Valuation of Employee Stock Options using the Exercise Multiple Approach and Life Tables

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Abstract
Employee stock options (ESOs) are highly exotic derivatives including various forms of call options and performance shares. Much effort in the academic literature has been devoted to modelling employee risk aversion and early exercise of ESOs and less attention has been paid to the effects of employee attrition during the lifetime of the ESO. We show that under the exercise multiple approach proposed by Hull and White (2004), an employee stock option can be decomposed into a gap call option and two partial time barrier options. Analytic formulae for these are derived using the Method of Images developed in Buchen (2001); Konstandatos (2003, 2008) and European exotic bivariate power options (Kyng, 2011). We propose an actuarial approach to incorporate employee attrition into the valuation method. Using exit probabilities obtained from empirically determined multiple decrement tables or life tables we model stock price independent causes of involuntary exercise or forfeiture of ESOs. This allows us to construct a portfolio of analytically tractable ESOs to obtain a valuation which correctly accounts for employee attrition. This is an alternative to the approaches used by Gerber et al. (2012) and Cvitanic et al. (2008) which attempt to model employee attrition without an empirically determined set of exit probabilities.

Keywords: employee stock options, Method of Images, life tables, partial-time barrier options
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1. Introduction
Employee stock options (ESO) are an integral part of executive and, increasingly, also of non-executive compensation. Due of their extensive use in corporate remuneration packages, these options represent substantial claims against issuing companies and it has been recognized that shareholders’ equity can be negatively impacted (Carpenter, 1998). Yet, no consensus has been reached about how to value these contracts correctly.

ESOs are essentially American-style call options on employer stock. However they are equipped with exotic features, which render them intractable to valuation by standard option pricing formulae. Among their most prominent particularities are:

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Non-Transferability: Grantees are not permitted to sell options. In order to realise a cash benefit, options must be exercised and the underlying shares sold.

Vesting Periods: The employee must survive in employment for a specified period of time, before being able to exercise the option. If the employment is terminated (voluntarily or involuntarily) during this vesting period, the option is forfeited. After the option vests, the employee may exercise at any time until contract maturity.

Automatic Exercise: Death, ill-health retirement, or severe accidents will induce automatic exercise, if the option vests. Other causes of termination of employment (notably if the employee is fired) usually result in forfeiture.

Due to non-transferability and possible constraints when transacting in employer stock, employees have very limited ability to hedge their option positions and will exercise ESOs earlier than the conventional risk-neutral setting would suggest, i.e. they will potentially sacrifice large parts of the (time-)value of their options.

Utility theory has been proposed as an explanatory approach for modelling employee’s early exercise behaviour (Carpenter, 1998; Carpenter et al., 2010). This approach, as explored in Lambert et al. (1991) and more recently in (Hall and Murphy, 2000, 2002), uses a certainty equivalent or utility-based framework to value a single ESO grant. At its core lies the observation that employees tend to be excessively exposed to firm-specific risk through their investments in employer securities, such as stocks and options. Given their inability to appropriately hedge this exposure, risk-averse employees will exercise ESOs early and sell the underlying shares, in order to benefit from diversification effects (Carpenter et al., 2010).

Whilst utility theory can be useful in providing some insight into the motivation behind early exercise by employees, it is unsuitable for ESO valuation from the perspective of the employer. In particular, from the employer’s perspective the justification of the choice of a specific utility function and the estimation of the necessary parameters will be difficult and not empirically based. The utility function of any employee will be essentially unobservable. A more serious objection to the utility approach is the implicit assumption that the ESO is the employee’s only asset, ignoring their complete asset portfolio. In reality employees are likely to have a large portfolio of assets such as cash, shares, real estate, superannuation retirement plan assets and other such financial assets which must also be included when applying utility considerations correctly. Valuing any ESO under this approach would therefore require a joint modelling of the employee’s entire portfolio of assets. Any utility based approach permitting analytically tractable formulae will furthermore require simplifying assumptions which will likely be unrealistic from the utility perspective. Any utility approach to valuation will also result in ESOs with the same structure having different valuations for different employees. Whilst the utility approach may be arguably suitable for ESO valuation from the employee’s perspective, from the employer’s perspective a standard approach to valuation will be preferred.

The standard accounting approach to incorporating employees’ early exercise behaviour is to adjust the life time of the option and then value the ESO using the conventional Black and Scholes (1973) model or some other market-based option pricing model. However, the derivation of the proper reduction in time to maturity proves to be quite difficult. A simple estimation based on average historical times until exercise for the company under consideration will depend on past stock returns and is thus prone to considerable error. During a phase of exceptionally poor stock performance for example, many ESOs might have been out-of-the-money and never exercised. Furthermore, the relationship between option value and time to maturity (theta) is non-linear, which renders the straightforward use of the expected time to maturity problematic.

A more applicable model for the valuation of ESOs was developed in Hull and White (2004), where the exercise behaviour of employees is emulated by assuming that voluntary exercise is triggered as soon as the option is sufficiently in the money, i.e. as soon as the stock price reaches a predefined multiple $M > 1$ of the exercise price $K$. Whereas conventional approaches emphasise numerical valuation, we extend the analysis
to demonstrate that the ESO may be decomposed into a gap call option and two partial-time barrier options, for which we provide closed-form analytic expressions in terms of the univariate and bivariate normal distributions in the conventional Black-Scholes economy. This is done by applying the PDE Method of Images first appearing in Buchen (2001) for standard call and put barrier options, and as extended in Konstandatos (2003, 2008) to barrier options with arbitrary payoffs and with both partial and full monitoring barrier windows. This contrasts with the analysis in Hull and White (2004) who employed a strictly numerical (binomial tree) approach to the valuation, without the decomposition into the above option components. Empirical findings on ESO exercise of Carpenter (1998) suggest an exercise multiple of around 2.75 for top executives. Huddart and Lang (1996) report a mean multiple of 2.22 for a broader range of employees including junior executives.

In order to account for involuntary early exercise and the forfeiture of options, a constant hazard rate process has been suggested in the literature. For example Carpenter (1998) extends the Cox et al. (1979) binomial model for American options by introducing an exogenous constant stopping rate at which the options are either exercised (voluntarily and involuntarily) or forfeited. Similarly, Hull and White (2004) use a constant annual exit rate at which employees are assumed to leave the company thus triggering involuntary exercise or forfeiture. Constant hazard rates however are only a poor proxy for expected employee attrition and do not account for heterogeneity among ESO holders. Moreover, the proposed estimation of such rates from company data is subject to a considerable selection bias.

