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HERDING, TREND CHASING AND MARKET VOLATILITY

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ABSTRACT. We introduce a heterogeneous agent asset pricing model in continuous-time to show that trend chasing, switching and herding all contribute to market volatility in price and return and volatility clustering, but their impact are different. On the one hand, the fluctuations of market price and return and the level of the significant autocorrelations (ACs) of the absolute and squared returns increase with herding and trend chasing based on long time horizon. On the other hand, the switching reduces the return volatility and an initial increase in switching reduces the price volatility and increases the level of the significant ACs, but the effect becomes opposite when the switching increases further. We also show that market noise plays more important role than fundamental noise on the power-law behavior.

Key words: Heterogeneous beliefs, herding, switching, stability, volatility, stochastic delay differential equations.

JEL Classification: C62, D53, D84, G12

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1. INTRODUCTION

Trend chasing, switching among different trading strategies and herding are the most commonly observed boundedly rational behaviors of investors in financial markets and this paper studies their impact on market volatility. Large fluctuations in market price, excess volatility in return, and volatility clustering are the most common stylized facts in financial markets. The question is how these boundedly rational behavior of investors contribute to market volatility differently. This paper introduces a heterogeneous agent asset pricing model in a continuous-time framework to address this question. We show that herding and trend chasing based on long time horizon increase market volatility in price and return. However, the effect of switching is different for price volatility and opposite for return volatility. We also show that these boundedly rational behavior of investors contribute to the power-law behavior, characterized by insignificant autocorrelations (ACs) in the returns and significant and decaying ACs in the absolute and squared returns; however, their effects are different. More precisely, the level of the significant ACs increases with the herding and trend chasing based on long time horizon, but increases initially and then decreases as the switching intensity increases. In general, it is the interaction of nonlinear dynamics and noises that generates realistic market price dynamics. We show that the market noise plays more important role than the fundamental noise in generating the power-law behavior. To our knowledge, this is the first paper showing that the herding and switching have opposite effect on the return volatility and different impact on the power-law behavior.

Over the last three decades, empirical evidence, unconvincing justifications of the assumption of unbounded rationality, and the recognition of the relevance of investor psychology have led to the incorporation of heterogeneous and boundedly rational behavior of investors, such as trend chasing, switching and herding, into asset price and financial market modeling. Cross-sectional and time series momentums based on trend chasing behavior have been well documented in empirical literature (see Moskowitz et al. 2012 and the reference cited there), in which the time horizon used to calculate the trend plays a very important role (He and Li, 2013). As for

the switching, in a context of less than perfect rationality, investors not necessarily change immediately their trading strategy when an alternative is performing better. Such boundedly rational choice of investors can lead to market price fluctuations, bubbles and crashes, see for example, Brock and Hommes (1997, 1998) in discrete-time and He and Li (2012) in continuous-time. Herding refers broadly to the tendency of many different agents to take similar actions at roughly the same time. Scharfstein and Stein (1990) attribute it to the reputational concerns and the unpredictable components to investment outcomes. Banerjee (1992) shows that herd behavior is rational in term of obtaining others' information.

This paper is closely related to the recent development of heterogeneous agent models (HAMs) considering financial markets as expectation feedback systems and hence asset price fluctuations can be caused by an endogenous mechanism with heterogeneity and bounded rationality. Various agent-based financial market models have been developed to incorporate trend chasing, switching and herding. For instance, by considering two types of traders, typically fundamentalists and trend followers, Beja and Goldman (1980) and Chiarella (1992) among many others have shown that interaction of agents with heterogeneous expectations may lead to market instability. More significantly, Brock and Hommes (1997, 1998) introduce the concept of an adaptively rational equilibrium in a discrete-time framework. Agents adapt their beliefs over time by choosing from different predictors or expectation functions based upon their past performance (such as realized profits). Such boundedly rational behavior of agents can lead to market instability. More recently, these models have been extended to a continuous-time framework in He et al. (2009), He and Zheng (2010) and He and Li (2012), which provide an uniformed approach in dealing with the effect of time horizon used for trend chasing. Within a continuous-time framework, Lux (1995) and Alfarano et al. (2008) model the herding behavior through the master equation and show that herding can give rise to realistic time series. Within a discrete-time framework, Hohnisch and Westerhoff (2008) show how herding behavior at the level of individual economic sentiment may lead to enduring business cycles, and Franke and Westerhoff (2012) show a strong role for a herding component when generating realistic moments in financial time series. Overall, these

models have successfully explained several market features (such as market booms and crashes, deviations of the market price from the fundamental price), the stylized facts (such as skewness, kurtosis, volatility clustering and fat tails of returns), and the power-law behavior. We refer the reader to Hommes (2006), LeBaron (2006), Chiarella et al. (2009), Lux (2009), and Chen et al. (2012) for surveys of the recent development in this literature.

This paper provides a unified framework in a continuous-time model to examine the joint impact of trend chasing, switching, and herding on market price dynamics and compare different roles they play in generating market volatility in price and return and the power-law behavior of stock return volatility. In order to explore the effect of different mechanism, we introduce a continuous-time financial market with two types of agents, the fundamentalists who trade on the fundamental value and the trend followers who extrapolate the market price trend based on weighted moving average price over a finite time horizon. The herding behavior among the agents is characterized by the master equation. The market price is determined by a market maker who adjusts the market price to the excess demand from the fundamentalists and trend followers, together with a noisy demand. The continuous-time setup chosen in the paper not only mathematically facilitates modeling the time horizon of the historical price information used by the trend followers, but also easily accommodates the herding behavior through the master equation with endogenously determined volatility.

We first examine the dynamics of the underlying deterministic model. Differently from the adaptive switching model in He and Li (2012), we find that the herding mechanism does not affect the local stability of the steady state fundamental price, although it does affect the nonlinear behavior. Based on the analysis of the deterministic dynamics, we then study the joint impact of trend chasing, herding, and switching on the volatility of market price and return of the stochastic model. We find that the trend chasing based on historical prices over a long time horizon always leads to high volatility in both market prices and stock returns, which is underlined by the destabilizing effect of the trend chasing. Also, herding and switching have very different effect on the market volatility. A strong herding contributes to high

fluctuation in market fractions and market price and hence generates high volatilities in prices and returns, while a more intense switching reduces the return volatility and has a non-monotonic effect on the volatilities of market fractions and prices. More interestingly, we observe a “hump” shaped volatility in the market fraction and an “U”-shaped price volatility as the switching intensity increases.

We explore further the potential of the model to generate the power-law behavior in volatility. Following the HAMs literature, we find that it is the interaction of the nonlinear dynamics of the underlying deterministic model and the noises that generates the power-law behavior. We examine the impact of different noises (including fundamental noise, market noise and market fraction noise), time horizon, switching and herding on the ACs of the returns, absolute returns, and squared returns. We find that market noise is the main driving force in generating the power-law behavior. The AC patterns become more significant as the time horizon and herding increase, but non-monotonic with the switching. Specifically, an initial increase in the switching intensity leads to an increase in the significant levels of ACs but a decrease as the switching increases further. In general, it is the combination of switching and herding, together with the market noise, that generates realistic power-law behavior.

The paper is organized as follows. We first introduce a stochastic HAM of asset pricing in continuous-time with trend chasing, herding, switching and heterogeneous beliefs in Section 2. In Section 3, we apply stability and bifurcation theory of delay differential equations, together with numerical analysis of the nonlinear system, to examine the impact of herding, switching and time horizon used by the trend followers on market stability. The effect of and different role played by trend chasing, herding and switching on market volatility and power-law behavior are then discussed in Sections 4 and 5, respectively. Section 6 concludes. All proofs and some additional results are given in the Appendices.

2. THE MODEL

Consider a financial market with a risky asset (such as stock market index) and let $P(t)$ be the (cum dividend) price of the risky asset at time t . Following the standard

approach of HAMs, see, for example, Brock and Hommes (1998), we assume that the market consists of fundamentalists who trade according to the fundamental value of the risky asset, trend followers who trade based on price trend of a weighted moving averages of historical prices over a time horizon, and a market maker who clears the market by providing liquidity. The behavior of the fundamentalists and trend followers is modeled as usual. Different from the discrete-time HAMs in the literature (for example, Chiarella and He, 2002 and 2003), we consider a continuous-time setup to accommodate different time horizon used by the trend followers and the stochastic master equation characterizing the herding behavior of agents. For completeness, we introduce the demand functions of the fundamentalists and the trend followers briefly and refer the reader to He and Li (2012) for details.

The fundamentalists believe that the market price $P(t)$ is mean-reverting to the fundamental value $F(t)$ that can be estimated based on fundamental analysis. They buy (sell) the stock when the current price $P(t)$ is below (above) the fundamental value $F(t)$. For simplicity, we assume that the demand of the fundamentalists, $Z_f(t)$ at time t , is proportional to the deviation of the market price $P(t)$ from the fundamental value $F(t)$, namely,

$$Z_f(t) = \beta_f[F(t) - P(t)], \quad (2.1)$$

where $\beta_f > 0$ is a constant, measuring the speed of mean-reversion of the market price to the fundamental value, weighted by the risk tolerance of the fundamentalists. To focus on the price dynamics, we simply assume that the fundamental value follows a stochastic process¹

$$dF(t) = \frac{1}{2}\sigma_F^2 F(t)dt + \sigma_F F(t)dW_F(t), \quad F(0) = \bar{F}, \quad (2.2)$$

where $\sigma_F > 0$ represents the volatility of the fundamental return and $W_F(t)$ is a standard Wiener process.

¹It follows from Eq. (2.2) that the fundamental return defined by $d(\ln(F(t))) = \sigma_F dW_F(t)$ is a pure white noise process following a normal distribution with mean of 0 and standard deviation of $\sigma_F \sqrt{dt}$. This implies that any non-normality and volatility clustering of market returns that generated by the model are not carried from the fundamental returns.

The trend followers believe that the future market price follows a price trend $u(t)$. When the current price is above (below) the trend, the trend followers believe the price will rise (fall) and hold a long (short) position of the risky asset. We assume that the demand of the trend followers is given by

$$Z_c(t) = \tanh [\beta_c(P(t) - u(t))]. \quad (2.3)$$

The S -shaped hyperbolic demand function capturing the trend following behavior is well documented in the HAM literature (see, for example, Chiarella et al. 2009), where the parameter β_c represents the extrapolation rate of the trend followers on the future price trend. The limited position may reflect the wealth constraint or the cautiousness of the trend followers when the price deviates from the trend significantly. Among various estimators of the price trend used in practice, we assume that the price trend $u(t)$ at time t is calculated by an exponentially decaying weighted average of historical prices over a time horizon $[t - \tau, t]$,

$$u(t) = \frac{k}{1 - e^{-k\tau}} \int_{t-\tau}^t e^{-k(t-s)} P(s) ds, \quad (2.4)$$

where time delay $\tau \in [0, \infty)$ represents a time horizon used to calculate the price trend, and $k > 0$ is a decay rate. Equation (2.4) implies that, when forming the price trend, the trend followers believe the more recent prices contain more information about the future price movement so that the weights associated to the historical prices decay exponentially with a decay rate of k . In particular, when $k \rightarrow 0$, the price trend $u(t)$ in equation (2.4) is simply given by the standard moving average with equal weights,

$$u(t) = \frac{1}{\tau} \int_{t-\tau}^t P(s) ds. \quad (2.5)$$

When $k \rightarrow \infty$, all the weights go to the current price so that $u(t) \rightarrow P(t)$. For the time delay, when $\tau \rightarrow 0$, the trend followers regard the current price as the price trend. When $\tau \rightarrow \infty$, they use all the historical prices to form the price trend

$$u(t) = \frac{1}{k} \int_{-\infty}^t e^{-k(t-s)} P(s) ds. \quad (2.6)$$

In general, for $0 < k, \tau < \infty$, equation (2.4) can be expressed as a delay differential equation with time delay τ ,

$$du(t) = \frac{k}{1 - e^{-k\tau}} [P(t) - e^{-k\tau} P(t - \tau) - (1 - e^{-k\tau})u(t)] dt. \quad (2.7)$$

Let $N_f(t)$ and $N_c(t)$ be the numbers of the fundamentalists and trend followers, respectively, at time t , satisfying $N_f(t) + N_c(t) = N$, a constant. Denote by $n_f(t) = N_f(t)/N$ and $n_c(t) = N_c(t)/N$ the market fractions of the fundamentalists and trend followers, respectively. The net profits of the fundamental and trend following strategies over a small time interval $[t, t + dt]$ are then measured by, respectively,

$$\pi_f(t)dt = Z_f(t)dP(t) - C_f dt, \quad \pi_c(t)dt = Z_c(t)dP(t) - C_c dt, \quad (2.8)$$

where $C_f, C_c \geq 0$ are constant costs of the strategies per time unit. The performances of the strategies are measured by cumulated and weighted net profits over time horizons $[t - \tau_i, t]$,

$$U_i(t) = \frac{\eta_i}{1 - e^{-\eta_i \tau_i}} \int_{t-\tau_i}^t e^{-\eta_i(t-s)} \pi_i(s) ds, \quad i = f, c, \quad (2.9)$$

where $\eta_i > 0$ and $\tau_i > 0$ for $i = f, c$ represent the decay parameter and time horizon used, respectively, to measure the performance of the fundamentalists and trend followers. Consequently,

$$dU_i(t) = \eta_i \left[\frac{\pi_i(t) - e^{-\eta_i \tau_i} \pi_i(t - \tau_i)}{1 - e^{-\eta_i \tau_i}} - U_i(t) \right] dt, \quad i = f, c. \quad (2.10)$$

