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Abstract

We consider the problem of hedging a European-type option in a market where asset prices have jump-diffusion dynamics. It is known that markets with jumps are incomplete in the context of Harrison and Pliska (1981) and that there are several risk-neutral measures one can use to price and hedge options (Cont and Tankov, 2004; Miyahara, 2012). As in Jensen (1999) and León et al. (2002), we approximate such a market by discretizing the jumps in an averaged sense, and complete it by including traded options in the model and hedge portfolio as utilized in Cont et al. (2007) and He et al. (2006). Under suitable conditions, we get a unique risk-neutral measure, which is used to determine the option price partial differential equation, along with the asset positions that will replicate the option payoff. This procedure is then implemented on a particular set of stock and option prices, and its performance is compared with the minimal variance and delta hedging strategies.

Key words: Incomplete markets, Jump-diffusion, Hedge portfolios, Compound Poisson processes, Integro-partial differential equation.

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1 Introduction

The presence of relatively large movements, called jumps, in stock prices (See Ball and Torous, 1985, for example) has led to the use of jump-diffusion models to better capture asset returns. Merton (1976) originally introduced lognormally-distributed Poisson jump components alongside the typical time and Brownian motion factors used in the classical Black and Scholes (1973) model. Given this shift in dynamics, updated option pricing and hedging results involving both Brownian motion and jumps driven by Poisson processes are popular in the literature and are necessary if one is to use jump-diffusion as the underlying assumption in making trading decisions.

Markets with jump components are known to be incomplete in the Harrison and Pliska (1981) sense. As such, there are many equivalent martingale measures which one can use to price options. Naik and Lee (1990) and Ahn (1992) provide option pricing formulas in jump-diffusion settings through utility functions. Also working with utility, Kou (2002) gives the option price when jumps follow an asymmetric double exponential distribution. An alternative approach is to find the best equivalent martingale measure in the sense of information theory, resulting in the Minimal Entropy Martingale Measure (MEMM), which is examined by Frittelli (2000) in the context of incomplete markets, and by Miyahara (1999) and Fujiwara and Miyahara (2003) for Lévy processes, of which models involving Brownian motion and Poisson processes are included.

The impossibility of perfectly replicating an option's payoff using a bond and the underlying asset (See Naik and Lee, 1990, for example) has led to various hedging techniques for incomplete markets. Hedging based on utility function arguments have been studied in Becherer (2004) and Ceci and Gerardi (2011) for assets with jumps. Superhedging, which solves for a self-financing strategy guaranteeing at least the same amount as the payoff at expiry, is examined in Bellamy and Jeanblanc (2000) for jump-diffusion processes. Another approach is finding the trading strategy which minimizes the risk based on quadratic criteria (See Föllmer and Sondermann, 1986; Föllmer and Schweizer, 1991, for example), such as the local risk-minimizing hedge (Schweizer, 1991) or the minimal variance hedge (Cont et al., 2007).

Although traditionally one tries to find the best way to equal an option's payoff by trading only in its underlying asset, the inclusion of options in the hedge portfolio has been suggested by Bates (1988) and also in Andersen and Andreasen (2000). Hedging using the underlying stock and one extra option has been studied in Cont et al. (2007) under a minimal variance criterion, while He et al. (2006) utilize several options to obtain a dynamic hedge that minimizes the jump risk.

This paper aims to address the pricing and hedging issues by utilizing an approximate complete market. Introducing extra assets such as term-structure related securities in order to complete the market has been studied in Jarrow and Madan (1995) and Björk et al. (1997), and this approach is modified by Jensen (1999) to approximately price options under jump-diffusion, while the hedge portfolio is derived in León et al. (2002) through Malliavin calculus. These previous approaches are related to ours in that by discretely estimating the jump sizes of a stock, the inclusion of additional assets such as market-traded options on the same underlying stock allows us to approximately account for the uncertainty due to the jumps. Under certain conditions, we get a unique risk-neutral measure with which we can price the option. This risk-neutral measure, combined with the additional assets, allows the construction of the hedge portfolio that replicates the option payoff in our approximate market. Our hedging result is also related to that in Elliott and Kopp (1990), who derive the hedge portfolio if an asset can only take jumps on a finite set of values.

This work is motivated by ideas in Cheang and Chiarella (2012), who price a European call option using the Esscher transform and examine the construction of a hedge portfolio with the jumps averaged out. In this paper, averaging the jumps is one of the choices by which we could discretize the jump size distribution, and we employ a classical hedge portfolio construction argument to replicate the option payoff.

The paper develops as follows. Section 2 describes the procedure for approximate market completion and derives the conditions for an almost surely unique risk-neutral measure. Section 3 details the construction of the hedge portfolio along with a formula for the hedge positions on the assets. In Section 4, we apply a simple approximation to a case study of Bank of America stock and option prices, and compare the hedging performance with the minimal variance hedge and a traditional delta hedge. Section 5 concludes and examines extensions to the procedure.

2 Approximate Market Completion

Let $S(t)$ be the price of a financial asset with return dynamics given by

$$\frac{dS(t)}{S(t-)} = (\mu(t) - \lambda\kappa)\, dt + \sigma(t)\, dW(t) + \left[e^{J} - 1\right] dN(t), \tag{1}$$

where $\mu(t)$ is the instantaneous expected return per unit time and $\sigma(t)$ is the instantaneous volatility per unit time. We assume that $\mu(t)$ and $\sigma(t)$ are deterministic. The stochastic process $W(t)$ is standard Brownian motion under the market measure $\mathbb{P}$ and the process $N(t)$ is a Poisson process, independent of $W(t)$ and the jump factor J, with arrival intensity λ per unit time under the measure $\mathbb{P}$. We further assume that λ is constant.

The quantity $J \in \mathbb{R}$ is the jump factor that affects the size of the jump in the sense that the jump size is given by $\left[e^{J} - 1\right] S(t-)$. We call $e^{J} - 1 \in (-1, \infty)$ the proportional jump size, and define the expected proportional jump size to be

$$\kappa \equiv \mathbf{E}_{\mathbb{P}} \left[e^{J} - 1\right].$$

The solution to (1) involves the total number of jumps up to time t, given by $N(t)$, and the corresponding jump factors. Let J_m be the mth jump factor affecting the size of the mth jump. We assume that the jump factors J_m are independent for each m and that each has the same distribution as J.

Theorem 2.1. *The stochastic differential equation (1) has a solution of the form*

$$S(t) = S(0) \exp\left[\int_0^t \left(\mu(s) - \lambda\kappa - \frac{\sigma^2(s)}{2}\right) ds + \int_0^t \sigma(s)\, dW(s) + \sum_{m=1}^{N(t)} J_m\right]. \tag{2}$$

Proof. See Runggaldier (2003) or Theorem 11.7.3 of Shreve (2004). ■

We work on a filtered probability measure space given by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, where $\{\mathcal{F}_t\}$ is the natural filtration generated by the Brownian motion $W(t)$ and the compound Poisson process $\sum_{m=1}^{N(t)} J_m$. For simplicity, we assume that stocks pay no dividends, and that $\mu(t)$ and $\sigma(t)$ are adapted to the filtration $\{\mathcal{F}_t\}$. We also assume that $\mu(t)$ and $\sigma(t)$ satisfy integrability conditions in (2).

We seek the hedge portfolio of a European-type option on $S(t)$. If the jump factor J is discrete, then we can completely hedge the option using a finite number of assets driven by the same simple Poisson processes that constitute $\sum_{m=1}^{N(t)} J_m$. We are assuming, however, that J could have a continuous distribution. If so, it would take an infinite number of assets to completely hedge the option (See He et al., 2006, for example). In this paper, we derive the hedge portfolio for the case when J has a continuous distribution by using approximate market completion as introduced by Jensen (1999).

Let $N_k(t)$, $k = 1, \ldots, n$, be independent Poisson processes, each with intensity λ_k, satisfying

$$N(t) = \sum_{k=1}^{n} N_k(t) \quad \text{and} \quad \lambda = \sum_{k=1}^{n} \lambda_k.$$

For a partition $\mathcal{A}_n \equiv \{A_1, \ldots, A_n\}$ over the values of the jump factor J, choose $d_k \in A_k$ for $k = 1, \ldots, n$.

There are several ways to choose d_k. A natural choice is to set d_k to be the average over each partition, that is,

$$d_k = \frac{\mathbf{E}_{\mathbb{P}}\left[J\,\mathbb{1}_{\{J \in A_k\}}\right]}{\mathbb{P}(A_k)}, \quad k = 1, \ldots, n, \tag{3}$$

where the notation $\mathbb{1}_B$ is used for the indicator function over a set B. An alternative is to set d_k to be the largest magnitude value over each partition. We could also view this partitioning as a discretization of the continuous density for J. So, d_k can also be chosen based on methods for discretely approximating probability distributions (See Miller and Rice, 1983; Hammond and Bickel, 2013).