We suggest an actuarial approach to the automatic exercise feature of ESOs, which allows for a more robust and empirically determined valuation. Using age specific survival functions, which are derived from multiple decrement tables tracking different causes of pre-vesting forfeiture and post-vesting involuntary exercise, we obtain weights for the construction of a portfolio of ESOs. The value of this portfolio correctly reflects employee attrition during the life time of the option and can be seen as a more robust and empirically determined valuation.

Our approach has some similarities with Gerber et al. (2012), where the authors present results on valuation of equity linked death benefits in terms of various plain vanilla and exotic options and other equity linked payoffs where the payoff happens at the time of death of an insured person. As in our analysis, these authors derive results assuming that stock prices follow geometric Brownian motion and that the GBM process is independent of the time of death, which is expressible as some continuous probability distribution or some weighted sum of continuous parametric distributions. These authors make the specific assumption that the time of death may be described by a specific continuous distribution, namely that of an exponentially distributed random variable or that the pdf may be expressed as a weighted sum of exponential RV’s, with weights adding to 1. These authors use the coefficients obtained in the exponential sum approximation of the pdf of time to death for their weighted sum. The authors then obtain the value of various option type contracts which have a maturity date equal to the time of death as a weighted sum of the values obtained via the assumption of GBM stock prices and exponentially distributed times of death. This approach is reminiscent of that of Dufresne (2013), which gives analytic approximations for the distribution of a stochastic life annuity contract, in which the future lifetime distribution is approximated as a combination of exponentials to which certain results relating to integrals of Geometric Brownian Motion are applied. This approach relies upon mathematical results which can be found in Dufresne (2007), where approximating distributions are described on the positive half-line based on Jacobi polynomials. The idea is to yield sequences of combinations of exponentials that converge to the true distribution being approximated, for the survival or involuntary exit of the ESO holder in our context.

Our analysis also has features in common with the approach of Cvitanic et al. (2008). These authors also treat the ESO valuation as a combination of barrier type options, relying however on standard approaches to their analytic valuation in terms of the reflection principle of Brownian motion rather than by the use of the Method of Images. Although the underlying approaches are mathematically equivalent (a comprehensive demonstration of this may be found in Konstandatos (2008)), the approach employed by these authors
result in analytic formulae which are much longer and more difficult to interpret than those developed here. Furthermore, these authors model the ESO involuntary exercise by use of a specific parametric process, namely a Poisson process rather than an empirically determined method. Though these authors specify closed-form analytical expressions which somewhat differ from ours in terms of length and elegance, a more substantial limitation of this approach is that the specification of a Poisson process to model involuntary exercise implicitly assumes a constant hazard rate for exits during the exercise window. The assumption of any parametric process other than a Poisson process also requires the estimation of the relevant parameters from real data. There is no discussion in their paper of how this parameter may be estimated. Our approach differs from this in allowing for a hazard rate that varies with age and gender, as provided in the empirical Life Tables. We also provide much briefer analytic formulae which are more easily understood, and are more readily amenable to implementation without error. We have refrained from following Cvitanic et al. (2008) who added an exponentially decaying barrier level, as we see no purpose to be served in the addition of yet another parameter which would need to be estimated from data. The addition of an exponentially time varying barrier level may be readily accomplished if so desired by the use of the generalisations of the Method of Images to time-varying barrier levels as developed in Buchen and Konstandatos (2009).

Both the Gerber et al. (2012) and Cvitanic et al. (2008) approaches consider the valuation from the perspective of the employer and don’t incorporate utility theory into the valuation. Our paper follows the same approach in that regard.

The formulae we develop here have the following important overall commonalities with these previous works as well as the important differences. In particular we provide a valuation formula for a complex form of barrier option with a fixed vesting date and maturity date and assume that the timing of death or involuntary exercise of the ESO is a random variable independent of the stock price process. Using this assumption we price the ESO as a weighted sum of fixed maturity ESOs which can be valued via our analytic approach, with weights which we obtain from the actuarial Life Tables obtained from the Australian Bureau of Statistics. Needless to say, our approach is readily extended to incorporate Life Tables and Survivorship functions from other National Bureaus, or to incorporate other sources of data about death or involuntary exercise probabilities. The weights we use are obtained from the probability distribution of the random time of death or involuntary exercise. The distribution distribution from the Life Tables is an input into the calculation procedure. The key difference of our approach is that we do not make any assumption about the distribution of the time of death nor about the involuntary exercise of the ESO, but rather rely upon the empirically obtained values represented by the life tables. Furthermore we also do not assume the ESO maturity extends to the whole of the life of the beneficiary. For example, the Gerber et al. (2012) approach implicitly assumes the maturity date of the contract may happen at any future time as the exponential distribution is defined for all positive values, as will be the case in assuming any other continuous distribution other than the Exponential distribution. Also, the assumption of any specific parametric distribution adds a further complication in the valuation process. Namely the required estimation of the distribution’s parameters from empirical data.

The Gerber et al. (2012) approach assumes that the distribution of the time of exercise is either an exponential distribution or alternatively that its PDF can be obtained as a weighted sum of exponential distribution PDFs. In this framework they derive analytic formulae for call options and other options where the payoff is made at the exponentially distributed time of death or a weighted sum of values given by such a formula. Implicitly this assumes the option payoff can be made at any time up to the time of death of the option holder. However for ESOs the maturity date of the ESO is usually before the expected time of death of the ESO holder. The final maturity date of the ESO imposes a discontinuity in the pdf of the timing of involuntary exercise of the ESO. This fact limits the applicability of the Gerber et al. (2012) method to the pricing of ESOs. The option pricing results given in Gerber et al. (2012) are not readily adaptable to price the kind of ESO we consider here.

The approach to pricing life contingent options adopted in our paper may be considered as an alternative to these approaches which is simpler to implement for some types of benefit payments. Our main result is a
readily implementable closed form analytic formula given in terms of Survival Functions and non-standard (exotic) barrier options, and the novel techniques which we use to express the price in terms of first and second order power options, which are simple options which pay some power of the underlying stock price at maturity.