Denote by $a(t)$ the transition probability of an agent switching from the trend follower to the fundamentalist and by $b(t)$ the probability of the inverse transition. Following Lux (1995), the probabilities can be quantified by

$$a(t) = v e^{\beta(U_f(t) - U_c(t))}, \quad b(t) = v e^{\beta(U_c(t) - U_f(t))}, \quad (2.11)$$

where β measures the switching intensity and $v > 0$ captures the intensity of herding explained in the following. Let $\zeta(t)$ denote the probability of observing a change of an agent from the trend follower to the fundamentalist and $\xi(t)$ denote the probability of recording the opposite transition. Both $\zeta(t)$ and $\xi(t)$ are assumed to be proportional

to the transition probability of the switching and the corresponding market fractions to capture the herding behavior. Then the transition rates can be expressed as

$$\zeta(t) = (1 - n_f(t))a(t) = v(1 - n_f(t))e^{\beta(U_f(t) - U_c(t))}, \quad (2.12)$$

$$\xi(t) = n_f(t)b(t) = vn_f(t)e^{\beta(U_c(t) - U_f(t))}. \quad (2.13)$$

Note that, when $\beta = 0$, a large v means a strong herding among the agents. Hereafter, we use β and v to measure the (performance based) switching and herding, respectively, among the agents. Following Lux (1995), the master equation measuring the variation of probability in a unit of time by taking the number of fundamentalists as a state variable is given by

$$\frac{dp(N_f, t)}{dt} = \zeta(t)p(N_f - 1, t) + \xi(t)p(N_f + 1, t) - [\zeta(t) + \xi(t)]p(N_f, t), \quad (2.14)$$

where $p(N_f, t)$ is the probability of recording a number of N_f fundamentalists at time t . Following Chiarella and Di Guilmi (2012), the dynamics of population evolution can be characterized by²

$$dn_f(t) = n_f(t)[-(\zeta(t) + \xi(t))n_f(t) + \zeta(t)]dt + \sigma_{n_f}dW_{n_f}(t), \quad (2.15)$$

where

$$\sigma_{n_f}(t) = \frac{\sqrt{\zeta(t)\xi(t)}}{\zeta(t) + \xi(t)}, \quad (2.16)$$

and $W_{n_f}(t)$ is the stochastic fluctuation component in the market population fraction of fundamentalists, which is assumed to be independent from the fundamental noises $W_F(t)$.

Finally, the price $P(t)$ at time t is adjusted by the market maker according to the aggregate market excess demand, that is,

$$dP(t) = \mu[n_f(t)Z_f(t) + n_c(t)Z_c(t)]dt + \sigma_M dW_M(t),$$

where $\mu > 0$ represents the speed of the price adjustment by the market maker, $\sigma_M > 0$ is a constant and $W_M(t)$ is a standard Wiener process capturing the random

²The derivation is given in Appendix A.

excess demand process either driven by unexpected market news or noise traders,³ which is independent of $W_F(t)$ and $W_{n_f}(t)$.

To sum up, the market price of the risky asset is determined according to the following stochastic delay differential system with three different time delays and three noise processes

$$\left\{ \begin{array}{l} dP(t) = \mu \left[n_f(t)Z_f(t) + (1 - n_f(t))Z_c(t) \right] dt + \sigma_M dW_M(t), \\ du(t) = \frac{k}{1 - e^{-k\tau}} \left[P(t) - e^{-k\tau}P(t - \tau) - (1 - e^{-k\tau})u(t) \right] dt, \\ dn_f(t) = vn_f(t) \left[(1 - n_f(t))^2 e^{\beta(U_f(t) - U_c(t))} - n_f^2(t) e^{\beta(U_c(t) - U_f(t))} \right] dt \\ \quad + \sigma_{n_f} dW_{n_f}(t), \\ dU_f(t) = \frac{\eta_f}{1 - e^{-\eta_f\tau_f}} \left[\pi_f(t) - e^{-\eta_f\tau_f}\pi_f(t - \tau_f) - (1 - e^{-\eta_f\tau_f})U_f(t) \right] dt, \\ dU_c(t) = \frac{\eta_c}{1 - e^{-\eta_c\tau_c}} \left[\pi_c(t) - e^{-\eta_c\tau_c}\pi_c(t - \tau_c) - (1 - e^{-\eta_c\tau_c})U_c(t) \right] dt, \\ dF(t) = \frac{1}{2}\sigma_F^2 F(t)dt + \sigma_F F(t)dW_F(t), \end{array} \right. \quad (2.17)$$

where $Z_f(t)$ and $Z_c(t)$ are defined by (2.1) and (2.3), respectively, and $\pi_i(t)$ is defined by (2.8) for $i = f, c$. The stochastic differential system (2.17) characterizes the market price dynamics with heterogeneity in trading strategies, trend chasing, switching, and herding.

In the following sections, we first conduct a stability analysis of the underlying deterministic model. Then we examine the impact of the interaction of the deterministic dynamics with the noises on the fluctuations of the market population fractions (of using different strategies) and market volatility in both prices and return. Furthermore, we explore the power-law behavior in volatility.

3. THE STABILITY ANALYSIS OF THE DETERMINISTIC MODEL

To understand the interaction of the nonlinear deterministic dynamics and the noise processes, we first study the local stability of the corresponding deterministic system. By assuming $F(t) = \bar{F}$, $\sigma_M = 0$ and considering the mean process of the

³The additive noise comes naturally when a demand from noise traders is introduced into the aggregated excess demand function.

market fraction of the fundamentalists, system (2.17) reduces to

$$\begin{cases} \frac{dP(t)}{dt} = \mu \left[n_f(t) \beta_f [F(t) - P(t)] + (1 - n_f(t)) \tanh [\beta_c (P(t) - u(t))] \right], \\ \frac{du(t)}{dt} = \frac{k}{1 - e^{-k\tau}} \left[P(t) - e^{-k\tau} P(t - \tau) - (1 - e^{-k\tau}) u(t) \right], \\ \frac{dn_f(t)}{dt} = v n_f(t) \left[(1 - n_f(t))^2 e^{\beta(U_f(t) - U_c(t))} - n_f^2(t) e^{\beta(U_c(t) - U_f(t))} \right], \\ \frac{dU_f(t)}{dt} = \frac{\eta_f}{1 - e^{-\eta_f \tau_f}} \left[\pi_f(t) - e^{-\eta_f \tau_f} \pi_f(t - \tau_f) - (1 - e^{-\eta_f \tau_f}) U_f(t) \right], \\ \frac{dU_c(t)}{dt} = \frac{\eta_c}{1 - e^{-\eta_c \tau_c}} \left[\pi_c(t) - e^{-\eta_c \tau_c} \pi_c(t - \tau_c) - (1 - e^{-\eta_c \tau_c}) U_c(t) \right], \end{cases} \quad (3.1)$$

where

$$\pi_i(t) = \mu Z_i(t) \left[n_f(t) Z_f(t) + (1 - n_f(t)) Z_c(t) \right] - C_i, \quad i = f, c.$$

The system has a steady state⁴

$$Q := (P, u, n_f, U_f, U_c) = (\bar{F}, \bar{F}, \frac{1}{1 + e^{\beta(C_f - C_c)}}, -C_f, -C_c),$$

in which the market price is given by the fundamental value. We call Q the fundamental steady state of the system (3.1). At the fundamental steady state, the market fraction of fundamentalists becomes $n_f^* = \frac{1}{1 + e^{\beta(C_f - C_c)}}$. When $C_f = C_c$, $n_f^* = n_c^* = 0.5$, meaning that the market fractions at the fundamental steady state is independent of the switching intensity β and the herding parameter v . However, when the fundamental strategy costs more, that is $C_f > C_c$, then $n_c^* > n_f^*$, meaning that there are more trend followers than the fundamentalists at the fundamental steady state.

Denote $\gamma_f = \mu n_f^* \beta_f$ and $\gamma_c = \mu(1 - n_f^*) \beta_c$. The local stability and bifurcation of the fundamental steady state with respect to the time delay of system (3.1) are summarized in the following proposition.⁵

Proposition 3.1. *The fundamental steady state Q of system (3.1) is*

⁴In addition, the line $P = u$, $n_f = 0$, $U_f = -C_f$, $U_c = -C_c$ is a steady state line of the system. This means that the system has infinite many steady states. Near the line, the solution with different initial values converge to different steady states on the line. Hence the line is locally attractive. A similar result is found in He et al. (2009) and He and Zheng (2010).

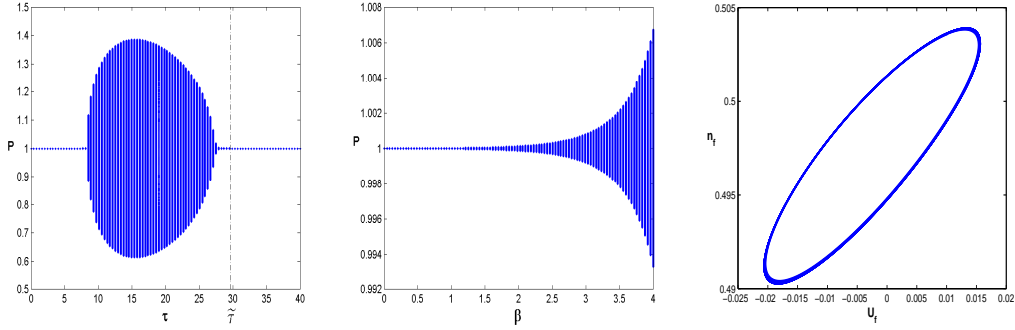
⁵The definitions of parameters $\tau_0, \tilde{\tau}$, functions $S_n^\pm(\tau)$ and the proof of Proposition 3.1 are given in Appendix B.

- (i) *asymptotically stable for $\tau \in [0, \tau_0)$;*
- (ii) *asymptotically stable for $\tau > \tilde{\tau}$ when $\gamma_f + k > \gamma_c$;*
- (iii) *unstable for $\tau > \tilde{\tau}$ when $\gamma_f + k < \gamma_c$.*

In addition, system (3.1) undergoes Hopf bifurcations at the zero solutions of functions $S_n^\pm(\tau)$.

Proposition 3.1 implies that the fundamental steady state is stable for either small or large time delay when the market is dominated by the fundamentalists (in the sense of $\gamma_f + k > \gamma_c$). Otherwise, when the trend followers become more active comparing to the fundamentalists (in the sense of $\gamma_c > \gamma_f + k$), the fundamental steady state becomes unstable through Hopf bifurcations when time delay increases. Meanwhile, when the trend followers put more weights to the most recent historical prices (so that k is large), the fundamental price is stabilized. In line with the discrete-time HAMs, when the time horizon is small, the insignificant price trend, resulting in weak trading signals for the trend followers, limits the destabilizing activity of the chartists. Consequently, the fundamentalists dominate the market and the market price is stabilized to the fundamental value. However, Proposition 3.1 also indicates a very interesting phenomenon of the continuous-time model that is not easy to obtain in discrete-time models, which is the stability switching when the fundamentalists dominate the market.⁶ That is, the system becomes unstable as time delay increases initially, but the stability can be recovered when the time delay becomes large enough. Intuitively, when time horizon is small, the price trend becomes less significant, which limits the activity of the trend followers. As the time horizon increases, the price trend becomes more sensitive to market price change and hence the trend followers become more active, which destabilizes the market. However, as the time horizon becomes very large, the price trend becomes smooth and less sensitive to price changes. Therefore the trend followers become less active and then, because of the dominance of the fundamentalists, the market becomes stable.

⁶This phenomenon is also observed in the continuous-time model in He et al. (2009), He and Zheng (2010), and He and Li (2012).



(a) Price bifurcation in τ (b) Price bifurcation in β (c) Phase plot of U_f and n_f

FIGURE 3.1. (a) The bifurcation of the market prices with respect to τ with $\beta = 1$; (b) The bifurcation of market price with respect to β with $\tau = 8$; (c) The phase plot of the relationship between the fitness U_f and the market fraction n_f with $\tau = 16$ and $\beta = 1$.

By simulating the nonlinear model (3.1),⁷ Fig. 3.1 verifies the stability results in Proposition 3.1. Fig. 3.1 (a) plots the bifurcation diagram of the market price of model (3.1) with respect to τ , showing that the fundamental steady state is stable for $\tau \in [0, \tau_0) \cup (\tau_1, \infty)$ and Hopf bifurcations occur at $\tau = \tau_0 \approx 8.5$ and $\tau = \tau_1 \approx 27$. Fig. 3.1 (b) plots the price bifurcation diagram with respect to the switching intensity parameter β . It shows that the steady state is stable when the switching intensity β is low, but becomes unstable as the switching intensity increases, bifurcating to stable periodic price with increasing fluctuations. This result shares the same spirit of rational routes to complicated price dynamics in the discrete-time Brock and Hommes (1997, 1998) framework. Fig. 3.1 (c) illustrates the phase plot of (U_f, n_f) , showing the positive relation between the fitness U_f and the market fraction n_f .

Fig. 3.2 provides further insights into the nonlinear dynamics of the market price and market fraction of the fundamentalists for $\tau = 16$ when the fundamental steady state is unstable. Fig. 3.2 (a) illustrates the time series of the market prices $P(t)$ and the market fraction $n_f(t)$ of fundamentalists. It shows that the market fractions

⁷Unless specified otherwise, the following set of parameters are used in all the simulations in this paper: $k = 0.05, \mu = 1, \beta_f = 1.4, \beta_c = 1.4, \beta = 1, C_f = 0.05, C_c = 0.03, \eta_f = 0.5, \eta_c = 0.6, \tau = 16, \tau_f = 10, \tau_c = 5, v = 0.5$ and $\bar{F} = 1$.

