . Otherwise, it depends on $F_4(x_4)$. With this decomposition, eighteen bivariate Copulas must be specified. Their parameters will be estimated via the ML method. In Section 5.2, the results of the estimation algorithm and the details involving the choices of the Pair-Copula functions will be given.

4.5. Maximum-Likelihood inference within the Copula-based logit-vMEM specification

This Section gives some computational details regarding the density/mass function from eq. (8). Its Log-Likelihood function (LLF) is

$$\begin{aligned} \text{LLF} &= \sum_{t=1}^T \ln f_{\sigma_t, r_t}(\mathbf{x}_t, y_t) = \sum_{t=1}^T \left\{ \underbrace{\sum_{i=1}^3 \ln f_{\sigma_{i_t}}(x_{i_t} | \mu_{i_t}) + \ln f_{|r_t|}(x_{4_t} | \mu_{4_t})}_{\text{LLF}_{1_t}} \right. \\ &\quad \left. + y_t \cdot \rho_t(F_{\sigma_{1_t}}(x_{1_t} | \mu_{1_t}), \dots, F_{\sigma_{3_t}}(x_{3_t} | \mu_{3_t}), F_{|r_t|}(x_{4_t} | \mu_{4_t})) \right. \\ &\quad \left. + (1 - y_t) \cdot \underbrace{[1 - \rho_t(F_{\sigma_{1_t}}(x_{1_t} | \mu_{1_t}), \dots, F_{\sigma_{3_t}}(x_{3_t} | \mu_{3_t}), F_{|r_t|}(x_{4_t} | \mu_{4_t}))]}_{\text{LLF}_{2_t}} \right\}. \end{aligned}$$

For the marginals contributions, it follows that

$$\text{LLF}_{1_t} = \sum_{i=1}^4 \left[\ln \frac{k_i^{k_i}}{\Gamma(k_i)} + \ln x_{i_t}^{k_i-1} - k_i \left(\ln \mu_{i_t}(\theta_i) + \frac{x_{i_t}}{\mu_{i_t}(\theta_i)} \right) \right], \quad (28)$$

with $k_i = 1, \forall i = \overline{2,4}$ and the conditional mean, μ_{i_t} from eq. (5). Alternatively, employing a Pair-Copula decomposition, the second part, i.e. the contribution of the direction model and Copula, can be computed as

$$\begin{aligned} \text{LLF}_{2_t} = & \ln c_{1,2;3,4,5}(u_{1|3,4,5}, u_{2|3,4,5}) + \ln c_{1,3;4,5}(u_{1|4,5}, u_{3|4,5}) + \ln c_{1,5;4}(u_{1|4}, u_{5|4}) + \ln c_{1,4}(u_1, u_4) \\ & + \ln c_{2,3;4,5}(u_{2|4,5}, u_{3|4,5}) + \ln c_{2,5;4}(u_{2|4}, u_{5|4}) + \ln c_{2,4}(u_2, u_4) \\ & + \ln c_{3,5;4}(u_{3|4}, u_{5|4}) + \ln c_{3,4}(u_3, u_4) \\ & + y \ln \eta_t(F_4(x_4)) + (1-y) \ln[1 - \eta_t(F_4(x_4))], \end{aligned} \quad (29)$$

where the function $\eta_t(z)$ is given in eq. (27). Hereby, each term containing a Copula function will be constructed based on the (i.i.d.) residuals of the marginal densities models. Noting that the probability integral transform of the realizations of the volatility indicators x_{i_t} yields the same results as the probability integral transform of their residuals ε_{i_t} , i.e. $\varepsilon_{i_t} \sim \text{Gamma}(a, 1/a) \Rightarrow x_{i_t} \sim \text{Gamma}(a, \mu_{i_t}/a)$, and hence,

$$F_{x_{i_t}}(x) = \frac{\gamma(a, ax/\mu_{i_t})}{\Gamma(a)} = \frac{\gamma(a, a\mu_{i_t}\varepsilon_{i_t}/\mu_{i_t})}{\Gamma(a)} = \frac{\gamma(a, a\varepsilon_{i_t})}{\Gamma(a)} = F_{\varepsilon}(\varepsilon_{i_t}),$$

where $\Gamma(a) = (a-1)!$ is the gamma function and $\gamma(s, x) = \int_0^x t^{s-1} \exp(-t) dt$ is the incomplete gamma function, the construction of the Copulas proceeds as follows:

- (a) The standardized residuals $\varepsilon_{i_t} \sim \text{Gamma}(k_i, \theta_i)$ are transformed using the cumulative distribution function of the Gamma distribution, i.e. $F_{\varepsilon_{i_t}}(\varepsilon_{i_t}, k_i, \theta_i)$, to yield the uniform distributed variables $u_{i_t} = F_{\varepsilon_{i_t}}(\varepsilon_{i_t}, k_i, \theta_i) \sim \mathcal{U}(0, 1)$. These variables are the arguments of the Pair-Copula densities $c_{1,4}$, $c_{2,4}$ and $c_{3,4}$, shown as edges in Fig. 2.¹⁶
- (b) Subsequently, the arguments of the Pair-Copula densities $c_{1,5;4}$, $c_{2,5;4}$ and $c_{3,5;4}$, shown as edges in Fig. 3, i.e. $u_{1|4}$, $u_{2|4}$, $u_{3|4}$ and $u_{5|4}$, which are the values of the conditional cumulative distribution functions $F_{1|4}(x_1|x_4)$, $F_{2|4}(x_2|x_4)$, $F_{3|4}(x_3|x_4)$ and $F_{5|4}(y|x_4)$, can be computed from the partial derivatives of the Pair-Copula functions $C_{1,4}(u_1, u_4)$, $C_{2,4}(u_2, u_4)$, $C_{3,4}(u_3, u_4)$ and $C_{4,5}(u_4, F_5(y))$, as $F_{1|4}(x_1|x_4) = \partial_2 C_{1,4}(u_1, u_4)$, $F_{2|4}(x_2|x_4) = \partial_2 C_{2,4}(u_2, u_4)$, $F_{3|4}(x_3|x_4) = \partial_2 C_{3,4}(u_3, u_4)$ and $F_{5|4}(y|x_4) = \partial_1 C_{4,5}(u_4, F_5(y))$.
- (c) The arguments of the next two Pair-Copulas, i.e. $c_{1,3;4,5}$ and $c_{2,3;4,5}$, shown as edges in Fig. 4, will be then constructed using the partial derivatives of the Pair-Copula functions from

¹⁶The arguments of the Pair-Copula are actually u_{i_t} , but we skip for simplicity the index notation t .

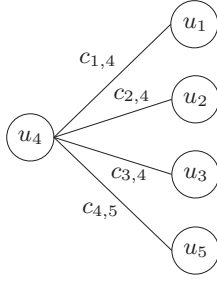


Figure 2: Step (a).

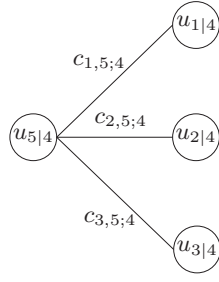


Figure 3: Step (b).

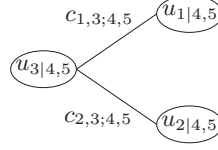


Figure 4: Step (c).

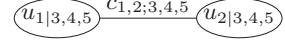


Figure 5: Step (d).

the last step. Thus, $u_{1|4,5} = F_{1|4,5}(x_1|x_4, y) = \partial_2 C_{1,5;4}(u_{1|4}, u_{5|4})$, $u_{2|4,5} = F_{2|4,5}(x_2|x_4, y) = \partial_2 C_{2,5;4}(u_{2|4}, u_{5|4})$ and $u_{3|4,5} = F_{3|4,5}(x_3|x_4, y) = \partial_2 C_{3,5;4}(u_{3|4}, u_{5|4})$.

- (d) The arguments of the last Pair-Copula, $c_{1,2;3,4,5}$, shown as edge in Fig. 5, are constructed using the partial derivatives of the Pair-Copula functions from the last step. Thus, it follows that $u_{1|3,4,5} = F_{1|3,4,5}(x_1|x_3, x_4, y) = \partial_2 C_{1,3;4,5}(F_{1|4,5}, F_{3|4,5})$ and $u_{2|3,4,5} = F_{2|3,4,5}(x_2|x_3, x_4, y) = \partial_2 C_{1,2;4,5}(F_{1|4,5}, F_{2|4,5})$.
- (e) The last component of the Log-Likelihood function is constructed using the function $\eta_t(u) = 1 - \partial_1 C_{4,5}(u, 1 - s_t)$.