This paper is organised as follows. Section 2 provides an overview of our approach to valuation of the ESO in which the ESO payoff is recognised as a combination of a gap call option and two partial time barrier options, following the methods for partial-time barrier options first described in Konstandatos (2003). This facilitates the application of the method of images as described in Konstandatos (2003, 2008) where the concept of an image solution is exploited to express the value of a barrier option in terms of European type options and their image functions. Using this approach we obtain an analytic formula for the ESO in terms of Bivariate European Binary Power options and their image functions which are also Bivariate European Binary Power options. Section 2 also provides an overview of our approach to barrier option valuation, in which the problem context is set up, and where the concept of an image solution is introduced. We draw from the discussion of The Method of Images as developed and described in Konstandatos (2003, 2008) which we apply to evaluate the underlying exotic option in terms of portfolios of European Binary Power Options and their image functions. Section 2.4 provides an overview of an efficient Linear Algebra approach to the calculation of the survival functions from the Life Tables data, and concludes with the expression for a closed form analytic formula for the ESO value adjusted for early exercise and forfeiture, namely Equation 13, which is the main result of this paper. Section 3 includes a case study of the limiting behaviour of our formula for various exercise multiples, and a discussion of the unadjusted ESO behaviour for the optimal exercise multiple maximising the ESO value. We conclude the paper by providing a table of computed values. We provide several proofs of results in the Appendix.

2. Valuation

Let $S_t$ denote the stock price process and $0 < T_1 < T_2$, where $T_1$ is the ESO vesting date and $T_2$ is the final contract maturity. Furthermore, let $\bar{S} = \max\{S_t : T_1 \leq t \leq T_2\}$ and $\bar{t} = \min\{T_1 \leq t \leq T_2 : S_t = B\}$, where we adopt the convention $t = \infty$ if the stock price never reaches the barrier in $[T_1, T_2]$.

Under the exercise multiple approach, the option will be exercised immediately, if and when $S_t$ crosses the barrier $B = M \cdot K$ from below. Such exercise may however only occur during the exercise window $[T_1, T_2]$.

If we ignore the possibilities of involuntary exercise and option forfeiture for now, the ESO payoff can thus be decomposed into three mutually exclusive and exhaustive scenarios:

1. The option is sufficiently in the money as it vests and the employee will therefore exercise immediately at time $T_1$.
2. The stock price is below the level $B$ at time $T_1$, but reaches the barrier $B$ before option maturity $T_2$.
   Thus, the option will be exercised early at time $\bar{t}$ for a payoff of $B - K = (M - 1)K$.
3. The stock price is below $B$ at $t = T_1$ and never reaches the barrier before contract maturity. The option is then exercised at time $T_2$, provided it is in the money of course.

Each of these scenarios can be captured by a corresponding conditional payoff:

\[
\begin{align*}
P_1(T_1) &= \mathbb{1}(S_{T_1} > B)(S_{T_2} - K) \\
P_2(\bar{t}) &= \mathbb{1}(S_{T_1} < B)\mathbb{1}(\bar{S} > B)(M - 1)K \\
P_3(T_2) &= \mathbb{1}(S_{T_1} < B)\mathbb{1}(\bar{S} < B)(S_{T_2} - K)^+
\end{align*}
\]

where $\mathbb{1}(x)$ denotes the unit step function ($1$ if $x > 0$ and $0$ else) and $(x)^+ = \max\{x, 0\}$ is the positive part of $x$.

The options and techniques presented in the following play an integral role in our approach to ESO valuation and allow us to considerably simplify later notation. In fact, payoffs 2 and 3 can be replicated by a portfolio of dual expiry binary power options using the method of images, both of which are introduced in the next sections.
2.1. The Method of Images for Barrier Options

In this section we follow the treatment in Konstandatos (2003, 2008). In the Black-Scholes framework for a dividend paying stock the asset price $S_t$ is assumed to follow a geometric Brownian motion with drift $r - q$ and volatility $\sigma$, where $r$ is the risk-free interest rate and $q$ the stock’s continuous dividend yield (Merton, 1973). Let us denote $V_0(s,t)$ as the time $t$ value of an option on the stock with current asset price $s$, payoff $f(s)$ and maturity $T$. $V_0(s,t)$ must satisfy the well-known diffusion PDE

$$
\begin{align*}
\mathcal{L} V_0(s,t) &= 0 \\
V_0(s,T) &= f(s) \quad 0 < s, \ t < T
\end{align*}
$$

(1)

where $\mathcal{L}$ is the BS operator with

$$
\mathcal{L} V = \frac{\partial V}{\partial t} - rV + (r - q)s \frac{\partial V}{\partial s} + \frac{1}{2} \sigma^2 s^2 \frac{\partial^2 V}{\partial s^2}
$$

(BS PDE)

A very handy property of the BS-PDE is, that in conjunction with a terminal condition it yields unique solutions (Black and Scholes, 1973).

The system in (1) holds true for all standard, i.e. non-path-dependent, options. However, we want to study a special type of option, whose payoff weakly depends on the realized asset price path over its lifetime, called barrier option. As the name suggests, its payoff at maturity depends on whether or not the realised stock price path has crossed a predefined barrier (in our case $B$). There are four fundamental types of barrier options: the up-and-out (UO), the down-and-out (DO), the up-and-in (UI) and the down-and-in (DI) barrier option.

The knock-out options, will expire worthless if the barrier is reached and pay the standard payoff $f(s)$ at maturity otherwise. Knock-in options on the other hand, initially pay nothing, but will be converted into their underlying standard type and pay $f(s)$ at maturity, if the stock price crosses the barrier. Down-options infer that the barrier is set below the initial asset price; up-options infer the opposite. The payoff $f(s)$ is usually that of a vanilla option, such as $f(s) = (s - K)^+$ for a call or $f(s) = (K - s)^+$ for a put.
With the barrier in place, (1) becomes a boundary value problem, which, for the DO barrier option, takes the form

\[
\begin{align*}
\mathcal{L} V_{DO}(s,t) &= 0 \\
V_{DO}(B,t) &= 0 \quad B < s, \ t < T \\
V_{DO}(s,T) &= f(s)
\end{align*}
\]

(2)

Note that the active domain changed from \(0 < s \) to \(B < s\). In fact, \(V_{DO}(s,t) \equiv 0\) for \(s < B\).