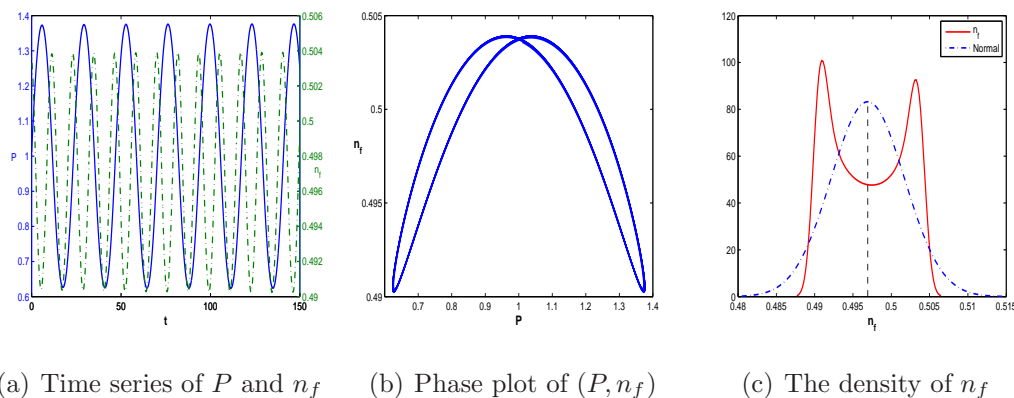


FIGURE 3.2. (a) The time series of the market prices $P(t)$ (the blue solid line) and the market fraction $n_f(t)$ of fundamentalists (the green dash dot line); (b) the phase plot of $(P(t), n_f(t))$; and (c) the density distribution of the market fraction $n_f(t)$. Here $\tau = 16$.

fluctuate with the market price. Fig. 3.2 (b) presents the phase plot of $(P(t), n_f(t))$ showing that the price and fraction converge to a *figure-eight shaped* attractor, a phenomenon also observed in the discrete-time model in Chiarella *et al* (2006). Fig. 3.2 (c) plots the corresponding density distribution of the market fraction $n_f(t)$ of the fundamentalists. The *twin-peak-shaped* density distribution implies that, when the fundamental price is unstable, the market fractions tend to stay away from the steady state market fraction level most of the time and a mean of $n_f(t)$ below 0.5 clearly indicates the dominance of the trend chasing strategy.

Interestingly, the herding parameter v does not affect the local stability of the fundamental price and the deterministic price dynamics are very similar to the results in He and Li (2012) (HL model hereafter) without herding. To illustrate the herding effect, we compare this model with the HL model. In Appendix C, corresponding to Figs. 3.1 and 3.2, we present Figs. C.1 and C.2, respectively, for the HL model. It is observed that both models exhibit similar deterministic dynamics in price and market fraction. However, the nonlinear dynamics can be affected by the herding parameter v . Comparing Fig. 3.2 with $v = 0.5$ and Fig. C.3 with $v = 0.1$ in Appendix C, we observe that, when herding among agents are not very strong indicating by a decrease in the parameter v , the fluctuations of market

price and, in particular, the market fractions of the fundamentalists are reduced. In other words, a strong herding among agents contributes to high fluctuations in market fractions, which then results in high volatility in market prices. This effect is further examined for the stochastic model in the following section.

4. PRICE BEHAVIOR OF THE STOCHASTIC MODEL

In this section, through numerical simulations, we examine the price dynamics of the stochastic model by focusing on the impact of the three parameters, the time horizon τ , herding v , and switching intensity β , and the two noisy processes characterized by σ_F and σ_M , on market volatility in both price and return. The analysis provides further insights into the different roles played by herding and switching in financial markets.

We first explore the interaction between the underlying deterministic dynamics and the two noisy processes by choosing two different values of time horizons. For the deterministic model (3.1), Fig. 3.1 (a) shows that the time horizon can affect the stability of the fundamental price. In particular, the fundamental steady state is stable for $\tau = 3$ and unstable for $\tau = 16$, leading to periodic fluctuations of the market price. For the stochastic model, we choose the volatility of the fundamental price $\sigma_F = 0.12$ and the volatility of the market noise $\sigma_M = 0.15$.⁸ With the same random draws of the fundamental price and market noise processes, we plot the fundamental price (the blue dotted line) and the market prices (the red solid line) in Fig. 4.1 for the two different values of τ under the same set of parameters for Fig. 3.1 (a). For $\tau = 3$ and $\tau = 16$, Figs. 4.1 (a) and (b) show that the market prices fluctuate around the fundamental prices and the fluctuations for $\tau = 16$ are significantly larger than that for $\tau = 3$. This observation is further supported by the distribution plots of the deviations of the market prices from the fundamental prices $P(t) - F(t)$ in Fig. 4.1 (c) for $\tau = 3$ and in Fig. 4.1 (d) for $\tau = 16$. With the standard deviations of 0.8158 for $\tau = 3$ and 1.2091 for $\tau = 16$, the

⁸The constraint $n_f(t) \in [0, 1]$ is imposed when simulating the stochastic system. In all the simulations, time unit is a year, and the time step corresponds to one day. Unless specified otherwise, an annual volatility of $\sigma_F = 0.12$ and a market noise volatility of $\sigma_M = 0.15$ are used in the paper.

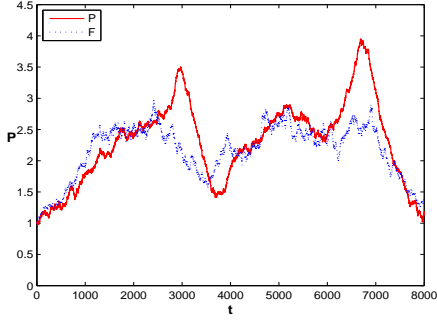
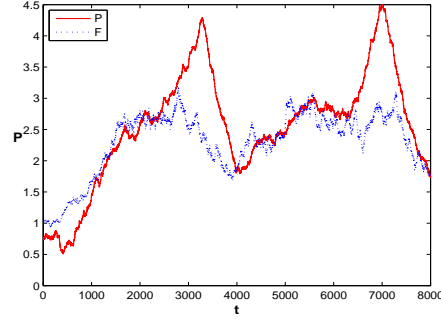
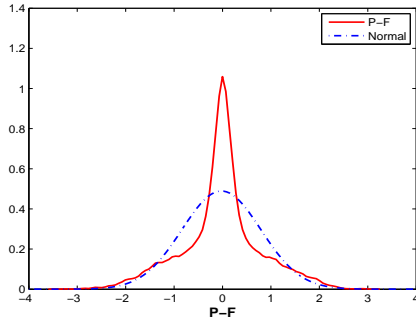
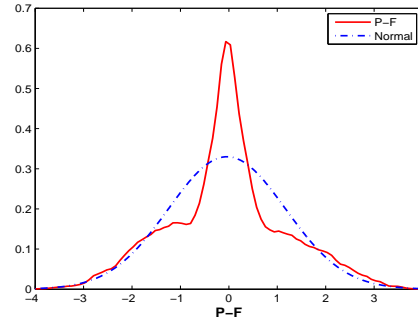
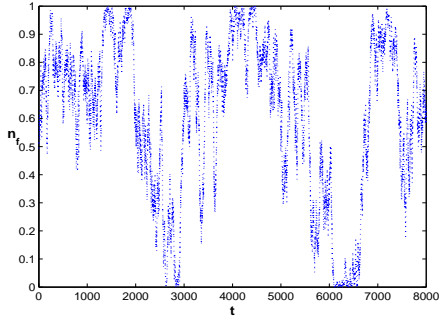
(a) Prices for $\tau = 3$ (b) Prices for $\tau = 16$ (c) Price deviation density for $\tau = 3$ (d) Price deviation density for $\tau = 16$

FIGURE 4.1. The time series of the fundamental price $F(t)$ (the blue dotted line) and the market prices $P(t)$ (the red solid line) with (a) $\tau = 3$ and (b) $\tau = 16$, and the distributions of the deviations of the market prices from the fundamental prices $P(t) - F(t)$ with (c) $\tau = 3$ and (d) $\tau = 16$. Here $\sigma_F = 0.12$ and $\sigma_M = 0.15$.

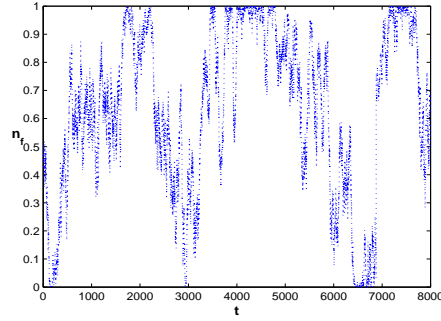
deviations are more spread for $\tau = 16$. This is partially underlined by the change in the stability of the underlying deterministic dynamics. Further simulations (not reported here) show that when the time horizon increases further to the stabilizing range indicated by Fig. 3.1 (a), the fluctuations of the market price deviations from the fundamental price become even more significant. This result illustrates that, when the underlying deterministic dynamics are stable, the stochastic dynamics can become very *unstable* with large fluctuations in price deviations. This is mainly due to the slow convergence of the market price to the fundamental price of the underlying deterministic model and its interaction with the fundamental and market

noises. Therefore, an increase in time horizon increases the deviations of the market price from the fundamental price and the fluctuations of the market price.

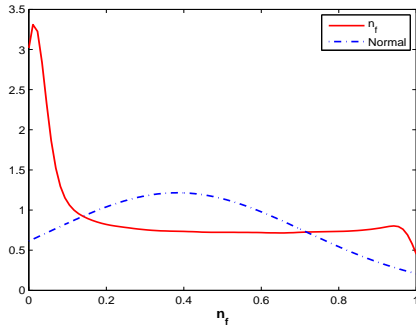
To examine the effect of herding, with the same parameters and random draws, Fig. D.1 in Appendix D illustrates the corresponding results of the HL model without herding. It displays similar price patterns but with less significant deviations in prices (with the standard deviations of 0.2477 for $\tau = 3$ and 0.4703 for $\tau = 16$). The comparison implies that the herding behavior contributes to the excess volatility of the market price, a higher acceleration to market highs and lows, and a quicker mean reversion to the fundamental price.



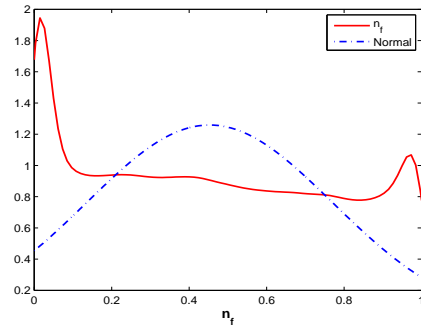
(a) Market fraction for $\tau = 3$



(b) Market fraction for $\tau = 16$



(c) Market fraction density for $\tau = 3$



(d) Market fraction density for $\tau = 16$

FIGURE 4.2. The time series of the fractions of the fundamentalists with (a) $\tau = 3$ and (b) $\tau = 16$ and the corresponding distributions with (c) $\tau = 3$ and (d) $\tau = 16$. Here $\sigma_F = 0.12$ and $\sigma_M = 0.15$.

We also plot the corresponding time series and distributions of the market fractions of the fundamentalists in Fig. 4.2 for both $\tau = 3$ (the left panel) and $\tau = 16$ (the right panel). Comparing to Fig. D.2 in Appendix D for the HL model (with

switching but without herding), we obtain two observations. (i) With herding, the market fractions fluctuate wildly between 0 and 1, are almost uniformly distributed except for the spike near 0, as shown in Fig. 4.2. However, without herding, the market fractions fluctuate around the steady state ($n_f = 0.495$) in a small range (from 35% to 75%), illustrated by both the time series and distribution plots in Fig. D.2. This implies that the herding effect dominates the switching effect in generating high fluctuations in the market fractions. (ii) For the market fraction, by comparing the time series and distributions of the market fractions in Fig. 4.2 for $\tau = 3$ (the left panel) and $\tau = 16$ (the right panel), the effect of the time horizon is not highly significant. However, note the small pick near 1 for $\tau = 16$ in Fig. 4.2 (d), the time horizon does affect the market fractions. In general we observe that herding leads to the dominance of the trend followers in the market, while an increase in time horizon reduces the dominance of the trend followers and increases the dominance of the fundamentalists.

One of the innovative features of the model is that the market fraction is determined by the stochastic master equation (2.15) with endogenously determined volatility (2.16). Because of the dependence of the transition rates on the time horizon τ , the switching β and herding v , the volatility also depends on τ, β and v . Therefore the variations in the market fractions can affect the deviation of the market price from the fundamental price and return volatility. In general, the impact on market volatility can be different for price and return. To provide further insights into the different roles of the herding, switching and time horizon on fluctuations of market fractions and market volatility, we consider three cases focusing each of the three parameters of τ, v and β with some typical choices of the other two parameters. In each case, we examine the impact on the endogenously determined volatility of the market fractions (of the fundamentalists), σ_{n_f} , the volatility of the price deviations, $\sigma(P - F)$, and the volatility of the market returns, $\sigma(r)$. Based on the common set of the parameters, we run 100 simulations for each parameter combinations and plot the averages of $\sigma_{n_f}, \sigma(P - F)$ and $\sigma(r)$ and denote by $\bar{\sigma}_{n_f}, \bar{\sigma}(P - F)$ and $\bar{\sigma}(r)$, respectively.

4.1. The effect of the time horizon. For $\tau \in [0, 20]$, we conduct Monte Carlo simulations and plot $\bar{\sigma}_{n_f}$, $\bar{\sigma}(P - F)$ and $\bar{\sigma}(r)$ in the upper panel for $\beta = 0.1, 1, 2$ and the lower panel for $v = 0.01, 0.1, 0.5$ in Fig. 4.3, from which we can draw two observations about the volatility.

