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Leveraged investments and agency conflicts when prices are mean reverting

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Abstract

We analyse the effect of differing uncertainty assumptions on the costs of shareholder-bondholder conflicts arising from partially debt-financed investments. A partial equilibrium model, valid for a large class of diffusion processes, is developed and then applied to the specific cases of a geometric Brownian motion (GBM) and a mean-reverting (MR) process. This allows for the comparison of the two scenarios and contributes to the ongoing discussion on the effects of mean reversion on investment and financing behaviour. We find that agency costs are much lower under MR dynamics and, through the application of a novel agency cost decomposition, we show that for a high expected growth in future profits (high growth GBM) agency costs are driven mainly by suboptimal *financing* decisions, as opposed to suboptimal (default and investment) *timing* decisions. The situation is reversed for lower growth assumptions and for an increase in the speed of mean reversion. Our results on the components and drivers of agency costs are valuable to both policy makers and regulators alike.

Keywords: investment, real option, mean reversion, agency conflicts
JEL classification: G13, G32, G33, G38.

1. Introduction

The bulk of the existing real option literature assumes uncertain output or input prices to follow geometric Brownian motion (GBM) (Dixit and Pindyck, 1994). While this modelling choice often provides tractable solutions it has been criticised in relation to its suitability for describing equilibrium price processes (Lund, 1993). It has also been suggested that such price dynamics, particularly in commodity markets, can be more accurately modelled using a mean-reverting (MR) process (Schwartz, 1997). Crucially, it has also been argued that the failure to account for the effects of mean reversion can lead to "systematic biases in capital budgeting decisions" (Bessembinder, Coughenour, Seguin, and Smoller, 1995).

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Motivated by the above, an important line of research, initiated by Metcalf and Hassett (1995), has attempted to assess the appropriateness of the use of GBM as a substitute for more realistic mean-reverting dynamics when considering firms' optimal investment decisions. The present paper continues this line of research by considering the effect of mean reversion on *leveraged* investment projects. The addition of leverage into the problem extends the previous analysis to a much more realistic and economically meaningful setting, however it requires the explicit consideration of the optionality of the equityholders to default on the levered project; resulting in a two-layered optimal stopping problem. Because of this two layered structure, and the assumption of strategic debt financing, the effect of MR on optimal investment in this setting is, unsurprisingly, more complex.

Furthermore, the inclusion of leverage into this framework, while complicating the analysis and introducing an equilibrium aspect to the model, does allow us to also evaluate the effect of mean reversion on the optimal *financing* decisions of firms and to investigate the *interaction* of the financing and investment timing decisions.[1] To our knowledge the effect of mean reversion on this interaction has not previously been studied.

This research therefore contributes to the literature on real options and stochastic price modelling as well as to the literature on corporate financial policy and related agency conflicts. The specific research questions we address are: (i) what are the characteristics and interaction of a firm's optimal investment, default, and financing strategies under the assumption of mean-reverting output prices? (ii) what are the implications of mean reversion for investment values and agency costs? and (iii) how do these results compare to those obtained under GBM dynamics?

To date, three effects of mean reversion on investments have been identified. Metcalf and Hassett (1995) identified the *variance effect* in which mean reversion reduces the long-run variance of a project's cash flow (compared to GBM), thus resulting in lower investment trigger prices and sooner investment. However, these authors also pointed out a second competing *realised price effect*. Here, the stationarity of the mean-reverting process implies that the probability of reaching a given level is also reduced, potentially offsetting the variance effect. Metcalf and Hassett (1995) concluded that GBM *could* be considered as an appropriate substitute for MR since the probability of investment under GBM and MR dynamics are comparable, resulting in no significant difference in cumulative investment.

Sarkar (2003) extended Metcalf and Hassett's arguments by incorporating a third so called *risk-discounting effect*. Under mean reversion, a lower cash flow variance also affects the project's risk-adjusted required rate of return and hence the discount rate used for valuation; affecting both the project value and the value of the real option to invest in the project. In contrast to Metcalf and Hassett (1995), Sarkar (2003) concluded that mean reversion *does* have a significant impact on investment when all three effects are correctly accounted for.

