Less Expensive Pricing and Hedging of Long-Dated Equity Index Options When Interest Rates are Stochastic

Kevin Fergusson and Eckhard Platen
LESS EXPENSIVE PRICING AND HEDGING OF LONG-DATED EQUITY INDEX OPTIONS WHEN INTEREST RATES ARE STOCHASTIC

KEVIN FERGUSSON AND ECKHARD PLATEN

Abstract. Many providers of variable annuities such as pension funds and life insurers seek to hedge their exposure to embedded guarantees using long-dated derivatives. This paper extends the benchmark approach to price and hedge long-dated equity index options using a combination of cash, bonds and equities under a variety of market models. The results show that when the discounted index is modelled as a squared Bessel process, as in Platen’s minimal market model, less expensive hedging is achieved irrespective of the short rate model.

1. Introduction

Long term savings products with embedded guarantees on capital, such as variable annuities, are popular among investors planning for retirement. Insurers who write such products are interested in hedging their risk exposure either through reinsurance, derivative markets or hedging programmes. Several frameworks of accounting standards such as US GAAP, IASB and IFRS prescribe that such products be marked-to-market and, therefore, hedging these products is paramount for insurers seeking stable earnings and high credit ratings.

Using the benchmark approach of Platen [2002], Platen [2006] and Platen and Heath [2006], pricing and hedging of long-dated claims on the S&P500 Total Return Index, when interest rates are deterministic, was demonstrated by Hulley and Platen [2008]. In the current article we extend this work under the benchmark approach to price and hedge long-dated equity index options when interest rates are stochastic. Some pricing and hedging of interest rate derivatives using the benchmark approach has been done by Fergusson and Platen [2014]. The pricing and hedging of equity options when share prices and interest rates are stochastic has previously been done by Scott [1997] who also incorporates a jump diffusion component and stochastic volatility to the stock price dynamics. Many approaches to pricing equity options with models involving stochastic interest rates employ inverse Fourier transforms, as done in Lee [2004]. However, in the current article we demonstrate less expensive pricing and hedging of long-dated equity index options and provide approximate pricing formulae involving either the cumulative distribution functions of the normal distribution or the non-central chi-squared distribution. Furthermore, we compute the cost of hedging an equity index put option, whose strike price is an exponential function of the spot price, for each of the considered...
market models and various terms to expiry and identify the best performing models as those involving a discounted GOP being modelled as a squared Bessel process as in Platen’s minimal market model.

In Section 2 we describe the models of the short rate and the discounted equity index used for option pricing and hedging. In Section 3 we show how to price various contingent claims such as call, put, asset-or-nothing and cash-or-nothing options. Supplementing these pricing formulae are approximations given in Appendix A used for the backtesting.

In Section 4 we describe how to hedge derivative securities and in Section 5 we articulate our way of assessing a hedging strategy. We describe in Section 6 the market data employed and how the models are fitted. In Sections 7, 8, 9 and 10 we compare the hedge performances of the deterministic, Vasicek, CIR and 3/2 short rate models, respectively, the numerical results being shown in Appendix B. Finally, in Section 11 we mention that modelling the discounted GOP with the MMM gives rise to significantly cheaper costs of hedging long-dated equity index put options.

2. Description of Models

The market models examined here are specified by the stochastic differential equation (SDE) of the short rate \( r_t \) and the SDE of the discounted GOP \( \bar{S}_t \). The short rate models considered are the deterministic short rate model, where the short rate is known for all times,

\[
(2.1) \quad r_t = r(t),
\]

the Vasicek short rate model described by Vasicek [1977],

\[
(2.2) \quad dr_t = \kappa(\bar{r} - r_t)dt + \sigma dZ_t,
\]

the Cox-Ingersoll-Ross (CIR) short rate model described by Cox et al. [1985],

\[
(2.3) \quad dr_t = \kappa(\bar{r} - r_t)dt + \sigma \sqrt{r_t} dZ_t
\]

and the 3/2 short rate model described by Ahn and Gao [1999],

\[
(2.4) \quad dr_t = (pr_t + qr_t^2)dt + \sigma r_t^{3/2} dZ_t.
\]

The discounted GOP models which are considered are the Black-Scholes model, equivalently the lognormal stock price model, employed by Black and Scholes [1973]

\[
(2.5) \quad d\bar{S}_t = \bar{S}_t \theta dt + \bar{S}_t^\gamma \theta dW_t,
\]

and the minimal market model described by Platen [2001] with SDE

\[
(2.6) \quad d\bar{S}_t = \tilde{\alpha}_t dt + \sqrt{\tilde{\alpha}_t \bar{S}_t} dW_t,
\]

where \( \tilde{\alpha}_t = \tilde{\alpha}_0 \exp(\eta t) \). Here \( Z_t \) and \( W_t \) are independent Wiener processes, \( r(t) \) is the realised value of the short rate at time \( t \) and \( \tilde{\alpha}, \kappa, \sigma, p, q, \theta, \tilde{\alpha}_0 \) and \( \eta \) are constants.

The cash account \( B_t \) is the accumulated value at time \( t \) of $1 deposited at initial time zero and we have the formula

\[
(2.7) \quad B_t = \exp \left( \int_0^t dr_s \right).
\]

The GOP \( \bar{S}_t^\gamma \) is obtained by multiplying the cash account \( B_t \) by the discounted GOP \( \bar{S}_t^\gamma \). The growth rate \( g_t \) of the GOP is equal to the drift term of the SDE.
of the logarithm of the GOP, which equals, for models involving the Black-Scholes
discounted GOP, \( g_t = r_t + \frac{1}{2} \sigma^2 \), and for models involving the MMM discounted
GOP \( g_t = r_t + \frac{1}{2} \alpha_t / S_t^\delta \).

For a given contingent claim \( H_T \) with maturity \( T \in (0, \infty) \), it has been shown
in Platen and Heath [2006] that the minimal possible price \( V_t^{H_T} \) for a replicating
hedge portfolio satisfies the real world pricing formula

\[
V_t^{H_T} = E_t \left( \frac{S_t^\delta}{S_t^\delta} H_T \right),
\]

where \( E_t \) denotes the real world conditional expectation under the real world prob-
ability measure given the information available at time \( t \). Furthermore, the GOP
\( S_t^\delta \) is taken here as the numéraire or benchmark. The numéraire is the portfolio
having maximal growth rate and is approximated well, see Platen and Heath [2006]
and Platen and Rendek [2012], by diversified equity market indices such as the
S&P/ASX 200 Total Return Index or S&P 500 Total Return Index.

3. Pricing of Contingent Claims

When the claim is \( H_T = 1 \) the real world pricing formula (2.8) gives at time \( t \)
the price \( P(t, T) \) of a zero coupon bond (ZCB). Further, when \( H_T = (S_t^\delta - K)^+ \)
formula (2.8) gives at time \( t \) the price \( c_{T,K}(t, S_t^\delta) \) of an equity index call option
having strike price \( K \), and when \( H_T = (K - S_t^\delta)^+ \) formula (2.8) gives the price
\( p_{T,K}(t, S_t^\delta) \) of an equity index put option having strike price \( K \). Because of the
following relation between payoffs

\[
(S_t^\delta - K)^+ = (K - S_t^\delta)^+ + S_t^\delta - K
\]

the put-call parity relation

\[
c_{T,K}(t, S_t^\delta) = p_{T,K}(t, S_t^\delta) + S_t^\delta - K P(t, T)
\]

holds. Additionally, when \( H_T = S_t^{\delta^+} 1_{S_t^{\delta^+} \geq K} \) formula (2.8) gives at time \( t \) the price
\( A_{T,K}^+(t, S_t^\delta) \) of an asset-or-nothing binary call option having strike price \( K \) and
when \( H_T = S_t^{\delta^-} 1_{S_t^{\delta^-} < K} \) formula (2.8) gives the price \( A_{T,K}^-(t, S_t^\delta) \) of an asset-or-
nothing binary put option having strike price \( K \). Finally, when \( H_T = 1_{S_t^{\delta^+} \geq K} \) for-

mula (2.8) gives the price \( B_{T,K}^+(t, S_t^\delta) \) of a cash-or-nothing binary call option having
strike price \( K \) and when \( H_T = 1_{S_t^{\delta^-} < K} \) formula (2.8) gives the price \( B_{T,K}^-(t, S_t^\delta) \)
of a cash-or-nothing binary put option having strike price \( K \).