(i) *All the volatilities, in terms of $\bar{\sigma}_{n_f}$, $\bar{\sigma}(P - F)$ and $\bar{\sigma}(r)$, increase as the time horizon and herding increase.*⁹ For $\tau = 0$, the trend followers are not participating in the market and the lower volatilities simply reflect the resulting volatilities of the market noise and fundamental noise. In this case, herding plays no role in market volatility in both the price and return (as indicated by the constant volatility for various β and v when $\tau = 0$ in the middle and right panels). As τ increases, all the volatilities increase significantly. This effect becomes less significant for the market fraction volatility $\bar{\sigma}_{n_f}$ after an initial increase in τ (from 0 to about 5). The increase in the price deviation volatility in the time horizon is underlined by the destabilizing effect of τ on the underlying deterministic dynamics. Because of the fluctuations in the market fractions, we observe an increase in the return volatility as well. We also observe the same effect as the herding parameter v increases in the low panel of Fig. 4.3. Due to the independence of the local stability to the herding, this result reflects more on the interaction of nonlinearity and noises.

(ii) *Switching has a non-monotonic impact on the volatilities of market fraction and market price, but reduces the return volatility as the switching intensity increases.* Figs. 4.3 (a) and (b) show that the volatilities of market fraction and market price deviations are non-monotonic as β increases. An initial increase in β leads to an increase in the volatility of market fraction but the effect reverses as β increases further, implying a “hump” shaped effect on the volatilities of market fraction. However, we also observe an “U”-shaped effect on the volatility of price deviation. This non-monotonic feature is explored further in the following discussion. Furthermore, Fig. 4.3 (c) shows that the volatilities of market return decrease in β , an opposite effect to the herding. Such different impact of the switching and

⁹Except $\bar{\sigma}_{n_f}$ for time horizon and large β that the volatility seems to decrease as time horizon increases further. This is related to the non-monotonic impact of β on the volatilities illustrated in Fig. 4.5.

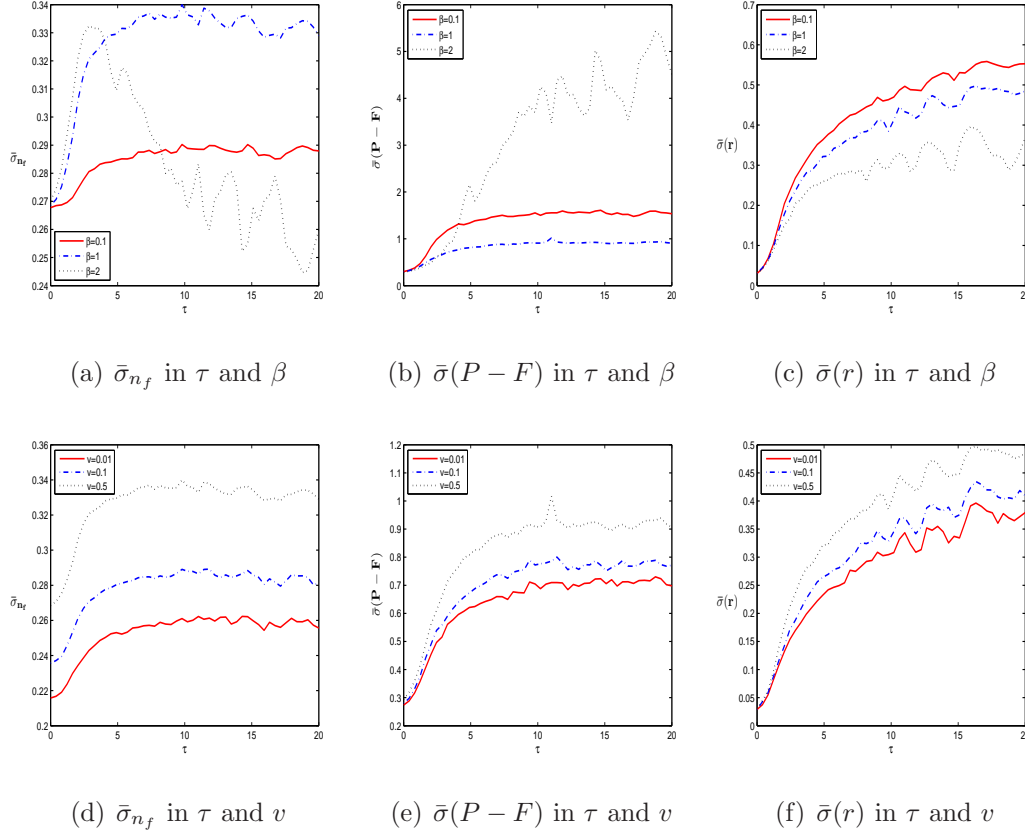


FIGURE 4.3. The average variations of market fraction volatility $\bar{\sigma}_{n_f}$ (a) and (d), price deviation volatility $\bar{\sigma}(P - F)$ (b) and (e), and return volatility $\bar{\sigma}(r)$ (c) and (f) with $\beta = 0.1, 1, 2$ (and $v = 0.5$) in the upper panel and $v = 0.01, 0.1, 0.5$ (and $\beta = 1$) in the lower panel, respectively, with respect to time horizon $\tau \in [0, 20]$.

herding on volatility of price and return has not been explored in the literature. This provides some insights into the distinct role played by switching and herding in market volatility. If one argues that switching is a more rational behavior than herding, the result indicates that herding can increase the return volatility, while switching can reduce the return volatility.

4.2. The effect of the herding. For $v \in [0, 1]$, we plot $\bar{\sigma}_{n_f}$, $\bar{\sigma}(P - F)$ and $\bar{\sigma}(r)$ for $\tau = 0, 3, 16$ in the upper panel and for $\beta = 0.1, 1, 2$ in the lower panel in Fig. 4.4 based on Monte Carlo simulations. It provides consistent observations as in the previous case. When the herding parameter v and the time horizon τ increase, all

the volatilities increase. However an increase in the switching parameter reduces the average volatility of return, as illustrated in Fig. 4.4 (f). Also Figs. 4.4 (d) and (e) show that the switching has a non-monotonic impact on the volatilities of market fraction and price. Consistently with the previous observations, the herding increases the fluctuations of the market price from the fundamental price and return fluctuations.

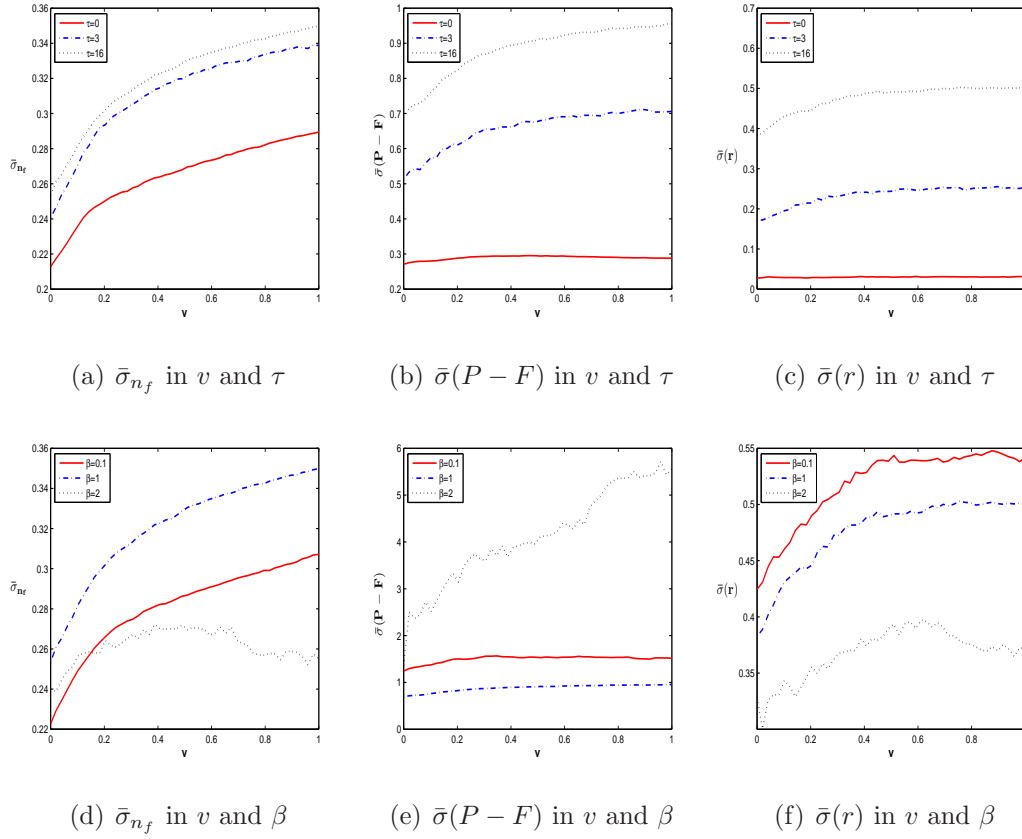


FIGURE 4.4. The average variations of market fraction volatility $\bar{\sigma}_{n_f}$ (a) and (d), price deviation volatility $\bar{\sigma}(P - F)$ (b) and (e), and return volatility $\bar{\sigma}(r)$ (c) and (f) with $\tau = 0, 3, 16$ (and $\beta = 1$) in the upper panel and $\beta = 0.1, 1, 2$ (and $\tau = 16$) in the lower panel, respectively, with respect to $v \in [0, 1]$.

4.3. The effect of switching. For $\beta \in [0, 2]$, we plot $\bar{\sigma}_{n_f}, \bar{\sigma}(P - F)$ and $\bar{\sigma}(r)$ for $\tau = 0, 3, 8, 16$ in the upper panel and for $v = 0.01, 0.1, 0.5$ in the lower panel in Fig. 4.5 based on Monte Carlo simulations. When $\beta = 0$, the market fractions are driven

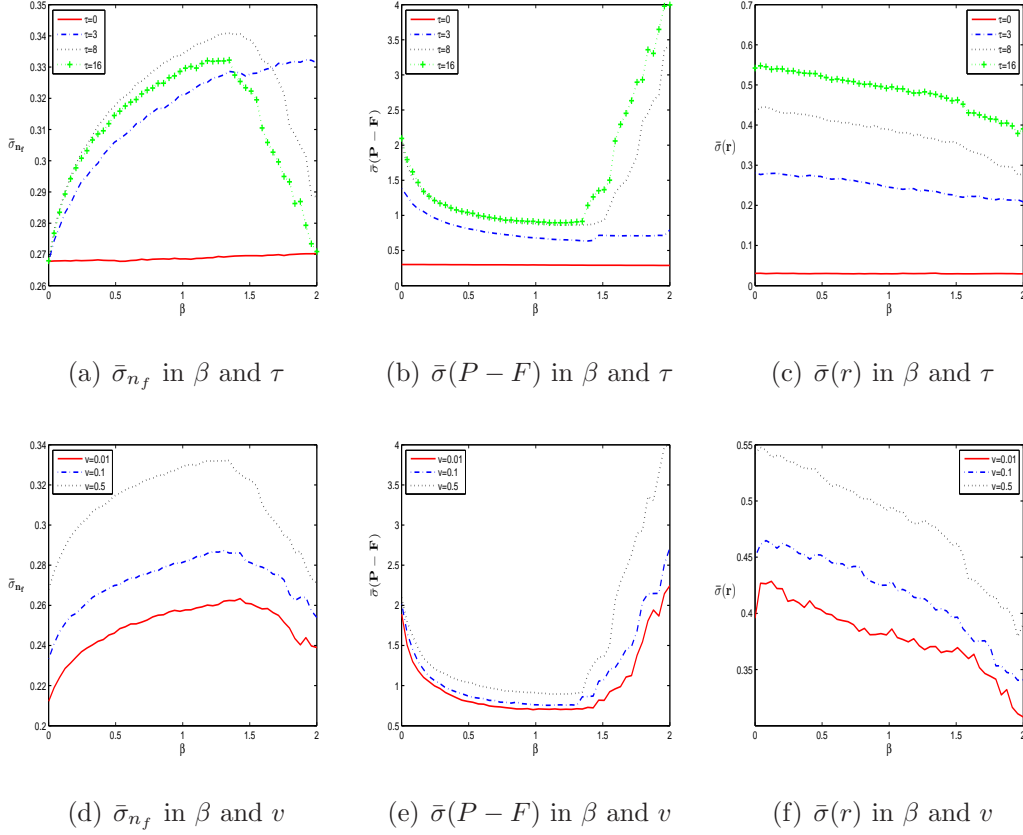


FIGURE 4.5. The average variations of market fraction volatility $\bar{\sigma}_{n_f}$ (a) and (d), price deviation volatility $\bar{\sigma}(P - F)$ (b) and (e), and return volatility $\bar{\sigma}(r)$ (c) and (f) with $\tau = 0, 3, 16$ (and $v = 0.5$) in the upper panel and $v = 0.01, 0.1, 0.5$ (and $\tau = 16$) in the lower panel, respectively, with respect to $\beta \in [0, 1]$.

purely by the herding behavior. In general, we observe consistent results in terms of the impact of time horizon and herding obtained in the previous two cases. However, there is a significantly non-monotonic relationship between the volatilities and the switching intensity β . We observe a “hump” shaped volatility in the market fraction (in Figs. 4.5 (a) and (d)), an “U”-shaped price volatility (in Figs. 4.5 (b) and (e)), and a decreasing volatility in returns (in Figs. 4.5 (c) and (f)) as the switching parameter β increases. Interestingly, an initial increase in the switching leads to higher market fraction volatility and lower market price volatility, following by the decreasing volatility in fractions and the increasing volatility in price deviations when β increases beyond certain threshold value. This result explains the phenomenon in

Fig. 4.3 (a) that for large β , an increase in the time horizon τ leads to an initial increase in the market fraction volatility, but a dramatic decline when τ increases further. It also implies that large fluctuations in the market fractions reduce the market price deviation from the fundamental price when the switching intensity is low, but the effect becomes opposite when the switching intensity is high. However, a strong switching always reduce return volatility. Therefore, we can have a market with high fluctuations in market price and low volatility in return volatility at the same time.