Now set

$$a_0(t) \equiv \mu(t) \quad \text{and} \quad b_0(t) \equiv \sigma(t).$$

We then have the approximation to (1) as

$$\frac{dS_{0,n}(t)}{S_{0,n}(t-)} = a_0(t)\,dt + b_0(t)\,dW(t) + \sum_{k=1}^{n} \left[e^{\,d_k} - 1\right] (dN_k(t) - \lambda_k\,dt)\,. \tag{4}$$

Now we introduce n extra assets $P_i(t)$, $i = 1, \ldots, n$, to complete this market. We propose that the extra assets be options on $S(t)$ so as to be driven by the same Brownian motion and Poisson process. For the ith extra asset, we let $a_i(t) \equiv a_i(t, S(t))$ be its instantaneous expected return per unit time and let $b_i(t) \equiv b_i(t, S(t))$ be its instantaneous volatility per unit time. Suppose these extra assets have return dynamics given by

$$\frac{dP_i(t)}{P_i(t-)} = (a_i(t) - \lambda\,\xi_i(t))\,dt + b_i(t)\,dW(t) + \left[e^{\,J^{(i)}(t)} - 1\right] dN(t), \tag{5}$$

where $J^{(i)}(t) \equiv J^{(i)}(t, S(t-))$ is the jump factor of $P_i(t)$. We assume that $J^{(i)}(t)$ is predictable and we define

$$\xi_i(t) \equiv \xi_i(t, S(t-)) = \mathbf{E}_{\mathbb{P}}\left[e^{\,J^{(i)}(t)} - 1\right], \quad i = 1, \ldots, n.$$

We remark that if $P_i(t)$ are European-type options on $S(t)$, then the representation (5) follows from Itô's formula (Cont et al., 2007).

Now for $i = 1, \ldots, n$, take a suitable partition $\mathcal{A}_n^{(i)}(t, S(t-)) \equiv \left\{A_1^{(i)}, \ldots, A_n^{(i)}\right\}$ over the values of the jump factor $J^{(i)}(t)$ such that

$$\mathbb{P}(A_k) = \mathbb{P}\left(A_k^{(i)}\right) = \frac{\lambda_k}{\lambda}\,, \quad k = 1, \ldots, n. \tag{6}$$

The above condition (6) is important for the matching of the Poisson processes in our approximate market. Using a procedure similar to the construction of $S_{0,n}(t)$, we approximate $P_i(t)$ by $S_{i,n}(t)$ for $i = 1, \ldots, n$, respectively, with dynamics given by

$$\frac{dS_{i,n}(t)}{S_{i,n}(t-)} = a_i(t)\,dt + b_i(t)\,dW(t) + \sum_{k=1}^{n} \left[e^{\,c_{i,k}(t)} - 1\right] (dN_k(t) - \lambda_k\,dt)\,, \tag{7}$$

where $c_{i,k}(t) \equiv c_{i,k}(t, S(t-)) \in A_k^{(i)}$ for $k = 1, \ldots, n$. Similar to d_k, choosing $c_{i,k}(t)$ can be done by the average over each partition, that is,

$$c_{i,k}(t) = \frac{\mathbf{E}_{\mathbb{P}}\left[J^{(i)}(t)\ \mathbb{1}_{\left\{J^{(i)}(t) \in A_k^{(i)}\right\}}\right]}{\mathbb{P}\left(A_k^{(i)}\right)}, \quad k = 1, \ldots, n,$$

by taking the largest magnitude values of each interval, or by existing methods of discrete approximations to continuous distributions (Miller and Rice, 1983; Hammond and Bickel, 2013).

Jensen (1999) calls the market containing $S_{i,n}(t)$, $i = 0, 1, \ldots, n$, to be the nth market. We omit the subscript n in (4) and (7) for ease in the succeeding computations, but include it when necessary.

We introduce additional notation and discuss the probability structure of the estimated jump factors. Let

$$y_{i,k}(t) \equiv y_{i,k}(t, S(t-)) = \begin{cases} e^{\,d_k} - 1 & \text{if } i = 0 \\ \\ e^{\,c_{i,k}(t)} - 1 & \text{if } i = 1, \ldots, n \end{cases}$$

for $k = 1, \ldots, n$ and define the random variable

$$Y_i(t) \equiv Y_i(t, S(t-)) \in \{y_{i,1}(t), \ldots, y_{i,n}(t)\}, \quad i = 0, 1, \ldots, n.$$

We then set probabilities

$$\mathbb{P}(y_{i,k}(t)) \equiv \begin{cases} \mathbb{P}(d_k) = \mathbb{P}(A_k) = \dfrac{\lambda_k}{\lambda} & \text{if } i = 0 \\ \\ \mathbb{P}(c_{i,k}(t)) = \mathbb{P}\left(A_k^{(i)}\right) = \dfrac{\lambda_k}{\lambda} & \text{if } i = 1, \ldots, n \end{cases}$$

for $k = 1, \ldots, n$. We also define

$$\beta_i(t) \equiv \beta_i(t, S(t-)) = \mathbf{E}_{\mathbb{P}}[Y_i(t)] = \frac{1}{\lambda} \sum_{k=1}^{n} \lambda_k\, y_{i,k}(t), \quad i = 0, 1, \ldots, n.$$

Although the jump sizes of the extra assets $S_1(t), \ldots, S_n(t)$, which are options on $S(t)$, may be different from those of the approximate underlying $S_0(t)$, it appears natural to assume that the extra asset values jump at the same times, and hence with the same approximate probabilities, as $S_0(t)$. Therefore, the nth market described by (4) and (7) can be written as

$$\begin{aligned} \frac{dS_i(t)}{S_i(t-)} &= a_i(t)\,dt + b_i(t)\,dW(t) + \sum_{k=1}^{n} y_{i,k}(t)\,(dN_k(t) - \lambda_k\,dt) \\ &= (a_i(t) - \lambda\,\beta_i(t))\,dt + b_i(t)\,dW(t) + \sum_{k=1}^{n} y_{i,k}(t)\,dN_k(t), \qquad (8) \\ &\qquad\qquad i = 0, 1, \ldots, n. \end{aligned}$$

Our nth market approximation is motivated by the fact that our approximate primary asset converges in distribution to the true dynamics, that is, $S_0(t) \xrightarrow{d} S(t)$ under $\mathbb{P}$. The same statement can be made for the extra assets in that $S_i(t) \xrightarrow{d} P_i(t)$ for $i = 1, \ldots, n$. Before we prove the convergence of the underlying asset, we state the following lemma.

Lemma 2.1. *Let $J \equiv J(\omega)$ be a random variable such that $|J| \leq U$ for all $\omega \in \Omega$ and for some $U > 0$. Partition J into $\mathcal{A} \equiv \mathcal{A}_n = \{A_1, \ldots, A_n\}$ represented by*

$$A_1 = (-\infty, f_1]\,,\ A_2 = (f_1, f_2]\,,\ \ldots\ ,\ A_{n-1} = (f_{n-2}, f_{n-1}]\,,\ A_n = (f_{n-1}, +\infty)$$

where $f_{l-1} < f_l$ for $l = 2, \ldots, n-1$. Choose $d_k \in A_k$ for $k = 1, \ldots, n$ and define the random variable

$$D(n) \equiv D(n)(\omega) \in \{d_1, \ldots, d_n\}$$

with $\mathbb{P}(d_k) = \mathbb{P}(A_k)$ for $k = 1, \ldots, n$. Let

$$a(n) \equiv f_1, \quad b(n) \equiv f_{n-1}, \quad \text{and} \quad |A_l| \equiv f_l - f_{l-1} < \epsilon(n) \text{ for } l = 2, \ldots, n-1.$$

If

$$a(n) \to -\infty, \quad b(n) \to +\infty, \quad \text{and} \quad \epsilon(n) \to 0 \quad \text{as } n \to \infty, \qquad (9)$$

then for all $\omega \in \Omega$,

$$D(n) \to J \quad \text{as } n \to \infty.$$

Proof. Given $\varepsilon > 0$. There is $M_1 > 0$ such that

$$a(n) < -U \quad \text{for all } n \geq M_1.$$

Also, there is $M_2 > 0$ such that

$$b(n) > U \quad \text{for all } n \geq M_2.$$

Finally, there is $M_3 > 0$ such that

$$\epsilon(n) < \varepsilon \quad \text{for all } n \geq M_3.$$

Now choose $M = \max\{M_1, M_2, M_3\}$. Then for $n \geq M$, each realization of $J(\omega)$ will be included in one of the A_l for $l = 2, \ldots, n-1$, and so

$$|D(n)(\omega) - J(\omega)| < \epsilon(n) < \varepsilon \quad \text{for all } \omega \in \Omega.$$

This completes the proof. ■

Theorem 2.2. *Define the compound Poisson process*

$$Q(t) \equiv \sum_{m=1}^{N(t)} J_m,$$

where J_m is the value of J at the mth jump time of $N(t)$. Under the same assumptions as Lemma 2.1, for a given n, let $D \equiv D(n)$ and define

$$Q_{\mathcal{A}}(t) \equiv \sum_{m=1}^{N(t)} D_m,$$

where D_m is the value of D at the mth jump time of $N(t)$. Then

$$Q_{\mathcal{A}}(t) \xrightarrow{d} Q(t) \quad \text{as } n \to \infty.$$

Proof. We recall that the characteristic function of $Q(t)$ is given by

$$\varphi_Q(u) = \mathbf{E}_{\mathbb{P}}\left[e^{iuQ(t)}\right] = \exp\{\lambda t\,(\varphi_J(u) - 1)\},$$

where $\varphi_J(u) = \mathbf{E}_{\mathbb{P}}\left[e^{iuJ}\right]$. Similarly, the characteristic function of $Q_{\mathcal{A}}(t)$ is given by

$$\varphi_{Q_{\mathcal{A}}}(u) = \mathbf{E}_{\mathbb{P}}\left[e^{iu\,Q_{\mathcal{A}}(t)}\right] = \exp\{\lambda t\,(\varphi_D(u) - 1)\},$$

where $\varphi_D(u) = \mathbf{E}_{\mathbb{P}}\left[e^{iuD}\right]$. So Lemma 2.1 implies that

$$\varphi_D(u) \to \varphi_J(u) \quad \text{as } n \to \infty$$

which implies that

$$\varphi_{Q_{\mathcal{A}}}(u) \to \varphi_Q(u) \quad \text{as } n \to \infty,$$

and the result follows. ■

Corollary 2.1. *Under the assumptions of Lemma 2.1, we have that*

$$S_0(t) = S_{0,n}(t) \xrightarrow{d} S(t) \quad \textit{as } n \to \infty.$$

Proof. This follows from Theorem 2.1 and Theorem 2.2. ■

We now state a version of Girsanov's Theorem applicable to our nth market. This result discusses the dynamics of changing from the real-world probability measure $\mathbb{P}$ to an equivalent probability measure $\mathbb{Q}$.