Under our considered market models the real-world pricing formula (2.8) gives
the price of a ZCB as

\[
P(t, T) = E_t \left( \frac{B_t}{B_T} \right) E_t \left( \frac{S_t^\delta}{S_T^\delta} \right)
\]

For the deterministic model of the short rate we have

\[
E_t \left( \frac{B_t}{B_T} \right) = \exp \left( - \int_t^T r(s) ds \right)
\]

For the Vasicek model of the short rate we have from Vasicek [1977]

\[
E_t \left( \frac{B_t}{B_T} \right) = A(t, T) \exp(-r_t B(t, T)),
\]
where
\begin{equation}
B(t, T) = \frac{1 - \exp(-\kappa(T - t))}{\kappa}
\end{equation}
and
\begin{equation}
A(t, T) = \exp\left(\left(\bar{r} - \frac{\sigma^2}{2\kappa^2}\right)(B(t, T) - T + t) - \frac{\sigma^2}{4\kappa}B(t, T)^2\right).
\end{equation}
For the CIR model of the short rate we have from Cox et al. [1985]
\begin{equation}
E_t(\frac{B_t}{B_T}) = A(t, T) \exp(-r_t B(t, T)),
\end{equation}
where
\begin{equation}
A(t, T) = \left(\frac{h \exp(\frac{1}{2} \kappa(T - t))}{\kappa \sinh(\frac{1}{2} h(T - t)) + \cosh(\frac{1}{2} h(T - t))}\right)^{2\bar{r}/\sigma^2}
\end{equation}
\begin{equation}
B(t, T) = \frac{2 \sinh(\frac{1}{2} h(T - t))}{\kappa \sinh(\frac{1}{2} h(T - t)) + \cosh(\frac{1}{2} h(T - t))}
\end{equation}
and
\begin{equation}
h = \sqrt{\kappa^2 + 2\sigma^2}.
\end{equation}
For the 3/2 model of the short rate we have from Ahn and Gao [1999]
\begin{equation}
E_t(\frac{B_t}{B_T}) = \frac{\Gamma(\gamma_1 - \alpha_1)}{\Gamma(\gamma_1)} \left(\frac{2}{\sigma^2 y(t, r_t)}\right)^{\alpha_1} M(\alpha_1, \gamma_1, -\frac{2}{\sigma^2 y(t, r_t)}),
\end{equation}
where
\begin{equation}
y(t, r_t) = \frac{r_t}{p} (\exp((T - t)p) - 1)
\end{equation}
\begin{equation}
\alpha_u = -\left(\frac{1}{2} - \frac{q}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{q}{\sigma^2}\right)^2 + \frac{2u}{\sigma^2}}
\end{equation}
\begin{equation}
\gamma_u = 2\left(\alpha_u + 1 - \frac{q}{\sigma^2}\right).
\end{equation}
Here \(M\) is the confluent hypergeometric function given by
\begin{equation}
M(\alpha, \gamma, z) = \sum_{n=0}^{\infty} \frac{\alpha_n \, z^n}{(\gamma)_n \, n!}
\end{equation}
and \(\Gamma(x) = \int_0^\infty u^{x-1} \exp(-u) \, du\) is the gamma function. For the Black-Scholes discounted GOP we have
\begin{equation}
E_t(\frac{\bar{S}_T}{\bar{S}_t}) = 1,
\end{equation}
whereas for the MMM discounted GOP we have from Platen and Heath [2006]
\begin{equation}
E_t\left(\frac{\bar{S}_T}{\bar{S}_t}\right) = 1 - \exp\left(-\frac{1}{2} \bar{S}_t^* / (\varphi_T - \varphi_t)\right),
\end{equation}
where
\begin{equation}
\varphi_t = \frac{1}{4} \delta_0 (\exp(\eta t) - 1)/\eta.
\end{equation}
Thus various combinations of (3.5), (3.8), (3.12), (3.15) and (3.16) inserted into (3.3) give explicit formulae for the real world prices of ZCBs under each considered market model.

Under our considered market models (2.8) gives the price of a call option as

$$c_{T,K}(t, S_t^\delta) = E_t \left( \frac{S_t^\delta}{S_T^\delta} (S_T^\delta - K)^+ \right)$$

and the price of a put option as

$$p_{T,K}(t, S_t^\delta) = E_t \left( \frac{S_T^\delta}{S_t^\delta} (K - S_T^\delta)^+ \right).$$

The prices of the asset-or-nothing and cash-or-nothing call and put options are

$$A_{T,K}^+(t, S_t^\delta) = S_t^\delta E_t \left( 1_{S_T^\delta > K} \right)$$
$$A_{T,K}^-(t, S_t^\delta) = S_t^\delta E_t \left( 1_{S_T^\delta \leq K} \right)$$
$$B_{T,K}^+(t, S_t^\delta) = E_t \left( \frac{S_T^\delta}{S_t^\delta} 1_{S_T^\delta > K} \right)$$
$$B_{T,K}^-(t, S_t^\delta) = E_t \left( \frac{S_T^\delta}{S_t^\delta} 1_{S_T^\delta \leq K} \right).$$

Let $f_{S_T^\delta}(x)$ denote the probability density function of the random variable $S_T^\delta$, and define the random variable $R_T^\delta$ as being related to $S_T^\delta$ having the probability density function

$$f_{R_T^\delta}(x) = \frac{S_t^\delta}{x} f_{S_T^\delta}(x) \left/ E_t \left( \frac{S_t^\delta}{S_T^\delta} \right) \right..$$

The pricing formulae of the various call and put options become

$$c_{T,K}(t, S_t^\delta) = S_t^\delta \left( 1 - F_{S_T^\delta}(K) \right) - P(t, T) K \left( 1 - F_{R_T^\delta}(K) \right)$$
$$p_{T,K}(t, S_t^\delta) = -S_t^\delta F_{S_T^\delta}(K) + P(t, T) K F_{R_T^\delta}(K)$$
$$A_{T,K}^+(t, S_t^\delta) = S_t^\delta \left( 1 - F_{S_T^\delta}(K) \right)$$
$$A_{T,K}^-(t, S_t^\delta) = S_t^\delta F_{S_T^\delta}(K)$$
$$B_{T,K}^+(t, S_t^\delta) = P(t, T) \left( 1 - F_{R_T^\delta}(K) \right)$$
$$B_{T,K}^-(t, S_t^\delta) = P(t, T) F_{R_T^\delta}(K),$$

where $F_{S_T^\delta}(x)$ and $F_{R_T^\delta}(x)$ denote the cumulative distribution functions of the random variables $S_T^\delta$ and $R_T^\delta$, respectively.

The following two theorems give exact expressions for the cumulative distribution functions $F_{S_T^\delta}(K)$ and $F_{R_T^\delta}(K)$ in terms of the cumulative distribution functions
of the lognormal distribution and the noncentral gamma distribution, a straightforward generalisation of the noncentral chi-squared distribution, these being

\[
\text{LN}(y; \mu, \sigma^2) = \int_0^y \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(\log x - \mu)^2\right) \, dx
\]

\[
\text{NCG}(y; \alpha, \gamma, \lambda) = \int_0^y \gamma\left(\frac{2\gamma x}{\lambda}\right)^{\alpha/2-1/2} \exp\left(-\frac{1}{2}(\lambda + 2\gamma x)\right) I_{\alpha-1}(\sqrt{2\lambda\gamma x}) \, dx,
\]

respectively, where \( I_\nu(x) \) is the modified Bessel function of the first kind with index \( \nu \), given by

\[
I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(\lambda/2)^j}{j!(\nu+1)} \left(\frac{z^2}{4}\right)^j.
\]

In the proof of Theorem 2 we make use of an equivalent expression for the cumulative distribution function of the noncentral gamma distribution as a Poisson mixture with a gamma distribution, analogous to the noncentral chi-squared distribution being a Poisson mixture with a chi-squared distribution, namely

\[
\text{NCG}(y; \alpha, \gamma, \lambda) = \sum_{j=0}^{\infty} \frac{(\lambda/2)^j}{j!} \exp(-\lambda/2) G(y; \alpha+j, \gamma),
\]

where \( G(y; \alpha, \gamma) \) is the cumulative distribution function of the gamma distribution

\[
G(y; \alpha, \gamma) = \frac{\gamma^\alpha}{\Gamma(\alpha)} \int_0^y x^{\alpha-1} \exp(-\gamma x) \, dx.
\]

Because a lognormal random variable \( X \sim \text{LN}(\mu, \sigma^2) \) is one for which its logarithm is normally distributed, that is \( \log X \sim N(\mu, \sigma^2) \), we have that the cumulative distribution function of the lognormal distribution satisfies \( \text{LN}(y; \mu, \sigma^2) = N((\log y - \mu)/\sigma) \) where \( N(x) \) denotes the cumulative distribution function of the standard normal distribution. Also, because a noncentral gamma random variable \( X \sim \text{NCG}(\alpha, \gamma, \lambda) \) is one for which product with \( 2\gamma \) is noncentral chi-squared distributed, that is \( 2\gamma X \sim \chi^2_{2\alpha, \lambda} \), we have that the cumulative distribution function of the noncentral gamma distribution satisfies \( \text{NCG}(y; \alpha, \gamma, \lambda) = \chi^2_{2\alpha, \lambda}(2\gamma y) \) where \( \chi^2_{\nu, \lambda}(x) \) denotes the cumulative distribution function of the noncentral chi-squared distribution having \( \nu \) degrees of freedom and noncentrality parameter \( \lambda \).