In summary, the impact on the volatility can be very different for price and return. The trend chasing over a long time horizon and herding always lead to high volatility. However the switching and herding have an opposite effect on the market return volatility. Although the switching has significant and non-monotonic impact on the market volatility, it can actually reduce return volatility. The analysis demonstrates different mechanisms of herding and switching in explaining volatilities in price and return.

5. POWER-LAW BEHAVIOR IN VOLATILITY

After exploring the impact of the time horizon, herding, and switching on market volatility in the previous section, we are now interested in their impact on the power-law behavior in volatility. It has been well explored in the HAM literature that it is the interaction of the nonlinear dynamics of the underlying deterministic model and the noises that generates the power-law behavior. Both switching and herding mechanisms have been explored, but a comparison among different mechanism is missing in the literature. This section is devoted to such a comparison. We first examine the impact of the noises and then of the time horizon, switching and herding on the ACs of the returns, absolute returns, and squared returns with the two noises.

To motivate the analysis, we first present the ACs of the returns, absolute returns, and squared returns for market daily closing price indices of the DAX 30, the FTSE 100, the NIKKEI 225, and the S&P 500 from 01/02/1984 to 31/07/2013 from Datastream in Fig. 5.1. Note that all the ACs for the returns are not significant, but they are significant and decaying for the absolute and squared returns.

This phenomenon is referred as the power-law behavior or long memory in market volatility in empirical literature, see He and Li (2007) and references cited there.

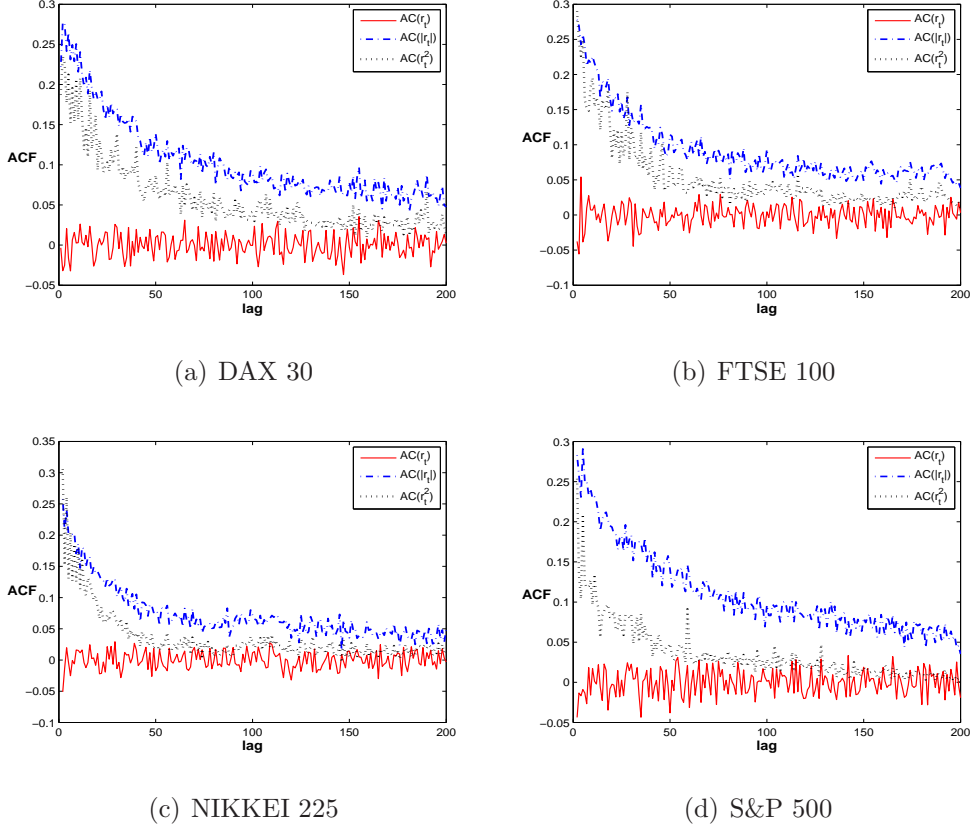


FIGURE 5.1. The ACs of the returns (the bottom lines), the squared returns (the middle lines) and the absolute returns (the upper lines) for (a) the DAX 30, (b) the FTSE 100, (c) the NIKKEI 225, and (d) the S&P 500.

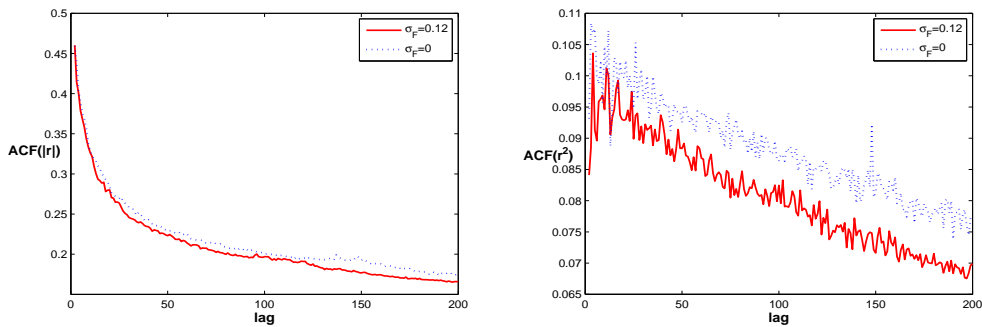
5.1. The effect of the noises. For the two exogenously given fundamental and market noises, we examine the impact of the noises by considering three combinations of (i) both the fundamental and market noises; (ii) the market noise only; and (iii) the fundamental noise only.

We first consider the effect of the fundamental and market noises with $\sigma_F = 0.12$ and $\sigma_M = 0.15$. Fig. E.1 in Appendix E represents the results of a typical simulation based on the same set of parameters in Fig. 4.1 with $\tau = 16$. The results demonstrate that the stochastic model established in this paper is able to generate market price

deviations from the fundamental value (in Fig. E.1 (a)), most of the stylized facts (including volatility clustering in Fig. E.1 (b), high kurtosis in in Fig. E.1 (c)), and the power-law behavior in volatility (insignificant autocorrelations (ACs) for returns in Fig. E.1 (d), but significant decaying ACs for the absolute returns and the squared returns in Figs. E.1 (e) and (f)) observed in financial markets.

Next we consider the effect of the market noise. With the same parameters and random seeds, Fig. E.2 in Appendix E shows that the model is able to generate a similar result to the previous case with the two noises, although the level of the significant ACs is lower. This implies that, even with a constant fundamental value, the model has a great potential in generating the power-law behavior. This result is significantly different from the switching HL model, in which the model is not able to generate the power-law behavior without fundamental noise.

Finally, we consider the effect of the fundamental noise. Fig. E.3 in Appendix E shows that the model is not able to generate the volatility clustering and the power-law behavior, which is consistent with the HL model. Meanwhile the market returns, absolute returns and squared returns exhibit highly significant ACs with strong decaying patterns, which is mainly due to the strong effect of the deterministic dynamics of the price process.



(a) The ACs of the absolute returns

(b) The ACs of the squared returns

FIGURE 5.2. The effect of the fundamental noise: the ACs of (a) the absolute returns and (b) the squared returns based on 100 simulations with $\tau = 16$, $v = 0.5$ for $\sigma_F = 0.12$ (the red solid line) and $\sigma_F = 0$ (the blue dash-dotted line).

To further investigate the effect of the fundamental noise on the AC patterns, Fig. 5.2 compares the ACs of the absolute returns and the squared returns for $\sigma_F = 0.12$ (the red solid line) and $\sigma_F = 0$ (the blue dash-dotted line) based on 100 simulations. The ACs of the absolute returns and squared returns are significant and decaying. However, with the market noise $\sigma_F = 0.12$, the ACs decay quickly than those for $\sigma_F = 0$. This indicates that the market noise plays a key role in generating the power-law behavior, although it is not the only factor, as argued in He and Li (2007).

The previous analysis on the effect of the time horizon, switching, and herding shows that they play different roles in generating volatility in market price and return. We now further investigate their effect on the the power-law behavior. Similarly, we consider three cases focusing each of the three parameters of τ, v and β . For each case, we examine the impact on the AC patterns of the absolute and squared returns. Based on the common set of the parameters, we run 100 simulations for each parameter combination and plot the averages ACs for the absolute and squared returns.¹⁰

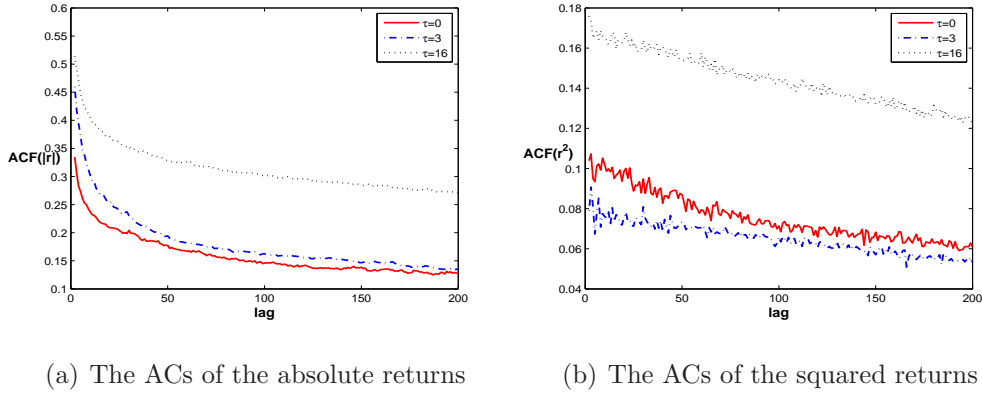


FIGURE 5.3. The effect of the time horizon: the ACs of (a) the absolute returns and (b) the squared returns based on 100 simulations with $v = 0.5$, $\beta = 1$, $\sigma_F = 0.12$, $\sigma_M = 0.15$ for $\tau = 0$, $\tau = 3$ and $\tau = 16$.

¹⁰The average ACs for the returns are insignificant in all three cases reported.

5.2. The effect of the time horizon. First, we present in Fig. 5.3 the effect of the time horizon on the AC patterns for the absolute returns (the left panel) and the squared returns (the right panel). We observe that the trend chasing based on different time horizons contributes to the significant decaying AC patters for both the absolute and squared returns. Also the significant levels of the ACs increase as the time horizon increases, in particular, when the time horizon is large. This suggests that a commonly observed slow decaying AC patterns in the discrete-time HAM literature (see for example He and Li (2007)) might be due to the long time horizons used for modeling the trend chasing. In other word, trend chasing based on short time horizons contributes to more realistic power-law behavior in volatility. Intuitively, technical analysis such as trend following strategy is mainly used for short-term investment comparing to the fundamental analysis for long-term investment. Therefore the trend chasing based on short-time horizon contributes to volatility in financial markets.

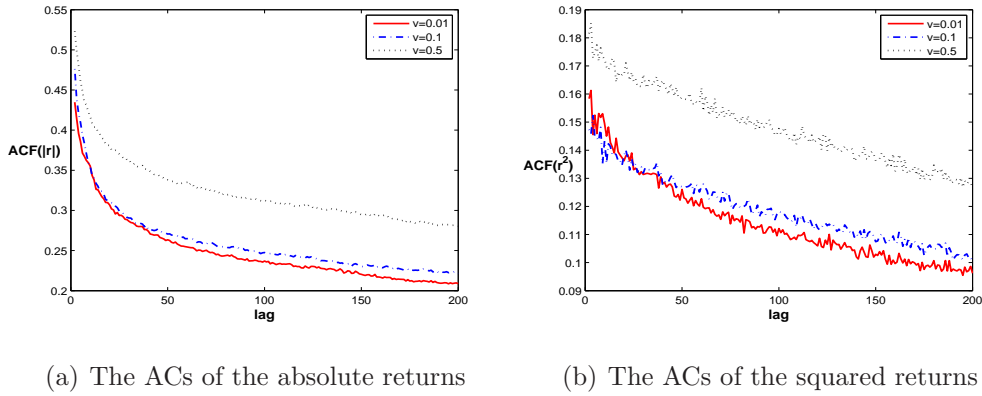
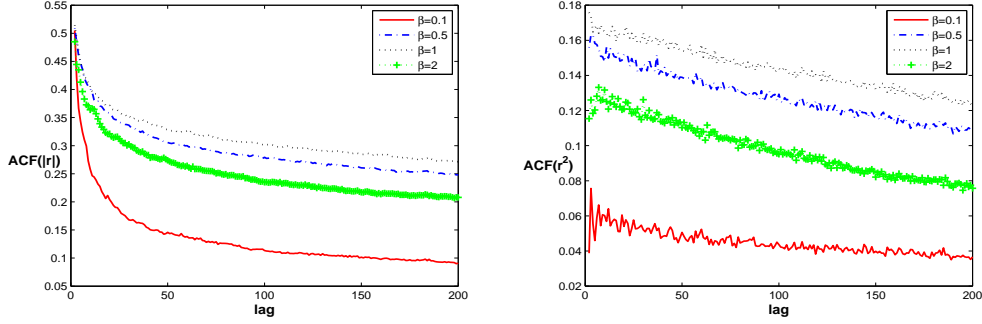


FIGURE 5.4. The effect of the herding: the ACs of (a) the absolute returns and (b) the squared returns based on 100 simulations with $\tau = 16$, $\beta = 1$, $\sigma_F = 0.12$, $\sigma_M = 0.15$ for $v = 0.01$, $v = 0.1$ and $v = 0.5$.