Finally, in a more recent contribution to this literature, Tsekrekos (2010) examined the effect of mean reversion on reversible entry *and* exit decisions of firms; thus incorporating the possibility

[1] Since Modigliani & Miller's ground-breaking work on optimal capital structure (Modigliani and Miller, 1959, Baxter, 1967) investment valuation has been closely linked to questions of optimal corporate financial policy. Financial structure is important for the valuation itself because it influences the policy that governs cash flow control, which in turn affects cash flows and the project value (Brennan and Trigeorgis, 2000).

of reversibility and disinvestment into the previous analysis. Similar to Sarkar (2003), Tsekrekos (2010) also reached the conclusion that it would be erroneous to use the more tractable GBM process as an approximation for a mean-reverting process in models of aggregate industry investment. We note that Tsekrekos (2010), nor the previous papers, considered a setting in which leverage was present.

An important consequence of the inclusion of leverage is that it introduces the potential for conflicting interests of shareholders (borrowers) and bondholders (lenders).[2] This introduces the concept of *agency costs* as a fundamental quantity in our investment and financing problem (Jensen and Meckling, 1976).[3] Existing literature has analysed the direction and magnitude of the agency costs resulting from over- or underinvestment. When new projects are financed solely by equity, some researchers have concluded that equityholders tend to *under*invest, because they bear all the cost of the investment while sharing the benefits with debtholders (Mauer and Ott, 2000, Moyen, 2007, Titman and Tsyplakov, 2007). In contrast to this, when projects are at least partially financed by new debt, equityholders tend to *over*invest due to the incentive to transfer wealth from the debtholders to themselves (Leland, 1998, Mauer and Sarkar, 2005).

In the context of leveraged investments, the model of Mauer and Sarkar (2005) is particularly appealing as it presents the agency conflicts using a two-layered real option framework; one being the project investment option, the other being the default option after investment. This setup allows the rational debtholders to incorporate the equityholders' strategy of equity value maximisation when deciding on how much debt to provide and at what price. Therefore, in light of the significance of the effects of mean reversion on investment timing decisions, we extend the (GBM based) model of Mauer and Sarkar (2005) in the present study to a more general analysis, incorporating the risk-discounting effect of Sarkar (2003), and allowing for the consideration of mean-reverting dynamics; thus providing insights into their effects on both investment *and* financing decisions.

The GBM based results of Mauer and Sarkar (2005) find that equityholders' incentive to overinvest significantly decreases firm value and optimal leverage, reporting a 9.4% *loss* in firm value and a reduction in optimal leverage from 66% to 39% for their base case parameters. Our analysis reveals similar results under GBM—8.4% loss in firm value and a reduction in leverage from 60% to 45%—but that under mean-reverting dynamics the reductions in firm value and optimal leverage are much smaller, finding only a 0.9% loss in firm value[4] and a reduction of optimal leverage from 47% to 43% for our base case parameters. These results indicate that the growth rate and stationarity assumptions of future cash-flows have a significant impact on the equilibrium effects of the agency conflict.

In sum, this research extends the current literature in the following ways. Firstly, we generalise the model of Mauer and Sarkar (2005) to a wide class of diffusion processes and to incorporate

[2]In the following, we shall use the terms *equity*holders and *debt*holders to maintain generality.

[3]Our model assumes that the investment decision-makers (managers) are the shareholders and so we focus on the agency conflict between shareholders and bondholders. When the decision makers are not the shareholders, there could be additional conflicts of interests between shareholders and managers (cf. Cadenillas, Cvitanic, and Zapatero, 2004, Morellec, 2004).

[4]Note that Leland (1998) also found around a 1% loss in firm value due to overinvestment for his base case parameters under GBM. However, Leland (1998) did not account for the effect of capital structure on firm valuation whereas Mauer and Sarkar (2005) and our analysis do.

the risk discounting effect as proposed by Sarkar (2003). We then apply GBM and MR dynamics to our general model to assess the effect of mean reversion on leveraged investments. Secondly, we present an alternative solution methodology to Mauer and Sarkar (2005) based upon diffusion theory, which provides important additional insights into existing and new results. Thirdly, when considering the agency costs of overinvestment, we propose a novel agency cost decomposition into the costs due to suboptimal *financing* decisions and those due to suboptimal (default and investment) *timing* decisions. Finally, we extend Mauer and Sarkar (2005) by parameterising our model using real (commodity) asset price data.

The remainder of the paper is structured as follows. We develop the extended version of the Mauer and Sarkar (2005) model in full generality in Section 2. We then apply both GBM and MR uncertainty processes to the general model presentation in Section 3 and provide results and conclusions in Sections 4 and 5 respectively.

2. Generalised Model

In the following we develop a generalised extension of the Mauer and Sarkar (2005) model. Our intention is to setup the model so as to allow for a presentation that is independent of the specific uncertainty process used to model output prices. Such a presentation illustrates which model results remain valid with a high degree of generality (i.e. independent of the chosen uncertainty process) and which are not so easily generalised.