4.6. Sampling from the (Pair-)Copula decomposition

The multivariate distribution, whose density function is given in eq. (26), can be used to obtain a new sample of realizations of the volatility indicators and of the sign of the returns. In the case of the Clayton and Gumbel Copulas, one can use the Marshall and Olkin algorithm, see Hofert (2008) for more details. In the case of the Pair-Copula decomposition, the generation of new realizations can be done by conditional sampling. First, five independent, uniformly distributed variables $W_{i_t} \sim U(0, 1)$, $i = \overline{1, 5}$, $t = \overline{1, T}$, with realizations w_{i_t} will be generated, and second, the quantile transformations

$$y_t^* = F_5^{-1}(w_{5_t}), \quad (30)$$

$$x_{4_t}^* = F_{4|5}^{-1}(w_{4_t}|y_t^*), \quad (31)$$

$$x_{3_t}^* = F_{3|4,5}^{-1}(w_{3_t}|x_{4_t}^*, y_t^*), \quad (32)$$

$$x_{2_t}^* = F_{2|3,4,5}^{-1}(w_{2_t}|x_{3_t}^*, x_{4_t}^*, y_t^*), \quad (33)$$

$$x_{1_t}^* = F_{1|2,3,4,5}^{-1}(w_{1_t}|x_{2_t}^*, x_{3_t}^*, x_{4_t}^*, y_t^*). \quad (34)$$

will be applied. The sequential inversion can be done using the following steps:¹⁷

¹⁷For simplicity, we disregard the index t from the variables.

Step 1: For eq. (31), it follows that

$$\begin{aligned}
w_4 = F_{4|5}(x_4^*|y^*) &= \frac{D_{y^*}[F_{4,5}(x_4^*, y^*)]}{D_{y^*}[F_5(y^*)]} = \frac{D_{y^*}[C_{4,5}(F_4(x_4^*), F_5(y^*))]}{D_{y^*}[F_5(y^*)]} \\
&= \frac{C_{4,5}(F_4(x_4^*), F_5(y^*)) - C_{4,5}(F_4(x_4^*), F_5(y^* - 1))}{F_5(y^*) - F_5(y^* - 1)} \\
&= \begin{cases} \frac{C_{4,5}(F_4(x_4^*), 1-s_t)}{1-s_t}, & y^* = 0, \\ \frac{F_4(x_4^*) - C_{4,5}(F_4(x_4^*), 1-s_t)}{s_t}, & y^* = 1, \end{cases} \\
&=: g_{4|5}(F_4(x_4^*), F_5(y^*), \alpha_{4,5}).
\end{aligned}$$

Hence,

$$x_4^* = F_4^{-1} \left(g_{4|5}^{-1}(w_4, F_5(y^*), \alpha_{4,5}) \right). \quad (35)$$

Depending on the choice of the Copula function $C_{4,5}(F_4(x_4^*), F_5(x_5^*))$, the inverse of the g-function, i.e. $g_{4|5}(F_4(x_4^*), F_5(y^*), \alpha_{4,5})$ can be computed analytically or numerically.

Step 2: We introduce in the following the function $h(u, v, \alpha) := \frac{\partial C(u, v)}{\partial v}$, where α is the Copula parameter. For eq. (32), it follows that

$$\begin{aligned}
w_3 &= F_{3|4,5}(x_3^*|x_4^*, y^*) \\
&= h_{3,5;4} \left(F_{3|4}(x_3^*|x_4^*), F_{5|4}(y^*|x_4^*), \alpha_{3,5;4} \right) \\
&= h_{3,5;4} \left(h_{3,4}(F_3(x_3^*), F_4(x_4^*), \alpha_{3,4}), F_{5|4}(y^*|x_4^*), \alpha_{3,5;4} \right).
\end{aligned}$$

Generating $F_{5|4}(y^*|x_4^*) =: u_{5|4}^*$ as in eq. (23), using y^* and x_4^* from eq. (30) resp. (35), it follows that

$$x_3^* = F_3^{-1} \left(h_{3,4}^{-1} \left(h_{3,5;4}^{-1}(w_3, u_{5|4}^*, \alpha_{3,5;4}), F_4(x_4^*), \alpha_{3,4} \right) \right), \quad (36)$$

where the inverse of the respective h-function can be computed analytically for the Clayton and Frank Copulas and numerically for the Gumbel copulas.

Step 3: For eq. (33), it follows analogously that

$$\begin{aligned}
w_2 &= F_{2|345}(x_2^*|x_3^*, x_4^*, y^*) \\
&= h_{2,3;4,5} \left(F_{2|45}(x_2^*|x_4^*, y^*), u_{3|45}^*, \alpha_{2,3;4,5} \right) \\
&= h_{2,3;4,5} \left(h_{2,5;4} \left(F_{2|4}(x_2^*|x_4^*), u_{5|4}^*, \alpha_{2,5;4} \right), u_{3|45}^*, \alpha_{2,3;4,5} \right) \\
&= h_{2,3;4,5} \left(h_{2,5;4} \left(h_{2,4}(F_2(x_2^*), F_4(x_4^*), \alpha_{2,4}), u_{5|4}^*, \alpha_{2,5;4} \right), u_{3|45}^*, \alpha_{2,3;4,5} \right).
\end{aligned}$$

Denoting $F_4(x_4^*)$ with u_4^* , we obtain

$$x_2^* = F_2^{-1} \left(h_{2,4}^{-1} \left(h_{2,5;4}^{-1} \left(h_{2,3;4,5}^{-1}(w_2, u_{3|45}^*, \alpha_{2,3;4,5}), u_{5|4}^*, \alpha_{2,5;4} \right), u_4^*, \alpha_{2,4} \right) \right). \quad (37)$$

Step 4: For eq. (34), we finally get

$$\begin{aligned}
w_1 &= F_{1|2345}(x_1^*|x_2^*, \dots, x_4^*, y^*) \\
&= h_{1,2;3,4,5} \left(F_{1|345}(x_1^*|x_3^*, x_4^*, y^*), u_{2|345}^*, \alpha_{1,2;3,4,5} \right) \\
&= h_{1,2;3,4,5} \left(h_{1,3;4,5} \left(F_{1|45}(x_1^*|x_4^*, y^*), u_{3|45}^*, \alpha_{1,3;4,5} \right), u_{2|345}^*, \alpha_{1,2;3,4,5} \right) \\
&= h_{1,2;3,4,5} \left(h_{1,3;4,5} \left(h_{1,5;4} \left(F_{1|4}(x_1^*|x_4^*), u_{5|4}^*, \alpha_{1,5;4} \right), u_{3|45}^*, \alpha_{1,3;4,5} \right), u_{2|345}^*, \alpha_{1,2;3,4,5} \right) \\
&= h_{1,2;3,4,5} \left(h_{1,3;4,5} \left(h_{1,5;4} \left(h_{1,4} (F_1(x_1^*), u_4^*, \alpha_{1,4}), u_{5|4}^*, \alpha_{1,5;4} \right), u_{3|4,5}^*, \alpha_{1,3;4,5} \right), u_{2|345}^*, \alpha_{1,2;3,4,5} \right).
\end{aligned}$$

It follows that

$$x_1^* = F_1^{-1} \left(h_{1,4}^{-1} \left(h_{1,5;4}^{-1} \left(h_{1,3;4,5}^{-1} \left(h_{1,2;3,4,5}^{-1} (w_1, u_{2|345}^*, \alpha_{1,2;3,4,5}), u_{3|45}^*, \alpha_{1,3;4,5} \right), u_{5|4}^*, \alpha_{1,5;4} \right), u_4^*, \alpha_{1,4} \right) \right). \quad (38)$$

The values x_i^* and y^* are thus new generated (*pseudo*) realizations of the volatility indicators and of the sign of the return from the joint distribution $F_{\sigma_t, r_t}(\mathbf{x}, y)$.

5. Estimation results

5.1. Maximum Likelihood (ML) inference for the univariate models

As implicated in Section 4.5, the ML estimation of the full model includes a large number of variables. To accelerate the estimation procedure, we start by estimating only the components of the first part of the Log-Likelihood function, i.e. LLF_i , which contain the model coefficients of the volatility indicators. We disregard also for simplicity the contemporaneous cross-terms in eq. (5), i.e. $\alpha_{ij} = 0, \beta_{ij} = 0, \forall i \neq j$. Furthermore, we use the equivalence of each model to a *meanless* GARCH model specification with Gaussian distributed error terms for the square root of the respective variable, i.e.

$$\sqrt{x_{i_t}} = h_{i_t} \cdot \zeta_{i_t}, \quad \zeta_{i_t} \sim N(0, 1), \quad (39)$$

where h_{i_t} is the standard deviation of the Gaussian distributed random variable $\sqrt{x_{i_t}} \sim N(0, h_{i_t})$. Its variance $h_{i_t}^2$ is equal to the conditional mean of x_{i_t} from eq. (4), i.e. μ_{i_t} , and, thus, the dynamic specification of μ_{i_t} can be used as the dynamic specification of the variance $h_{i_t}^2$ of $\sqrt{x_{i_t}}$. Furthermore, since ζ_{i_t} is standard Gaussian distributed, $\zeta_{i_t}^2$ will be chi-squared distributed with one degree of freedom, i.e. $\zeta_{i_t}^2 \sim \chi_1^2$. But, since the realization $\zeta_{i_t}^2$ is the same realization as ε_{i_t} , it follows that ε_{i_t} will be also chi-squared distributed, which is equivalent to the unit-mean Gamma distribution with $k_i = 1/2$. Hence, the variable x_{i_t} will be *Gamma*($k_i, \mu_{i_t}/k_i$) distributed, with $k_i = 1/2$.