5.3. The effect of the herding. Second, we present in Fig. 5.4 the effect of the herding on the AC patterns for the absolute returns (the left panel) and the squared returns (the right panel), showing the contribution of the herding to the power-law behavior in volatility. Similar to the effect of the time horizon, an increase in the

herding increases the level of the significant ACs for both the absolute and squared returns. However, differently from the effect of the time horizon, the ACs decay quickly under the herding.



(a) The ACs of the absolute returns

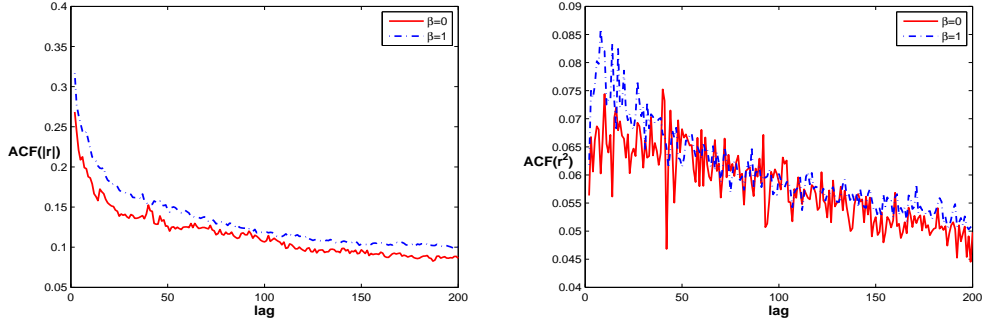
(b) The ACs of the squared returns

FIGURE 5.5. The effect of the switching: the ACs of (a) the absolute returns and (b) the squared returns based on 100 simulations with $v = 0.5$, $\tau = 16$, $\sigma_F = 0.12$, $\sigma_M = 0.15$ for $\beta = 0.1$, $\beta = 0.5$, $\beta = 1$ and $\beta = 2$.

5.4. The effect of the switching. Third, we present in Fig. 5.5 the effect of the switching on the AC patterns for the absolute and squared returns. It shows that the switching contributes to the power-law behavior. Interestingly, different from the effect of the time horizon and herding, the level of the significant ACs for both the absolute and squared returns is not monotonic with respect to the switching intensity β . The level increases significantly when β increases from 0.1 to 0.5, and then less significantly when β increases to 1, but decreases when β increases further to 2. In particular, the ACs for the absolute returns decay very quickly, comparing to the effect of the herding. This observation, together with the discussion in Section 4.3, suggests that an increase in the switching can reduce the return volatility and generate the power-law behavior at the same time. This provides further support on the explanatory power of the adaptive switching in financial markets initiated in Brock and Hommes (1998).

In summary, we have explored different mechanisms of the switching and herding on the market volatility and power-law behavior in particular. We show that both

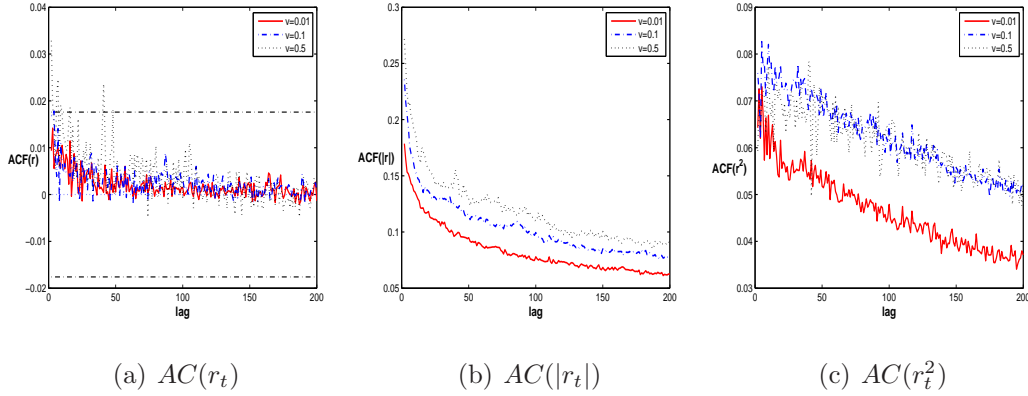
contribute to the power-law behavior, however the effect is monotonically increasing with the herding, but not monotonic for the switching.



(a) The ACs of the absolute returns

(b) The ACs of the squared returns

FIGURE 5.6. The ACs of (a) the absolute returns and (b) the squared returns based on 100 simulations with $v = 0.5$, $\tau = 0$, $\sigma_F = 0$, $\sigma_M = 0.15$ for $\beta = 0$ and $\beta = 1$.



(a) $AC(r_t)$

(b) $AC(|r_t|)$

(c) $AC(r_t^2)$

FIGURE 5.7. The ACs of (a) the market returns; (b) the absolute returns and (c) the squared returns based on 100 simulations with $\tau = 0$, $\beta = 0$, $\sigma_F = 0$, $\sigma_M = 0.15$ for $v = 0.01$, $v = 0.1$ and $v = 0.5$.

Finally, we are interested in the question that if a pure herding is enough to explain the power-law behavior by considering the case with $\sigma_F = 0$ and $\tau = 0$. In this case, the fundamental value becomes constant and the trend followers become naive traders who take the current price as the expected future market price and hence do not trade in the market anymore. Thus the market is driven by pure herding

mechanism. Fig. 5.6 illustrates the significant and decaying AC patterns in both the absolute and squared returns, although the AC level for the squared returns is significantly lower comparing to the cases discussed previously. Interestingly, there is no significant difference in the AC patterns of the absolute return between no switching ($\beta = 0$) and the switching ($\beta = 1$). This result is consistent with Alfarano et al. (2005) who show that a pure herding model with fundamentalists and noise traders can generate the power-law behavior. As a robustness check, we present Fig. 5.7 with three different values of v . Similar to Fig. 5.4, we observe more significant AC patterns as the herding parameter v increases.

Comparing to the AC patterns of the market indices in Fig. 5.1, we may argue that the switching generates similar AC patterns to the NIKKEI 225 and the S&P 500 with quickly decaying AC patterns, while the herding generates similar AC patterns to the DAX 30 and FTSE 300 with relatively slow decaying AC patterns. Whether this observation suggests different market behavior in different markets is an empirical and challenging question. Our analysis explores different mechanism in explaining volatility in different markets.

6. CONCLUSION

Market volatility is one of the most important features in financial markets and the question is what drives it. To answer this question, the most obvious way is to consider how agents behave in financial markets. Among various agents' behaviors in financial markets, trend chasing, adaptive switching, and herding are the most important ones as well documented and studied in the empirical literature. This paper incorporates these three behavioral elements into an asset pricing model in a continuous-time framework and shows that they all contribute to market volatility in different manner.

Most of the asset pricing models with heterogeneous agents are in discrete-time framework focusing on trend chasing over short time horizon and adaptive switching. Herding is commonly modeled by the master equation in a continuous-time setting. Therefore the roles of trend chasing, switching and herding in market volatility have been studied in separate frameworks. Within a continuous-time framework, this is

the first paper, to our knowledge, combining trend chasing based on different time horizon, switching and herding together to examine their roles on market volatility in price and return. We show that, both the herding and trend chasing based on long time horizon increase the fluctuations of market price deviation from the fundamental price and volatility of market return. With respect to the switching, it reduces the volatility in returns but leads to an “U”-shaped price volatility as the switching intensity increases. Therefore herding and switching have opposite effect on return volatility.

We also examine the explanatory power of the model in generating the power-law behavior in return volatility. We show that, although the trend chasing, switching and herding all contribute to the power-law behavior, the significant levels for the ACs increase in the time horizon and herding, but an initial increase and then decrease when the switching intensity increases. In addition, with the herding, the market noise plays an essential role in generating the power-law behavior.

The model proposed in this paper provides a unified framework to deal with trend chasing, switching and herding in financial markets. The results provide some further insights into different mechanism of generating bubbles and crashes, excess volatility, and power-law behavior in volatility. Whether a particular market is dominated by herding or switching is an empirical question which is left for future research.

APPENDIX A. ANALYTICAL SOLUTION FOR THE MASTER EQUATION

We solve the master equation using the approximation method introduced by Aoki (2002). Assume the fraction of fundamentalists in a given moment is determined by its expected mean (m), the drift, and, an additive fluctuation component s of order $1/N^{1/2}$ around this value. Thus we can write

$$\frac{N^f}{N} = m + \frac{1}{\sqrt{N}}s, \quad (\text{A.1})$$

where s is a standard white noise. The asymptotically approximate solution of the master equation is given by the system of coupled differential equations

$$\begin{aligned} \frac{dm}{dt} &= \zeta(t)m - [\zeta(t) + \xi(t)]m^2, \\ \frac{\partial Q}{\partial t} &= [2(\zeta(t) + \xi(t))m - \zeta(t)]\frac{\partial}{\partial s}(sQ(s, t)) \\ &\quad + \frac{m[\zeta(t) + m(\xi(t) - 1)]}{2}\frac{\partial^2}{\partial s^2}Q(s, t), \end{aligned} \quad (\text{A.2})$$

where $Q(s, t)$ is the transition density function of the spread s at time t . The first equation of (A.2) is a deterministic ordinary differential equation which displays logistic dynamics for the trend. The second equation is a second order stochastic partial differential equation, known as the *Fokker-Planck* equation that drives the spread component (i.e. the fluctuations around the trend) of the probability flow. By letting m equal to its steady state $m^* = \frac{\zeta}{\zeta + \xi}$, we have the distribution function θ for the spread s is given by

$$\theta(s) = Ce^{-\frac{s^2}{2\sigma^2}} \quad \text{with} \quad \sigma^2 = \frac{\zeta\xi}{(\zeta + \xi)^2}, \quad (\text{A.3})$$

which is a Gaussian density. Therefore, the two components of the dynamics of the proportion of fundamentalists as represented by (A.1) are quantified. Accordingly the evolution of the proportion of fundamentalists is given by the trend, described by (A.2), plus a stochastic noise distributed according to (A.3). So we have (2.15). For more details, we refer to Chiarella and Di Guilmi (2011).

APPENDIX B. DEFINITIONS OF THE PARAMETERS AND FUNCTIONS USED IN
PROPOSITION 3.1

The characteristic equation of the system (3.1) at the fundamental steady state is given by¹¹ $\Delta(\lambda) := (\lambda + 2vn_f^*)\tilde{\Delta}(\lambda) = 0$, where

$$\tilde{\Delta}(\lambda) = (\lambda + \eta_f)(\lambda + \eta_c) \left[\lambda^2 + (k + \gamma_f - \gamma_c)\lambda + k\gamma_f - k\gamma_c + \frac{k\gamma_c}{1 - e^{-k\tau}} - \frac{k\gamma_c e^{-(\lambda+k)\tau}}{1 - e^{-k\tau}} \right]. \quad (\text{B.1})$$

Note that equation (B.1) has the same form as the characteristic equation of the model studied in He and Li (2012). Hence we can apply Proposition 3.1 in He and Li (2012) to system (3.1). It follows that the stability of the steady state does not change for time delay $\tau > \tilde{\tau}$ with

$$\tilde{\tau} = \frac{1}{k} \ln \left[1 + \frac{2k\gamma_c}{(k + \gamma_f - \gamma_c)^2 + 2 |k + \gamma_f - \gamma_c| \sqrt{k\gamma_f}} \right].$$

That is, there is an upper bound on the time delay for stability change. The change in stability happens only for $\tau \in [0, \tilde{\tau}]$ if there exists a non-negative integer n such that $S_n^+(\tau) = 0$ or $S_n^-(\tau) = 0$. Here¹²

$$S_n^\pm(\tau) = \tau - \frac{\theta_\pm(\tau) + 2n\pi}{\omega_\pm(\tau)}, \quad \tau \in (0, \tilde{\tau}], \quad n = 0, 1, 2, \dots,$$

$$\omega_\pm = \left(\frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} \right)^{\frac{1}{2}}, \quad \theta_\pm(\tau) = \begin{cases} \arccos(a_{4\pm}), & \text{for } a_{3\pm} \geq 0; \\ 2\pi + \arcsin(a_{3\pm}), & \text{for } a_{3\pm} < 0, a_{4\pm} \geq 0; \\ 2\pi - \arccos(a_{4\pm}), & \text{for } a_{3\pm} < 0, a_{4\pm} < 0 \end{cases}$$

and

$$\begin{aligned} a_1 &= k^2 + \gamma_f^2 + \gamma_c^2 - 2\gamma_f\gamma_c - \frac{2k\gamma_c}{1 - e^{-k\tau}}, & a_2 &= k^2\gamma_f^2 + \frac{2k^2\gamma_f\gamma_c e^{-k\tau}}{1 - e^{-k\tau}}, \\ a_{3\pm} &= \frac{-\omega_\pm(\tau)(1 - e^{-k\tau})(k + \gamma_f - \gamma_c)}{k\gamma_c e^{-k\tau}}, & a_{4\pm} &= 1 - \frac{(1 - e^{-k\tau})(\omega_\pm^2(\tau) - k\gamma_f)}{k\gamma_c e^{-k\tau}}. \end{aligned}$$

Denote

$$\tau_0 = \inf \left\{ \{\tilde{\tau}\} \cup \{\tau \in (0, \tilde{\tau}] \mid \exists n \in \{0, 1, 2, \dots\}, S_n^+(\tau) = 0 \text{ or } S_n^-(\tau) = 0\} \right\}.$$

¹¹Interestingly, the time delays τ_f, τ_c introduced in the performance measures in (2.9) do not appear in the characteristic equation, hence they do not affect the local stability and bifurcation analysis. This is due to the fact that they are in higher order terms and they affect the nonlinear dynamics, rather than the dynamics of the linearized system.