We begin by modelling the *inner option*, representing the value of the unlevered project, or in the presence of debt financing that of the levered project, *after* investment. Given the investor's ability to abandon the project and file for bankruptcy the valuation of the inner option requires the determination of the optimal abandonment strategy. In the case of the levered project, such abandonment is labeled as default. Next, we evaluate the *outer option*, which represents the value of the investment project to the investor *before* investment. This option must account for uncertain future output prices, the investor's optimal timing decisions, and the lender's optimal decision on providing debt. As such, a strategic equilibrium (under complete information) between investor (equityholders) and lender (debtholders) is determined.

The investment project is assumed to represent a production facility for a commodity that produces one unit of the commodity per year at a constant cost C per unit, which can be sold at the (uncertain) price $(X_t)_{t\geq 0}$. Per period profits $X_t - C$ are taxed at the constant effective tax rate τ.[5] The project is subject to initial investment costs I and both the underlying project and the option to invest are assumed to have infinite time-horizons. Financing of the project is assumed to be undertaken by a mixture of both equity and perpetual debt, where the latter is denoted K. In exchange for the financing amount K, equityholders are required to pay a periodic coupon payment denoted by R. The debt amount K and coupon payment R are pre-negotiated from a 'revolving line of credit' type of loan commitment, which equity- and debtholders have agreed upon at $t = 0$, before the investment decision is taken.[6] Debtholders are assumed to be rational

[5] Note that the presence of tax in the model is crucial since the existence of a tax shield is important to induce equityholders to employ debt financing.

[6] This type of commitment allows the equityholders to borrow, on pre-negotiated terms, at any time during the life of the commitment. For more details see Kashyap, Rajan, and Stein (2002).

and set the equityholders' coupon payment R not only based on the level of debt provided K, but also on their expectation of the equityholders' behaviour regarding project default. In the case of default the equity value is assumed to be zero and the bankruptcy costs amount to b percent of the value of the unlevered project at time of default, with debtholders receiving the remainder.

2.1. Uncertainty dynamics

We model the price process $(X_t)_{t\geq 0}$ as a general non-negative, time-homogeneous, and regular diffusion living on the filtered probability space $(\Omega, \mathrm{P}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F})$ and described by the SDE

$$dX_t = \alpha(X_t)dt + \sigma(X_t)dW_t^{\mathrm{P}}, \quad X_0 = x, \tag{1}$$

where α and σ are assumed to be continuous and dW_t^{P} denotes the increment of the Wiener process under the *real-world* measure P. Furthermore, we assume that $\sigma(x) > 0$ for $x \in (0, \infty)$ and that the upper boundary at infinity is a natural one (and hence is unattainable in finite time). In addition, from a practical perspective we are only interested in processes for which zero is unattainable after the process has started. This restricts our attention to processes for which zero is either *natural* or an *entrance (not exit)* boundary. These boundary classifications are an important distinction to make since the often applied GBM process has zero as a natural boundary, whereas the mean-reverting process employed, and further discussed, in Section 3 has zero as an entrance (not exit) boundary. Eq. (1) encompasses many of the well known processes used in modern finance, such as the GBM, CIR and CEV models (see Black and Scholes, 1973, Cox, Ingersoll, and Ross, 1985, Cox, 1975, respectively).

In this paper, and consistent with the literature, we will assume the existence of a suitable spanning asset (resulting in a complete market) and hence we apply contingent claims analysis to price the various real options introduced.[7] Such an analysis requires that expectations be taken under the equivalent *risk-neutral* measure Q and so we find that the dynamics under this measure are given by[8]

$$dX_t = (\alpha(X_t) - \lambda\sigma(X_t))dt + \sigma(X_t)dW_t^{\mathrm{Q}}, \quad X_0 = x, \tag{2}$$

where we have effectively subtracted a risk-premium ($\lambda\sigma$) from the drift of the real-world price dynamics. Here λ represents the (justified) Sharpe ratio of the commodity X.

2.2. Unlevered project value

To start, we consider the (inner) option determining the unlevered project value $V_u(x)$ after investment. The project derives value from the expectation of future cashflows (in present value terms), subject to optimal abandonment. The project value is therefore

$$V_u(x) := \sup_{T_a} \mathbb{E}_x^{\mathrm{Q}} \int_0^{T_a} e^{-rt}\pi_u(X_t)dt, \tag{3}$$

[7] In Section 4 we use an oil producing firm as our illustrative example and hence, since oil is traded, the market is complete and the use of contingent claims analysis can be easily justified.