For a different parameter k_i of the Gamma distribution, the estimated coefficients of the model in eq. (39) will still be the same as the coefficients of the model in eq. (4). This follows

from the same first-order optimality conditions, as shown in Engle and Gallo (2006), which are independent of the parameter k_i . The computed significance of the coefficients from the model in eq. (39) would however be generally different to the one in (4). An exception is the coefficient significance computed using the robust variance-covariance matrix of Bollerslev and Wooldridge (1992), since this estimator is independent of k_i .

We check further if the χ_1^2 or unit-mean Gamma($k_i, 1/k_i$) distributions are correct assumptions. If they are not, then one would have to run an ML estimation for θ_i with other distributions, like the ones mentioned in Section 4.1. An exception builds the daily range, which has strictly positive realizations. For the daily range, one can model the logarithm of the realizations using wider conditional variance specifications, e.g. EGARCH, GJR etc., as well as wider conditional mean specifications, e.g. ARMAX. We check the assumption of the chi-squared or unit-mean Gamma distribution of the residuals after applying the probability integral transform to the residuals, i.e. $u_{i_t} = F_i(\varepsilon_{i_t})$, where $F_i(\cdot)$ is the respective cumulative distribution function, and testing whether the transformed residuals u_{i_t} are uniformly and i.i.d. distributed. This is done using the Kolmogorov-Smirnov (K-S) test for the specification of the distribution and the Ljung-Box Q-test for residual autocorrelation in the first four moments. Table 4 shows the results of these tests as well as the lag dimensions of the mean specifications for each variable and each series.

The coefficients of the direction model are estimated via the ML-method. The number of lagged variables in eq. (7) has been determined based on a randomness test for residual autocorrelation of the transformed realizations $u_{5_t} = F_{\mathbb{I}[r_t > 0]}(y)$ from eq. (13). Table 5 shows the results, i.e. the model order and distribution tests for the direction model.

5.2. Maximum Likelihood (ML) inference within the (Pair-)Copula-based Logit-vMEM model

The estimation of the copula parameters is done based on the standardized residuals of the univariate models. Even if this multistage method (known as the inference functions for margins (IFM) method) is not fully efficient, it is consistent, see Joe (1997) for a detailed analysis and efficiency comparisons, also Patton (2006) and Aas et al. (2009) for the same approach. Furthermore, we analyse both types of specifications regarding the Copula functions. From the first model, which employs a five dimensional Copula function, we compare the results from a one parameter Clayton Copula, shown in eq. (15), with the ones from a Gumbel Copula, shown in eq. (16) and with the ones from the Frank Copula, shown in eq. (17). We compare furthermore these results with the ones from the second model, which employs the Pair-Copula decomposition. Table 6 shows the Copula parameters of the different five-dimensional Copula models, their statistical significance and log likelihood values. The results of the optimizations show that neither of the five-variate Copulas is significantly different than the independence copula, i.e. for

Table 4: Marginal distributions tests. The Ljung-Box Q-test for residual autocorrelation tests the *null*-hypothesis that the first $m = \ln(T) \sim 10$ autocorrelations are jointly zero. The Ljung-Box test statistic $Q(m) = N(N+2) \sum_{h=1}^m \rho_h^2 / (N-h)$, where ρ_h is the autocorrelation for lag h , follows a χ_{m-g}^2 distribution, where $g(=0)$ is the number of model parameters used to compute the respective residuals.

	σ_{HL_t}	σ_{HC_t}	σ_{HO_t}	$ r_t $
GBP				
Modelled variable	$\ln(\sigma_{HL_t})$	$\sqrt{\sigma_{HC_t}}$	$\sqrt{\sigma_{HO_t}}$	$\sqrt{ r_t }$
Model specification	ARMA(1,2)-GARCH(1,1)	ARMA(0,0)-EGARCH(1,1)	ARMA(0,0)-GARCH(1,1)	ARMA(0,0)-GARCH(1,1)
Error Distribution (ε_{it})	-	χ_1^2	χ_1^2	χ_1^2
Distribution tests ($u_{it} = F_\varepsilon(\varepsilon_{it})$), p-values				
KS-test (u_{it})	0.1080	0.1004	0.5396	0.1755
LBQ-test $((u_{it} - \bar{u})^1)$	0.0880	0.3184	0.2600	0.2335
LBQ-test $((u_{it} - \bar{u})^2)$	0.3027	0.0218	0.2833	0.6404
LBQ-test $((u_{it} - \bar{u})^3)$	0.0440	0.9372	0.6785	0.2209
LBQ-test $((u_{it} - \bar{u})^4)$	0.0798	0.0279	0.1968	0.7259
CHF				
Modelled variable	$\ln(\sigma_{HL_t})$	$\sqrt{\sigma_{HC_t}}$	$\sqrt{\sigma_{HO_t}}$	$\sqrt{ r_t }$
Model specification	ARMA(1,1)-GARCH(1,1)	ARMA(0,0)-EGARCH(1,1)	ARMA(0,0)-GARCH(1,1)	ARMA(0,0)-GARCH(1,1)
Error Distribution (ε_{it})	-	χ_1^2	χ_1^2	χ_1^2
Distribution tests ($u_{it} = F_\varepsilon(\varepsilon_{it})$), p-values				
KS-test (u_{it})	0.0138	0.0827	0.5501	0.0138
LBQ-test $((u_{it} - \bar{u})^1)$	0.6056	0.0931	0.3291	0.1792
LBQ-test $((u_{it} - \bar{u})^2)$	0.0312	0.3982	0.9341	0.6741
LBQ-test $((u_{it} - \bar{u})^3)$	0.3874	0.1314	0.7344	0.6722
LBQ-test $((u_{it} - \bar{u})^4)$	0.0219	0.2768	0.8378	0.9285
JPY				
Modelled variable	$\ln(\sigma_{HL_t})$	$\sqrt{\sigma_{HC_t}}$	$\sqrt{\sigma_{HO_t}}$	$\sqrt{ r_t }$
Model specification	ARMA(2,1)-GARCH(3,1)	ARMA(0,0)-EGARCH(1,1)	ARMA(0,0)-GARCH(1,1)	ARMA(0,0)-GARCH(1,1)
Error Distribution (ε_{it})	-	χ_1^2	χ_1^2	χ_1^2
Distribution tests ($u_{it} = F_\varepsilon(\varepsilon_{it})$), p-values				
KS-test (u_{it})	0.0129	0.1652	0.0667	0.0721
LBQ-test $((u_{it} - \bar{u})^1)$	0.2572	0.7330	0.2161	0.4923
LBQ-test $((u_{it} - \bar{u})^2)$	0.0383	0.6592	0.3397	0.5180
LBQ-test $((u_{it} - \bar{u})^3)$	0.2612	0.8667	0.2672	0.2177
LBQ-test $((u_{it} - \bar{u})^4)$	0.0259	0.0794	0.3600	0.5672

Table 5: Marginal distribution tests for the direction model. The distribution test checks the *null*-Hypothesis that the realizations come in random order, against the alternative that they do not. The test statistic is computed as the difference between the observed number of *runs*, i.e. the number of segments with consecutive values above or below the median of u_{5t} , and the expected number of *runs*, i.e. $\hat{\mu} = 2N_- N_+ / N + 1$, divided by the standard deviation $\hat{\sigma} = \sqrt{(\hat{\mu} - 1)(\hat{\mu} - 2) / (N - 1)}$, with $N = N_- + N_+$. Accordingly, the test statistic is asymptotically standard normal distributed.

	GBP	CHF	JPY
Q_d	1	1	2
P_d	1	1	2
Distribution test (for $u_{5t} = F_{\mathbb{I}[r_t > 0]}(y)$), p-values			
	0.6037	0.0576	0.2005

any futures contract is the Copula-parameter statistically significant different from zero (Clayton, Frank Copulas) or one (Gumbel Copula).