**Theorem 1.** For the Black-Scholes discounted GOP \( \bar{S}_t^{\delta_t} \) and random variable \( L = \log(B_T/B_t) \) we have

\[
F_{\delta_t}(K) = E(\text{LN}(K; \log S_t^{\delta_t} + L + \frac{1}{2}\theta^2(T-t), \theta^2(T-t))) = E(N(-d_1(L)))
\]

\[
F_{\theta_t}(K) = E(\exp(-L) \text{LN}(K; \log S_t^{\delta_t} + L - \frac{1}{2}\theta^2(T-t), \theta^2(T-t))/E(\exp(-L))
\]

\[
= \frac{E(\exp(-L)N(-d_2(L)))}{E(\exp(-L))}
\]
where

\[
d_1(L) = \frac{L + \frac{1}{2} \theta^2(T - t) + \log \frac{S^*}{K}}{\sqrt{\theta^2(T - t)}}
\]
\[
d_2(L) = d_1(L) - \sqrt{\theta^2(T - t)}.
\]

Proof. We know that under the Black-Scholes model of the discounted GOP $\bar{S}^*_T$ is lognormally distributed, that is

\[
\bar{S}^*_T \sim LN(\log \bar{S}^*_t + \frac{1}{2} \theta^2(T - t), \theta^2(T - t)),
\]

and, therefore, $S^*_T$ conditioned on the random variable $L = \log(B_T/B_t)$ is also lognormally distributed, that is

\[
S^*_T \sim LN(\log B_T + \log \bar{S}^*_t + \frac{1}{2} \theta^2(T - t), \theta^2(T - t)),
\]

which can be rewritten as

\[
S^*_T \sim LN(L + \log S^*_t + \frac{1}{2} \theta^2(T - t), \theta^2(T - t)).
\]

Hence

\[
F_{S^*_T}(K) = E(1_{S^*_T \leq K})
= E(E(1_{S^*_T \leq K} | L))
= E(LN(K; L + \log S^*_t + \frac{1}{2} \theta^2(T - t), \theta^2(T - t)))
= E(N(-d_1(L))).
\]

Also the cumulative distribution function of the random variable $R^*_T$ is computed to be

\[
F_{R^*_T}(K) = E\left(E(S^*_T \mid S^*_T \leq K) \right) / E\left(E(S^*_T \mid S^*_T \leq K) \right).
\]

But

\[
E(S^*_T \mid S^*_T \leq K) = S^*_t \exp\left(- (L + \log S^*_t + \frac{1}{2} \theta^2(T - t) + \frac{1}{2} \theta^2(T - t))
\right)
= \exp(-L)
\]

and

\[
E(S^*_T \mid S^*_T \leq K) = S^*_t \int_0^K \frac{1}{x} f_{S^*_T \mid L}(x) dx
= S^*_t \int_0^K \exp(-x) f_{S^*_T \mid L}(x) dx.
\]
Inserting the explicit expression for the lognormal density function $f_{S_T^δ}(x)$ gives

\[(3.36)\]

$$E(S_t^δ; (S_T^δ)^{-1} S_t^δ \leq K | L)$$

$$= S_t^δ \int_0^K \frac{1}{x \sqrt{2\pi \theta^2(T-t)}} \times \exp \left\{ - \frac{1}{2\theta^2(T-t)} \left[ (\log x - (L + \log S_t^δ - \frac{1}{2}\theta^2(T-t))^2 + 2\theta^2(T-t)\log x \right] \right\} dx$$

and completing the square in the exponential in the integrand above gives

\[(3.37)\]

$$S_t^{\delta^*} \int_0^K \frac{1}{x \sqrt{2\pi \theta^2(T-t)}} \times \exp \left\{ - \frac{1}{2\theta^2(T-t)} \left[ (\log x - (L + \log S_t^{\delta^*})^2 + 2(L + \log S_t^{\delta^*})\theta^2(T-t) \right] \right\} dx$$

$$\times \exp \left\{ - \frac{1}{2\theta^2(T-t)} \left[ (\log x - (L + \log S_t^{\delta^*} - \frac{1}{2}\theta^2(T-t))^2 \right] \right\} dx$$

$$= \exp(-L) LN(K; L + \log S_t^{\delta^*} - \frac{1}{2}\theta^2(T-t), \theta^2(T-t))$$

Therefore,

\[(3.38)\]

$$F_{R_T^δ}(K) = E(\exp(-L) LN(K; L + \log S_t^{\delta^*} - \frac{1}{2}\theta^2(T-t), \theta^2(T-t))/E(\exp(-L))$$

$$= E(\exp(-L) N(-d_2(L))/E(\exp(-L)),$$

as required.

\[\square\]

**Theorem 2.** For the MMM discounted GOP $\tilde{S}_T^δ$, and the random variable $L = \log(B_T/B_t)$ we have

\[(3.39)\]

$$F_{R_T^δ}(K) = E(NCG(K; 2, \exp(-L)/2(\varphi_T - \varphi_t)B_t), \lambda) = E(\chi^2_{\lambda,\lambda}(u(L)))$$

$$F_{R_T^δ}(K) = \frac{E(\exp(-L)NCG(K; 0, \exp(-L)/2(\varphi_T - \varphi_t)B_t), \lambda - \exp(-\lambda/2)))}{1 - \exp(-\lambda/2))E(\exp(-L))}$$

$$= \frac{E(\exp(-L)(\chi^2_{\lambda,\lambda}(u(L)) - \exp(-\lambda/2))}{1 - \exp(-\lambda/2))E(\exp(-L))},$$

where

\[(3.40)\]

$$u(L) = \frac{K}{B_t(\varphi_T - \varphi_t) \exp(L)}.$$

\[(3.41)\]

$$\varphi_t = \frac{1}{4} \alpha_0 (\exp(\eta t) - 1)/\eta.$$
Proof. We know that under the MMM, the discounted GOP $\bar{S}^{\delta^*_T}$ is noncentral gamma distributed, that is
\begin{equation}
\bar{S}^{\delta^*_T} \sim \text{NCG}(2, 1/(2(\varphi_T - \varphi_t)), \lambda),
\end{equation}
and therefore $S^\delta_T$ conditioned on the random variable $L = \log(B_T/B_t)$ is also noncentral gamma distributed, that is
\begin{equation}
S^\delta_T \sim \text{NCG}(2, \exp(-L)/(2B_t(\varphi_T - \varphi_t)), \lambda),
\end{equation}
which can be rewritten as
\begin{equation}
S^\delta_T/(\exp(L)B_t(\varphi_T - \varphi_t)) \sim \chi^2_{4, \lambda}.
\end{equation}
Hence
\begin{equation}
F_{S^\delta_T}(K) = E(1_{S^\delta_T \leq K})
= E(E(1_{S^\delta_T \leq K} | L))
= E(\text{NCG}(K; 2, \exp(-L)/(2B_t(\varphi_T - \varphi_t)), \lambda))
= E(\chi^2_{4, \lambda}(u(L))).
\end{equation}
Also the cumulative distribution function of the random variable $R^\delta_T$ is computed to be
\begin{equation}
F_{R^\delta_T}(K) = E\left(E(S^\delta_T (S^\delta_T)^{-1} 1_{S^\delta_T \leq K} | L)\right) / E\left(E(S^\delta_T (S^\delta_T)^{-1} | L)\right).
\end{equation}
But
\begin{equation}
E(S^\delta_T (S^\delta_T)^{-1} | L) = \bar{S}^\delta_T \exp(-L)E((\bar{S}^\delta_T)^{-1})
= \exp(-L)(1 - \exp(-\lambda/2))
\end{equation}
and
\begin{equation}
E\left(\frac{S^\delta_T}{\bar{S}^\delta_T} 1_{S^\delta_T \leq K} | L\right)
= \exp(-L)E\left(\frac{S^\delta_T}{\bar{S}^\delta_T} 1_{S^\delta_T \leq K/B_T} \right)
= \exp(-L)E\left(\frac{\lambda}{\bar{S}^\delta_T / (\varphi_T - \varphi_t)} 1_{S^\delta_T / (\varphi_T - \varphi_t) \leq K \exp(-L)/(B_t(\varphi_T - \varphi_t))} \right)
= \exp(-L) \int_0^K \exp(-L)/(B_t(\varphi_T - \varphi_t)) \frac{\lambda}{x} f_{\chi^2_{4, \lambda}}(x) dx.
\end{equation}
The random variable $\bar{S}^\delta_T / (\varphi_T - \varphi_t)$ is distributed as $\chi^2_{4, \lambda}$, which has probability density function
\begin{equation}
f(x) = \sum_{j=0}^{\infty} \exp(-\lambda/2) \frac{(\lambda/2)^j}{j!} f_{\chi^2_{4, \lambda}}(x),
\end{equation}
where the probability density function of the chi-squared distribution having \(4 + 2j\) degrees of freedom has the formula

\[
(3.51) \quad f_{\chi^2_{4+2j}}(x) = \frac{(1/2)^{2+j}}{\Gamma(2+j)} x^{1+j} \exp(-x/2).
\]

Therefore, the integrand in the RHS of (3.49) can be written as

\[
\begin{align*}
\frac{\lambda}{x} f_{S^t_T / (\varphi_T - \varphi_t)}(x) &= \frac{\lambda}{x} \sum_{j=0}^{\infty} \exp(-\lambda/2) \frac{\left(\lambda/2\right)^j}{j!} \frac{(1/2)^{2+j}}{\Gamma(2+j)} x^{1+j} \exp(-x/2) \\
&= \frac{\lambda}{x} \sum_{j=0}^{\infty} \exp(-\lambda/2) \frac{\left(\lambda/2\right)^j}{j!} \frac{(1/2)^{2+j}}{\Gamma(2+j)} x^{1+j} \exp(-x/2) \\
&= \frac{\lambda}{x} \sum_{j=0}^{\infty} \exp(-\lambda/2) \frac{\left(\lambda/2\right)^j}{j!} \frac{(1/2)^{2+j}}{\Gamma(2+j)} \Gamma(1+j) f_{\chi^2_{2+2j}}(x) \\
&= \frac{\lambda}{x} \sum_{j=0}^{\infty} \exp(-\lambda/2) \frac{\left(\lambda/2\right)^j}{j!} \frac{1}{1+j} f_{\chi^2_{2+2j}}(x) \\
&= \sum_{j=1}^{\infty} \exp(-\lambda/2) \frac{\left(\lambda/2\right)^j}{j!} f_{\chi^2_{2+2j}}(x) \\
&= f_{\chi^2_{0,\lambda}}(x) - \exp(-\lambda/2) 1_{x=0}.
\end{align*}
\]