Using the linearity of \(\mathcal{L}\), it is a simple exercise to derive the following set of symmetry relations, which are also known as the in-out parity relations:

\[
\begin{align*}
V_{DO}(s,t) + V_{DI}(s,t) &= V_0(s,t) \quad \text{for } s > B \\
V_{UO}(s,t) + V_{UI}(s,t) &= V_0(s,t) \quad \text{for } s < B
\end{align*}
\]

(3)

Now, the traditional and rather involved approach to valuing barrier options uses the discounted expectations method in conjunction with the reflection principle (Rubinstein and Reiner, 1991). However, by exploiting inherent symmetry properties of the BS PDE and with the concept of image solutions, Buchen (2001) and Konstandatos (2003, 2008) were able to derive barrier option prices in an elegant manner using the PDE approach to option pricing.

Buchen (2001) defines the image of any solution \(V(s,t)\) of the BS PDE with respect to the barrier \(B\) as the function

\[I_B[V](s,t) = \left(\frac{B}{s}\right)^\alpha V\left(\frac{B^2}{s},t\right)\]

with \(\alpha = 2\frac{r-q}{\sigma^2} - 1\).

\(I_B[V]\) is the reflection of the original solution \(V\) about the barrier \(B\). It coincides with \(V\) at \(s = B\) and has active domain \(B \leq s\), if \(V\) has active domain \(B \geq s\). Furthermore \(\mathcal{L} V = 0\) implies \(\mathcal{L} I_B[V] = 0\) and \(I_B[I_B[V]] = V\).

With the concept of images, the solution to (2) can now be expressed as

\[V_{DO}(s,t) = V_B(s,t) - I_B[V_B](s,t) \quad \text{for } s > B\]

(4)

\(V_B(s,t)\) is the price of a so-called high-pass binary option. Its price is the solution of a modified problem, which is related to (1) and given by

\[
\begin{align*}
\mathcal{L} V_B(s,t) &= 0 \\
V_B(s,T) &= f(s)1(s > B) \quad 0 < s, \ t < T
\end{align*}
\]

(5)

System (5) is easier to solve compared to the original barrier option problem. For example, the adjustment \(1(s > B)\) in the option payout has no effect on vanilla European calls with a strike price above \(B\) and yields a common gap call option otherwise.

That the right hand side in equation (4) does indeed coincide with the solution of the down-and-out barrier option price in its active domain \(\{s : B < s\}\) can be verified by comparing the boundary and terminal conditions. A discussion of this and relevant proofs may be found in Konstandatos (2008).

The concept of image solutions also gives rise to a new up-down parity relation between the knock-in and the knock-out barrier option types, first noticed in Buchen (2001) for barrier options with the standard call/put payoffs. This up-down parity relation was shown in Konstandatos (2003, 2008) to apply to barrier options with arbitrary payoffs:

\[I_B[V_{UI}](s,t) = V_{DI}(s,t)\]
Together with the symmetry relations described in equations (3), Konstandatos (2008) demonstrates that all four barrier option types can be expressed in terms of the standard option price \( V_0(s,t) \), the high-pass solution \( V_B(s,t) \) and their respective images, for any barrier payoff \( f(s) \):

\[
\begin{align*}
V_{DO} &= V_B - \mathcal{I}_B[V_B] \\
V_{DI} &= V_0 - (V_B - \mathcal{I}_B[V_B]) \\
V_{UI} &= (V_0 - \mathcal{I}_B[V_0]) - (V_B - \mathcal{I}_B[V_B]) \\
V_{UO} &= \mathcal{I}_B[V_0] + (V_B - \mathcal{I}_B[V_B])
\end{align*}
\] (6)

The formulae in Equations 6 are collectively referred to as the Method of Images for pricing barrier options with arbitrary payoffs for the purposes of this paper. In particular, the interested reader should note that the above solutions to all four barrier option types follow from the properties of the image operator, see Konstandatos (2003, 2008) for details. These works also contain more standard proofs based on Green’s function approaches to the solution of the barrier option pricing PDE problem. Extensions of this approach may be found on Buchen and Konstandatos (2009) for single and double barrier option pricing problems where the barrier levels are exponential and time-varying.

The Method of Images will be particularly useful in the valuation of payoff \( P_2 \) and \( P_3 \). In fact, together they form an up-and-out barrier call option with a rebate \((M - 1)K\), which is paid if and when the underlying vanilla call gets knocked out. The small but non-trivial detail is that the barrier is only partially active, i.e. from time \( T_1 \) to \( T_2 \).

2.2. European Binary Power Options

Binary options form the building blocks of a wide class of exotic options, and it transpires that this is true for the exotic options we encounter here. For example Buchen (2004) expressed prices for numerous dual-expiry European options with compound option features in terms of standardised components. Konstandatos (2003, 2008) extended the analysis to express prices for the whole class of weakly path dependent exotic options in terms of standardised European option instruments. The remarkable fact which should strike the reader is that the expressions for Barrier and Lookback options may be expressed in terms of portfolios of European options. This includes precisely the types of barrier options we encounter in our analysis here.

For the purposes of our analysis, we define a European binary power option (BPO) with power \( n \), strike \( K \), sign \( \xi = \pm \) and maturity \( T \) as the European option with terminal payoff \( S_T^n \mathbb{I}(\xi S_T > \xi K) \). In the standard BS framework its value at time \( t < T \) is given by

\[
P^K(\xi=S_t,t,T;n) = S_t^n e^{\gamma(n)\tau} \mathcal{N}(d^K_{\xi,n})
\] (7)

with

\[
\gamma(n) = \frac{1}{2} \sigma^2 n^2 + \left( r - q - \frac{1}{2} \sigma^2 \right) n - r
\]

and

\[
d^K_{\xi,n} = \xi \left( \frac{1}{\sigma \sqrt{\tau}} \left( \ln \frac{S_t}{K} + \left( r - q - \frac{1}{2} \sigma^2 \right) \tau \right) + n \sigma \sqrt{\tau} \right)
\]

As usual \( \mathcal{N} \) denotes the standard normal distribution function and \( \tau = T - t \) the remaining time to maturity. The sign \( \xi \) indicates the option type: \( \xi = + \) for a call-type and \( \xi = - \) for a put-type BPO. A simple (i.e. non-binary) European power option with payoff \( S_T^n \) can be obtained by a portfolio of BPOs with opposite signs:

\[
P(S_t,t,T;n) = P^K(\xi=S_t,t,T;n) + P^{-K}(S_t,t,T;n)
= S_t^n e^{\gamma(n)\tau}
\]

where the identity \( \mathcal{N}(-d) = 1 - \mathcal{N}(d) \) was used. The image of a BPO with respect to the barrier \( B \) is given by

\[
\mathcal{I}_B[P^K](S_t,t,T;n) = B^{2n+\alpha} P^{-\xi}_{B^{2}/K}(S_t,t,T;-(n + \alpha))
\]
which is a multiple of a related BPO with power $-(n + \alpha)$, strike $B^2/K$ and opposite sign.