Table 6: Estimation parameters and summary statistics for the five dimensional Copula specifications. The p-value corresponds to the outcome of the Wald test for the *null*-Hypothesis that the Copula parameter is zero (for Clayton and Frank Copulas) or one (for the Gumbel Copula). The estimator of the asymptotic covariance matrix was the outer product of gradients (OPG), calculated analytically for the optimal parameter value.

Copula	Parameter	p-Value	LLF	Parameter	p-Value	LLF	Parameter	p-Value	LLF
GBP				CHF			JPY		
Clayton Copula	0.0653	0.8816	-6079.5	0.0631	0.8646	-6063.5	0.0516	0.9035	-6118.6
Gumbel Copula	1.0568	0.8432	-5880.7	1.0635	0.8291	-5746.8	1.0506	0.8570	-5854.2
Frank Copula	0.5545	0.8116	-5801.9	0.5761	0.7979	-5734.7	0.4830	0.8228	-5884.4

For the Pair-Copula decomposition ten Copulas must be specified. The choice of the Copula

functions will be made partly based on a Goodness-of-Fit measure (for the three bivariate Copulas associated with continuous distributions) and partly by trial-and-error (for the remaining seven bivariate Copulas). All Pair-Copula parameters will be computed within the same ML-estimation. The set of analysed Copulas contains the Clayton, Gumbel, Frank and the product Copula.

The Goodness-of-Fit measure for the Copulas associated with continuous bivariate distributions is based on the computation of empirical Copulas, which are the most objective benchmarks since they are entirely non-parametric, see Berg (2009) for an analysis of different Goodness-of-Fit tests. An empirical Copula for a random vector (X_1, X_2) with a continuous bivariate distribution, can be constructed as

$$C_{\text{emp}}(u_1, u_2) = \frac{1}{T+1} \sum_{j=1}^T \mathbb{I}[Z_{j1} \leq u_1, Z_{j2} \leq u_2], \quad (40)$$

where for all $j = \overline{1, T}$ and $i = \overline{1, 2}$, the realization of Z_{ji} is the normalized rank of the observed variable, i.e. $z_{ji} = \frac{r_{ji}}{T+1}$, where r_{ji} is the rank of x_{ji} amongst $x_{1i}, \dots, x_{Ti}$. This empirical estimator of the true Copula is consistent, whether or not the true Copula belongs to the chosen parametric Copula family, see Genest et al. (2009) for details. Therefore a simple test consists of comparing the empirical Copula with a fitted parametric Copula, using a Cramér-von Mises statistic, i.e.

$$S_T = \sum_{j=1}^T [C_{\text{emp}}(\mathbf{z}_j) - C_{\theta}(\mathbf{z}_j)]^2. \quad (41)$$

The smallest value S_T gives the most suitable Copula from the analysed set of Copula families. Table 7 shows the results for the Pair-Copulas corresponding to continuous bivariate distributions. The table shows, that for any of the future price series is the product copula a well suited choice. In each case, the parameter value of the chosen Pair-Copula is used as the starting value for the ML estimation of the Pair-Copula decomposition.

Table 8 shows further the results of the chosen Pair-Copula model. It can be seen that for the Pair-Copula decomposition the Log-Likelihood value (LLF) is significantly increased, resulting also in highly significant coefficients for almost all bivariate Copula parameters. Since the Pair-Copula decomposition proves to be a better model for the underlying volatility indicators, we will apply the parametric bootstrap using only this *null*-model.

Table 7: The Goodness-of-Fit test consists of comparing the fitted parametric Copula with the respective empirical Copula, using a Cramér-von Mises statistic from eq. (41).

Goodness-of-Fit test									
	$C_{1,4}$	$C_{2,4}$	$C_{3,4}$	$C_{1,4}$	$C_{2,4}$	$C_{3,4}$	$C_{1,4}$	$C_{2,4}$	$C_{3,4}$
	GBP			CHF			JPY		
Clayton	3.193	1.082	1.076	2.961	1.017	1.076	2.522	0.936	0.855
Frank	1.759	0.595	0.552	1.826	0.484	0.700	1.641	0.526	0.681
Gumbel	1.321	0.478	0.399	1.543	0.306	0.675	1.559	0.392	0.842
Product	3.536	1.405	1.316	3.066	1.695	1.197	2.522	1.458	0.883

Table 8: Estimation parameters and summary statistics for the Pair-Copula specifications. The outcomes of the Wald-tests for statistical significance of each parameter (given in parantheses) indicate in almost all cases highly significant departures from the respective independence Copulas, i.e. from parameter values of zero (Clayton, Frank Copulas) resp. one (Gumbel Copulas).

Pair-Copula Parameters										LLF
$C_{1,4}$	$C_{2,4}$	$C_{3,4}$	$C_{4,5}$	$C_{1,5;4}$	$C_{2,5;4}$	$C_{3,5;4}$	$C_{1,3;4,5}$	$C_{2,3;4,5}$	$C_{1,2;3,4,5}$	
GBP										
gumbel	gumbel	gumbel	product	product	frank	clayton	product	product	frank	
1.160	1.117	1.090	-	-	-0.270	11.284	-	-	2.085	-4978.3
(0.000)	(0.000)	(0.000)	(-)	(-)	(0.000)	(0.000)	(-)	(-)	(0.001)	
CHF										
gumbel	gumbel	gumbel	product	product	frank	clayton	product	product	frank	
1.162	1.129	1.103	-	-	-0.264	12.469	-	-	2.030	-4933.2
(0.000)	(0.000)	(0.000)	(-)	(-)	(0.000)	(0.000)	(-)	(-)	(0.006)	
JPY										
gumbel	gumbel	frank	product	product	frank	clayton	product	product	frank	
1.148	1.116	4.452	-	-	-0.314	9.624	-	-	1.643	-5249.0
(0.000)	(0.000)	(0.000)	(-)	(-)	(0.792)	(0.000)	(-)	(-)	(0.361)	

5.3. Parametric bootstrap results

To check the statistical significance of the candlestick profitability, we employ the parametric bootstrap. Thus, we generate iteratively up to 10000 samples from the joint distribution of the full model, based on which, new *pseudo*- OLHC-prices are built. This is done as follows:

$$\begin{aligned}
C_t^* &= C_{t-1}^* \cdot \exp(\mathbb{I}[r_t > 0]^* \cdot |r_t|^*), \\
H_t^* &= C_t^* \cdot \exp(\sigma_{hc_t}^*), \\
O_t^* &= \frac{1}{H_t^*} \cdot \exp(\sigma_{ho_t}^*), \\
L_t^* &= \frac{1}{H_t^*} \cdot \exp(\sigma_{hl_t}^*),
\end{aligned} \tag{42}$$

where $*$ denotes a newly generated realization. Based on these *pseudo*-prices, the profitability of the candlesticks will again be computed. The proportion between the number of times that the candlesticks based on these *pseudo*-prices are more profitable than the candlesticks based on the original series and the total number of samples, acts as the p-value of the hypothesis tests from eq. (2) and (3). Tables 9 and 10 present the p-values for the candlesticks and for each futures price series. The results show that only few candlesticks have statistically significant predictive value. The reliability of the results is checked by analysing the simulated p-value for a increasing number of samples. Figure 6 shows the results for each type of candlestick based trading rule.

Overall, only the *White/Black Paper Umbrella* and the *Long White/Black Candle* have statistically significant predictive values on all analysed currency futures markets. Most of the remaining candlesticks single-lines or patterns are either not found in the bootstrap samples or have p-values greater than 0.05, indicating that the hypothesis tests from eq. (2) resp. (3) cannot be rejected, i.e. that the candlesticks have no statistically significant predictive values. Exceptions are the *Bullish Engulfing* and *Three Outside Up* patterns, for which the Null-Hypothesis of no predictive value can be rejected in the case of the Japanese futures contract.

In the case of *White/Black Marubozus* and *Dragonfly/Gravestone Doji*, as well as of the *Hammer/Hanging Man* and *Tweezer Top/Bottom* candlestick patterns, it is clear that a higher number of bootstrap samples is needed to compute the p-values. This is due to the special form of the candlesticks. The *White/Black Marubozus* have no shadows, meaning that for a certain amount of times t the *pseudo-realization* $\sigma_{ho_t}^*$ is exactly zero and $\sigma_{hl_t}^* = \sigma_{hc_t}^*$, which is highly unlikely. This unlikeliness can be seen also in the case of the *Closing/Opening White/Black Marubozu* (CWM/OBM), where, on average, only 0.003 candlesticks of each type were found in each bootstrap sample. Fig. 6 shows in this context that a greater number of samples is needed to assert the convergence of the p-value simulation in the cases CWM/OBM.

6. Conclusions

In this paper we have analysed the profitability of candlestick based trading rules in currency futures markets. Because the trading rules are not precise and mainly based on causal knowledge of traders, we have used a fuzzy-system to mathematize and classify them. To check if the identified candlesticks have any statistically significant predictive value, we have applied the parametric bootstrap, employing a new multivariate *null*-model, which was constructed by means of Pair-Copulas. The results show, that not all candlestick-based trading rules are profitable and only very few have a statistically significant predictive value in all analysed currency futures markets.