Hence (3.49) becomes

\[
\begin{align*}
\exp(-L) \int_0^K &\exp(-L)/(B_t(\varphi_T - \varphi_t)) \frac{\lambda}{x} f_{S^t_T / (\varphi_T - \varphi_t)}(x) dx \\
&= \exp(-L) \left( \int_0^K \exp(-L)/(B_t(\varphi_T - \varphi_t)) f_{\chi^2_{0,\lambda}}(x) dx - \exp(-\lambda/2) \right) \\
&= \exp(-L) \left( \chi^2_{0,\lambda} \left( K \exp(-L)/(B_t(\varphi_T - \varphi_t)) \right) - \exp(-\lambda/2) \right),
\end{align*}
\]

which leads to the result. \(\square\)

For a deterministic short rate the cumulative distribution functions \(F_{S^t_T}(x)\) and \(F_{R^t_T}(x)\) are readily computed to be, under a BS discounted GOP,

\[
\begin{align*}
F_{S^t_T}(x) &= \text{LN}(x; \log S^t_T + \int_t^T r(s) ds + \frac{1}{2} \theta^2(T-t), \theta^2(T-t)) \\
F_{R^t_T}(x) &= \text{LN}(x; \log S^t_T + \int_t^T r(s) ds - \frac{1}{2} \theta^2(T-t), \theta^2(T-t))
\end{align*}
\]
and, under the MMM discounted GOP,

\begin{align}
F_{S_{t}^4}(x) &= \left( NCG(x; 2, 1/(2(\varphi(T) - \varphi(t)))B_t \exp(\int_{t}^{T} r(s)ds), \lambda) \right) \\
F_{R_{t}^4}(x) &= \frac{NCG(x; 0, 1/2(\varphi(T) - \varphi(t)))B_t \exp(\int_{t}^{T} r(s)ds), \lambda) - \exp(-\lambda/2)}{1 - \exp(-\lambda/2)}.
\end{align}

For a Vasicek model of the short rate

\begin{align}
L &\sim N(m(t, T), v(t, T))
\end{align}

where

\begin{align}
m(t, T) &= (r_t - \bar{r})B(t, T) + \bar{r}(T - t) \\
v(t, T) &= \frac{\sigma^2}{\kappa^2}(T - t - B(t, T) - \frac{1}{2}\kappa B(t, T)^2)
\end{align}

and

\begin{align}
B(t, T) &= \frac{1 - \exp(-\kappa(T - t))}{\kappa}.
\end{align}

So for a Vasicek short rate and BS discounted GOP

\begin{align}
S_{t}^4 &\sim LN(\log S_t + \frac{1}{2}\sigma^2(T - t) + m(t, T), \theta^2(T - t) + v(t, T)) \\
R_{t}^4 &\sim LN(\log S_t - \frac{1}{2}\sigma^2(T - t) + m(t, T) - v(t, T), \theta^2(T - t) + v(t, T))
\end{align}

and the cumulative distribution functions \( F_{S_{t}^4}(x) \) and \( F_{R_{t}^4}(x) \) are readily computed to be

\begin{align}
F_{S_{t}^4}(x) &= LN(x; \log S_t + \frac{1}{2}\sigma^2(T - t) + m(t, T), \theta^2(T - t) + v(t, T)) \\
F_{R_{t}^4}(x) &= LN(x; \log S_t - \frac{1}{2}\sigma^2(T - t) + m(t, T) - v(t, T), \theta^2(T - t) + v(t, T)).
\end{align}

Also for a Vasicek short rate and MMM discounted GOP

\begin{align}
F_{S_{t}^4}(x) &= \int_{-\infty}^{\infty} \chi_{4,\lambda}(u(z))n(z)dz \\
F_{R_{t}^4}(x) &= \left\{ \int_{-\infty}^{\infty} \exp(-m(t, T) - \sqrt{v(t, T)}z)\chi_{0,\lambda}(u(z))n(z)dz \\
&\quad - \exp(-\lambda/2)\exp(-m(t, T) + \frac{1}{2}v(t, T)) \right\}^{-1} \\
&\quad \left\{ (1 - \exp(-\lambda/2))\exp(-m(t, T) + \frac{1}{2}v(t, T)) \right\}^{-1},
\end{align}

where \( m(t, T) \) and \( v(t, T) \) are given in (3.57), \( u(z) \) is given in (3.40) and \( n(z) \) is the probability density function of the standard normal distribution.

For the CIR short rate model and the 3/2 short rate model the probability density function of \( L \) is computed as the inverse Fourier transform of the moment
generating function (MGF), that is

\[(3.62)\quad f_L(x) = \int_{-\infty}^{\infty} \exp(2\pi ixs) MGF_L(-2\pi is) \, ds.\]

Here the MGF of $L$ under the CIR short rate model is

\[(3.63)\quad MGF_L(u) = \left( \frac{h_u \exp(\frac{1}{2}\kappa(T-t))}{\kappa \sinh \frac{1}{2}(T-t)h_u + h_u \cosh \frac{1}{2}(T-t)h_u} \right)^{2\sigma^2/\kappa^2} \times \exp \left( u \frac{2 \sinh \frac{1}{2}(T-t)h_u}{\kappa \sinh \frac{1}{2}(T-t)h_u + h_u \cosh \frac{1}{2}(T-t)h_u} \right),\]

where $h_u = \sqrt{\kappa^2 - 2u\sigma^2}$. From Theorem 3 of Carr and Sun [2007] the MGF of $L$ under the $3/2$ short rate model is

\[(3.64)\quad MGF_L(-u) = \frac{\Gamma(\gamma_u - \alpha_u)}{\Gamma(\gamma_u)} \left( \frac{2}{\sigma^2 y(t, r_t)} \right)^{\alpha_u} M(\alpha_u, \gamma_u, -\frac{2}{\sigma^2 y(t, r_t)}),\]

where the variables are as in (3.13). The cumulative distribution functions $F_{S_T^\delta}(x)$ and $F_{R_T^\delta}(x)$ under the BS discounted GOP become

\[(3.65)\quad F_{S_T^\delta}(x) = \int_0^\infty N(-d_1(x)) f_L(x) \, dx\]

\[(3.66)\quad F_{R_T^\delta}(x) = \int_0^\infty e^{-x} N(-d_2(x)) f_L(x) \, dx \]

and, under the MMM discounted GOP, become

\[(3.67)\quad F_{S_T^\delta}(x) = \int_0^\infty \chi_{4,\lambda}(u(x)) f_L(x) \, dx\]

\[(3.68)\quad F_{R_T^\delta}(x) = \int_0^\infty e^{-x} \left( \chi_{4,\lambda}(u(x)) - e^{-\lambda/2} \right) f_L(x) \, dx,\]

where $u(x)$ is given by

\[(3.69)\quad u(x) = \frac{K}{B_t \exp(x)(\varphi - \varphi_t)},\]

and $\varphi_t = \frac{1}{4} \tilde{a}_0(\exp(\eta t) - 1)/\eta$.

Thus we have demonstrated how the various cumulative distribution functions can be computed and, combined with (3.22), how prices of various call, put, asset-or-nothing and cash-or-nothing options can be computed. In Appendix A we provide approximate formulae for the various cumulative distribution functions, which lead to simplified and rapid computations.

4. DESCRIPTION OF HEDGING METHODOLOGY

Beyond pricing of long-dated put options on an equity index, our aim is to demonstrate cheaper costs of hedging such options. We focus on hedging a long-dated put option expiring at time $T$, whose strike price $K$ keeps pace with the level of the equity index by way of the formula

\[(4.1)\quad K = S_t^{\delta_S} \exp((\eta + \mu_r)(T-t)).\]

Here $t$ is the time at which the put option is written and $\eta = 0.046841$ is the net market growth rate given in Table 1 and $\mu_r = \frac{1}{11} \sum_s r(s) = 0.045726$ is the average
of the one year continuously compounded cash rates over the 141 year period of the data.

In respect of a put option, and more generally, for any derivative security, a hedging strategy is a trading strategy involving a portfolio of hedge securities whose value at a prescribed payoff date is intended to replicate the value of the derivative security.

When the market values of securities are driven by a deterministic short rate and stochastic discounted GOP, then we have only one random factor in our market and we can hedge a suitable derivative security using a managed self-financing portfolio $\pi$ of cash (the savings account) and the GOP. The value of the hedge portfolio can be written as

\[ V_{\pi}^{(\pi)}_t = \delta^{(0)}_t B_t + \delta^{(1)}_t S_t^{\delta^*_t}, \]

where $\delta^{(0)}_t$ is the number of units of the cash account and $\delta^{(1)}_t$ is the number of units of the GOP account at time $t \in [0, T]$. The respective fractions invested at time $t \geq 0$ are $\pi_t = (\pi^{(0)}_t, \pi^{(1)}_t)$ with $\pi^{(0)}_t = \delta^{(0)}_t B_t / V_{\pi}^{(\pi)}_t$ and $\pi^{(1)}_t = 1 - \pi^{(0)}_t = \delta^{(1)}_t S_t^{\delta^*_t} / V_{\pi}^{(\pi)}_t$. We have some flexibility in our choice of hedge securities and we could have used instead the savings account and futures on the GOP, for example.