[8] See Appendix A for a detailed derivation of the risk-neutral dynamics given in Eq. (2).

where T_a is the abandonment time for the manager and π_u represents the after-tax profit flow of the unlevered project given by

$$\pi_u(x) = (1-\tau)(x-C). \tag{4}$$

Without loss of generality we assume, for simplicity, that only one unit of X is produced (at a cost C). To obtain the solution to the optimal stopping problem defined in Eq. (3) the Markovian nature of the process allows us to reformulate the problem to obtain (see Appendix B)

$$V_u(x) = f_u(x) + \sup_{T_a} \mathbb{E}_x^{\mathrm{Q}} \left[e^{-rT_a} \left(-f_u(X_{T_a})\right) \right], \tag{5}$$

where

$$f_u(x) := \mathbb{E}_x^{\mathrm{Q}} \int_0^\infty e^{-rt} \pi_u(X_t) dt = (1-\tau) \left(\int_0^\infty e^{-rt} \mathbb{E}_x^{\mathrm{Q}} [X_t] dt - \frac{C}{r} \right), \tag{6}$$

representing the present value of the total expected profits of the unlevered project if the project was *never* abandoned. The second term in Eq. (5) can therefore be interpreted economically as the value of the option to abandon the project. Furthermore, the project manager would only abandon at prices for which the expected future profits are negative, i.e. $f_u(X_t) < 0$.

Given the infinite-horizon of the optimal stopping problem in Eq. (5) it would seem intuitive that the optimal stopping rule be independent of time and hence take the form of a threshold strategy, i.e. $T_a^* = \inf\{t \geq 0 \,|\, X_t = x_a^*\}$, the first hitting time of the commodity price X of the level x_a^*. However, the optimality of such a threshold strategy should not be assumed *a priori*.[9] However, necessary and sufficient conditions on the payoff function f_u and the process X for the optimality of threshold strategies are provided by Villeneuve (2007). Furthermore, it can be shown that for all cases considered in Section 3, f_u takes on a simple linear form and hence easily satisfy the (fairly weak) conditions required.

Once the optimality of a threshold strategy is shown, to proceed with the solution two methods are commonly used. One can formulate the associated free-boundary problem and solve it using the general methods of second-order ordinary differential equations (ODEs) and the principle of smooth fit;[10] this was the methodology employed by Mauer and Sarkar (2005). Here, we choose an alternative probabilistic method, based on diffusion theory (see Rogers and Williams, 2000), which we believe to be more direct and illuminating when applied to the wider class of uncertainty assumptions under consideration in the present paper.

Being justified to use simple threshold strategies, it is clear that our optimisation over stopping times in Eq. (5) now becomes an optimisation over threshold levels x_a, in other words

$$V_u(x) = f_u(x) - \inf_{T_a} \mathbb{E}_x^{\mathrm{Q}} \left[e^{-rT_a} f_u(X_{T_a}) \right] = f_u(x) - \min_{x_a} \left\{ f_u(x_a) \mathbb{E}_x^{\mathrm{Q}} \left[e^{-rT_a} \right] \right\}, \tag{7}$$

where we have used the continuity of the process X in the last equality above. To proceed, an expression for the expected discount factor, $\mathbb{E}_x^{\mathrm{Q}}[e^{-rT_a}]$, is needed. This object can be identified as

[9] Such a priori assumptions of optimality are abundant in the literature as was noted by Villeneuve (2007) who provides illuminating cautionary examples.

[10] See Chapter 10, Section 4 of Oksendal (2003).

simply the Laplace transform of the hitting time T_a for which, in the class of time-homogeneous diffusions under consideration, we have a very general expression, namely

$$\mathbb{E}_x^{\mathbb{Q}}\left[e^{-rT_a}\right] = \begin{cases} \phi(x)/\phi(x_a) & \text{for } x \geq x_a, \\ \psi(x)/\psi(x_a) & \text{for } x < x_a, \end{cases} \tag{8}$$

where $\phi(x)$ and $\psi(x)$ are the unique (up to a linear scaling), positive, decreasing and increasing solutions, respectively, of the linear second-order ODE[11]

$$\frac{1}{2}\sigma^2(x)u''(x) + (\alpha(x) - \lambda\sigma(x))\, u'(x) - ru(x) = 0. \tag{9}$$