The advantage of our employed fuzzy-system is that it mathematizes the causal definition of the candlesticks and also gives a measure of strength of the buy/sell signal, which could be used further in the computation of their profitability. The advantage of our testing procedure is that it copes better with the multivariate premise of the analysed trading rules, modelling directly the correlations between the different time series by means of the highly flexible Pair-Copula approach. Furthermore, although the model employs a certain amount of parameters, another advantage of our approach is that the parameters can be relatively easily estimated with standard GARCH-software.

A more detailed analysis, for example an analysis of the identified multivariate distribution by means of a Goodness-of-Fit test, which is very complex due to the mixed margins and would exceed the scope of our paper, could however bring more insight into the profitability of candlesticks and the performance of the testing procedure.

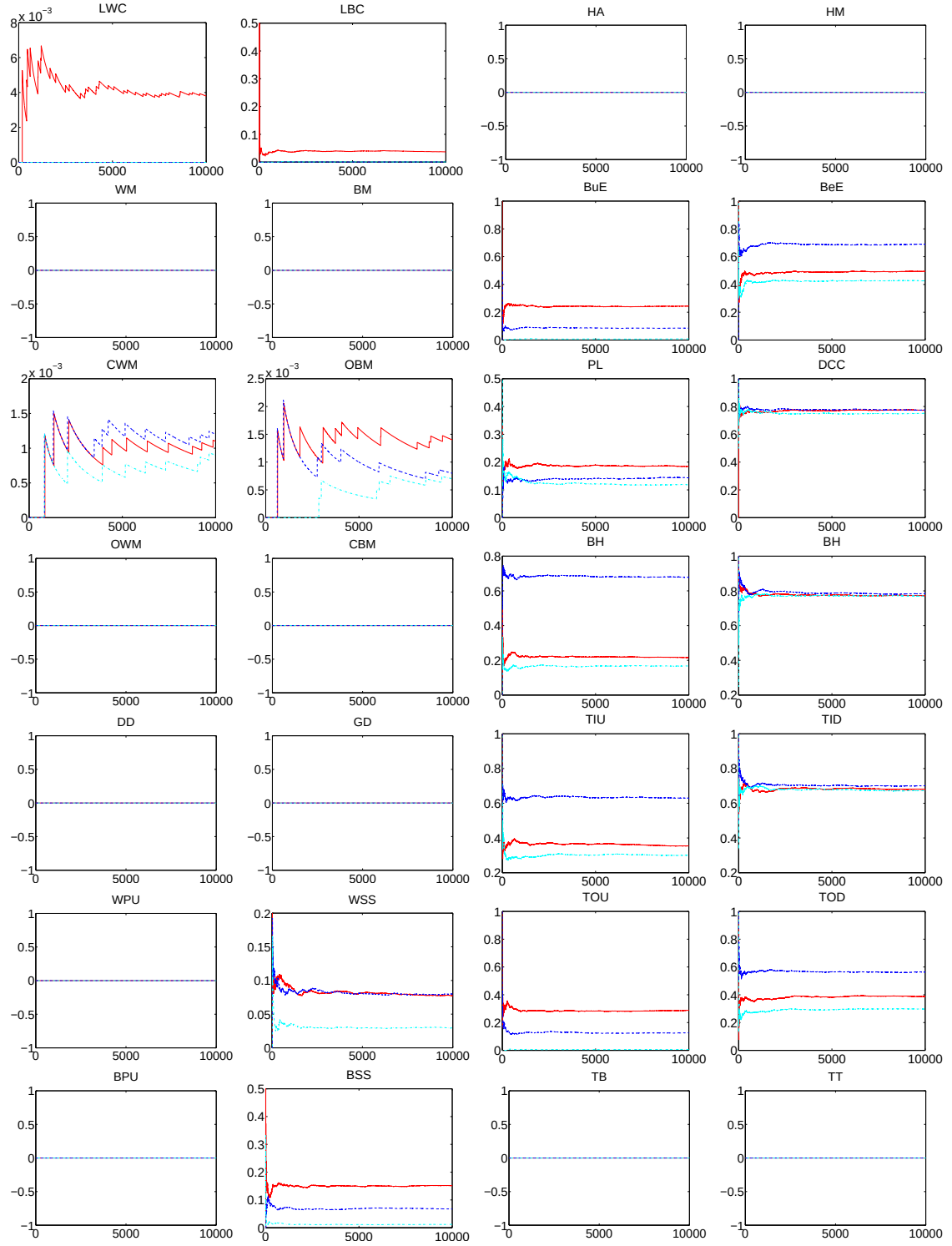


Figure 6: Simulated p-value variation with sample size for the GBP (-), CHF (-) and JPY (-) futures contracts.

Table 9: Bullish single lines/patterns bootstrap statistics for the JPY, CHF and GBP futures contracts: Column 2 shows the sample average number of identified candlestick forms, Column 3 shows the sample average largest output value, Column 4 shows the sample average smallest output value, Column 5 and 6 show the sample average mean and median output values of the identified candlesticks, Column 7 shows the proportion between the number of times that the candlesticks based on the *pseudo*-prices are more profitable than the candlesticks based on the original series and the total number of samples. NaN (not-a-number) means that there were no identified candlesticks in the bootstrap samples.

Bullish Single Lines	$N(y \neq 0)$	max y	min y	mean y	median y	p-value
GBP futures						
<i>Long White Candle</i> (LWC)	428.051	0.999	0.200	0.559	0.600	0.004
<i>White Marubozu</i> (WM)	NaN	NaN	NaN	NaN	NaN	NaN
<i>Closing White Marubozu</i> (CWM)	0.003	0.615	0.615	0.615	0.615	0.001
<i>Opening White Marubozu</i> (OWM)	NaN	NaN	NaN	NaN	NaN	NaN
<i>Dragonfly Doji</i> (DD)	NaN	NaN	NaN	NaN	NaN	NaN
<i>White Paper Umbrella</i> (WPU)	3645.802	1.000	0.200	0.765	0.800	0.000
<i>Black Paper Umbrella</i> (BPU)	3643.214	1.000	0.200	0.784	0.848	0.000
CHF futures						
<i>Long White Candle</i> (LWC)	527.942	0.999	0.200	0.556	0.600	0.000
<i>White Marubozu</i> (WM)	NaN	NaN	NaN	NaN	NaN	NaN
<i>Closing White Marubozu</i> (CWM)	0.003	0.654	0.654	0.654	0.654	0.001
<i>Opening White Marubozu</i> (OWM)	NaN	NaN	NaN	NaN	NaN	NaN
<i>Dragonfly Doji</i> (DD)	NaN	NaN	NaN	NaN	NaN	NaN
<i>White Paper Umbrella</i> (WPU)	3476.589	1.000	0.200	0.743	0.769	0.000
<i>Black Paper Umbrella</i> (BPU)	3468.169	1.000	0.200	0.762	0.800	0.000
JPY futures						
<i>Long White Candle</i> (LWC)	455.195	0.999	0.200	0.549	0.600	0.000
<i>White Marubozu</i> (WM)	NaN	NaN	NaN	NaN	NaN	NaN
<i>Closing White Marubozu</i> (CWM)	0.003	0.595	0.595	0.595	0.595	0.001
<i>Opening White Marubozu</i> (OWM)	NaN	NaN	NaN	NaN	NaN	NaN
<i>Dragonfly Doji</i> (DD)	NaN	NaN	NaN	NaN	NaN	NaN
<i>White Paper Umbrella</i> (WPU)	3657.536	1.000	0.200	0.761	0.800	0.000
<i>Black Paper Umbrella</i> (BPU)	3526.869	1.000	0.200	0.776	0.808	0.000
Bullish Patterns	$N(y \neq 0)$	max y	min y	mean y	median y	p-value
GBP futures						
<i>Hammer</i> (HA)	NaN	NaN	NaN	NaN	NaN	NaN
<i>Bullish Engulfing</i> (BuE)	25.579	0.014	0.005	0.006	0.005	0.245
<i>Piercing Line</i> (PL)	40.579	0.015	0.005	0.006	0.005	0.184
<i>Bullish Harami</i> (BH)	18.328	0.013	0.005	0.006	0.005	0.216
<i>Three Inside Up</i> (TIU)	6.571	0.010	0.005	0.006	0.005	0.355
<i>Three Outside Up</i> (TOU)	10.291	0.011	0.005	0.006	0.005	0.289
<i>Tweezer Bottom</i> (TB)	NaN	NaN	NaN	NaN	NaN	NaN
CHF futures						
<i>Hammer</i> (HA)	NaN	NaN	NaN	NaN	NaN	NaN
<i>Bullish Engulfing</i> (BuE)	43.119	0.011	0.005	0.005	0.005	0.085
<i>Piercing Line</i> (PL)	69.422	0.012	0.005	0.005	0.005	0.143
<i>Bullish Harami</i> (BH)	29.131	0.011	0.005	0.006	0.005	0.678
<i>Three Inside Up</i> (TIU)	10.212	0.009	0.005	0.006	0.005	0.629
<i>Three Outside Up</i> (TOU)	17.056	0.009	0.005	0.005	0.005	0.126
<i>Tweezer Bottom</i> (TB)	NaN	NaN	NaN	NaN	NaN	NaN
JPY futures						
<i>Hammer</i> (HA)	NaN	NaN	NaN	NaN	NaN	NaN
<i>Bullish Engulfing</i> (BuE)	50.065	0.013	0.005	0.006	0.005	0.008
<i>Piercing Line</i> (PL)	60.884	0.013	0.005	0.006	0.005	0.119
<i>Bullish Harami</i> (BH)	32.800	0.011	0.005	0.006	0.005	0.168
<i>Three Inside Up</i> (TIU)	11.668	0.009	0.005	0.005	0.005	0.301
<i>Three Outside Up</i> (TOU)	20.485	0.010	0.005	0.006	0.005	0.007
<i>Tweezer Bottom</i> (TB)	NaN	NaN	NaN	NaN	NaN	NaN