Theorem 2.3 (Girsanov, discrete jump factor, single-asset)**.** *Given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ on which is defined a Brownian motion $W(t)$ and n independent Poisson processes $N_1(t), \ldots, N_n(t)$, each with intensity λ_k, $k = 1, \ldots, n$. Let $\theta(t)$ be an adapted process and let $\tilde{\lambda}_k(t) \geq 0$ be predictable for $k = 1, \ldots, n$. Also, let $T_{k,m}$ be the time of the mth jump of $N_k(t)$. Define*

$$Z_0(t) \equiv \exp\left\{-\int_0^t \theta(s)\,dW(s) - \frac{1}{2}\int_0^t \theta^2(s)\,ds\right\},$$

$$Z_k(t) \equiv \exp\left\{\lambda_k\,t - \int_0^t \tilde{\lambda}_k(s)\,ds + \sum_{m=1}^{N_k(t)} \ln\left(\frac{\tilde{\lambda}_k(T_{k,m})}{\lambda_k}\right)\right\}, \quad k = 1, \ldots, n,$$

$$Z(t) \equiv Z_0(t)\prod_{k=1}^{n} Z_k(t),$$

$$\mathbb{Q}(A) \equiv \int_A Z(T)\, d\mathbb{P} \quad \textit{for all } A \in \mathcal{F}.$$

Suppose that

$$\mathbf{E}_{\mathbb{P}}[Z_0(t)] = 1, \quad \mathbf{E}_{\mathbb{P}}\left[\prod_{k=1}^{n} Z_k(t)\right] = 1, \quad \textit{and} \quad \int_0^t \tilde{\lambda}_k(s)\, ds < \infty.$$

Then under the probability measure $\mathbb{Q}$, the process

$$\widetilde{W}(t) = W(t) + \int_0^t \theta(s)\, ds$$

is a Brownian motion, each $N_k(t)$ is a Poisson process with intensity $\tilde{\lambda}_k(t)$, and $\widetilde{W}(t)$ and $N_1(t), \ldots, N_n(t)$ are independent of one another.

Proof. See Theorem 2.5 of Runggaldier (2003). ■

Now define

$$\tilde{\lambda}(t) \equiv \sum_{k=1}^{n} \tilde{\lambda}_k(t)$$

and probabilities

$$\mathbb{Q}(y_{i,k}(t)) \equiv \frac{\tilde{\lambda}_k(t)}{\tilde{\lambda}(t)} \quad \text{for } i = 0, 1, \ldots, n, \ k = 1, \ldots, n.$$

Under $\mathbb{Q}$, the process $N(t) = \sum_{k=1}^{n} N_k(t)$ is Poisson with intensity $\tilde{\lambda}(t)$. We also define

$$\tilde{\beta}_i(t) \equiv \tilde{\beta}_i(t, S(t-)) = \mathbf{E}_{\mathbb{Q}}[Y_i(t)] = \frac{1}{\tilde{\lambda}(t)} \sum_{k=1}^{n} \tilde{\lambda}_k(t)\, y_{i,k}(t), \quad i = 0, 1, \ldots, n.$$

The probability measure $\mathbb{Q}$ is risk-neutral if and only if the mean rate of return of each asset is the instantaneous continuously compounded risk-free interest rate $r(t)$. That is,

$$\begin{aligned}\frac{dS_i(t)}{S_i(t-)} &= r(t)\,dt + b_i(t)\,d\widetilde{W}(t) + \sum_{k=1}^{n} y_{i,k}(t)\left(dN_k(t) - \tilde{\lambda}_k(t)\,dt\right)\\ &= \left(r(t) - \tilde{\lambda}(t)\,\tilde{\beta}_i(t) + b_i(t)\,\theta(t)\right)dt + b_i(t)\,dW(t) + \sum_{k=1}^{n} y_{i,k}(t)\,dN_k(t),\end{aligned} \tag{10}$$

$$i = 0, 1, \ldots, n.$$

From (8) and (10), we get the market price of risk equations given by

$$a_i(t) - \lambda\,\beta_i(t) = r(t) - \tilde{\lambda}(t)\,\tilde{\beta}_i(t) + b_i(t)\,\theta(t), \quad i = 0, 1, \ldots, n. \tag{11}$$

Theorem 2.4. *The market price of risk equations (11) have an almost surely unique solution for $\theta(t)$ and $\tilde{\lambda}_1(t), \ldots, \tilde{\lambda}_n(t)$ when*

$$\begin{bmatrix} b_0(t) & -y_{0,1}(t) & \ldots & -y_{0,n}(t) \\ b_1(t) & -y_{1,1}(t) & \ldots & -y_{1,n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ b_n(t) & -y_{n,1}(t) & \ldots & -y_{n,n}(t) \end{bmatrix}$$

is almost surely nonsingular. An almost surely unique solution implies an almost surely unique risk-neutral measure $\mathbb{Q}$, thereby giving a complete and arbitrage-free approximate market with probability 1.

Proof. In matrix notation, we can write (11) as

$$\begin{bmatrix} a_0(t) - r(t) - \lambda\,\beta_0(t) \\ a_1(t) - r(t) - \lambda\,\beta_1(t) \\ \vdots \\ a_n(t) - r(t) - \lambda\,\beta_n(t) \end{bmatrix} = \begin{bmatrix} b_0(t) & -y_{0,1}(t) & \ldots & -y_{0,n}(t) \\ b_1(t) & -y_{1,1}(t) & \ldots & -y_{1,n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ b_n(t) & -y_{n,1}(t) & \ldots & -y_{n,n}(t) \end{bmatrix} \begin{bmatrix} \theta(t) \\ \tilde{\lambda}_1(t) \\ \vdots \\ \tilde{\lambda}_n(t) \end{bmatrix}$$

and the result follows. ■

Remark. The market price of risk equations (11) above are a generalization of Example 11.7.4 in Shreve (2004). A different form of (11) is provided in Jensen (1999). ■

3 Approximate Complete Market Hedge

We now undertake the construction of the self-financing hedge portfolio that replicates the option payoff at maturity time T. Let $c(t, S_0(t))$ be the price at time t of a European-type option on $S_0(t)$ and let $V_H(t)$ be the value of the portfolio at time t. Suppose we have initial capital

$$V_H(0) = c(0, S_0(0)).$$

We set $H_i(t)$ to be the number of units of asset $S_i(t)$ held in the portfolio at time t. We assume that $H_i(t)$ is predictable, that is, $H_i(t)$ is $\mathcal{F}_{t-}$–measurable. So, $H_i(t-) = H_i(t)$. Also, we express the cash in the money market account as

$$V_H(t) - \sum_{i=0}^{n} H_i(t)\, S_i(t).$$

Then,

$$V_H(t) = \sum_{i=0}^{n} H_i(t)\, S_i(t) + \left[V_H(t) - \sum_{i=0}^{n} H_i(t)\, S_i(t) \right]$$

and

$$dV_H(t) = \sum_{i=0}^{n} H_i(t)\, dS_i(t) + r(t) \left[V_H(t) - \sum_{i=0}^{n} H_i(t)\, S_i(t) \right] dt. \tag{12}$$

Our goal is to have

$$dc(t, S_0(t)) = dV_H(t).$$

To compute the differential of functions on stochastic processes, we use Itô's formula. Standard references for cases that may involve jumps are Protter (2004) and Applebaum (2009). For our purposes, we state the version in Runggaldier (2003).

Theorem 3.1 (Generalized Itô formula)**.** *Let $X(t)$ be a process of the form*

$$dX(t) = X(t-) \big(\alpha(t)\, dt + \zeta(t)\, dW(t) + \gamma(t)\, dN(t) \big),$$

where $\alpha(t), \zeta(t)$ are adapted processes with

$$\mathbf{E}_{\mathbb{P}} \left[\int_0^T \zeta^2(t)\, dt \right] < \infty$$

and $\gamma(t)$ is predictable with $\gamma(t) > -1$. Then given a function $f(t,x)$ with continuous partial derivatives f_t, f_x and f_{xx}, we have that

$$\begin{aligned} df(t,X(t)) = {} & f_t(t,X(t))\,dt + f_x(t,X(t))\,X(t)\,\alpha(t)\,dt \\ & + \frac{1}{2}\,f_{xx}(t,X(t))\,X^2(t)\,\zeta^2(t)\,dt + f_x(t,X(t))\,X(t)\,\zeta(t)\,dW(t) \\ & + \big[f\big(t,(1+\gamma(t))\,X(t-)\big) - f\big(t,X(t-)\big)\big]\,dN(t). \end{aligned}$$

Proof. See Proposition 8.14 of Cont and Tankov (2004) or Theorem 11.5.1 of Shreve (2004). ■

We now give the integro-partial differential equation which the price of a European-type option must satisfy in our nth market. First, we define the random variable

$$D \in \{d_1, \ldots, d_n\}$$

which represents our approximated jump factors.