When the market values of securities are driven by a stochastic short rate and a stochastic discounted GOP, then we have two random factors in our market and we can hedge any derivative security using a managed portfolio of cash $B_t$, the GOP index $S^{\delta^*_t}_t$ and, for instance, a $(T-t)$-year zero coupon bond $F(t, T)$. In practice, because liquidity is essential for any hedge strategy, we would choose to hedge using a managed portfolio of cash, S&P 500 Index Futures and 10Y US Treasury Bonds. The value of the hedge portfolio $\pi$ can be written as

\[ V_{\pi}^{(\pi)}_t = \delta^{(0)}_t B_t + \delta^{(1)}_t S_t^{\delta^*_t} + \delta^{(2)}_t F(t, T), \]

where $\delta^{(0)}_t$ and $\delta^{(1)}_t$ describe numbers of units as before, and $\delta^{(2)}_t$ is the number of units of the $T$-maturity zero coupon bond at time $t \in [0, T]$.

The cost $C_t$ at time $t$ of hedging a derivative since initial time 0 is equal to the cost of the derivative at time $t$ less any gains from trading the hedge portfolio. We write

\[ C_t = V^{\delta^*_t}_t - \int_0^t \delta^{(0)}_u dB_u - \int_0^t \delta^{(1)}_u dS_u^{\delta^*_u} = V^{\delta^*_t}_t - \int_0^t dV_u^{(\pi)} \]

where $V^{\delta^*_t}_t$ is the value of the derivative at time $t$ and $V_t^{(\pi)}$ is the value of the hedge portfolio at time $t$.

This equation can be rewritten as

\[ C_t = V^{\delta^*_t}_t - (V^{(\pi)}_t - V^{(\pi)}_0) \]

\[ = V^{(\pi)}_0 + (V^{\delta^*_t}_t - V^{(\pi)}_t) \]

and we can see that the cost of hedging can be expressed alternatively as the cost of the hedge portfolio at outset, namely $V_0^{(\pi)}$, plus additional funds needed at time $t$ to purchase the derivative in excess of the value of the hedge portfolio.

At the payoff date $T$ the cost of hedging is

\[ C_T = V^{\delta^*_T}_T - \int_0^T dV_u^{(\pi)}. \]
Because we are interested in the real world price of hedging, as given in (2.8), we consider the benchmarked cost of hedging, computed as

\[
\hat{C}_T = \frac{C_T}{S^\delta_T} = \hat{V}_T^\delta_T - \int_t^T d\hat{V}^{(\pi)}_u = \hat{V}_t^{(\pi)} + \hat{V}_T^\delta_T - \hat{V}_T^{(\pi)}.
\]

According to (2.8) the average of the benchmarked costs of hedging performed over a large number of backtests ought to approximate the real world price of the derivative with payoff \(H_T\).

Given a fully specified model with known parameters, we backtest hedging of the derivative over the time interval \([0, T]\) by setting the \(n-1\) rebalancing times 
\[t_1 < t_2 < \ldots < t_{n-1}\]

satisfying \(0 = t_0 < t_1\) and \(t_{n-1} < t_n = T\). The hedge portfolio \(V^{(\pi)}\) is adjusted at the rebalancing times and is computed iteratively using the formula

\[
V_{t_i}^{(\pi)} = \delta_{t_{i-1}}^{(0)} B_{t_i} + \delta_{t_{i-1}}^{(1)} S_{t_i}^\delta + \delta_{t_{i-1}}^{(2)} F(t_i, T)
\]

for \(i = 1, 2, \ldots, n\) with initial condition

\[
V_0^{(\pi)} = V_0^\delta_T,
\]

where, for \(i = 1, 2, \ldots, n-1\), the numbers of units held in the GOP and the ZCB at time \(t_i\) are computed as

\[
\delta_{t_i}^{(1)} = \frac{\partial}{\partial S_s} V_s^{\delta_T} (r_s, S_s^\delta_s) |_{s=t_i} - \delta_{t_i}^{(2)} \frac{\partial}{\partial S_s} F(s, T) |_{s=t_i},
\]

\[
\delta_{t_i}^{(2)} = \frac{\partial}{\partial r_s} V_s^{\delta_T} (r_s, S_s^\delta_s) |_{s=t_i} / \frac{\partial}{\partial r_s} F(s, T) |_{s=t_i},
\]

and the number of units held in the cash account at time \(t_i\) is computed as

\[
\delta_{t_i}^{(0)} = \left( V_{t_i}^{(\pi)} - \delta_{t_i}^{(1)} S_{t_i}^\delta - \delta_{t_i}^{(2)} F(t_i, T) \right) / B_{t_i}.
\]

5. Assessing the Performance of a Hedging Strategy

A perfect hedge strategy is one for which

\[
C_t = V_0^{(\pi)}
\]

for all times \(t \in [0, T]\). That is to say, the hedge portfolio replicates the value of the derivative over the life of the hedging strategy.

However, perfect hedging is not possible for many reasons and we are interested in strategies which generate the payoff at expiry date \(T\), with “minimum” cost.

Therefore, for a given market model, a given data set and a given term to expiry we compute the benchmarked costs of hedging a put option at expiry over all possible periods within the data set. From this the \(p\)-th percentile of the set of benchmarked costs is computed. The best hedge strategy is derived from the market model which gives the minimum percentile benchmarked cost of hedging. Consequently, our task in this article is to compare the percentile benchmarked costs of hedging across all mentioned market models.
Table 1. Maximum likelihood estimates of model parameters fitted to US data 1871-2012.

<table>
<thead>
<tr>
<th>Model</th>
<th>Parameters</th>
<th>Standard Errors</th>
<th>Log Likelihood</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vasicek</td>
<td>$\hat{\bar{r}} = 0.042294$</td>
<td>0.0080023</td>
<td>399.7019</td>
</tr>
<tr>
<td></td>
<td>$\kappa = 0.162953$</td>
<td>0.053703</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\sigma = 0.015384$</td>
<td>0.00099592</td>
<td></td>
</tr>
<tr>
<td>CIR</td>
<td>$\hat{\bar{r}} = 0.041078$</td>
<td>0.011421</td>
<td>427.8116</td>
</tr>
<tr>
<td></td>
<td>$\kappa = 0.092540$</td>
<td>0.038668</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\sigma = 0.064670$</td>
<td>0.0040761</td>
<td></td>
</tr>
<tr>
<td>$3/2$</td>
<td>$p = 0.038506$</td>
<td>0.042499</td>
<td>406.2713</td>
</tr>
<tr>
<td></td>
<td>$q = 0.877908$</td>
<td>1.190438</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\sigma = 2.0681$</td>
<td>0.13425</td>
<td></td>
</tr>
<tr>
<td>Black-Scholes</td>
<td>$\theta = 0.177297$</td>
<td>0.059087</td>
<td>-267.4135</td>
</tr>
<tr>
<td>MMM</td>
<td>$\alpha_0 = 0.010028$</td>
<td>0.0023389</td>
<td>-260.6433</td>
</tr>
<tr>
<td></td>
<td>$\eta = 0.046841$</td>
<td>0.0028769</td>
<td></td>
</tr>
</tbody>
</table>

6. Market Data and Fitting the Models

The data set used for backtesting has been the annual series of US 1Y deposit rates, 10Y treasury bond yields and S&P Composite Stock Index from 1871 to 2012, shown in Chapter 26 of Shiller [1989] and subsequently updated on Shiller’s website http://aida.wss.yale.edu/shiller/data/chapt26.xls. The 141 year length of this data series makes it a most useful series for analysing the hedging of long-dated ZCBs because we are able to backtest any given hedge strategy over the large term to maturity of the ZCB. Also, because there are 10Y bond yields accompanying the 1Y deposit rates and stock index values we are able to construct and backtest a hedge portfolio which immunises against movements in both the stock index and short rate.

Each short rate model and discounted GOP model has an explicit formula for the transition density function and this has allowed us to fit the models to the historical data using maximum likelihood estimation (MLE). The maximum likelihood estimates (MLEs) of the parameters of all models fitted to US data are shown in Table 1.

The backtests of the hedging strategies were performed using in-sample estimation of parameters. Of course in reality one would backtest a hedge strategy using out-of-sample parameter estimates but by employing in-sample estimates any poorly performing model is readily falsified.

7. Hedging Costs under a Deterministic Short Rate

We present the costs of hedging GOP options under market models having a deterministic short rate.

In Table 2 the percentile benchmarked costs of hedging GOP options of various terms to expiry and levels of probability are shown for the deterministic short rate and Black-Scholes discounted GOP model.
In Table 3 the percentile benchmarked costs of hedging GOP options of various terms to expiry and levels of probability are shown for the deterministic short rate and MMM discounted GOP model.

For hedging GOP options with terms to expiry up to 10 years the BS discounted GOP model and MMM discounted GOP model perform similarly. Beyond GOP option terms to expiry of 10 years the MMM discounted GOP model outperforms the BS discounted GOP model. For example, hedging a 50Y GOP option at the 99% probability level incurs a cost of 5.0172 under the MMM discounted GOP model, which is significantly less than the corresponding cost of 23.995 under the BS discounted GOP model.

8. Hedging Costs under a Vasicek Short Rate

We present the costs of hedging GOP options under Vasicek short rate models. In Table 4 the percentile benchmarked costs of hedging GOP options of various terms to expiry and levels of probability are shown for the Vasicek short rate and Black-Scholes discounted GOP model. In Table 5 the percentile benchmarked costs of hedging GOP options of various terms to expiry and levels of probability are shown for the Vasicek short rate and MMM discounted GOP model.

For hedging GOP options with terms to expiry up to 15 years the BS discounted GOP model and MMM discounted GOP model perform similarly. However, beyond 15 years the MMM discounted GOP model outperforms the BS discounted GOP model. In particular, the cost of hedging a 50Y GOP option at the 99% probability level is 10.537 under the MMM discounted GOP model, which is significantly less than the corresponding cost of 17.235 under the BS discounted GOP model.