¹²We refer to Theorem 3.3 in He et al. (2009) for the properties of functions $S_n^\pm(\tau)$.

APPENDIX C. COMPARISON TO THE HL MODEL AND NONLINEAR EFFECT OF HERDING

To examine the effect of herding, we present the corresponding results of no-herding model in He and Li (2012), the HL model. Unless specified otherwise, we choose the parameter values $k = 0.05$, $\mu = 1$, $\beta_f = 1.4$, $\beta_c = 1.4$, $C_f = 0.05$, $C_c = 0.03$, $\eta_f = 0.5$, $\eta_c = 0.6$, $\tau_f = 10$, $\tau_c = 5$, and $\bar{F} = 1$.

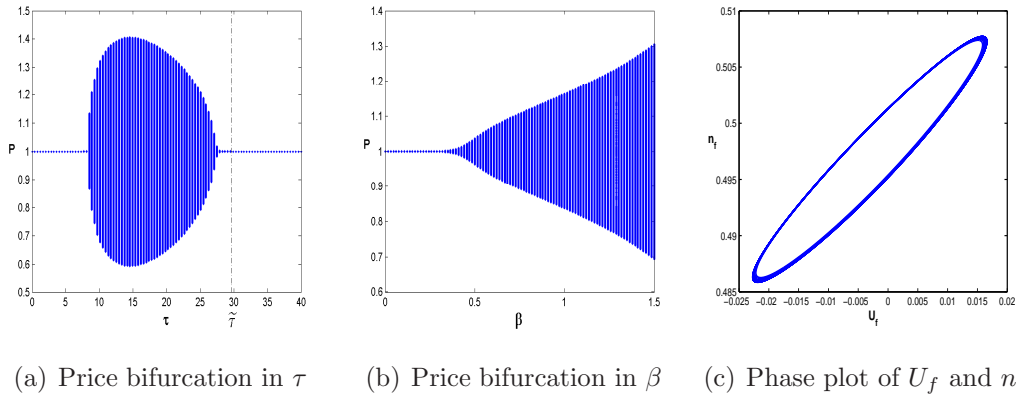


FIGURE C.1. (a) The bifurcation of the market prices with respect to τ with $\beta = 1$; (b) The bifurcation of market price with respect to β with $\tau = 8$; (c) The phase plot of the relationship between the fitness U_f and the market fraction n_f with $\tau = 16$ and $\beta = 1$ for HL model.

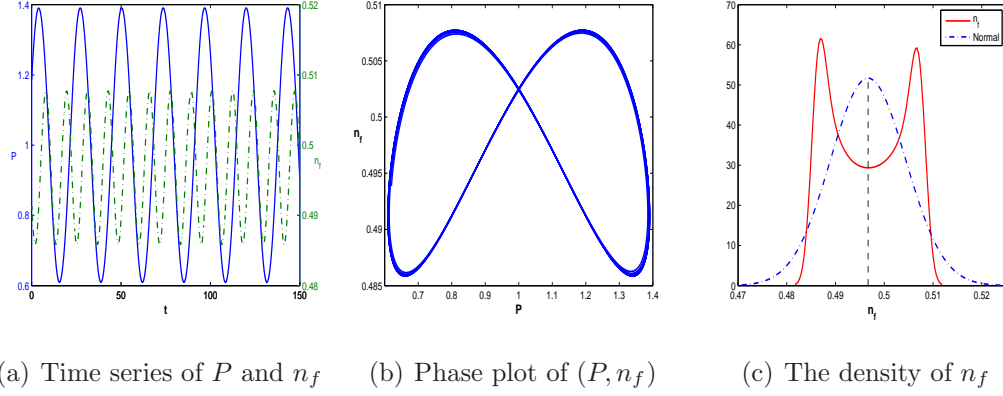


FIGURE C.2. (a) The time series of the market prices $P(t)$ (the blue solid line) and the market fraction $n_f(t)$ of fundamentalists (the green dash dot line); (b) the phase plot of $(P(t), n_f(t))$; and (c) the density distribution of the market fraction $n_f(t)$.

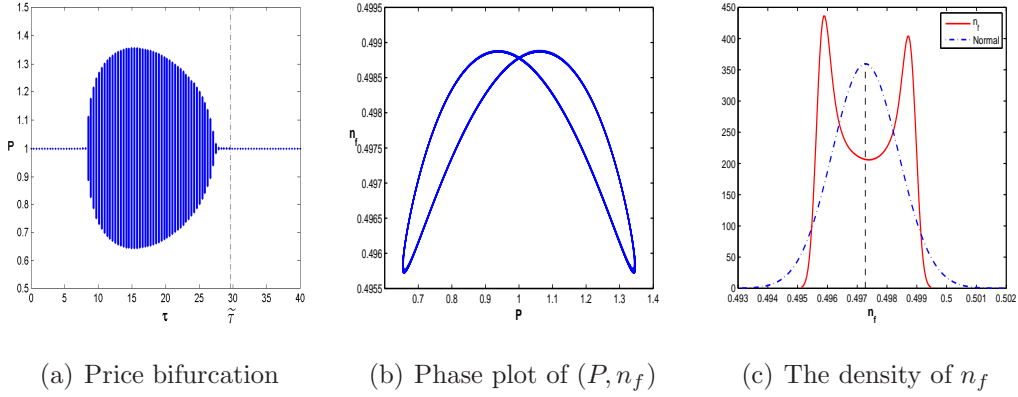


FIGURE C.3. (a) The bifurcation diagram of the market prices with respect to τ for model (3.1); (b) the corresponding phase plot of $(P(t), n_f(t))$ and (c) the density distribution of the market fraction $n_f(t)$ of the fundamentalists. Here $v = 0.1$ and $\tau = 16$.

APPENDIX D. PRICE VOLATILITY COMPARISON TO THE HL MODEL

This appendix presents some results from the HL model and price volatility of the herding model.

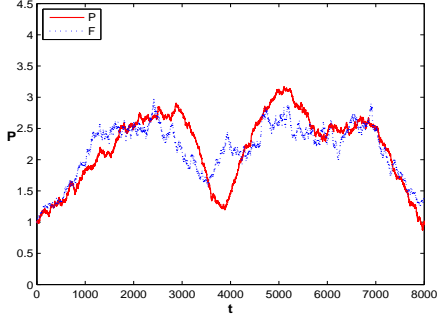
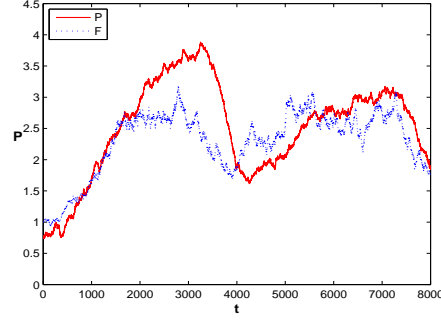
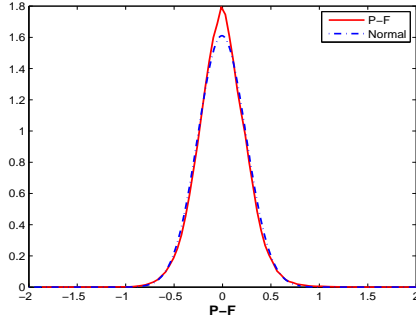
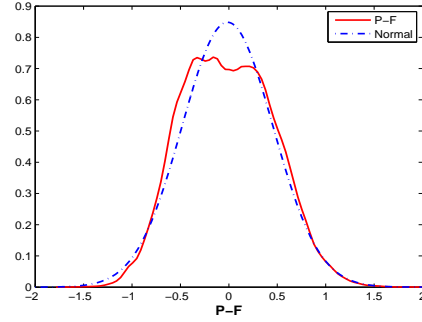
(a) Prices for $\tau = 3$ (b) Prices for $\tau = 16$ (c) Price deviation density for $\tau = 3$ (d) Price deviation density for $\tau = 16$

FIGURE D.1. The time series of the fundamental price $F(t)$ (the blue dotted line) and the market prices $P(t)$ (the red solid line) with (a) $\tau = 3$ and (b) $\tau = 16$, and the distributions of the deviations of the market prices from the fundamental prices $P(t) - F(t)$ with (c) $\tau = 3$ and (d) $\tau = 16$ for the HL model. Here $\sigma_F = 0.12$ and $\sigma_M = 0.15$.

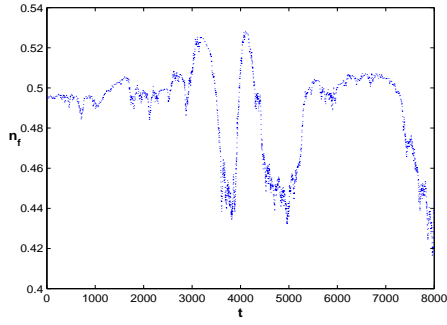
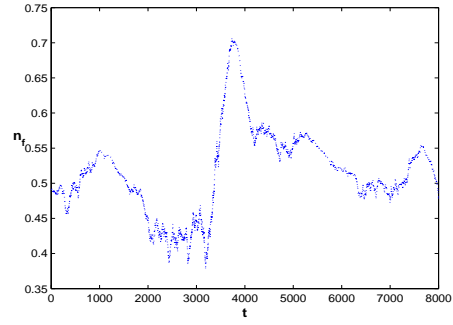
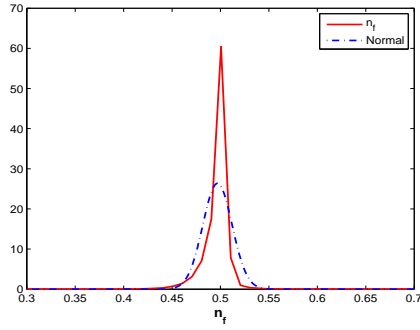
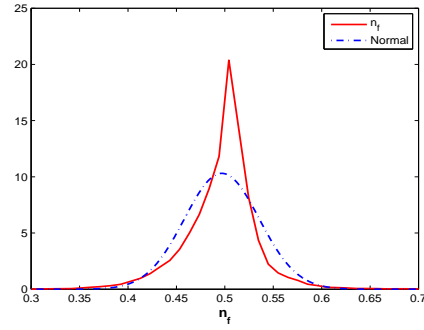
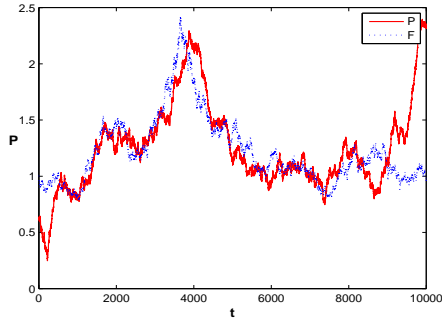
(a) Market fraction for $\tau = 3$ (b) Market fraction for $\tau = 16$ (c) Market fraction density for $\tau = 3$ (d) Market fraction density for $\tau = 16$

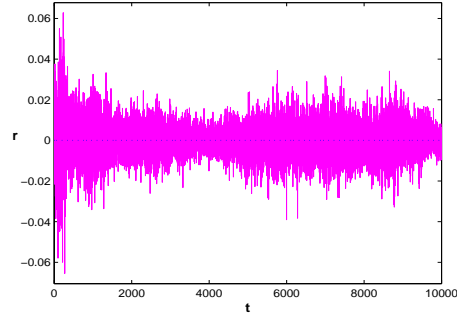
FIGURE D.2. The time series of the fractions of the fundamentalists with (a) $\tau = 3$ and (b) $\tau = 16$ and the corresponding distributions with (c) $\tau = 3$ and (d) $\tau = 16$ for the HL model. Here $\sigma_F = 0.12$ and $\sigma_M = 0.15$.

APPENDIX E. NOISES AND THE STYLIZED FACTS

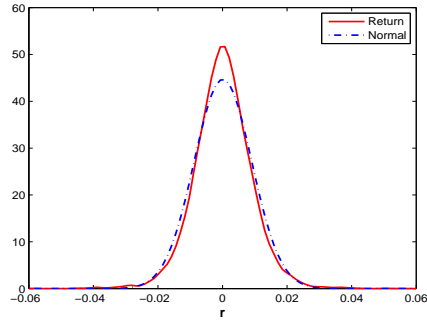
This appendix presents the time series properties and AC patterns of the return, absolute return, and squared return of the model with both the market and fundamental noises in Fig. E.1, the market noise only in Fig. E.2, and the fundamental noise only in Fig. E.3.