Following Konstandatos (2003, 2008) and Buchen (2004), we define a dual-expiry version of binary power options as a binary option expiring at $T_1$ on an underlying binary power option with maturity $T_2$. Its payoff at $T_1$ therefore is $P_{B,T_2}^{\xi_1} (S_{T_1}, T_1, T_2; n) \mathbb{1}(\xi_1 S_{T_1} > \xi_1 K_1)$ and its value at time $t < T_1$ is given by

$$P_{B,T_2}^{\xi_1} (S_t, t, T_1, T_2; n) = S_t^n e^{n \gamma (n) \tau_2} \mathcal{N}_2 (d_{K_1,\tau_1}, d_{K_2,\tau_2}^n; \xi_1 \xi_2 \rho)$$

(8)

where $\tau_1 = T_1 - t$, $\tau_2 = T_2 - t$ and $\rho = \sqrt{\tau_1/\tau_2}$. $\mathcal{N}_2 (x, y; \rho)$ is the bivariate standard normal distribution with correlation coefficient $\rho$ evaluated at point $(x, y)$.

Formulas (7) and (8) are derived in the appendix.

2.3. An Analytical Formula for the ESO

$P_1(T_1)$ represents the payoff of a common gap call option with maturity $T_1$, i.e. a call option whose strike price is above its exercise price. It can be statically replicated fairly easy using a portfolio of two BPOs:

$$P_1(T_1) = \mathbb{1}(S_{T_1} > B) (S_{T_1} - K)$$
$$= S_{T_1} \mathbb{1}(S_{T_1} > B) - K \cdot \mathbb{1}(S_{T_1} > B)$$

(9)

In order to avoid arbitrage, its value at time $t < T_1$ must thus be given by

$$P_1(t) = P_{B,T_2}^+(S_t, t, T_1; 1) - K \cdot P_{B,t}^+(S_t, t, T_1; 0)$$

The valuation of $P_2$ needs considerably more work. We will focus on pricing the payoff $P_{2}^{+}(t) = \mathbb{1}(\bar{S} > B) (M - 1) K$ first and deal with the extra condition $\mathbb{1}(S_{T_1} < B)$ in a second step. $P_{2}^{+}(t)$ is the payoff of an (American) up-and-in digital contract, that pays the fixed amount $(M - 1) K$ if and when the stock price crosses the barrier $B$ from below and whose price $U(s,t)$ is described by

$$\begin{align*}
\mathcal{L} U(s,t) &= 0 \\
U(B,t) &= (M - 1) K \quad s < B, \ t < T_2 \\
U(s,T_2) &= 0
\end{align*}$$

(10)

We will solve System (10) by constructing a BPO portfolio which satisfies the required terminal and boundary conditions. Consider the following multiple of a European power option:

$$V_0(S_t, t) = \frac{(M - 1) K}{B^\beta_1} \cdot P(S_t, t, T_2; \beta_1)$$
$$= (M - 1) K \cdot \left(\frac{T_2}{B}\right)^{\beta_1}$$

where

$$\beta_1 = - \left( r - q - \frac{1}{2} \sigma^2 \right) + \sqrt{\left(r - q - \frac{1}{2} \sigma^2 \right)^2 + 2 r \sigma^2} > 0$$

is the positive root of $\gamma$. By virtue of the choice of the power $\beta_1$, $V_0$ is a time-independent solution of the BSPDE and $V_0(B,t) = (M - 1) K$. The corresponding high-pass binary option is the same multiple of a related BPO with positive sign and strike $B$:

$$V_B(S_t, t) = \frac{(M - 1) K}{B^\beta_1} \cdot P_{B,t}^+(S_t, t, T_2; \beta_1)$$

Using (6) for the up-and-in version of $V_0$ with barrier $B > S_t$, we obtain

$$V_{UI}(S_t, t) = \mathcal{I}_B [V_0] (S_t, t) + V_B(S_t, t) - \mathcal{I}_B [V_B] (S_t, t)$$
$$= \mathcal{I}_B [V_0 - V_B] (S_t, t) + V_B(S_t, t)$$
$$= \frac{(M - 1) K}{B^\beta_1} \cdot \left( \mathcal{I}_B [P_{B,t}^+] (S_t, t, T_2; \beta_1) + P_{B,t}^+(S_t, t, T_2; \beta_1) \right)$$
$$= \frac{(M - 1) K}{B^\beta_1} \cdot \left( B^{\beta_1 - \beta_2} P_{B,t}^+(S_t, t, T_2; \beta_2) + P_{B,t}^+(S_t, t, T_2; \beta_1) \right)$$
where $\beta_2 = -(\beta_1 + \alpha) < 0$ is the negative root of $\gamma$. Clearly $V_{UI}(s,T_2) = 0$ for $s < B$ and $V_{UI}(B,t) = V_0(B,t) = (M - 1)K$ for $t < T_2$. Therefore $V_{UI} = U$ in the domain $\{(s,t) : s < B, t < T_2\}$ as desired.