Theorem 3.2. *Let $c(t,x)$ be the price of a European-type option with expiry time T on a stock with dynamics given by (4). Suppose the partial derivatives c_t, c_x, c_{xx} exist and are continuous. Then $c(t,x)$ satisfies the integro-partial differential equation given by*

$$\begin{aligned} -\,r(t)\,c(t,x) + c_t(t,x) + \Big(r(t) - \tilde{\lambda}(t)\,\tilde{\beta}_0(t)\Big)\;x\,c_x(t,x) + \frac{1}{2}\,b_0^2(t)\,x^2\,c_{xx}(t,x) & \\ + \tilde{\lambda}(t)\,\mathbf{E}_{\mathbb{Q}}^{D}\left[c\big(t,e^D x\big) - c\big(t,x\big)\right] = 0, \quad 0 \le t < T, x \ge 0, & \quad (13) \end{aligned}$$

and the terminal condition that $c(T,x)$ is the payoff function at expiry, assuming (13) has a $\mathcal{C}^2$–solution, where $\mathcal{C}^2$ is the set of all functions with continuous first and second-order derivatives. The expectation $\mathbf{E}_{\mathbb{Q}}^{D}$ is taken on the discrete approximate jump factor D under the risk-neutral measure $\mathbb{Q}$.

Proof. From (10), we have that

$$\frac{dS_0(t)}{S_0(t-)} = \Big(r(t) - \tilde{\lambda}(t)\,\tilde{\beta}_0(t)\Big)\,dt + b_0(t)\,d\widetilde{W}(t) + \sum_{k=1}^{n} y_{0,k}(t)\,dN_k(t).$$

We apply Itô's formula (Theorem 3.1) to the option price $c(t,S_0(t))$ and get

$$dc(t,S_0(t)) = c_t(t,S_0(t))\,dt + c_x(t,S_0(t))\,S_0(t)\;\Big(r(t) - \tilde{\lambda}(t)\,\tilde{\beta}_0(t)\Big)\;dt$$

$$
\begin{aligned}
&+ \frac{1}{2}\, c_{xx}(t, S_0(t))\, S_0^2(t)\, b_0^2(t)\, dt + c_x(t, S_0(t))\, S_0(t)\, b_0(t)\, d\widetilde{W}(t) \\
&+ \sum_{k=1}^{n} \Big[c\big(t, (1 + y_{0,k}(t))\, S_0(t-)\big) - c\big(t, S_0(t-)\big) \Big]\, dN_k(t) \\
= &\Bigg\{ c_t(t, S_0(t)) + c_x(t, S_0(t))\, S_0(t)\, \Big(r(t) - \tilde{\lambda}(t)\, \tilde{\beta}_0(t) \Big) \\
&+ \frac{1}{2}\, c_{xx}(t, S_0(t))\, S_0^2(t)\, b_0^2(t) \\
&+ \sum_{k=1}^{n} \tilde{\lambda}_k(t)\, \Big[c\big(t, (1 + y_{0,k}(t))\, S_0(t-)\big) - c\big(t, S_0(t-)\big) \Big] \Bigg\}\, dt \\
&+ c_x(t, S_0(t))\, S_0(t)\, b_0(t)\, d\widetilde{W}(t) \\
&+ \sum_{k=1}^{n} \Big[c\big(t, (1 + y_{0,k}(t))\, S_0(t-)\big) - c\big(t, S_0(t-)\big) \Big] \cdot \\
&\qquad \Big(dN_k(t) - \tilde{\lambda}_k(t)\, dt \Big)\,. \qquad (14)
\end{aligned}
$$

We now consider the hedge portfolio $V_H(t)$. Applying (8) with

$$
dW(t) = d\widetilde{W}(t) - \theta(t)\, dt
$$

on (12), we have

$$
\begin{aligned}
dV_H(t) = &\Bigg\{ \sum_{i=0}^{n} H_i(t)\, S_i(t-)\, \Big(a_i(t) - r(t) - \lambda\, \beta_i(t) + \tilde{\lambda}(t)\, \tilde{\beta}_i(t) - \theta(t)\, b_i(t) \Big) \\
&+ r(t)\, V_H(t) \Bigg\} dt + \left[\sum_{i=0}^{n} H_i(t)\, S_i(t-)\, b_i(t) \right] d\widetilde{W}(t) \\
&+ \sum_{i=0}^{n} H_i(t)\, S_i(t-) \left[\sum_{k=1}^{n} y_{i,k}(t)\, \Big(dN_k(t) - \tilde{\lambda}_k(t)\, dt \Big) \right] \\
= &\; r(t)\, V_H(t)\, dt + \left[\sum_{i=0}^{n} H_i(t)\, S_i(t-)\, b_i(t) \right] d\widetilde{W}(t)
\end{aligned}
$$

$$+\sum_{k=1}^{n}\left[\sum_{i=0}^{n} H_i(t)\, S_i(t-)\, y_{i,k}(t)\right]\Big(dN_k(t)-\tilde{\lambda}_k(t)\,dt\Big) \quad \text{by (11).} \tag{15}$$

So setting $V_H(t) = c(t, S_0(t))$ and equating the dt-term coefficients of (14) and (15) results in the integro-partial differential equation given by

$$c_t(t,S_0(t)) + c_x(t,S_0(t))\,S_0(t)\,\Big(r(t)-\tilde{\lambda}(t)\,\tilde{\beta}_0(t)\Big) + \frac{1}{2}\,c_{xx}(t,S_0(t))\,S_0^2(t)\,b_0^2(t)$$
$$+\sum_{k=1}^{n}\tilde{\lambda}_k(t)\,\Big[c\big(t,(1+y_{0,k}(t))\,S_0(t-)\big)-c\big(t,S_0(t-)\big)\Big] = r(t)\,c(t,S_0(t)). \tag{16}$$

Now the last term on the left-hand side of (16) can be written as

$$\tilde{\lambda}(t)\sum_{k=1}^{n}\frac{\tilde{\lambda}_k(t)}{\tilde{\lambda}(t)}\,\Big[c\big(t,e^{\,d_k}S_0(t-)\big)-c\big(t,S_0(t-)\big)\Big]$$

$$=\tilde{\lambda}(t)\,\mathbf{E}_{\mathbb{Q}}^{D}\,\Big[c\big(t,e^{D}S_0(t-)\big)-c\big(t,S_0(t-)\big)\Big]\,,$$

where the expectation is taken on the discrete approximate jump factor D using its density at time t. ■

Remark. An alternative proof of Theorem 3.2 can be found in Theorem 11.7.7 of Shreve (2004). ■

We now discuss the solution for the asset positions $H_i(t)$, $i = 0, \ldots, n$, that will replicate the option payoff at expiry.

Theorem 3.3. *Let*

$$G \equiv \begin{bmatrix} b_0(t)\,S_0(t-) & b_1(t)\,S_1(t-) & \cdots & b_n(t)\,S_n(t-) \\ y_{0,1}(t)\,S_0(t-) & y_{1,1}(t)\,S_1(t-) & \cdots & y_{n,1}(t)\,S_n(t-) \\ \vdots & \vdots & \ddots & \vdots \\ y_{0,n}(t)\,S_0(t-) & y_{1,n}(t)\,S_1(t-) & \cdots & y_{n,n}(t)\,S_n(t-) \end{bmatrix}.$$

If G is almost surely nonsingular, then we have an almost surely unique solution for the hedge positions given by

$$\begin{bmatrix} H_0(t) \\ H_1(t) \\ \vdots \\ H_n(t) \end{bmatrix} = G^{-1} \begin{bmatrix} c_x(t, S_0(t-))\, S_0(t-)\, b_0(t) \\ c\big(t, (1+y_{0,1}(t))\, S_0(t-)\big) - c\big(t, S_0(t-)\big) \\ \vdots \\ c\big(t, (1+y_{0,n}(t))\, S_0(t-)\big) - c\big(t, S_0(t-)\big) \end{bmatrix}.$$

Proof. We equate the coefficients of the $d\widetilde{W}(t)$ and $\Big(dN_k(t) - \tilde{\lambda}_k(t)\, dt\Big)$ terms of (14) and (15). From the $d\widetilde{W}(t)$-terms, we have

$$c_x(t, S_0(t-))\, S_0(t-)\, b_0(t) = \sum_{i=0}^{n} H_i(t)\, S_i(t-)\, b_i(t). \tag{17}$$

For each $k \in \{1, \dots, n\}$, we have from the $\Big(dN_k(t) - \tilde{\lambda}_k(t)\, dt\Big)$-term that

$$c\big(t, (1+y_{0,k}(t))\, S_0(t-)\big) - c\big(t, S_0(t-)\big) = \sum_{i=0}^{n} H_i(t)\, S_i(t-)\, y_{i,k}(t). \tag{18}$$

In matrix notation, we can write (17) and (18) as

$$\begin{bmatrix} c_x(t, S_0(t-))\, S_0(t-)\, b_0(t) \\ c\big(t, (1+y_{0,1}(t))\, S_0(t-)\big) - c\big(t, S_0(t-)\big) \\ \vdots \\ c\big(t, (1+y_{0,n}(t))\, S_0(t-)\big) - c\big(t, S_0(t-)\big) \end{bmatrix} = G \begin{bmatrix} H_0(t) \\ H_1(t) \\ \vdots \\ H_n(t) \end{bmatrix},$$

which has an almost surely unique solution for the hedge positions when G is almost surely nonsingular. ■

Remark. We shall call (17) and (18) the hedge condition equations. If there are no jumps, then no extra assets are required and the result of Theorem 3.3 reduces to the classical delta hedge of holding $c_x(t, S_0(t))$ units of the underlying asset at time t. ■

4 Application

We consider the implementation of our proposed approximate complete market hedge on actual data. We use Bank of America (BAC) hourly open stock and option prices for the period 2 January 2013 to 7 March 2013 as our historical data, and we examine hourly pricing and hedging of a call option for the period 7 March 2013 to 15 March 2013. A comparison with the minimal variance hedge in Cont et al. (2007) and the delta hedge suggested in Shreve (2004) is included.