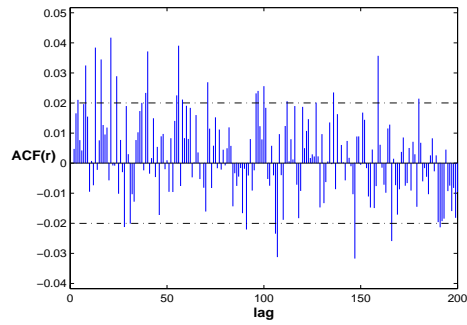
(a) The market price and the fundamental price



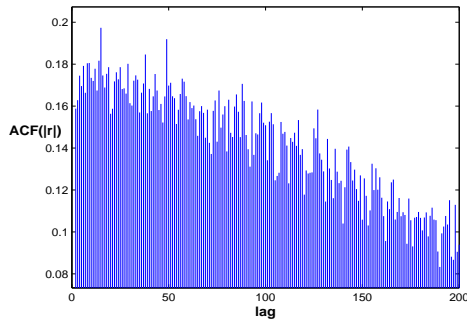
(b) The market returns (r)



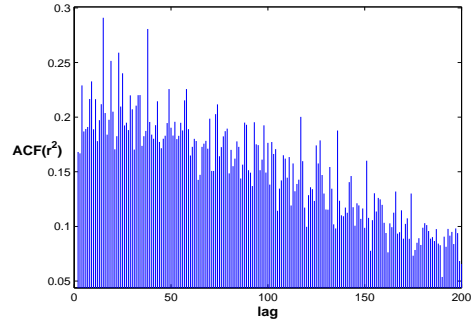
(c) The density of the market returns



(d) The ACs of the market returns

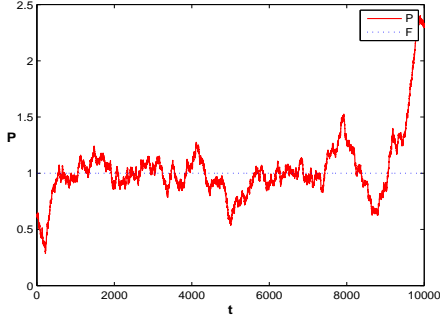


(e) The ACs of the absolute returns

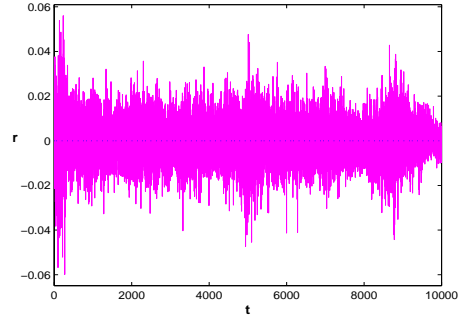


(f) The ACs of the squared returns

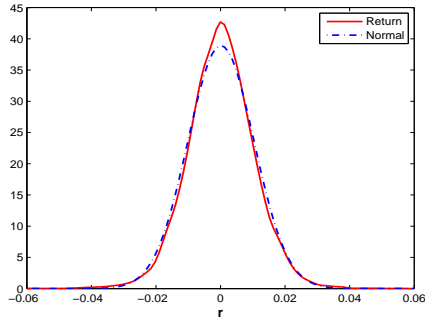
FIGURE E.1. The effect of the two noises: the time series of (a) the market price (red solid line) and the fundamental price (blue dotted line) and (b) the market returns; (c) the return distribution; the ACs of (d) the returns; (e) the absolute returns, and (f) the squared returns. Here $\sigma_F = 0.12$ and $\sigma_M = 0.15$.



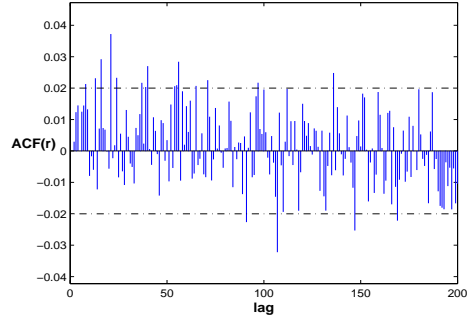
(a) The market price and the fundamental price



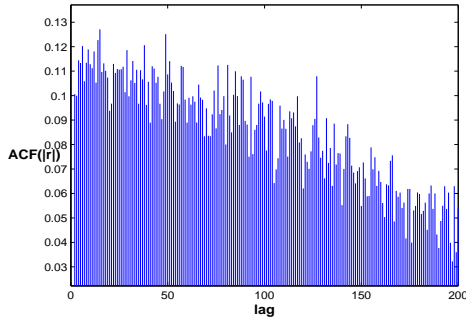
(b) The market returns (r)



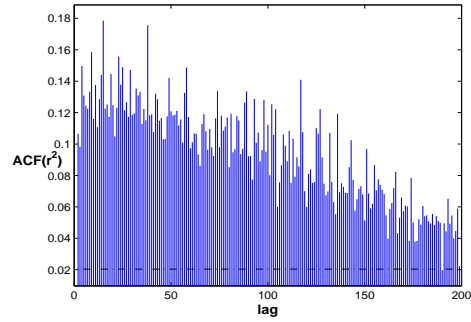
(c) The density of the market returns



(d) The ACs of the market returns

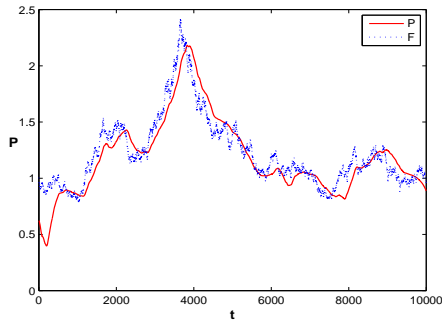


(e) The ACs of the absolute returns

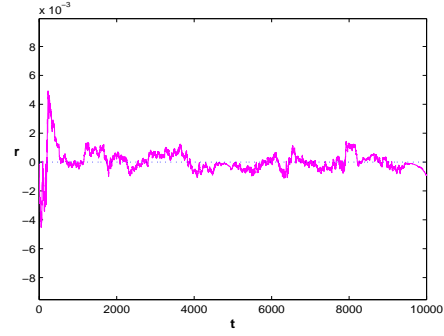


(f) The ACs of the squared returns

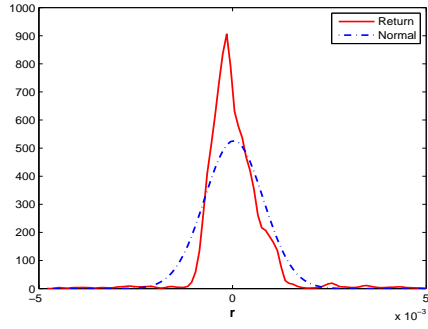
FIGURE E.2. The effect of the market noise only: the time series of (a) the market price (red solid line) and the fundamental price (blue dotted line) and (b) the market returns; (c) the return distribution; the ACs of (d) the returns; (e) the absolute returns, and (f) the squared returns. Here $\sigma_F = 0$ and $\sigma_M = 0.15$.



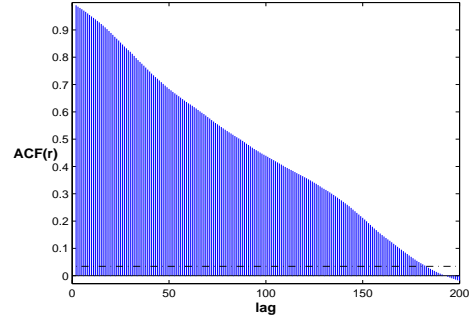
(a) The market price and the fundamental price



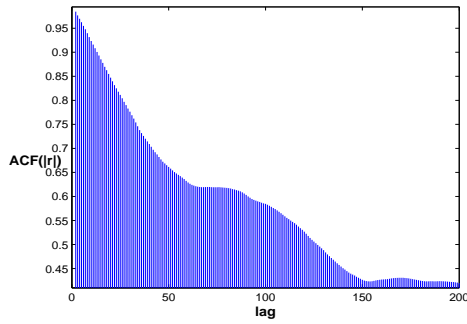
(b) The market returns (r)



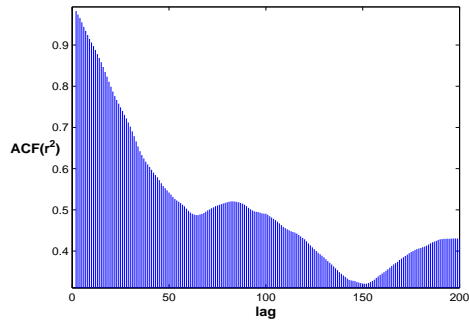
(c) The density of the market returns



(d) The ACs of the market returns



(e) The ACs of the absolute returns



(f) The ACs of the squared returns

FIGURE E.3. The effect of the fundamental noise only: the time series of (a) the market price (red solid line) and the fundamental price (blue dotted line) and (b) the market returns; (c) the return distribution; the ACs of (d) the returns; (e) the absolute returns, and (f) the squared returns. Here $\sigma_F = 0.12$ and $\sigma_M = 0$.

REFERENCES

- Alfarano, S., Lux, T. and Wagner, F. (2005), ‘Estimation of agent-based models: The case of an asymmetric herding model’, *Computational Economics* **26**, 19–49.
- Alfarano, S., Lux, T. and Wagner, F. (2008), ‘Time variation of higher moments in a financial market with heterogeneous agents: An analytical approach’, *Journal of Economic Dynamics and Control* **32**, 101–136.
- Aoki, M. (2002), *Modeling Aggregate Behavior and Fluctuations in Economics*, Cambridge University Press.
- Banerjee, A. (1992), ‘A simple model of herd behavior’, *Quarterly Journal of Economics* **107**, 797–817.
- Beja, A. and Goldman, M. (1980), ‘On the dynamic behavior of prices in disequilibrium’, *Journal of Finance* **35**, 235–247.
- Brock, W. and Hommes, C. (1997), ‘A rational route to randomness’, *Econometrica* **65**, 1059–1095.
- Brock, W. and Hommes, C. (1998), ‘Heterogeneous beliefs and routes to chaos in a simple asset pricing model’, *Journal of Economic Dynamics and Control* **22**, 1235–1274.
- Chen, S.-H., Chang, C. and Du, Y. R. (2012), ‘Agent-based economic models and econometrics’, *Knowledge Engineering Review* **27**, 187–219.
- Chiarella, C. (1992), ‘The dynamics of speculative behavior’, *Annals of Operations Research* **37**, 101–123.
- Chiarella, C., Dieci, R. and He, X. (2009), *Heterogeneity, Market Mechanisms and Asset Price Dynamics*, Elsevier, pp. 277–344. in *Handbook of Financial Markets: Dynamics and Evolution*, Eds. Hens, T. and K.R. Schenk-Hoppe.
- Chiarella, C. and Di Guilmi, C. (2011), ‘The financial instability hypothesis: A stochastic micro-foundation framework’, *Journal of Economic Dynamics and Control* **35**, 1151–1171.
- Chiarella, C. and Di Guilmi, C. (2012), ‘Limit distribution of evolving strategies in financial markets’. Working paper.
- Chiarella, C. and He, X. (2002), ‘Heterogeneous beliefs, risk and learning in a simple asset pricing model’, *Computational Economics* **19**, 95–132.
- Chiarella, C. and He, X. (2003), ‘Dynamics of beliefs and learning under a_t -processes – The Heterogeneous case’, *Journal of Economic Dynamics and Control* **27**, 503–531.
- Chiarella, C., He, X. and Hommes, C. (2006), ‘A dynamic analysis of moving average rules’, *Journal of Economic Dynamics and Control* **30**, 1729–1753.
- Franke, R. and Westerhoff, F. (2012), ‘Structural stochastic volatility in asset pricing dynamics: Estimation and model contest’, *Journal of Economic Dynamics and Control* **38**, 1193–1211.
- He, X. and Li, K. (2012), ‘Heterogeneous beliefs and adaptive behavior in a continuous-time asset price model’, *Journal of Economic Dynamics and Control* **36**, 973–987.

- He, X. and Li, K. (2013), ‘Time series momentum and market stability’, *Working Paper*.
- He, X., Li, K., Wei, J. and Zheng, M. (2009), ‘Market stability switches in a continuous-time financial market with heterogeneous beliefs’, *Economic Modelling* **26**, 1432–1442.
- He, X. and Li, Y. (2007), ‘Power law behavior, heterogeneity, and trend chasing’, *Journal of Economic Dynamics and Control* **31**, 3396–3426.
- He, X. and Zheng, M. (2010), ‘Dynamics of moving average rules in a continuous-time financial market model’, *Journal of Economic Behavior and Organization* **76**, 615–634.
- Hohnisch, M. and Westerhoff, F. (2008), ‘Business cycle synchronization in a simple Keynesian macro-model with socially transmitted economic sentiment and international sentiment spillover’, *Structural Change and Economic Dynamics* **19**, 249–259.
- Hommes, C. (2006), *Heterogeneous Agent Models in Economics and Finance*, Vol. 2 of *Handbook of Computational Economics*, North-Holland, pp. 1109–1186. in *Agent-based Computational Economics*, Eds. Tesfatsion, L. and K.L. Judd.
- LeBaron, B. (2006), *Agent-based Computational Finance*, Vol. 2 of *Handbook of Computational Economics*, North-Holland, pp. 1187–1233. in *Agent-based Computational Economics*, Eds. Tesfatsion, L. and K.L. Judd.
- Lux, T. (1995), ‘Herd behavior, bubbles and crashes’, *Economic Journal* **105**, 881–896.
- Lux, T. (2009), *Stochastic Behavioral Asset Pricing and Stylized Facts*, Elsevier, pp. 161–215. in *Handbook of Financial Markets: Dynamics and Evolution*, Eds. Hens, T. and K.R. Schenk-Hoppé.
- Moskowitz, T., Ooi, Y. H. and Pedersen, L. H. (2012), ‘Time series momentum’, *Journal of Financial Economics* **104**, 228–250.
- Scharfstein, D. and Stein, J. (1990), ‘Herd behavior and investment’, *American Economic Review* **80**, 465–479.