9. Hedging Costs under a CIR Short Rate

We present the costs of hedging GOP options under CIR short rate models. In Table 6 the percentile benchmarked costs of hedging GOP options of various terms to expiry and levels of probability are shown for the CIR short rate and Black-Scholes discounted GOP model. In Table 7 the percentile benchmarked costs of hedging GOP options of various terms to expiry and levels of probability are shown for the CIR short rate and MMM discounted GOP model.

For GOP option terms to expiry up to 15 years the BS discounted GOP model provides a significantly lower cost of hedging than under the MMM discounted GOP model. However, beyond a GOP option term to expiry of 15 years the MMM discounted GOP model outperforms the BS discounted GOP model. For example, the cost of hedging a GOP option is significantly reduced for a 50 year term to expiry at the 99% probability level, the cost being 12.392 under the MMM discounted GOP model, which is significantly less than the corresponding cost of 22.827 under the BS discounted GOP model.

10. Hedging Costs under a 3/2 Short Rate

We present the costs of hedging GOP options under 3/2 short rate models. In Table 8 the percentile benchmarked costs of hedging GOP options of various terms to expiry and levels of probability are shown for the 3/2 short rate model and Black-Scholes discounted GOP model.
In Table 9 the percentile benchmarked costs of hedging GOP options of various terms to expiry and levels of probability are shown for the 3/2 short rate model and MMM discounted GOP model.

For GOP option terms to expiry shorter than 15 years the BS discounted GOP model is about the same as or lower than the MMM discounted GOP model. Beyond a GOP option term to expiry of 15 years the MMM discounted GOP model outperforms the BS discounted GOP model. For example, the cost of hedging a 50Y GOP option at the 99% probability level is 13.838 under the MMM discounted GOP model, which is significantly less than the corresponding cost of 51.438 under the BS discounted GOP model.

11. Conclusions on the Hedge Performances

In Figure 1 the percentile costs of hedging GOP options of varying terms to expiry are graphed. Each model for which the discounted GOP is modelled by the MMM has significantly cheaper costs of hedging long-dated GOP options. In particular, we find that among the models having a stochastic short rate, the Vasicek short rate and MMM discounted GOP model provides the cheapest hedging strategy for long-dated GOP put options. We remark on the effect of stochastic versus determinsitic interest rates that Jensen’s Inequality gives

$$E\left(\exp\left(-\int_t^T r_s \, ds\right)\right) \geq \exp\left(-E\left(\int_t^T r_s \, ds\right)\right),$$

since the function $f(x) = \exp(-x)$ is convex. This indicates what we have also seen and we see that stochastic interest rates will give rise to higher derivative prices than those from deterministic interest rates if everything else is modelled analogously.

References


Appendix A: Approximate Pricing of Equity Index Options

The calculation of inverse Fourier transforms is computationally intensive on a computer and a faster computational method approximates the distribution of the GOP with a probability distribution having the same moments up to the second or third order and having the same form as the distribution of the discounted GOP. So for a model involving a BS discounted GOP the distribution of the GOP is approximated by a lognormal distribution that matches the first two moments of the GOP. Also, for a model involving a MMM discounted GOP the distribution of the GOP is approximated by a noncentral gamma distribution that matches the first three moments of the GOP.
Because of the independence of the driving Wiener processes $Z$ and $W$ of the short rate and the discounted GOP, respectively, the moments of the GOP $S_T^{\delta^*}$ are the product of the corresponding moments of the savings account $B_T$ and discounted GOP $\bar{S}_T^{\delta^*}$. Also, the $k$-th moment of the related random variable $R_T^{\delta^*}$ is

$$E\left(\frac{S_T^{\delta^*}(S_T^{\delta^*})^k}{\bar{S}_T^{\delta^*}}\right) = \frac{S_T^{\delta^*}}{P(t, T)} E\left((\bar{S}_T^{\delta^*})^{k-1}\right)$$

and therefore can be computed from the $k-1$-th moment of $S_T^{\delta^*}$.

The $k$-th moment of $B_T$ is computed as

$$B_T^k MGF_k(k).$$

When the discounted GOP obeys the BS model, the $k$-th moment of $\bar{S}_T^{\delta^*}$ is

$$\langle \bar{S}_T^{\delta^*}\rangle^k \exp\left(\frac{k}{2} \vartheta^2(T - t) + \frac{k^2}{2} \theta^2(T - t)\right).$$

When the discounted GOP obeys the MMM model, the first, second and third moments of $\bar{S}_T^{\delta^*}$, which is distributed as $NCG(2, 1/(2(\varphi_T - \varphi_t)), \lambda)$, are

$$E(\bar{S}_T^{\delta^*}) = (\varphi_T - \varphi_t)(4 + \lambda)$$
$$E((\bar{S}_T^{\delta^*})^2) = (\varphi_T - \varphi_t)^2(8 + 4\lambda)$$
$$E((\bar{S}_T^{\delta^*})^3) = (\varphi_T - \varphi_t)^3(32 + 24\lambda).$$

Having computed the moments of $S_T^{\delta^*}$ as the product of corresponding moments of $B_T$ and $\bar{S}_T^{\delta^*}$ the lognormal approximations to the distributions of $S_T^{\delta^*}$ and $R_T^{\delta^*}$ are

$$S_T^{\delta^*} \sim LN(m, v)$$
$$R_T^{\delta^*} \sim LN(m - v, v),$$

where

$$v = \log \left(1 + VAR(S_T^{\delta^*})/E(S_T^{\delta^*})^2\right)$$
$$m = \log E(S_T^{\delta^*}) - \frac{1}{2} v.$$ 

Also, the noncentral gamma approximations to the distributions of $S_T^{\delta^*}$ and $R_T^{\delta^*}$ are

$$S_T^{\delta^*} \sim NCG(\alpha, \gamma, \lambda)$$
$$R_T^{\delta^*} \sim NCG(\alpha', \gamma', \lambda'),$$

where $\alpha$, $\gamma$, $\lambda$ are given by

$$\gamma = 2 \frac{VAR(S_T^{\delta^*})}{SKEW(S_T^{\delta^*})} + \sqrt{4 \left(\frac{VAR(S_T^{\delta^*})}{SKEW(S_T^{\delta^*})}\right)^2 - 2 \frac{E(S_T^{\delta^*})}{SKEW(S_T^{\delta^*})}}$$
$$\alpha = 2\gamma E(S_T^{\delta^*}) - \gamma^2 VAR(S_T^{\delta^*})$$
$$\lambda = -2\gamma E(S_T^{\delta^*}) + 2\gamma^2 VAR(S_T^{\delta^*})$$

and $\alpha'$, $\gamma'$, $\lambda'$ have the corresponding formulae in terms of moments of $R_T^{\delta^*}$. 
Using these approximations we can compute prices of various options straightforwardly from (3.22).

**APPENDIX B: TABLES OF PERCENTILE COSTS OF HEDGING**

**Table 2.** Percentile costs of hedging put options under a deterministic short rate & Black-Scholes discounted GOP based on US data 1871 - 2012.

<table>
<thead>
<tr>
<th>Term to Expiry of Put Option</th>
<th>99-th Percentile</th>
<th>95-th Percentile</th>
<th>90-th Percentile</th>
<th>85-th Percentile</th>
<th>80-th Percentile</th>
</tr>
</thead>
<tbody>
<tr>
<td>1Y</td>
<td>0.27397</td>
<td>0.20183</td>
<td>0.18254</td>
<td>0.15424</td>
<td>0.13648</td>
</tr>
<tr>
<td>2Y</td>
<td>0.39842</td>
<td>0.29176</td>
<td>0.24874</td>
<td>0.22419</td>
<td>0.20839</td>
</tr>
<tr>
<td>3Y</td>
<td>0.44751</td>
<td>0.36636</td>
<td>0.34749</td>
<td>0.30787</td>
<td>0.29391</td>
</tr>
<tr>
<td>4Y</td>
<td>0.56623</td>
<td>0.48567</td>
<td>0.43258</td>
<td>0.41133</td>
<td>0.38396</td>
</tr>
<tr>
<td>5Y</td>
<td>0.68747</td>
<td>0.60918</td>
<td>0.54247</td>
<td>0.50127</td>
<td>0.46287</td>
</tr>
<tr>
<td>7Y</td>
<td>0.94346</td>
<td>0.89324</td>
<td>0.82469</td>
<td>0.73077</td>
<td>0.66711</td>
</tr>
<tr>
<td>10Y</td>
<td>1.4014</td>
<td>1.3563</td>
<td>1.2211</td>
<td>1.0961</td>
<td>1.0314</td>
</tr>
<tr>
<td>15Y</td>
<td>2.5887</td>
<td>2.4553</td>
<td>2.2317</td>
<td>1.9554</td>
<td>1.7858</td>
</tr>
<tr>
<td>20Y</td>
<td>4.184</td>
<td>3.9995</td>
<td>3.5928</td>
<td>3.2131</td>
<td>2.8099</td>
</tr>
<tr>
<td>25Y</td>
<td>6.3033</td>
<td>5.8811</td>
<td>5.5006</td>
<td>4.9965</td>
<td>4.1317</td>
</tr>
<tr>
<td>30Y</td>
<td>8.76</td>
<td>8.5493</td>
<td>7.9183</td>
<td>7.1568</td>
<td>6.0602</td>
</tr>
<tr>
<td>40Y</td>
<td>15.45</td>
<td>15.067</td>
<td>13.972</td>
<td>13.166</td>
<td>12.26</td>
</tr>
<tr>
<td>50Y</td>
<td>23.995</td>
<td>23.444</td>
<td>22.838</td>
<td>22.538</td>
<td>21.216</td>
</tr>
</tbody>
</table>

**Table 3.** Percentile costs of hedging put options under a deterministic short rate & MMM discounted GOP based on US data 1871 - 2012.