Suppose we write an at-the-money Bank of America call option at 9am of 7 March 2013, with expiry time 3pm of 15 March 2013. Such an option has strike price $K = \$12$. We assume no dividends during this period, so we can apply the results of Section 3, which pertain to European options, to American options which are the commonly traded options in exchanges. We set $r(t) = 0$ and $b_0(t) = \sigma(t)$ to be constant, but allow updating of this volatility value at each hour when a new price becomes observable. We also assume no transaction costs and that our assets could be liquidly traded at any point.

4.1 Approximate Complete Market Hedge

To construct our approximate complete market detailed in Section 2, we opt for a simple approximation where $n = 2$. So, we partition J into

$$A_1 = (-\infty, 0) \quad \text{and} \quad A_2 = (0, +\infty),$$

that is, we discretize J into either the downward or upward jump factors in stock price movements. We now choose d_1 to be the average of the observed negative jump factors until the current time t, and we let d_2 be the average of the observed positive jump factors up to t. For a more realistic hedge, we allow d_1 and d_2 to take new values when new information becomes available. A conservative choice would be to choose d_1 to be the largest observed negative jump factor and d_2 to be the largest observed positive jump factor, although we do not include the results of this implementation in this paper. Finally, our choice of $n = 2$ requires two extra assets $S_1(t)$ and $S_2(t)$, which we choose to be call options on Bank of America with strike prices $K_1 = \$10$

and $K_2 = \$11$, respectively, and with expiry 15 March 2013.

To implement our hedge, we first require a formula for the call option price.

Theorem 4.1. *Let $\tau \equiv T - t$ and $\tilde{\lambda}_1$, $\tilde{\lambda}_2$ be constants. Assume that*

$$r(t) = r, \quad \sigma(t) = \sigma, \quad \tilde{\lambda}(t) = \tilde{\lambda} = \tilde{\lambda}_1 + \tilde{\lambda}_2, \quad \tilde{\beta}_0(t) = \tilde{\beta}_0$$

are constants. For $n = 2$, define

$$D(\eta, \alpha) \equiv \alpha\, d_1 + (\eta - \alpha)\, d_2.$$

Then the price at time t of a European call option on $S_0(t)$ with strike price K is

$$c(t, S_0(t)) = \sum_{\eta=0}^{\infty} \sum_{\alpha=0}^{\eta} \left\{ e^{-\tilde{\lambda}\,\tau} \frac{\left(\tilde{\lambda}\,\tau\right)^{\eta}}{\eta!} \binom{\eta}{\alpha} \left(\frac{\tilde{\lambda}_1}{\tilde{\lambda}}\right)^{\alpha} \left(\frac{\tilde{\lambda}_2}{\tilde{\lambda}}\right)^{\eta-\alpha} \cdot \left[S_0(t)\, e^{D(\eta,\alpha) - \tilde{\lambda}\,\tilde{\beta}_0\,\tau}\, \Phi\left(\delta_{+}^{\eta,\alpha}\right) - K e^{-r\tau}\, \Phi\left(\delta_{-}^{\eta,\alpha}\right) \right] \right\}$$

where Φ is the cumulative standard normal distribution and

$$\delta_{\pm}^{\eta,\alpha} \equiv \frac{\ln \frac{S_0(t)}{K} + \left(r - \tilde{\lambda}\,\tilde{\beta}_0 \pm \frac{\sigma^2}{2}\right)\tau + D(\eta, \alpha)}{\sigma\sqrt{\tau}}.$$

Proof. Without loss of generality, we derive the formula for $t = 0$. We recall that we had the random variable $D \in \{d_1, d_2\}$ with

$$\mathbb{Q}(d_1) = \frac{\tilde{\lambda}_1}{\tilde{\lambda}} \quad \text{and} \quad \mathbb{Q}(d_2) = \frac{\tilde{\lambda}_2}{\tilde{\lambda}}.$$

Now let D_m be the value of D at the mth jump time. By (10) and Theorem 2.1, we have that

$$S_0(T) = S_0(0) \exp\left[\left(r - \tilde{\lambda}\,\tilde{\beta}_0 - \frac{\sigma^2}{2}\right)T + \sigma\,\widetilde{W}(T) + \sum_{m=1}^{N(T)} D_m\right].$$

First,

$$\mathbf{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{S_0(T)>K\}}\right]$$

$$\begin{aligned}
&= \mathbb{Q}(S_0(T) > K) \\
&= \sum_{\eta=0}^{\infty} \mathbb{Q}\Bigg(S_0(0)\exp\left[\left(r - \tilde{\lambda}\,\tilde{\beta}_0 - \frac{\sigma^2}{2}\right)T + \sigma\,\widetilde{W}(T) + \sum_{m=1}^{\eta} D_m\right] > K\Bigg)\cdot \\
&\qquad\qquad e^{-\tilde{\lambda}\,T}\,\frac{\left(\tilde{\lambda}\,T\right)^{\eta}}{\eta!} \\
&= \sum_{\eta=0}^{\infty}\sum_{\alpha=0}^{\eta} \mathbb{Q}\left(\widetilde{W}(T) > \frac{\ln\frac{K}{S_0(0)} - \left(r - \tilde{\lambda}\,\tilde{\beta}_0 - \frac{\sigma^2}{2}\right)T - D(\eta,\alpha)}{\sigma}\right)\cdot \\
&\qquad\qquad \binom{\eta}{\alpha}\left(\frac{\tilde{\lambda}_1}{\tilde{\lambda}}\right)^{\alpha}\left(\frac{\tilde{\lambda}_2}{\tilde{\lambda}}\right)^{\eta-\alpha} e^{-\tilde{\lambda}\,T}\,\frac{\left(\tilde{\lambda}\,T\right)^{\eta}}{\eta!} \\
&= \sum_{\eta=0}^{\infty}\sum_{\alpha=0}^{\eta} \Phi\left(\delta_-^{\eta,\alpha}\right)\binom{\eta}{\alpha}\left(\frac{\tilde{\lambda}_1}{\tilde{\lambda}}\right)^{\alpha}\left(\frac{\tilde{\lambda}_2}{\tilde{\lambda}}\right)^{\eta-\alpha} e^{-\tilde{\lambda}\,T}\,\frac{\left(\tilde{\lambda}\,T\right)^{\eta}}{\eta!}.
\end{aligned}$$

Now define the event

$$E \equiv \left\{ S_0(0)\exp\left[\left(r - \tilde{\lambda}\,\tilde{\beta}_0 - \frac{\sigma^2}{2}\right)T + \sigma\,\widetilde{W}(T) + D(\eta,\alpha)\right] > K\right\},$$

and introduce the probability measure space $\widehat{\mathbb{Q}}$ such that

$$\widehat{W}(t) \equiv \widetilde{W}(t) - \sigma t$$

is Brownian motion under $\widehat{Q}$. By Theorem 2.3, this new measure induces the Radon-Nikodým derivative given by

$$\frac{d\widehat{\mathbb{Q}}}{d\mathbb{Q}} = \exp\left[\sigma\,\widetilde{W}(T) - \frac{\sigma^2}{2}\,T\right].$$

Now we have that

$$\mathbf{E}_{\mathbb{Q}}\left[\exp\left[\sigma\,\widetilde{W}(T)-\frac{\sigma^2}{2}\,T\right]\cdot\mathbb{1}_E\right]$$

$$\begin{aligned}
&=\widehat{\mathbb{Q}}(E)\\
&=\widehat{\mathbb{Q}}\left(\widetilde{W}(T)>\frac{\ln\frac{K}{S_0(0)}-\left(r-\tilde{\lambda}\,\tilde{\beta}_0-\frac{\sigma^2}{2}\right)T-D(\eta,\alpha)}{\sigma}\right)\\
&=\widehat{\mathbb{Q}}\left(\widehat{W}(T)>\frac{\ln\frac{K}{S_0(0)}-\left(r-\tilde{\lambda}\,\tilde{\beta}_0+\frac{\sigma^2}{2}\right)T-D(\eta,\alpha)}{\sigma}\right)\\
&=\Phi\left(\delta_+^{\eta,\alpha}\right).
\end{aligned}$$

So,

$$\mathbf{E}_{\mathbb{Q}}\left[S_0(T)\cdot\mathbb{1}_{\{S_0(T)>K\}}\right]$$

$$\begin{aligned}
&=\sum_{\eta=0}^{\infty}\sum_{\alpha=0}^{\eta}S_0(0)\,e^{(r-\tilde{\lambda}\,\tilde{\beta}_0)\,T+D(\eta,\alpha)}\,\mathbf{E}_{\mathbb{Q}}\left[\exp\left[\sigma\,\widetilde{W}(T)-\frac{\sigma^2}{2}\,T\right]\cdot\mathbb{1}_E\right]\cdot\\
&\qquad\binom{\eta}{\alpha}\left(\frac{\tilde{\lambda}_1}{\tilde{\lambda}}\right)^{\alpha}\left(\frac{\tilde{\lambda}_2}{\tilde{\lambda}}\right)^{\eta-\alpha}e^{-\tilde{\lambda}\,T}\,\frac{\left(\tilde{\lambda}\,T\right)^{\eta}}{\eta!}\\
&=\sum_{\eta=0}^{\infty}\sum_{\alpha=0}^{\eta}S_0(0)\,e^{(r-\tilde{\lambda}\,\tilde{\beta}_0)\,T+D(\eta,\alpha)}\,\Phi\left(\delta_+^{\eta,\alpha}\right)\binom{\eta}{\alpha}\left(\frac{\tilde{\lambda}_1}{\tilde{\lambda}}\right)^{\alpha}\left(\frac{\tilde{\lambda}_2}{\tilde{\lambda}}\right)^{\eta-\alpha}\cdot\\
&\qquad e^{-\tilde{\lambda}\,T}\,\frac{\left(\tilde{\lambda}\,T\right)^{\eta}}{\eta!}.
\end{aligned}$$