Table 10: Bearish single lines/patterns statistics for the JPY, CHF and GBP futures contracts: Column 2 shows the sample average number of identified candlestick forms, Column 3 shows the sample average largest output value, Column 4 shows the sample average smallest output value, Column 5 and 6 show the sample average mean and median output values of the identified candlesticks, Column 7 shows the proportion between the number of times that the candlesticks based on the *pseudo*-prices are more profitable than the candlesticks based on the original series and the total number of samples. NaN (not-a-number) means that there were no identified candlesticks in the bootstrap samples.

Bearish Single Lines	$N(y \neq 0)$	max y	min y	mean y	median y	p-value
GBP futures						
<i>Long Black Candle</i> (LBC)	252.872	0.997	0.200	0.537	0.600	0.037
<i>Black Marubozu</i> (BM)	NaN	NaN	NaN	NaN	NaN	NaN
<i>Opening Black Marubozu</i> (OBM)	0.003	0.592	0.592	0.592	0.592	0.001
<i>Closing Black Marubozu</i> (CBM)	NaN	NaN	NaN	NaN	NaN	NaN
<i>Gravestone Doji</i> (GD)	NaN	NaN	NaN	NaN	NaN	NaN
<i>White Shooting Star</i> (WSS)	43.181	0.854	0.200	0.349	0.236	0.077
<i>Black Shooting Star</i> (BSS)	38.151	0.852	0.200	0.358	0.257	0.151
CHF futures						
<i>Long Black Candle</i> (LBC)	316.872	0.998	0.200	0.535	0.600	0.001
<i>Black Marubozu</i> (BM)	NaN	NaN	NaN	NaN	NaN	NaN
<i>Opening Black Marubozu</i> (OBM)	0.002	0.610	0.610	0.610	0.610	0.001
<i>Closing Black Marubozu</i> (CBM)	NaN	NaN	NaN	NaN	NaN	NaN
<i>Gravestone Doji</i> (GD)	NaN	NaN	NaN	NaN	NaN	NaN
<i>White Shooting Star</i> (WSS)	61.018	0.899	0.200	0.354	0.237	0.080
<i>Black Shooting Star</i> (BSS)	53.453	0.898	0.200	0.362	0.259	0.068
JPY futures						
<i>Long Black Candle</i> (LBC)	291.772	0.997	0.200	0.524	0.600	0.001
<i>Black Marubozu</i> (BM)	NaN	NaN	NaN	NaN	NaN	NaN
<i>Opening Black Marubozu</i> (OBM)	0.003	0.609	0.609	0.609	0.609	0.001
<i>Closing Black Marubozu</i> (CBM)	NaN	NaN	NaN	NaN	NaN	NaN
<i>Gravestone Doji</i> (GD)	NaN	NaN	NaN	NaN	NaN	NaN
<i>White Shooting Star</i> (WSS)	60.426	0.909	0.200	0.359	0.249	0.029
<i>Black Shooting Star</i> (BSS)	55.730	0.905	0.200	0.365	0.265	0.012
Bearish Patterns	$N(y \neq 0)$	max y	min y	mean y	median y	p-value
GBP futures						
<i>Hanging Man</i> (HM)	NaN	NaN	NaN	NaN	NaN	NaN
<i>Bearish Engulfing</i> (BeE)	3.188	0.006	0.005	0.005	0.005	0.494
<i>Dark Cloud Cover</i> (DCC)	7.541	0.007	0.005	0.005	0.005	0.773
<i>Bearish Harami</i> (BH)	10.964	0.008	0.005	0.005	0.005	0.772
<i>Three Inside Down</i> (TID)	5.845	0.007	0.005	0.005	0.005	0.680
<i>Three Outside Down</i> (TOD)	1.779	0.006	0.005	0.005	0.005	0.387
<i>Tweezer Top</i> (TT)	NaN	NaN	NaN	NaN	NaN	NaN
CHF futures						
<i>Hanging Man</i> (HM)	NaN	NaN	NaN	NaN	NaN	NaN
<i>Bearish Engulfing</i> (BeE)	3.358	0.006	0.005	0.005	0.005	0.689
<i>Dark Cloud Cover</i> (DCC)	8.797	0.006	0.005	0.005	0.005	0.774
<i>Bearish Harami</i> (BH)	12.544	0.007	0.005	0.005	0.005	0.784
<i>Three Inside Down</i> (TID)	6.764	0.006	0.005	0.005	0.005	0.700
<i>Three Outside Down</i> (TOD)	1.908	0.005	0.005	0.005	0.005	0.564
<i>Tweezer Top</i> (TT)	NaN	NaN	NaN	NaN	NaN	NaN
JPY futures						
<i>Hanging Man</i> (HM)	NaN	NaN	NaN	NaN	NaN	NaN
<i>Bearish Engulfing</i> (BeE)	1.719	0.005	0.005	0.005	0.005	0.427
<i>Dark Cloud Cover</i> (DCC)	4.801	0.005	0.005	0.005	0.005	0.751
<i>Bearish Harami</i> (BH)	6.984	0.006	0.005	0.005	0.005	0.770
<i>Three Inside Down</i> (TID)	3.342	0.005	0.005	0.005	0.005	0.672
<i>Three Outside Down</i> (TOD)	0.874	0.005	0.005	0.005	0.005	0.298
<i>Tweezer Top</i> (TT)	NaN	NaN	NaN	NaN	NaN	NaN

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Appendix A. Description of the candlestick recognition algorithm

The candlestick recognition algorithm consists of five different fuzzy-systems. Three fuzzy-systems are used for the identification of candlestick single lines, one fuzzy-system for the identification of candlestick patterns and one for the identification of a buy/sell signal.

The inputs of the first three fuzzy-systems are the body ($\mathcal{B}_t$), the lower ($\mathcal{S}_{l_t}$) and the upper shadow ($\mathcal{S}_{u_t}$) of the candlestick, which are normalized by the total length ($\mathcal{L}_t$) of the candlestick. These are expressed in terms of the available daily prices, as follows:

$$\mathcal{B}_t = \frac{|O_t - C_t|}{H_t - L_t}, \quad \mathcal{S}_{u_t} = \frac{H_t - \max(C_t, O_t)}{H_t - L_t}, \quad \mathcal{S}_{l_t} = \frac{\min(C_t, O_t) - L_t}{H_t - L_t}. \quad (\text{A.1})$$

A fourth input is chosen to be the color of the candlestick, i.e. $\mathcal{C}_t = \text{sgn}(O_t - C_t)$.

In the first design step of the fuzzy-system, the input signals are mapped into some fuzzy sets via some eligible fuzzy-membership functions $\mu_{\mathbf{A}} : X \rightarrow [0, 1]$. Thus, to a certain value of the input signal $x \in \mathbf{X}$ (in our case $\mathbf{X} = [0, 1]$ due to the normalized lengths) is given some degrees $\mu_{\mathbf{A}}(x)$ of belonging to some sets $\mathbf{A}$, which are defined by means of linguistic values, for example *very short*, *short*, *medium*, *long* and/or *very long*. Fig. A.7 and A.8 show the fuzzy-sets of the input variables of the first fuzzy system.