It immediately follows that the original payoff $P_2$ represents a $T_1$-expiry binary option on an underlying portfolio of BPOs expiring at $T_2$ and can thus be priced using dual-expiry BPOs:

$$P_2(T_1) = \mathbb{1}(S_{T_1} < B) \cdot V_{UI}(S_{T_1}, T_1)$$

$$= \mathbb{1}(S_{T_1} < B) \cdot (M - 1)K \cdot \left( B^{-\beta_2} P_{B}^{-}(S_{T_1}, T_1, T_2; \beta_2) + B^{-\beta_1} P_{B}^{+}(S_{T_1}, T_1, T_2; \beta_1) \right)$$

Assuming the following application of equation (11):

$$P_2(t) = (M - 1)K \cdot \left( B^{-\beta_2} P_{B}^{-}(S_{T_1}, t, T_1, T_2; \beta_2) + B^{-\beta_1} P_{B}^{+}(S_{T_1}, t, T_1, T_2; \beta_1) \right)$$

for $t < T_1$. Following the rationale used in the derivation of $P_2$ and equation (11) in particular, we can think of $P_3$ as being the value of a $T_1$-expiry binary option on a $T_2$-expiry up-and-out barrier call option with barrier $B$ and payoff $f(s) = (s - K)^+$. We can price this with a portfolio of BPOs by using (6) and the decomposition of a gap call option seen in (9). Static replication then yields

$$P_3(t) = \left( P_{B,K}^{-}(S_t, t, T_1, T_2; 1) - K \cdot P_{B,K}^{+}(S_t, t, T_1, T_2; 0) \right) -$$

$$\left( P_{B,B}^{-}(S_t, t, T_1, T_2; 1) - K \cdot P_{B,B}^{+}(S_t, t, T_1, T_2; 0) \right) +$$

$$\left( B^{2+\alpha} P_{K,B}^{-}(S_t, t, T_1, T_2; -(1 + \alpha)) - K \cdot B^{\alpha} P_{B,B}^{-}(S_t, t, T_1, T_2; -\alpha) \right) -$$

$$\left( B^{2+\alpha} P_{K,B}^{+}(S_t, t, T_1, T_2; -(1 + \alpha)) - K \cdot B^{\alpha} P_{B,B}^{+}(S_t, t, T_1, T_2; -\alpha) \right)$$

for $t < T_1$. Because the payoffs $P_1$, $P_2$ and $P_3$ correspond to mutually exclusive and exhaustive scenarios, we obtain the time $t < T_1$ value of the ESO under the condition $S_t < K \cdot M$ by simple summation:

$$ESO_{K,M}^{T_1, T_2}(t, S_t) = P_1(t) + P_2(t) + P_3(t)$$

Note, that above formula does not account for the automatic exercise feature of ESOs. The following section will derive the appropriate adjustment of (12) to allow for this possibility.

2.4. Automatic Exercise and Forfeiture

Certain events over the lifetime of the ESO will result in pre-vesting forfeiture, or trigger post-vesting automatic exercise. Such events include, but are not limited to, death, retirement due to ill-health and severe accidents, all of which would result in an unintentional and unforced termination of employment. As the former events are strictly independent of stock price movements, they readily lend themselves to a life table approach similar to the modelling of defined benefit pension schemes.

Life tables and their extension, multiple decrement tables, are a key concept in insurance and pension fund mathematics. Given a population of $l_0$ individuals all aged 0, life tables provide the number of individuals $l_x$ out of the original $l_0$, who survive up to a specific (integer) age $x \in \{0, 1, \ldots, X\}$. The exact starting number $l_0$ is arbitrary and may not matter much for subsequent calculations.

The term survival is used quite loosely and its meaning depends on the application at hand. For a pension fund with compulsory retirement at age 65 for example, surviving up to age 65 means being a paying member of the fund at age 65 and retiring after age $X = 65$ at the latest. In the context of ESO valuation, survival refers to the grantee surviving in employment.

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Whereas simple life tables track only one attrition factor, in multiple decrement tables population membership can be terminated by two or more factors. Tables for the common attrition factors under consideration in this paper can be easily obtained from the trusted national statistical bureau of your choice. In the following we assume a multiple decrement table for causes of pre-vesting forfeiture and post-vesting automatic exercise (namely death, ill-health retirement and severe accidents) to be available.

Given such data \( l_x \) it is an easy exercise to derive the age specific survival rates \( p_x \), which represent the probability of surviving one more year being aged \( x \):

\[
p_x = \frac{l_{x+1}}{l_x}
\]

where we adopt the convention \( p_x = 0 \) for \( x \geq X \). Similarly, the probability of remaining in the population for further \( n \) years at age \( x \) can be calculated as

\[
np_x = \frac{l_{x+n}}{l_x} = p_x p_{x+1} \cdots p_{x+n-1}
\]

The so called survival functions \( np_x \) may be obtained ensemble for each age \( x \) using an elegant linear algebra approach. Let the matrix \( P \) have the one year survival probabilities \( p_x \) down the diagonal below the main diagonal and be 0 everywhere else:

\[
P = \begin{pmatrix}
p_{x,y} & 0 & \cdots & 0 \\
0 & p_{x,y} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & p_{x,y}
\end{pmatrix}
\]

for \( 0 \leq x, y \leq X \) and the relation \( P^X = 0 \) was used. The lower triangular matrix \( S = (I - P)^{-1} = (I + P + P^2 + \cdots + P^{X-1}) \cdot (I - P) = I \)

where the identity matrix

\[
I = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}
\]

and the relation \( P^X = 0 \) was used. The lower triangular matrix \( S = (I - P)^{-1} = (I + P + P^2 + \cdots + P^{X-1}) \)

now contains the survival functions for all ages \( 0 \leq y \leq X \) as columns:

\[
S = \begin{pmatrix}
l_x/l_y = (x-y)p_y \\
\end{pmatrix}
\]

That is, given an age \( y \), the probability to survive up to age \( x \geq y \) is \( S_{x,y} \). A MATLAB implementation of the procedure described above may be found in the appendix together with the ESO valuation functions.

The survival function \( np_x \) for the age \( \bar{x} \) of the ESO grantee now gives the probability for the contract lasting at least \( n \) years. The probability of the employee exiting the population (i.e. terminating employment due to the attrition factors under consideration) in the interval \( [\bar{x}+n, \bar{x}+n+1] \) is easily obtained by differencing the survival function: \( np_x - (n+1)p_x \). Under standard actuarial assumptions such exit occurs approximately in the middle of the interval, i.e. at time \( \bar{x} + n + \frac{1}{2} \).

Note, that for \( n, s.t. \ (t + n) < T_1 \), the ESO will be forfeited and thus expire worthless. In the case of \( T_1 \leq (t + n) \leq T_2 \), it will be exercised involuntarily, which corresponds to the option having a reduced time
to maturity of \( t + n + \frac{1}{2} \).