By the risk-neutral valuation principle, the European call option price is then given by

$$c(0,S_0(0))=e^{-rT}\,\mathbf{E}_{\mathbb{Q}}\left[(S_0(T)-K)^+\right]$$

$$
\begin{aligned}
&= e^{-rT}\, \mathbf{E}_{\mathbb{Q}} \left[(S_0(T) - K)\, \mathbb{1}_{\{S_0(T)>K\}} \right] \\
&= e^{-rT}\, \mathbf{E}_{\mathbb{Q}} \left[S_0(T) \cdot \mathbb{1}_{\{S_0(T)>K\}} \right] - K e^{-rT}\, \mathbf{E}_{\mathbb{Q}} \left[\mathbb{1}_{\{S_0(T)>K\}} \right]
\end{aligned}
$$

and the result follows. ■

Corollary 4.1. *Under the same assumptions as Theorem 4.1, we have that*

$$
\begin{aligned}
c_x(t, S_0(t)) = \sum_{\eta=0}^{\infty} \sum_{\alpha=0}^{\eta} \Bigg\{ & e^{-\tilde{\lambda}\,\tau} \frac{\left(\tilde{\lambda}\,\tau\right)^{\eta}}{\eta!} \binom{\eta}{\alpha} \left(\frac{\tilde{\lambda}_1}{\tilde{\lambda}}\right)^{\alpha} \left(\frac{\tilde{\lambda}_2}{\tilde{\lambda}}\right)^{\eta-\alpha} \cdot \\
& e^{D(\eta,\alpha) - \tilde{\lambda}\,\tilde{\beta}_0\,\tau}\, \Phi\left(\delta_{+}^{\eta,\alpha}\right) \Bigg\}.
\end{aligned}
$$

Proof. This follows from differentiating $c(t, S_0(t))$ in Theorem 4.1. ■

Table A1 illustrates the application of the hedge portfolio result, Theorem 3.3, over the life (9am, 7 March 2013 – 3pm, 15 March 2013) of our target option with strike price $K = \$12$. We observe that on the first hour (9am, 7 March 2013), our hedge portfolio needs to hold 1.88 units of Bank of America (BAC) shares, -2.23 units of $S_1(t)$, 1.14 units of $S_2(t)$, and $-\$18.95$ in our money market account. Applying the hedge at each hour eventually leads to a position, one hour before expiry, of 1 unit of stock, none of the extra assets, and $-\$12.01$ in cash. This provides a portfolio value of \$0.63 at expiry time, which we observe is the exact target option's payoff given that the stock price at expiry is \$12.63.

We include the evolution of the hedge portfolio's value at each time point. The pre-adjust column shows the new portfolio value once the assets change price, given the positions during the previous hour. The post-adjust column shows the portfolio value after adjusting positions given the current hour's asset prices. The value in the post-adjust column should also just be equal to the call option price at that time, whose formula is given by Theorem 4.1.

We also examine the hedging errors at each hour, which could be interpreted as the profit/loss of the hedge portfolio relative to the current price of the target option. Each entry in the Error column is done by taking the difference of the corresponding pre-adjusted and post-adjusted portfolio values. For a perfectly self-financing portfolio, the errors at each hour should be zero. It turns out that the approximate complete market hedge is not, in reality, perfectly self-financing, but we observe that in this case study, our hedge generally gives a profit when the stock price goes up. This does compensate for the losses of the call option writer due to rises in the stock's value.

4.2 Minimal Variance Hedge

We now consider the minimal variance hedge studied in Cont et al. (2007), whose results we state in the notation of Sections 2 and 3. Suppose $V(t)$ is the value of the hedge portfolio at time t where $H_i(t)$ is the number of units of the ith asset held at this time. We recall our approximate market whose dynamics under the risk-neutral measure $\mathbb{Q}$ were given by (10). We also defined random variables

$$Y_i(t) \in \{y_{i,1}(t), \ldots, y_{i,n}(t)\}, \quad i = 0, 1, \ldots, n.$$

Let

$$\mathbf{b}(t) \equiv \begin{bmatrix} b_0(t)\, S_0(t-) \\ b_1(t)\, S_1(t-) \\ \vdots \\ b_n(t)\, S_n(t-) \end{bmatrix} \text{ and } \boldsymbol{\gamma}(t) \equiv \begin{bmatrix} Y_0(t)\, S_0(t-) \\ Y_1(t)\, S_1(t-) \\ \vdots \\ Y_n(t)\, S_n(t-) \end{bmatrix}.$$

Also, define

$$\begin{aligned} \mathbf{H}(t) &\equiv \begin{bmatrix} H_0(t) & H_1(t) & \cdots & H_n(t) \end{bmatrix}^\top \\ \mathbf{S}(t) &\equiv \begin{bmatrix} S_0(t) & S_1(t) & \cdots & S_n(t) \end{bmatrix}^\top \\ b^0(t) &\equiv c_x(t, S_0(t-))\, b_0(t)\, S_0(t-) \\ \gamma^0(t) &\equiv c(t, (1 + Y_0(t))\, S_0(t-)) - c(t, S_0(t-)). \end{aligned}$$

Theorem 4.2. *Suppose the matrix*

$$M(t) \equiv \mathbf{b}(t) \cdot \mathbf{b}(t)^\top + \tilde{\lambda}(t)\, \mathbf{E}_{\mathbb{Q}}^{Y} \left[\boldsymbol{\gamma}(t) \cdot \boldsymbol{\gamma}(t)^\top \right]$$

is almost surely nonsingular for all $t \in [0,T]$. Then under some general assumptions (See Cont et al., 2007), the minimal variance hedge is the solution to

$$\inf_{(V(0),\,\mathbf{H}(t))} \mathbf{E}_{\mathbb{Q}} \left[c(T,S(T)) - \left(V(0) + \int_0^T \mathbf{H}(t)^\top \, d\mathbf{S}(t) \right) \right]^2$$

which is given by

$$\begin{aligned} \hat{V}(0) &= \mathbf{E}_{\mathbb{Q}}[c(T,S(T))] \\ \hat{\mathbf{H}}(t) &= M(t)^{-1} \left(b^0(t)\,\mathbf{b}(t) + \tilde{\lambda}(t)\,\mathbf{E}^Y_{\mathbb{Q}}\left[\gamma^0(t)\,\boldsymbol{\gamma}(t)\right] \right), \end{aligned}$$

where the expectation is taken using the probability structure of $Y_i(t)$ for $i = 0, 1, \ldots, n$ under the risk-neutral measure $\mathbb{Q}$.

Proof. See Cont et al. (2007). ■

Table A2 shows the construction of the minimal variance hedge applied to our Bank of America (BAC) scenario. Compared to the approximate complete market hedge, the minimal variance hedge mostly employs positions on the extra assets, which are our call options with strike prices $K_1 = \$10$ and $K_2 = \$11$, and virtually no position in the underlying stock. As indicated by the cash position column, the minimal variance hedge involves considerably less cash to implement. We observe, however, that the minimal variance hedge does not do that well in replicating our target option during the immediate hours prior to expiry.

4.3 Delta Hedge

We also consider the performance of a traditional delta-hedge on the underlying stock only. Shreve (2004) suggests holding $c_x(t, S_0(t))$ units of stock at time t in order to hedge our call option on average. This proposal is motivated by the fact that the convexity of the call option value $c(t, S_0(t))$ allows the hedge portfolio to outperform the option between jumps. At jump times, however, the option outperforms the hedge portfolio.

Table A3 shows the implementation of the delta hedge in our Bank of America (BAC) scenario. Like the approximate complete market hedge, the delta hedge indicates that as we get closer to maturity, we need to hold 1 unit of the underlying asset. Replication of the target option's payoff at maturity is also accurate. We do observe, however, that very large changes in the underlying stock price negatively affect the hedge portfolio's performance.

4.4 Comparison

We summarize and compare some statistics from the three hedging strategies considered in this case study. These are presented in Table 1. Taking the sum of the values in the Error columns of each strategy yields the Sum of Errors, which is interpreted as the net profit/loss from the implementation. Although all three strategies are profitable, we find that for this scenario, an approximate complete market hedge gives the most profit, followed by the delta hedge, then the minimal variance hedge.

Table 1: Comparison of Approximate Complete Market Hedge (ACMH), Minimal Variance Hedge (MVH), and Delta Hedge

	ACMH	MVH	Delta Hedge
Mean Error	0.00620	0.00039	0.00203
Std Dev of Errors	0.06012	0.04055	0.01079
Sum of Errors	0.29748	0.01859	0.09749

We also study the self-financing property by taking the mean of the errors for each strategy. A strategy that is self-financing on average would have a mean error of zero. Hence, we conclude that in our case study, the minimal variance hedge comes closest to being self-financing on average, followed by the delta hedge.

Possibly the best balance between profitability, replication, and the self-financing property is the delta hedge. For a moderate amount of net profit ($0.09749), the delta hedge is close to being self-financing with a mean error of 0.00203 along with error values that vary the least with a standard deviation of 0.01079.