Based on the degrees $\mu_{\mathbf{A}}(x)$ of belonging to the fuzzy-sets $\mathbf{A}$, the fuzzy inference step incorporates at first the *aggregation* of the premises by means of the min-operator. By this procedure, the premise of the r -th candlestick-rule receives a value, which is based on the respective degrees of membership corresponding to the employed fuzzy-sets ($\mathbf{A}_i, \mathbf{A}_j, \mathbf{A}_k, \mathbf{A}_l$), i.e.

$$\mu_{\text{agg},r}(\mathcal{B}_t, \mathcal{S}_{u_t}, \mathcal{S}_{l_t}, \mathcal{C}_t) = \min\{\mu_{\mathbf{A}_i}(\mathcal{B}_t), \mu_{\mathbf{A}_j}(\mathcal{S}_{u_t}), \mu_{\mathbf{A}_k}(\mathcal{S}_{l_t}), \mu_{\mathbf{A}_l}(\mathcal{C}_t)\}.$$

The resulting value gives the total degree of *truth* of the computed premise. Thus, a value of 1 means that the premise is completely fulfilled, whereas a value smaller than 1 means that at least one part of the premise is not completely fulfilled, and a value of 0 means that the premise is not

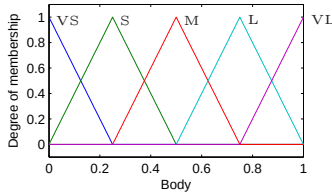


Figure A.7: Fuzzy-sets for the input variables $\mathcal{B}_t$ ($\mathbf{A}_i$), $\mathcal{S}_{u_t}$ ($\mathbf{A}_j$) and $\mathcal{S}_{l_t}$ ($\mathbf{A}_k$), *very small* (VS), *small* (S), *medium* (M), *large* (L) and *very large* (VL).

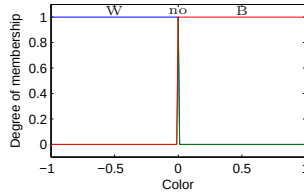


Figure A.8: Classical sets ($\mathbf{A}_l$) for the input variable $\mathcal{C}_t$ (color), *white* (W), *black* (B) and *no color*.

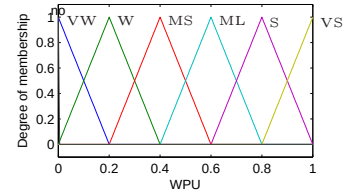


Figure A.9: Fuzzy-sets for the output variables ($\mathbf{B}_p$), *very weak* (VW), *weak* (W), *medium small* (MS), *medium large* (ML), *strong* (S), *very strong* (VS) and *no output*.

fulfilled. Second, it evaluates the fuzzy-*implication*, also by means of the min-operator, i.e

$$\mu_r(\mathcal{B}_t, \mathcal{S}_{u_t}, \mathcal{S}_{l_t}, \mathcal{C}_t, y) = \min\{\mu_{\text{agg},r}(\mathcal{B}_t, \mathcal{S}_{u_t}, \mathcal{S}_{k_t}, \mathcal{C}_t), \mu_{\mathcal{B}_p}(y)\}, \forall y \in [0, 1],$$

where $\mathcal{B}_p$ are the fuzzy-sets of the output variable. These are shown in Fig. A.9. The min-operator of the fuzzy-implication indicates that the degree of *truth* of the r -th rule cannot be larger than the one of the premise or the one of the output set $\mathcal{B}_p$ corresponding to any output value.

All candlesticks with the same form are grouped in the same class, for example the class of *Hammer/Hanging Man/Paper Umbrella*. For a certain class of candlestick forms, one or more rules can be implemented. This is possible due to the relatively dense nesting of the fuzzy sets. For each input variable of the first three fuzzy-systems we have chosen five fuzzy sets and for the fourth input, the color, three classical sets. Since each fuzzy set from one input variable can be combined with all other fuzzy sets from the remaining input variables, in order to state a premise, one can mathematize a total of $5^3 \times 3 = 375$ rules. In this way, a large number of candlestick forms can be identified.

In the last step of the fuzzy-*inference*, the *accumulation* of the different rules for a certain candlestick class is computed by means of the max-operator, i.e. in the case of R rules, it follows

$$\mu_{\text{res}}(\mathcal{B}_t, \mathcal{S}_{u_t}, \mathcal{S}_{l_t}, \mathcal{C}_t, y) = \max\{\mu_1(\mathcal{B}_t, \mathcal{S}_{u_t}, \mathcal{S}_{l_t}, \mathcal{C}_t, y), \dots, \mu_R(\mathcal{B}_t, \mathcal{S}_{u_t}, \mathcal{S}_{l_t}, \mathcal{C}_t, y)\}, \forall y \in [0, 1].$$

The resulting area under this function in y is at last *defuzzified* by means of MOM-method (mean of maximum), which basically gives the mean value of the upper limit, in order to get the computed strength $y^* \in [0, 1]$ of the candlestick form, 1 meaning that the candlestick form fully belongs to the candlestick class, a number in the interval $(0, 1)$ meaning that the candlestick form does not fully belong to the specific class, and 0 meaning that it does not belong to the candlestick class. Nevertheless, the smallest degree greater than zero would still fulfill the basic definition of the candlestick class.

For brevity, the single lines definitions, which are based on Nison (1991) and Marshall et al. (2006), are given as the output surfaces of the first four fuzzy systems. Since each fuzzy system has four inputs, two inputs have been set to a certain value, in order to generate these output surfaces. For example, the input *color* was set to the value of the specific candlestick. Fig. A.10 and A.11 represent for example the output surfaces for the *White Paper Umbrella*, with $\mathcal{S}_{u_t} \in \text{very short}$ and $\mathcal{C}_t \in \text{white}$ (left) and $\mathcal{S}_{u_t} \in \text{short}$ and $\mathcal{C}_t \in \text{white}$ (right). In the same manner, both figures represent the *Black Paper Umbrella* for $\mathcal{C}_t \in \text{black}$. Also, the *White/Black Shooting Star*¹⁸ can be described by the same figures using the upper shadow (US) instead of the

¹⁸The *White/Black Shooting Stars* have the form of inverted *White/Black Paper Umbrellas*.

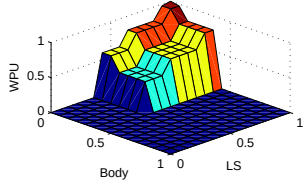


Figure A.10: Output surface for WPU for $S_{u_t} \equiv \text{very small}$ and $C_t \in \text{white}$.

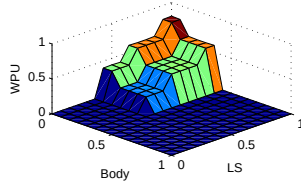


Figure A.11: Output surface for WPU for $S_{u_t} \equiv \text{small}$ and $C_t \in \text{white}$.

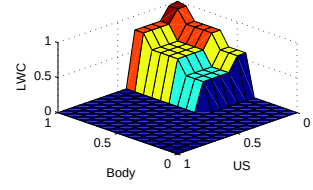


Figure A.12: Output surface for LWC for $S_{l_t} = \text{very small}$ and $C_t \in \text{white}$.

lower shadow (LS). The output surfaces for the *Long White/Black Candle* are shown in A.12 for $S_{l_t} \in \text{very short}$ and $C_t \in \text{white/black}$.

A second fuzzy-system is constructed for the more simpler single-lines, the *White/Black Marubozu*, *Dragonfly Doji* and *Gravestone Doji*. The fuzzy-sets of the first three inputs, body, lower and upper shadow and for all outputs are linear, i.e. $\mu_A : [0, 1] \rightarrow [0, 1]$, $\mu_A(x) = x$, $\forall i = 1, \dots, 3$. The output surface for the *White Marubozu* is shown in Fig. A.13. The same output surface is generated for the *Black Marubozu* if the color is set to $C_t \in \text{black}$. The *Dragonfly Doji* can be described by the same surface using the lower shadow (LS) instead of the body and the upper shadow instead of the lower shadow. Finally, the *Gravestone Doji* can also be described by the same surface using the upper shadow (LS) instead of the body. The last single-line type candlesticks, the *Opening/Closing White/Black Marubozus* are identified using a third fuzzy-system with membership functions in trapezoidal form, see eq. (A.4) and output surfaces, shown in Fig. A.14.

A fourth fuzzy system is constructed for the identification of candlestick patterns. Its structure is similar to that of the second fuzzy system. The inputs however differ. The inputs used here are the normalized differences between the opening and closing prices from two consecutive candlesticks, i.e.

$$CO_t = \frac{C_t - O_{t-1}}{\mathcal{L}_2}, \quad CC_t = \frac{C_t - C_{t-1}}{\mathcal{L}_2}, \quad OO_t = \frac{O_t - O_{t-1}}{\mathcal{L}_2}, \quad OC_t = \frac{O_t - C_{t-1}}{\mathcal{L}_2}. \quad (\text{A.2})$$

The normalizing length is the half of the difference between the maximum and minimum prices

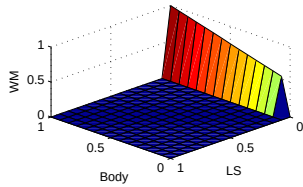


Figure A.13: Output surface for WM for $S_{u_t} \in \text{very small}$ and $C_t \in \text{white}$.