Altogether, the time \( t < T_1 \) value of the ESO adjusted for pre-vesting forfeiture and post-vesting involuntary exercise is given by the following probability weighted portfolio of BPOs:

\[
\text{adj} \text{ESO}^{K,M,t}(t,S_t) = \sum_{n=\lfloor T_1 - t \rfloor}^{\lfloor T_2 - t \rfloor - 1} \left(n p_x - (n+1)p_x\right) \cdot \text{ESO}^{K,M}_{T_1,(t+n+\frac{1}{2})}(t,S_t) + (\lfloor T_2 - t \rfloor)p_x \cdot \text{ESO}^{K,M}_{T_1,(t+\lfloor T_2 - t \rfloor)}(t,S_t)
\]  

(13)

where \( \lfloor \cdot \rfloor \) and \( \lceil \cdot \rceil \) are the floor and ceiling function respectively. Formula (13) constitutes the main result of this paper and there are some straightforward ways of extending it:

1. Using \( n p_x \) with coarse (yearly) increments \( n \), might yield imprecise results for short vesting and exercise periods. By interpolating the survival functions to obtain probabilities at quarterly or even monthly increments, the accuracy of formula (13) can be improved. The MATLAB implementation in the appendix derives higher frequency survival functions using common log-linear interpolation.

2. Including additional attrition factors, such as forced termination of employment (which results in forfeiture at any time) or personal financial liquidity crises (which result in post-vesting involuntary exercise), will improve robustness and accuracy. The inclusion of further factors is really only restricted by the availability of reliable data.

3. Case Study

The analytical sensitivity of the ESO value towards the exercise multiple \( M \) and the employee age \( \bar{x} \) can be inferred from formula (12) and (13) respectively, but the corresponding calculations are very tedious. We thus use the last paragraphs of this paper to provide some numerical results on ESO valuation. Throughout this section we consider the common type of ESO, which is granted \( (t = 0) \) at the money with an initial stock price \( S = \$1 \), is equipped with a vesting period of \( T_1 = 3 \) years and matures in \( T_2 = 10 \) years. Moreover, we assume a continuous yearly dividend yield of \( q = 2\% \), a risk-free interest rate \( r = 3\% \) and a stock volatility of \( \sigma = 30\% \). Life table data is obtained from the Australian Bureau of Statistics.

Clearly, for large \( M \) the chance of the employee exercising the option early diminishes. The values \( P_1 \) of the gap call and \( P_2 \) of the up-and-in barrier power option thus approach zero, whereas the up-and-out barrier call option represented by \( P_3 \) starts to behave like a common European call. Therefore, we should expect the (unadjusted) ESO value to approach the price of a \( T_2 \)-maturity European call option, whose value \( \$0.324836 \) is readily obtained using the Black-Scholes formula for dividend paying assets (Merton, 1973).

More interesting are the results for lower values of \( M \) as depicted in Figure 3. It is well known that for assets paying a continuous dividend yield, it is optimal to exercise an American call option prior to maturity and at sufficiently high asset value. This fact manifests itself in the ESO values for exercise multiples \( M \) of 1.9 or more, which exceed the Black-Scholes value. The ESO value for the above choice of parameters reaches its maximum of \( \$0.339663 \) for an exercise multiple of around 2.85, which is close to the multiple of 2.75 for top executives reported in Carpenter (1998) for the chosen parameters. The value for the unadjusted ESO price is bounded above by the American value for a call option, and it clearly approaches the American price at the optimal exercise multiple at which early exercise is optimal. The American upper bound may be readily approximated using a slight alteration of the Cox et al. (1979) binomial tree model for American call options, which allows early exercise only during the time window \( [T_1,T_2] \). We obtain a value of \( \$0.340455 \).

The ESO value is comparably sensitive to a change in the exercise multiple, if \( M \) is below 2. This effect is robust under reasonable changes in the the other ESO parameters, but empirical studies place \( M \) well
above this threshold even under consideration of measurement error, suggesting that the exercise multiple approach is indeed applicable.

The impact of employee age on the value of an ESO can be inferred from an inspection of the survival functions (Figure 2). Given two ages $\bar{x} < \bar{y}$, we can expect $nP_x - (n+1)p_y < nP_y - (n+1)p_y$ to hold for $n > 1$. In other words: the older the grantee, the higher is his or her probability of exit from the population in the coming years (in general). Thus, the ESO value will decrease at an accelerating rate with increasing employee age (see table 1).

<table>
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<th>2.2</th>
<th>2.3</th>
<th>2.4</th>
<th>2.5</th>
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Table 1: ESO Values (in A$) for different exercise multiples and employee ages.

4. Conclusions

Based on the exercise multiple approach of Hull and White (2004), which allows a simple parametrization of the early exercise behaviour of risk-averse employees, we decomposed an employee stock option into three component exotic options: a European gap-call option, an up-and-in partial-time barrier power option and an up-and-out partial-time barrier call option which we priced using the novel PDE methods described in (Buchen, 2001; Konstandatos, 2003). Working from the perspective of the employer, and considering
a conventional Black Scholes economy, we expressed our prices as portfolios of (single and dual-expiry) European options which we refer to as first and second order Binary Power Options, for which we obtained analytical pricing formulae, and which are readily implementable European option standardised instruments. By considering common causes of employee attrition which are independent from stock price movements, the analytical value of an ESO is appropriately adjusted using standard actuarial assumptions regarding survival probabilities derived from multiple decrement tables obtained from Life Tables from the Australian Bureau of Statistics. Our work has some important similarities with previous works by treating ESO valuation in terms of exotic option pricing theory. Whereas previous works employ parametric distributions in accounting for the attrition probabilities whilst employing standard techniques for the exotic option pricing, our approach differs by allowing for non-constant hazard rates implicit in the empirically determined factors for attrition under consideration, which may vary with age, sex or other factors not considered here, and for which we presuppose no a-priori specification of a distributional form for the attrition probabilities. The approach to pricing life contingent options adopted in our paper may be considered as an alternative and more robust empirically determined valuation, which is also simpler to implement. This leads to simpler expressions with more transparent software implementations which are less prone to coding error. Our main result is a readily implementable closed form analytic formula given in terms of Survival Functions and non-standard (exotic) barrier options, with prices expressed solely in terms of standardised European contracts and their images. We also avoid the complications inherent in estimating the distributional parameters from real data for the attrition probabilities. Our approach may be readily extended to include a greater variety of attrition factors than presented in this paper.