5 Conclusion

This paper has extended the complete market approximation proposed by Jensen (1999) to hedging a European option on a single asset. Aided by a discrete approximation of the jump factors, introducing extra assets driven by the same noise processes as the underlying allows the specification of a unique risk-neutral measure which we can use to approximately price the option. In addition, a classical replicating portfolio argument gives a formula for the asset positions in our hedge portfolio. We then compare the performance of our approximate complete market hedge to some existing hedging strategies for jump-diffusion models, such as minimal variance and delta hedging.

We have not examined risk-neutral measure calibration nor volatility smile fitting in our work. Procedures for calibrating or fitting the local volatility (See Andersen and Andreasen, 2000; He et al., 2006, for example) could produce an approximate market that is closer to the actual realizations of stock and option prices, and hence could result in more accurate prices and hedges. We also have not experimented on the impact of using more than two extra assets, although Andersen and Andreasen (2000) and He et al. (2006) notice improvements in hedging performance if more options are included in the hedge portfolio.

The tractability of our approximate complete market permits its extension to options on several assets, such as exchange or basket options. Although this approach would require the use of considerably more extra single-asset options to complete the market, discretizing the jumps could allow for reasonable pricing and hedging of multi-asset options in jump-diffusion settings.

Appendix: Hedge Portfolio Implementations

Table A1: Approximate Complete Market Hedge

BAC Data			Hedge Positions				Portfolio Value		Error
Stock $S_0(t)$	Call1 $S_1(t)$	Call2 $S_2(t)$	Units Stock	Units Call1	Units Call2	Cash	Pre-Adjust	Post-Adjust	Pre–Post
11.99	2.00	1.01	1.88	-2.23	1.14	-18.95	NA	0.32312	NA
12.15	2.15	1.19	1.62	-1.46	0.61	-16.90	0.50180	0.37454	0.12725
12.19	2.18	1.20	1.64	-1.44	0.60	-17.15	0.39346	0.39246	0.00101
12.19	2.20	1.23	1.65	-1.44	0.59	-17.23	0.38335	0.39113	-0.00778
12.21	2.18	1.23	1.69	-1.50	0.61	-17.75	0.45120	0.39158	0.05962
12.16	2.15	1.18	1.67	-1.54	0.66	-17.42	0.31974	0.36388	-0.04414
12.19	2.19	1.22	1.69	-1.52	0.63	-17.68	0.38193	0.37989	0.00203
12.44	2.41	1.44	1.50	-0.91	0.31	-16.40	0.60846	0.53598	0.07248
12.18	2.16	1.16	1.76	-1.42	0.46	-18.56	0.28563	0.36198	-0.07635
12.07	2.08	1.13	1.74	-1.54	0.51	-18.03	0.26800	0.29932	-0.03133
12.10	2.05	1.09	1.80	-1.63	0.55	-18.78	0.37717	0.30417	0.07300
12.07	2.05	1.09	1.79	-1.64	0.56	-18.55	0.25006	0.28864	-0.03858
12.05	2.05	1.07	1.78	-1.64	0.58	-18.38	0.24164	0.27856	-0.03692
12.09	2.11	1.11	1.69	-1.51	0.57	-17.59	0.27414	0.30120	-0.02706
12.08	2.05	1.06	1.85	-1.68	0.57	-19.20	0.34657	0.28325	0.06331
12.17	2.14	1.20	1.91	-1.65	0.54	-19.99	0.37804	0.32948	0.04857
12.15	2.15	1.16	1.88	-1.64	0.56	-19.68	0.24387	0.31845	-0.07458
12.11	2.07	1.12	1.95	-1.77	0.58	-20.27	0.36143	0.28512	0.07631
12.13	2.07	1.15	2.00	-1.83	0.59	-20.90	0.34153	0.28850	0.05303
12.14	2.14	1.15	1.96	-1.68	0.54	-20.54	0.18041	0.30023	-0.11981
12.17	2.17	1.17	1.99	-1.67	0.53	-20.88	0.31928	0.31450	0.00478
12.10	2.09	1.07	2.00	-1.82	0.62	-20.81	0.25606	0.27015	-0.01409
12.15	2.13	1.15	2.06	-1.83	0.60	-21.53	0.34706	0.29241	0.05466
12.09	2.01	1.03	2.13	-2.04	0.67	-22.12	0.31219	0.24444	0.06775
12.00	1.99	1.00	2.04	-2.09	0.77	-20.85	0.07313	0.20358	-0.13045
12.01	2.00	1.02	2.09	-2.14	0.77	-21.41	0.22243	0.20406	0.01837
12.00	1.98	1.01	2.13	-2.23	0.80	-21.78	0.21620	0.19235	0.02385
12.00	2.04	1.01	2.10	-2.09	0.75	-21.44	0.05430	0.19489	-0.14058
12.04	2.03	1.03	2.23	-2.24	0.79	-22.89	0.32085	0.20537	0.11549
12.03	2.05	1.01	2.22	-2.22	0.81	-22.77	0.11809	0.19908	-0.08099
12.06	2.03	1.02	2.35	-2.38	0.84	-24.19	0.32264	0.20470	0.11794
12.02	2.02	1.04	2.36	-2.47	0.88	-24.16	0.15129	0.18013	-0.02884
12.07	2.07	1.07	2.45	-2.45	0.85	-25.15	0.20107	0.20221	-0.00114
12.07	2.07	1.08	2.52	-2.54	0.87	-25.86	0.21074	0.19630	0.01445
12.05	2.03	1.04	2.60	-2.72	0.95	-26.63	0.21251	0.17797	0.03453
12.12	2.11	1.11	2.66	-2.58	0.86	-27.54	0.20926	0.21389	-0.00463
12.17	2.16	1.14	2.68	-2.46	0.80	-27.92	0.23035	0.23812	-0.00777

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Table A1 – continued from previous page

BAC Data			Hedge Positions				Portfolio Value		Error
Stock $S_0(t)$	Call1 $S_1(t)$	Call2 $S_2(t)$	Units Stock	Units Call1	Units Call2	Cash	Pre-Adjust	Post-Adjust	Pre–Post
12.14	2.14	1.14	2.80	-2.66	0.86	-29.04	0.22041	0.21562	0.00479
12.14	2.15	1.14	2.88	-2.73	0.88	-29.88	0.19462	0.21163	-0.01701
12.15	2.15	1.14	2.97	-2.81	0.89	-30.91	0.23467	0.21029	0.02438
12.12	2.13	1.13	3.16	-3.11	0.99	-32.63	0.17430	0.18523	-0.01093
12.16	2.16	1.15	3.18	-2.97	0.91	-33.04	0.23205	0.20442	0.02764
12.50	2.49	1.48	1.28	-0.25	0.02	-14.88	0.60477	0.50168	0.10310
12.49	2.49	1.48	1.23	-0.20	0.01	-14.39	0.48889	0.49109	-0.00220
12.57	2.57	1.57	1.06	-0.05	0.00	-12.65	0.57419	0.57018	0.00402
12.51	2.52	1.50	1.07	-0.05	0.00	-12.72	0.50899	0.51015	-0.00117
12.59	2.60	1.58	1.01	0.00	0.00	-12.07	0.59544	0.59401	0.00144
12.57	2.56	1.56	1.00	0.00	0.00	-12.01	0.57004	0.57000	0.00004
12.63	2.65	1.63	NA	NA	NA	NA	0.62999	0.63000	-0.00001

Table A2: Minimal Variance Hedge

BAC Data			Hedge Positions				Portfolio Value		Error
Stock $S_0(t)$	Call1 $S_1(t)$	Call2 $S_2(t)$	Units Stock	Units Call1	Units Call2	Cash	Pre-Adjust	Post-Adjust	Pre–Post
11.99	2.00	1.01	0.00	1.34	-1.00	-1.37	NA	0.32312	NA
12.15	2.15	1.19	0.00	1.32	-0.83	-1.52	0.34476	0.37454	-0.02979
12.19	2.18	1.20	0.00	1.33	-0.84	-1.55	0.40586	0.39246	0.01340
12.19	2.20	1.23	0.00	1.34	-0.84	-1.57	0.39383	0.39113	0.00270
12.21	2.18	1.23	0.00	1.35	-0.83	-1.59	0.36449	0.39158	-0.02709
12.16	2.15	1.18	0.00	1.35	-0.87	-1.57	0.39219	0.36388	0.02831
12.19	2.19	1.22	0.00	1.37	-0.87	-1.61	0.38322	0.37989	0.00333
12.44	2.41	1.44	0.01	1.60	-0.96	-2.04	0.49106	0.53598	-0.04491
12.18	2.16	1.16	0.01	1.42	-0.94	-1.68	0.40316	0.36198	0.04118
12.07	2.08	1.13	0.01	1.38	-0.93	-1.58	0.27600	0.29932	-0.02333
12.10	2.05	1.09	0.01	1.40	-0.94	-1.60	0.29527	0.30417	-0.00889
12.07	2.05	1.09	0.01	1.40	-0.96	-1.59	0.30400	0.28864	0.01536
12.05	2.05	1.07	0.01	1.40	-0.99	-1.59	0.30770	0.27856	0.02914
12.09	2.11	1.11	0.01	1.42	-1.00	-1.66	0.32304	0.30120	0.02184
12.08	2.05	1.06	0.01	1.39	-0.97	-1.60	0.26575	0.28325	-0.01751
12.17	2.14	1.20	0.01	1.53	-1.00	-1.81	0.27282	0.32948	-0.05665
12.15	2.15	1.16	0.01	1.54	-1.07	-1.82	0.38474	0.31845	0.06629
12.11	2.07	1.12	0.01	1.51	-1.04	-1.75	0.23773	0.28512	-0.04739
12.13	2.07	1.15	0.01	1.54	-1.03	-1.79	0.25413	0.28850	-0.03437