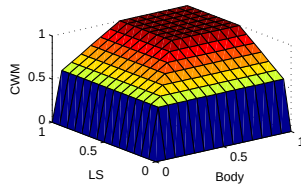


Figure A.14: Output surface for CWM for $S_{u_t} \in \text{very small}$ and $C_t \in \text{white}$.

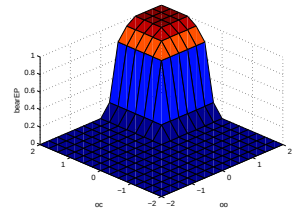


Figure A.15: Output surface for $bearEP$ for $CO_t \in \text{neg}$, $CC_t \in \text{neg}$ and $C_t \in \text{white}$.

of both days, i.e. $2\mathcal{L}_2 = \max(H_t, H_{t-1}) - \min(L_t, L_{t-1})$. Three more inputs are chosen for the color of the first candlestick ($\mathcal{C}_t$) and the normalized differences between the highs ($\mathcal{H}\mathcal{H}_t$) and lows ($\mathcal{L}\mathcal{L}_t$) of both days, i.e.

$$\mathcal{C}_t = \text{sign}(O_{t-1} - C_{t-1}), \quad \mathcal{L}\mathcal{L}_t = \frac{L_t - L_{t-1}}{\mathcal{L}_2}, \quad \mathcal{H}\mathcal{H}_t = \frac{H_t - H_{t-1}}{\mathcal{L}_2}. \quad (\text{A.3})$$

All input signals can thus take values in the interval $[-2, 2]$. The fuzzy membership functions are of trapezoidal type for the fuzzy sets *negative* and *positive*, i.e. linear in the interval $[-1, 0]$, resp. $[0, 1]$, and constant in the interval $[-2, -1]$, resp. $[1, 2]$ ($\mu_{\mathbf{A}} : [-2, 0] \rightarrow [0, 1]$, $\mu_{\mathbf{A}}(x) = -1$ for $-2 \leq x < -1$ and $\mu_{\mathbf{A}}(x) = -x$ for $-1 \leq x \leq 0$ for the fuzzy-set *negative*, $\mu_{\mathbf{A}} : [0, 2] \rightarrow [0, 1]$, $\mu_{\mathbf{A}}(x) = x$, $0 \leq x \leq 1$ and $\mu_{\mathbf{A}}(x) = 1$, $1 < x \leq 2$ for the fuzzy-set *positive*). The fuzzy membership function for the fuzzy sets *zero*, *black* and *white* are $\mu_{\mathbf{A}} : \mathbf{X} \rightarrow [0, 1]$, $\mu_{\mathbf{A}}(x) = 1$ ($\mathbf{X} = \{0\}$), *black* ($\mathbf{X} = \{1\}$) and *white* ($\mathbf{X} = \{-1\}$). The rule basis for the fuzzy-system is given in Table A.11. An example of the output surface is given in Fig. A.15. A finer tuning of the candlestick patterns can be reached by inserting more inputs regarding the form of each candlestick in the pattern.

Table A.11: Summarized rule basis for the fourth fuzzy-system (candlestick pattern forms). The linguistic variables are: *positive* (pos), *negative* (neg), *zero* (z), *white* (w) and *black* (b). The operator $\vee$ is logical OR. The linguistic variables for the output, e.g. bearEP, correspond to the analysed candlestick patterns.

$\mathcal{CO}_t$	$\mathcal{CC}_t$	$\mathcal{OO}_t$	$\mathcal{OC}_t$	$\mathcal{C}_t$	$\mathcal{L}\mathcal{L}_t$	$\mathcal{H}\mathcal{H}_t$	Candlestick pattern form
neg	neg	pos	pos	$w \vee z$	-	-	bearEP (<i>Bearish Engulfing Pattern</i>)
neg	neg	pos	pos	$b \vee z$	-	-	bullEP (<i>Bullish Engulfing Pattern</i>)
pos	neg	pos	neg	$w \vee z$	-	-	bearHarami (<i>Bearish Harami</i>)
pos	neg	pos	neg	$b \vee z$	-	-	bullHarami (<i>Bullish Harami</i>)
neg	pos	neg	neg	$b \vee z$	-	-	PierLine (<i>Piercing Line</i>)
pos	neg	pos	pos	$w \vee z$	-	-	DarkCloud (<i>Dark Cloud Cover</i>)
-	-	-	-	-	z	-	TweezerBottom (<i>Tweezer Bottom</i>)
-	-	-	-	-	-	z	TweezerTop (<i>Tweezer Top</i>)

Finally, a fifth fuzzy-system uses the outputs of the first fuzzy-systems, i.e. the strength of the candlestick single line/pattern form, and two other variables, which determine the prior trend ($\mathcal{T}_t$) and the next day confirmation ($\mathcal{OO}_{t+1}$ and $\mathcal{OC}_{t+1}$). As indicated in Nison (1991), the methods for determining the prior trend can be based on different moving average values, i.e.

1. Short-term overbought (oversold) indicator: $\mathcal{T}_t = \frac{C_t - \text{MA}_{5t}}{\text{MA}_{5t}} > 0 (< 0)$,
2. Long-term overbought (oversold) indicator: $\mathcal{T}_t = \frac{C_t - \text{MA}_{65t}}{\text{MA}_{65t}} > 0 (< 0)$.

The fuzzy membership functions for the input variable determining the prior trend are set to

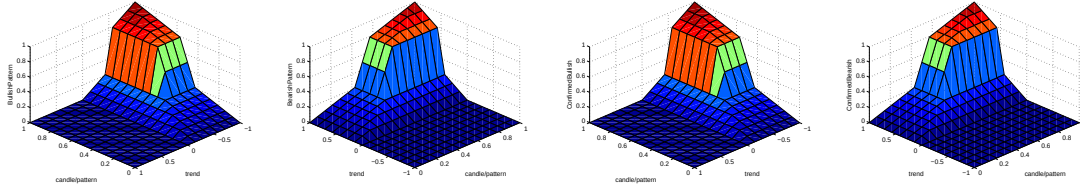


Figure A.16: Outputs surfaces of the third fuzzy system. Since the fuzzy system has four inputs, two inputs have been set to a certain value, in order to generate these output surfaces. Here, the next day confirmations $\mathcal{CO}_{t+1}$ and $\mathcal{OC}_{t+1}$ were set to 1 (-1) for the *confirmed bullish (bearish) patterns*.

trapezoidal form, i.e.

$$\mu_{\text{pos}}(x) = \begin{cases} 0, & x \leq 0 \\ x, & 0 < x \leq 1, \\ 1, & x > 1, \end{cases} \quad \mu_{\text{neg}}(x) = \begin{cases} -1, & x \leq -1, \\ -x, & -1 < x \leq 0, \\ 0, & x > 0. \end{cases} \quad (\text{A.4})$$

Using the fuzzy-inference method, each day can be classified into a buy/sell/neutral signal. The output gives not only the signal-class but also the strength of the signal or, in other words, the degree of membership to the buy/sell class. Table A.12 shows the rule basis and Fig. A.16 shows the output surfaces of the last fuzzy-system. From the single line category, only the candlesticks *Hammer* and *Hanging Man* need both a prior trend and a next day confirmation. All other single lines generate a buy/sell signal directly from the first fuzzy system. All candlestick patterns need a prior trend, and each of the last three bullish/bearish patterns, i.e. *Three Inside Down*, *Three Outside Down* and *Tweezer Top*, as well as *Three Inside Up*, *Three Outside Up* and *Tweezer Bottom* need the next day confirmation.

Table A.12: Summarized rule basis for the third fuzzy-system (buy/sell signals). The linguistic variables are the output variable of the first or second fuzzy-system (y_c), *positive* (pos) and *negative* (neg). The linguistic variables for the output, e.g. *bullp*, correspond to the bullish/bearish signals and a value greater than zero corresponds to a buy/sell signal.

single line/pattern form	$\mathcal{CO}_{t+1}$	$\mathcal{OC}_{t+1}$	$\mathcal{T}_t$	Buy/Sell signal
y_c	-	-	neg	bullp (<i>bullish pattern</i>)
y_c	-	-	pos	bearp (<i>bearish pattern</i>)
y_c	pos	pos	neg	confBullP (<i>confirmed bullish pattern</i>)
y_c	neg	neg	pos	confBearishP (<i>confirmed bearish pattern</i>)