<table>
<thead>
<tr>
<th>Term to Expiry of Put Option</th>
<th>99-th Percentile</th>
<th>95-th Percentile</th>
<th>90-th Percentile</th>
<th>85-th Percentile</th>
<th>80-th Percentile</th>
</tr>
</thead>
<tbody>
<tr>
<td>1Y</td>
<td>0.26256</td>
<td>0.20971</td>
<td>0.17928</td>
<td>0.15455</td>
<td>0.13933</td>
</tr>
<tr>
<td>2Y</td>
<td>0.38844</td>
<td>0.28898</td>
<td>0.25646</td>
<td>0.22633</td>
<td>0.21036</td>
</tr>
<tr>
<td>3Y</td>
<td>0.44883</td>
<td>0.37276</td>
<td>0.34775</td>
<td>0.30562</td>
<td>0.28377</td>
</tr>
<tr>
<td>4Y</td>
<td>0.55028</td>
<td>0.50237</td>
<td>0.43545</td>
<td>0.41342</td>
<td>0.37323</td>
</tr>
<tr>
<td>5Y</td>
<td>0.6583</td>
<td>0.60488</td>
<td>0.54859</td>
<td>0.50968</td>
<td>0.45911</td>
</tr>
<tr>
<td>7Y</td>
<td>0.8764</td>
<td>0.82385</td>
<td>0.77834</td>
<td>0.73707</td>
<td>0.68101</td>
</tr>
<tr>
<td>10Y</td>
<td>1.2785</td>
<td>1.1621</td>
<td>1.0663</td>
<td>1.0022</td>
<td>0.95727</td>
</tr>
<tr>
<td>15Y</td>
<td>1.9878</td>
<td>1.7257</td>
<td>1.4987</td>
<td>1.3902</td>
<td>1.3662</td>
</tr>
<tr>
<td>20Y</td>
<td>3.5324</td>
<td>2.1929</td>
<td>1.7767</td>
<td>1.5438</td>
<td>1.4853</td>
</tr>
<tr>
<td>25Y</td>
<td>4.7702</td>
<td>2.117</td>
<td>1.9963</td>
<td>1.8997</td>
<td>1.8677</td>
</tr>
<tr>
<td>30Y</td>
<td>5.5783</td>
<td>2.6459</td>
<td>2.5039</td>
<td>2.4681</td>
<td>2.2429</td>
</tr>
<tr>
<td>40Y</td>
<td>3.8298</td>
<td>3.6076</td>
<td>3.5365</td>
<td>2.1097</td>
<td>1.3734</td>
</tr>
<tr>
<td>50Y</td>
<td>5.0172</td>
<td>3.217</td>
<td>2.3311</td>
<td>1.8324</td>
<td>1.2442</td>
</tr>
</tbody>
</table>
Table 4. Percentile costs of hedging put options under a Vasicek short rate & Black-Scholes discounted GOP based on US data 1871 - 2012.

<table>
<thead>
<tr>
<th>Term to Expiry of Put Option</th>
<th>99-th Percentile</th>
<th>95-th Percentile</th>
<th>90-th Percentile</th>
<th>85-th Percentile</th>
<th>80-th Percentile</th>
</tr>
</thead>
<tbody>
<tr>
<td>1Y</td>
<td>0.26209</td>
<td>0.20057</td>
<td>0.18518</td>
<td>0.15539</td>
<td>0.1384</td>
</tr>
<tr>
<td>2Y</td>
<td>0.38887</td>
<td>0.29589</td>
<td>0.25043</td>
<td>0.22619</td>
<td>0.2102</td>
</tr>
<tr>
<td>3Y</td>
<td>0.42445</td>
<td>0.38483</td>
<td>0.33276</td>
<td>0.30434</td>
<td>0.28799</td>
</tr>
<tr>
<td>4Y</td>
<td>0.56939</td>
<td>0.47183</td>
<td>0.42084</td>
<td>0.40336</td>
<td>0.36581</td>
</tr>
<tr>
<td>5Y</td>
<td>0.73917</td>
<td>0.5589</td>
<td>0.50985</td>
<td>0.49118</td>
<td>0.45009</td>
</tr>
<tr>
<td>7Y</td>
<td>0.9217</td>
<td>0.75279</td>
<td>0.72856</td>
<td>0.69505</td>
<td>0.67038</td>
</tr>
<tr>
<td>10Y</td>
<td>1.1019</td>
<td>1.0746</td>
<td>1.0463</td>
<td>1.0302</td>
<td>0.98135</td>
</tr>
<tr>
<td>15Y</td>
<td>1.7703</td>
<td>1.7526</td>
<td>1.7253</td>
<td>1.6592</td>
<td>1.6007</td>
</tr>
<tr>
<td>20Y</td>
<td>2.6505</td>
<td>2.6003</td>
<td>2.5506</td>
<td>2.4401</td>
<td>2.3668</td>
</tr>
<tr>
<td>30Y</td>
<td>5.2499</td>
<td>5.1718</td>
<td>5.127</td>
<td>4.9874</td>
<td>4.8277</td>
</tr>
<tr>
<td>50Y</td>
<td>17.235</td>
<td>17.059</td>
<td>16.905</td>
<td>16.788</td>
<td>16.218</td>
</tr>
</tbody>
</table>

Table 5. Percentile costs of hedging put options under a Vasicek short rate & MMM discounted GOP based on US data 1871 - 2012.

<table>
<thead>
<tr>
<th>Term to Expiry of Put Option</th>
<th>99-th Percentile</th>
<th>95-th Percentile</th>
<th>90-th Percentile</th>
<th>85-th Percentile</th>
<th>80-th Percentile</th>
</tr>
</thead>
<tbody>
<tr>
<td>1Y</td>
<td>0.26891</td>
<td>0.21077</td>
<td>0.18833</td>
<td>0.15075</td>
<td>0.13911</td>
</tr>
<tr>
<td>2Y</td>
<td>0.34535</td>
<td>0.29305</td>
<td>0.26402</td>
<td>0.23253</td>
<td>0.21133</td>
</tr>
<tr>
<td>3Y</td>
<td>0.47236</td>
<td>0.37087</td>
<td>0.35022</td>
<td>0.3161</td>
<td>0.29035</td>
</tr>
<tr>
<td>4Y</td>
<td>0.59532</td>
<td>0.49272</td>
<td>0.43524</td>
<td>0.41383</td>
<td>0.3879</td>
</tr>
<tr>
<td>5Y</td>
<td>0.70438</td>
<td>0.59369</td>
<td>0.53894</td>
<td>0.50715</td>
<td>0.46685</td>
</tr>
<tr>
<td>7Y</td>
<td>0.9257</td>
<td>0.7902</td>
<td>0.75692</td>
<td>0.71941</td>
<td>0.69043</td>
</tr>
<tr>
<td>10Y</td>
<td>1.3062</td>
<td>1.089</td>
<td>1.0271</td>
<td>0.99164</td>
<td>0.96337</td>
</tr>
<tr>
<td>15Y</td>
<td>1.8235</td>
<td>1.6769</td>
<td>1.595</td>
<td>1.528</td>
<td>1.4794</td>
</tr>
<tr>
<td>20Y</td>
<td>2.4199</td>
<td>2.3152</td>
<td>2.2524</td>
<td>2.2031</td>
<td>2.1564</td>
</tr>
<tr>
<td>30Y</td>
<td>4.5578</td>
<td>4.2153</td>
<td>3.9152</td>
<td>3.4983</td>
<td>3.341</td>
</tr>
<tr>
<td>40Y</td>
<td>7.7437</td>
<td>6.1303</td>
<td>5.7859</td>
<td>5.2006</td>
<td>4.6009</td>
</tr>
<tr>
<td>50Y</td>
<td>10.537</td>
<td>8.7545</td>
<td>7.3966</td>
<td>6.9161</td>
<td>6.0211</td>
</tr>
</tbody>
</table>
Table 6. Percentile costs of hedging put options under a CIR short rate & Black-Scholes discounted GOP based on US data 1871 - 2012.