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Table A2 – continued from previous page

BAC Data			Hedge Positions				Portfolio Value		Error
Stock $S_0(t)$	Call1 $S_1(t)$	Call2 $S_2(t)$	Units Stock	Units Call1	Units Call2	Cash	Pre-Adjust	Post-Adjust	Pre–Post
12.14	2.14	1.15	0.01	1.52	-1.05	-1.82	0.39627	0.30023	0.09604
12.17	2.17	1.17	0.01	1.54	-1.05	-1.87	0.32498	0.31450	0.01048
12.10	2.09	1.07	0.01	1.56	-1.16	-1.82	0.29624	0.27015	0.02609
12.15	2.13	1.15	0.01	1.64	-1.16	-1.95	0.24015	0.29241	-0.05226
12.09	2.01	1.03	0.01	1.64	-1.23	-1.86	0.23397	0.24444	-0.01047
12.00	1.99	1.00	0.01	1.62	-1.30	-1.79	0.24794	0.20358	0.04436
12.01	2.00	1.02	0.01	1.65	-1.31	-1.84	0.19382	0.20406	-0.01024
12.00	1.98	1.01	0.01	1.66	-1.32	-1.83	0.18403	0.19235	-0.00832
12.00	2.04	1.01	0.01	1.65	-1.37	-1.86	0.29173	0.19489	0.09684
12.04	2.03	1.03	0.01	1.70	-1.35	-1.93	0.15120	0.20537	-0.05417
12.03	2.05	1.01	0.01	1.72	-1.43	-1.96	0.26641	0.19908	0.06733
12.06	2.03	1.02	0.01	1.76	-1.41	-2.01	0.15060	0.20470	-0.05410
12.02	2.02	1.04	0.01	1.76	-1.42	-1.99	0.15876	0.18013	-0.02137
12.07	2.07	1.07	0.01	1.82	-1.44	-2.12	0.22616	0.20221	0.02395
12.07	2.07	1.08	0.01	1.86	-1.46	-2.16	0.18781	0.19630	-0.00849
12.05	2.03	1.04	0.01	1.87	-1.51	-2.14	0.18033	0.17797	0.00236
12.12	2.11	1.11	0.01	1.97	-1.53	-2.35	0.22241	0.21389	0.00852
12.17	2.16	1.14	0.01	2.02	-1.53	-2.46	0.26699	0.23812	0.02887
12.14	2.14	1.14	0.01	2.06	-1.58	-2.49	0.19761	0.21562	-0.01801
12.14	2.15	1.14	0.01	2.12	-1.65	-2.57	0.23627	0.21163	0.02464
12.15	2.15	1.14	0.01	2.19	-1.70	-2.66	0.21170	0.21029	0.00141
12.12	2.13	1.13	0.01	2.26	-1.80	-2.71	0.18331	0.18523	-0.00192
12.16	2.16	1.15	0.01	2.35	-1.84	-2.87	0.21747	0.20442	0.01305
12.50	2.49	1.48	0.01	1.81	-1.01	-2.61	0.37588	0.50168	-0.12579
12.49	2.49	1.48	0.01	1.79	-0.99	-2.60	0.50160	0.49109	0.01051
12.57	2.57	1.57	0.01	1.62	-0.81	-2.41	0.54620	0.57018	-0.02398
12.51	2.52	1.50	0.01	1.69	-0.88	-2.49	0.54545	0.51015	0.03529
12.59	2.60	1.58	0.01	1.58	-0.78	-2.36	0.57507	0.59401	-0.01894
12.57	2.56	1.56	0.01	1.59	-0.78	-2.36	0.54615	0.57000	-0.02385
12.63	2.65	1.63	NA	NA	NA	NA	0.65912	0.63000	0.02912

Table A3: Delta Hedge

BAC Data			Hedge Positions				Portfolio Value		Error
Stock $S_0(t)$	Call1 $S_1(t)$	Call2 $S_2(t)$	Units Stock	Units Call1	Units Call2	Cash	Pre-Adjust	Post-Adjust	Pre–Post
11.99	2.00	1.01	0.51	0.00	0.00	-5.74	NA	0.32312	NA
12.15	2.15	1.19	0.59	0.00	0.00	-6.83	0.40601	0.37454	0.03147
12.19	2.18	1.20	0.61	0.00	0.00	-7.07	0.39529	0.39246	0.00283
12.19	2.20	1.23	0.61	0.00	0.00	-7.08	0.39307	0.39113	0.00194
12.21	2.18	1.23	0.63	0.00	0.00	-7.27	0.40278	0.39158	0.01120
12.16	2.15	1.18	0.60	0.00	0.00	-6.91	0.35957	0.36388	-0.00431
12.19	2.19	1.22	0.62	0.00	0.00	-7.14	0.38302	0.37989	0.00313
12.44	2.41	1.44	0.76	0.00	0.00	-8.88	0.53413	0.53598	-0.00185
12.18	2.16	1.16	0.61	0.00	0.00	-7.13	0.33912	0.36198	-0.02286
12.07	2.08	1.13	0.55	0.00	0.00	-6.34	0.29434	0.29932	-0.00498
12.10	2.05	1.09	0.57	0.00	0.00	-6.59	0.31582	0.30417	0.01165
12.07	2.05	1.09	0.55	0.00	0.00	-6.36	0.28707	0.28864	-0.00157
12.05	2.05	1.07	0.54	0.00	0.00	-6.21	0.27762	0.27856	-0.00094
12.09	2.11	1.11	0.56	0.00	0.00	-6.51	0.30008	0.30120	-0.00111
12.08	2.05	1.06	0.56	0.00	0.00	-6.47	0.29557	0.28325	0.01231
12.17	2.14	1.20	0.62	0.00	0.00	-7.22	0.33354	0.32948	0.00407
12.15	2.15	1.16	0.60	0.00	0.00	-6.99	0.31398	0.31845	-0.00447
12.11	2.07	1.12	0.58	0.00	0.00	-6.77	0.29739	0.28512	0.01227
12.13	2.07	1.15	0.60	0.00	0.00	-6.99	0.29678	0.28850	0.00828
12.14	2.14	1.15	0.60	0.00	0.00	-7.03	0.29449	0.30023	-0.00573
12.17	2.17	1.17	0.63	0.00	0.00	-7.31	0.31835	0.31450	0.00385
12.10	2.09	1.07	0.58	0.00	0.00	-6.72	0.27064	0.27015	0.00050
12.15	2.13	1.15	0.62	0.00	0.00	-7.21	0.29902	0.29241	0.00661
12.09	2.01	1.03	0.57	0.00	0.00	-6.69	0.25415	0.24444	0.00971
12.00	1.99	1.00	0.50	0.00	0.00	-5.81	0.19280	0.20358	-0.01077
12.01	2.00	1.02	0.51	0.00	0.00	-5.93	0.20960	0.20406	0.00554
12.00	1.98	1.01	0.50	0.00	0.00	-5.83	0.19844	0.19235	0.00609
12.00	2.04	1.01	0.50	0.00	0.00	-5.80	0.19134	0.19489	-0.00354
12.04	2.03	1.03	0.54	0.00	0.00	-6.25	0.21637	0.20537	0.01101
12.03	2.05	1.01	0.53	0.00	0.00	-6.12	0.19893	0.19908	-0.00015
12.06	2.03	1.02	0.56	0.00	0.00	-6.50	0.21589	0.20470	0.01119
12.02	2.02	1.04	0.52	0.00	0.00	-6.07	0.18246	0.18013	0.00233
12.07	2.07	1.07	0.57	0.00	0.00	-6.65	0.20611	0.20221	0.00390
12.07	2.07	1.08	0.57	0.00	0.00	-6.69	0.20221	0.19630	0.00591
12.05	2.03	1.04	0.55	0.00	0.00	-6.47	0.18489	0.17797	0.00692
12.12	2.11	1.11	0.63	0.00	0.00	-7.41	0.21661	0.21389	0.00272
12.17	2.16	1.14	0.68	0.00	0.00	-8.04	0.24218	0.23812	0.00406
12.14	2.14	1.14	0.66	0.00	0.00	-7.82	0.22110	0.21562	0.00548
12.14	2.15	1.14	0.67	0.00	0.00	-7.93	0.21695	0.21163	0.00531

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Table A3 – continued from previous page

BAC Data			Hedge Positions				Portfolio Value		Error
Stock $S_0(t)$	Call1 $S_1(t)$	Call2 $S_2(t)$	Units Stock	Units Call1	Units Call2	Cash	Pre-Adjust	Post-Adjust	Pre–Post
12.15	2.15	1.14	0.69	0.00	0.00	-8.17	0.21699	0.21029	0.00671
12.12	2.13	1.13	0.66	0.00	0.00	-7.87	0.19097	0.18523	0.00574
12.16	2.16	1.15	0.73	0.00	0.00	-8.64	0.21048	0.20442	0.00607
12.50	2.49	1.48	0.98	0.00	0.00	-11.78	0.45185	0.50168	-0.04983
12.49	2.49	1.48	0.99	0.00	0.00	-11.84	0.49185	0.49109	0.00076
12.57	2.57	1.57	1.00	0.00	0.00	-11.97	0.57010	0.57018	-0.00007
12.51	2.52	1.50	1.00	0.00	0.00	-11.97	0.51032	0.51015	0.00016
12.59	2.60	1.58	1.00	0.00	0.00	-12.00	0.59396	0.59401	-0.00004
12.57	2.56	1.56	1.00	0.00	0.00	-12.00	0.57001	0.57000	0.00001
12.63	2.65	1.63	NA	NA	NA	NA	0.63000	0.63000	0.00000

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