Figure 3: The relationship between (unadjusted) ESO value and exercise multiple. The vertical axis shows the ESO value and the horizontal axis the the multiple \( M \) ranging from 1 to 10. The dashed red line indicates the value of an American call option with early exercise in \([T_1, T_2]\). The solid red line indicates the BS-value of a \(T_2\)-maturity European call option. The blue line depicts the ESO value for the different exercise multiples.
Appendix

The purpose of this appendix is to derive the pricing formulas for (dual expiry) binary power options and to enable the interested reader to reproduce the results in section 3 using MATLAB implementations of the relevant formulae.

The proofs of (7) and (8) have been adapted from Buchen (2004); Konstandatos (2008) and will make use of the handy Gaussian shift theorem (Konstandatos (2008)): let $Z$ be a $k$-variate standard Gaussian random variable with (symmetric and positive definite) covariance/correlation matrix $R$, $c$ a constant vector and $F(Z)$ any measurable function, then

$$E\left[e^{c^\top Z}F(Z)\right] = e^{c^\top Rc}E\left[F(Z+Rc)\right]$$

The proof is straightforward:

$$E\left[F(Z+Rc)\right] = \int_{\mathbb{R}^k} F(Z+Rc) \phi(Z) \, dZ$$
$$= \int_{\mathbb{R}^k} F(V) \phi(V-Rc) \, dV$$
$$= \int_{\mathbb{R}^k} e^{c^\top V}e^{-\frac{1}{2}c^\top Rc} \phi(V) \, dV$$
$$= e^{-\frac{1}{2}c^\top Rc} \int_{\mathbb{R}^k} F(Z) e^{c^\top Z} \phi(Z) \, dZ$$
$$= e^{-\frac{1}{2}c^\top Rc} E\left[e^{c^\top Z}F(Z)\right]$$

where the substitution $V = Z + Rc$, the density function $\phi(Z) = (2\pi)^{-\frac{k}{2}}|R|^{-\frac{1}{2}}e^{-\frac{1}{2}Z^\top RZ}$ of the multivariate standard normal distribution and some simple algebra were used.

Appendix A. Binary power option value

In the BS framework for dividend paying stocks (Black and Scholes, 1973; Merton, 1973), the asset price at time $T > t$ conditional on $S_t$ is given by the log-normal random variable

$$S_T \overset{d}{=} S_te^{(r-q-\frac{1}{2}\sigma^2)\tau + \sigma W_t},$$

where $W_t$ is the risk-neutral Wiener process with variance $\tau$. We thus have

$$S_T^n \overset{d}{=} S_t^n e^{(r-q-\frac{1}{2}\sigma^2)\tau + n\sigma W_t}.$$ 

Let $Z$ be a standard Gaussian random variable with $Z = \frac{1}{\sqrt{\tau}}W_t$. Then

$$\xi S_T > \xi K$$
$$\xi S_t e^{(r-q-\frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}Z} > \xi K$$
$$\xi \left( (r-q-\frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}Z \right) > \xi \ln \left( \frac{K}{\xi} \right)$$

$$-\xi Z < \frac{\xi}{d_{K,\tau}}.$$ 

By the fundamental theorem of asset pricing we obtain

$$P_K^\tau(S_t, t; T; n) = e^{-r\tau}E_t\left[S_T^n\mathbb{1}(\xi S_T > \xi K)\right]$$
$$= S_t^n e^{(r-q-\frac{1}{2}\sigma^2)\tau - r\tau}E_t\left[e^{\alpha\sigma\sqrt{\tau}Z}\mathbb{1}(\xi Z < d_{K,\tau})\right]$$
$$= S_t^n e^{(\gamma(n)\tau)\mathbb{1}\{\xi Z < K\}}$$
$$= S_t^n e^{(\gamma(n)\tau)\mathbb{1}\{\xi Z < d_{K,\tau}\}}$$

where $E_t[.]$ is the expectation taken under the risk-neutral measure and conditional on $S_t$. The third equation above follows from the Gaussian shift theorem with $c = n\sigma\sqrt{\tau}$. 

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AppendixB. Dual-expiry binary power option value

The proof of formula (8) is along the lines of the former one. We have

\[ S_T^n \overset{d}{=} S_t^n e^{n(r-q-\frac{1}{2} \sigma^2)\tau_i + n \sigma W_{\tau_i}} \]

where \( W_{\tau_1} \) and \( W_{\tau_2} \) have covariance \( \sigma_{1,2} = \min\{\tau_1, \tau_2\} = \tau_1 \) and correlation \( \rho = \sqrt{\tau_1/\tau_2} \). Let \( Z_i \) be such that

\[ Z_i = \frac{1}{\sqrt{\tau_i}} W_{\tau_i}, \]

then \( \text{corr}(Z_1, Z_2) = \rho \) and \( \xi_i S_{T_i} > \xi_i K_i \) is equivalent to \( -\xi_i Z_i < d_{K_i,\tau_i}^{(0)} \). The claimed pricing relation now readily follows from the fundamental theorem of asset pricing, the Gaussian shift theorem with \( Z = (Z_1, Z_2)^T \) and \( c = (0, n \sigma \sqrt{\tau_2})^T \), and the observation that \( \text{corr}(\xi_1 Z_1, -\xi_2 Z_2) = \xi_1 \xi_2 \rho \).

AppendixC. Matlab Implementation

This section provides MATLAB implementations of formulas (12) and (13). Please note that the code given below does not sanitize input parameters. Proper and responsible use of the functions is left to the discretion of the reader.

\textit{ESO Price} implements the analytical ESO pricing formula in (12). It makes use of the normal cumulative distribution function \texttt{normcdf} and the multivariate normal cumulative distribution function \texttt{mvncdf}, both of which are part of the statistics toolbox. We refer the reader to the work of Genz (1992), if this toolbox can not be accessed\(^1\).

\textit{ESO Price Adjusted} implements formula (13). It employs log-linear interpolation to increase the frequency of the survival functions \( n_p \) up to a specified frequency.

References


\(^1\)Corresponding MATLAB code is provided at \texttt{www.math.wsu.edu/faculty/genz/software/matlab/qsimvnv.m}