<table>
<thead>
<tr>
<th>Term to Expiry of Put Option</th>
<th>99-th Percentile</th>
<th>95-th Percentile</th>
<th>90-th Percentile</th>
<th>85-th Percentile</th>
<th>80-th Percentile</th>
</tr>
</thead>
<tbody>
<tr>
<td>1Y</td>
<td>0.26361</td>
<td>0.20099</td>
<td>0.18623</td>
<td>0.15548</td>
<td>0.13815</td>
</tr>
<tr>
<td>2Y</td>
<td>0.38399</td>
<td>0.29298</td>
<td>0.2529</td>
<td>0.22631</td>
<td>0.20976</td>
</tr>
<tr>
<td>3Y</td>
<td>0.42767</td>
<td>0.38364</td>
<td>0.3379</td>
<td>0.30484</td>
<td>0.28674</td>
</tr>
<tr>
<td>4Y</td>
<td>0.57874</td>
<td>0.47589</td>
<td>0.42838</td>
<td>0.413</td>
<td>0.37579</td>
</tr>
<tr>
<td>5Y</td>
<td>0.7401</td>
<td>0.57269</td>
<td>0.5157</td>
<td>0.50415</td>
<td>0.4559</td>
</tr>
<tr>
<td>7Y</td>
<td>0.91691</td>
<td>0.76812</td>
<td>0.74329</td>
<td>0.72131</td>
<td>0.67822</td>
</tr>
<tr>
<td>10Y</td>
<td>1.1679</td>
<td>1.1198</td>
<td>1.0954</td>
<td>1.0659</td>
<td>1.0108</td>
</tr>
<tr>
<td>15Y</td>
<td>1.9723</td>
<td>1.9333</td>
<td>1.8381</td>
<td>1.7431</td>
<td>1.6647</td>
</tr>
<tr>
<td>20Y</td>
<td>3.0453</td>
<td>2.9837</td>
<td>2.9111</td>
<td>2.6859</td>
<td>2.5507</td>
</tr>
<tr>
<td>25Y</td>
<td>4.4507</td>
<td>4.36</td>
<td>4.2831</td>
<td>4.0354</td>
<td>3.7726</td>
</tr>
<tr>
<td>30Y</td>
<td>6.3147</td>
<td>6.2158</td>
<td>6.0923</td>
<td>5.7914</td>
<td>5.5315</td>
</tr>
<tr>
<td>40Y</td>
<td>12.208</td>
<td>12.021</td>
<td>11.877</td>
<td>11.386</td>
<td>10.894</td>
</tr>
<tr>
<td>50Y</td>
<td>22.827</td>
<td>22.505</td>
<td>22.362</td>
<td>21.817</td>
<td>20.354</td>
</tr>
</tbody>
</table>

Table 7. Percentile costs of hedging put options under a CIR short rate & MMM discounted GOP based on US data 1871 - 2012.

<table>
<thead>
<tr>
<th>Term to Expiry of Put Option</th>
<th>99-th Percentile</th>
<th>95-th Percentile</th>
<th>90-th Percentile</th>
<th>85-th Percentile</th>
<th>80-th Percentile</th>
</tr>
</thead>
<tbody>
<tr>
<td>1Y</td>
<td>0.38674</td>
<td>0.27452</td>
<td>0.22708</td>
<td>0.1934</td>
<td>0.17269</td>
</tr>
<tr>
<td>2Y</td>
<td>0.52285</td>
<td>0.41077</td>
<td>0.35496</td>
<td>0.32561</td>
<td>0.28225</td>
</tr>
<tr>
<td>3Y</td>
<td>0.64996</td>
<td>0.55456</td>
<td>0.45693</td>
<td>0.39492</td>
<td>0.36086</td>
</tr>
<tr>
<td>4Y</td>
<td>0.81779</td>
<td>0.62385</td>
<td>0.57292</td>
<td>0.53832</td>
<td>0.46281</td>
</tr>
<tr>
<td>5Y</td>
<td>0.85147</td>
<td>0.79164</td>
<td>0.68805</td>
<td>0.60938</td>
<td>0.53912</td>
</tr>
<tr>
<td>7Y</td>
<td>1.1763</td>
<td>0.98336</td>
<td>0.88747</td>
<td>0.83298</td>
<td>0.7417</td>
</tr>
<tr>
<td>10Y</td>
<td>1.5361</td>
<td>1.3964</td>
<td>1.3001</td>
<td>1.131</td>
<td>1.0229</td>
</tr>
<tr>
<td>15Y</td>
<td>2.0624</td>
<td>1.9721</td>
<td>1.8385</td>
<td>1.7358</td>
<td>1.6192</td>
</tr>
<tr>
<td>20Y</td>
<td>2.6329</td>
<td>2.4387</td>
<td>2.3819</td>
<td>2.3443</td>
<td>2.2961</td>
</tr>
<tr>
<td>25Y</td>
<td>3.4261</td>
<td>3.3294</td>
<td>3.1933</td>
<td>3.1383</td>
<td>3.0093</td>
</tr>
<tr>
<td>30Y</td>
<td>4.9173</td>
<td>4.5382</td>
<td>4.3104</td>
<td>4.0591</td>
<td>3.8871</td>
</tr>
<tr>
<td>40Y</td>
<td>8.6307</td>
<td>6.8582</td>
<td>6.2608</td>
<td>6.0561</td>
<td>5.803</td>
</tr>
</tbody>
</table>
Table 8. Percentile costs of hedging put options under a 3/2 short rate & Black-Scholes discounted GOP based on US data 1871 - 2012.

<table>
<thead>
<tr>
<th>Term to Expiry of Put Option</th>
<th>99-th Percentile</th>
<th>95-th Percentile</th>
<th>90-th Percentile</th>
<th>85-th Percentile</th>
<th>80-th Percentile</th>
</tr>
</thead>
<tbody>
<tr>
<td>1Y</td>
<td>0.2664</td>
<td>0.21034</td>
<td>0.18632</td>
<td>0.15655</td>
<td>0.13641</td>
</tr>
<tr>
<td>2Y</td>
<td>0.383</td>
<td>0.29462</td>
<td>0.2604</td>
<td>0.22775</td>
<td>0.21844</td>
</tr>
<tr>
<td>3Y</td>
<td>0.44263</td>
<td>0.37739</td>
<td>0.34415</td>
<td>0.32066</td>
<td>0.29338</td>
</tr>
<tr>
<td>4Y</td>
<td>0.60574</td>
<td>0.494</td>
<td>0.43825</td>
<td>0.41616</td>
<td>0.3963</td>
</tr>
<tr>
<td>5Y</td>
<td>0.7494</td>
<td>0.6122</td>
<td>0.53268</td>
<td>0.51847</td>
<td>0.48492</td>
</tr>
<tr>
<td>7Y</td>
<td>0.89428</td>
<td>0.82897</td>
<td>0.80913</td>
<td>0.75714</td>
<td>0.70008</td>
</tr>
<tr>
<td>10Y</td>
<td>1.3188</td>
<td>1.2501</td>
<td>1.1929</td>
<td>1.1476</td>
<td>1.0268</td>
</tr>
<tr>
<td>15Y</td>
<td>2.4691</td>
<td>2.3606</td>
<td>2.1138</td>
<td>1.8398</td>
<td>1.7279</td>
</tr>
<tr>
<td>20Y</td>
<td>4.1914</td>
<td>3.946</td>
<td>3.5957</td>
<td>2.8854</td>
<td>2.6338</td>
</tr>
<tr>
<td>25Y</td>
<td>6.715</td>
<td>6.1678</td>
<td>5.6977</td>
<td>4.8608</td>
<td>4.0434</td>
</tr>
<tr>
<td>30Y</td>
<td>10.283</td>
<td>9.5079</td>
<td>8.6169</td>
<td>7.4723</td>
<td>5.9803</td>
</tr>
<tr>
<td>40Y</td>
<td>22.815</td>
<td>21.099</td>
<td>19.133</td>
<td>17.088</td>
<td>14.239</td>
</tr>
<tr>
<td>50Y</td>
<td>51.438</td>
<td>48.154</td>
<td>44.121</td>
<td>39.216</td>
<td>32.847</td>
</tr>
</tbody>
</table>

Table 9. Percentile costs of hedging put options under a 3/2 short rate & MMM discounted GOP based on US data 1871 - 2012.

<table>
<thead>
<tr>
<th>Term to Expiry of Put Option</th>
<th>99-th Percentile</th>
<th>95-th Percentile</th>
<th>90-th Percentile</th>
<th>85-th Percentile</th>
<th>80-th Percentile</th>
</tr>
</thead>
<tbody>
<tr>
<td>1Y</td>
<td>0.26558</td>
<td>0.21062</td>
<td>0.18689</td>
<td>0.15239</td>
<td>0.14139</td>
</tr>
<tr>
<td>2Y</td>
<td>0.34809</td>
<td>0.29062</td>
<td>0.26405</td>
<td>0.22823</td>
<td>0.20994</td>
</tr>
<tr>
<td>3Y</td>
<td>0.47585</td>
<td>0.39805</td>
<td>0.35755</td>
<td>0.31779</td>
<td>0.29356</td>
</tr>
<tr>
<td>4Y</td>
<td>0.60302</td>
<td>0.51428</td>
<td>0.45006</td>
<td>0.43793</td>
<td>0.41451</td>
</tr>
<tr>
<td>5Y</td>
<td>0.75283</td>
<td>0.62405</td>
<td>0.56998</td>
<td>0.53439</td>
<td>0.51137</td>
</tr>
<tr>
<td>7Y</td>
<td>1.7135</td>
<td>0.84799</td>
<td>0.82889</td>
<td>0.7829</td>
<td>0.75554</td>
</tr>
<tr>
<td>10Y</td>
<td>2.1669</td>
<td>1.2285</td>
<td>1.1974</td>
<td>1.1496</td>
<td>1.1172</td>
</tr>
<tr>
<td>15Y</td>
<td>2.4737</td>
<td>1.9552</td>
<td>1.8517</td>
<td>1.7445</td>
<td>1.7174</td>
</tr>
<tr>
<td>20Y</td>
<td>3.102</td>
<td>2.7326</td>
<td>2.5879</td>
<td>2.4996</td>
<td>2.4446</td>
</tr>
<tr>
<td>30Y</td>
<td>5.0805</td>
<td>4.7434</td>
<td>4.5609</td>
<td>4.4722</td>
<td>4.2388</td>
</tr>
<tr>
<td>40Y</td>
<td>9.1205</td>
<td>7.6594</td>
<td>7.2593</td>
<td>6.944</td>
<td>6.8233</td>
</tr>
</tbody>
</table>