From Eqs. (7) and (8) we see that the unlevered firm value thus becomes

$$V_u(x) = \begin{cases} f_u(x) - f_u(x_a^*)\frac{\phi(x)}{\phi(x_a^*)}, & \text{for } x \geq x_a^*, \\ 0, & \text{for } x < x_a^*, \end{cases} \tag{10}$$

where the optimal abandonment trigger price x_a^* solves the following equation

$$\frac{\phi'(x_a^*)}{\phi(x_a^*)} = \frac{f_u'(x_a^*)}{f_u(x_a^*)}, \tag{11}$$

which is obtained from the first-order condition of the minimisation in Eq. (7).[12] Note that depending on the specification of the uncertainty process X, Eq. (11) may or may not allow us to solve for x_a^* explicitly. Only in the very simplest cases will such explicit solutions be available, however in all cases x_a^* can be found numerically very easily using any standard root-finding algorithm.[13] In the latter case, knowledge of the existence and uniqueness of the solution to Eq. (11) is of practical interest as will be discussed in the context of Proposition 1.

2.3. *Levered project value*

Next, consider the availability of debt funding, where interest payments are assumed to be tax deductible. Due to the resulting tax-shield of debt-financing, equityholders have the incentive to take on debt to increase the total equity value of the investment. In the presence of coupon payments $R \geq 0$, the profit function of the levered project changes Eq. (4) to

$$\pi_\ell(x) = (1 - \tau)(x - C - R). \tag{12}$$

The levered project value $V_\ell(x)$ after investment is simply the sum of the values of equity and debt

$$V_\ell(x) := E(x) + D(x). \tag{13}$$

[11]These functions are often called the *fundamental* solutions to such ODEs. For more details see Chapter II, Part 11 of Borodin and Salminen (2002).

[12]The second-order condition can be verified, obtaining a minimum provided $f_u''(x_a^*)\phi(x_a^*) - f_u(x_a^*)\phi''(x_a^*) > 0$.

[13]Such algorithms are built in to most software packages such as *MATLAB* or *Mathematica*.

Analogous to the value of the unlevered firm, the equity value of the levered project, $E(x)$, is

$$E(x) := \sup_{T_d} \mathbb{E}_x^{\mathrm{Q}} \int_0^{T_d} e^{-rt} \pi_\ell(X_t) dt, \tag{14}$$

where $T_d = \inf\{t \geq 0 \,|\, X_t = x_d\}$, the first hitting time of the default trigger x_d, the price at which the equityholders chose to default. Similar to the unlevered case, the optimal stopping problem (14) can be reformulated as[14]

$$E(x) = f_\ell(x) + \sup_{T_d} \mathbb{E}_x^{\mathrm{Q}} \left[e^{-rT_d} \left(-f_\ell(X_{T_d}) \right) \right] \tag{15}$$

where

$$f_\ell(x) = \mathbb{E}_x^{\mathrm{Q}} \int_0^{\infty} e^{-rt} \pi_\ell(X_t) dt = (1-\tau) \left(\int_0^{\infty} e^{-rt} \mathbb{E}_x^{\mathrm{Q}}[X_t] dt - \frac{C+R}{r} \right) \tag{16}$$

represents the present value of the total expected profits of the levered project if the project was never abandoned by the equityholders. Note that by inspection of Eqs. (6) and (16) it follows that

$$f_\ell(x) = f_u(x) - \frac{R(1-\tau)}{r}, \tag{17}$$

meaning that the total expected profit function of the levered project equals that of the unlevered project less the expected present value of the after-tax coupon payment stream. In addition, inspection of Eq. (16) and the continuity of X indicate that $f_\ell(x)$ (and hence $f_u(x)$) is increasing in x.

Using the same solution techniques as before yields the following equity value

$$E(x) = \begin{cases} f_\ell(x) - f_\ell(x_d^*) \frac{\phi(x)}{\phi(x_d^*)}, & \text{for } x \geq x_d^*, \\ 0, & \text{for } x < x_d^*, \end{cases} \tag{18}$$

where x_d^* solves the equation

$$\frac{\phi'(x_d^*)}{\phi(x_d^*)} = \frac{f_\ell'(x_d^*)}{f_\ell(x_d^*)}. \tag{19}$$

As previously mentioned, an important question arising from the above analysis is under what conditions do Eqs. (11) and (19) have a solution and when is such a solution unique? The answer is provided by the following proposition.

Proposition 1. *Under the standing assumptions on the process X, there exists a solution to Eqs.* (11) *and* (19) *provided that $f_u(z)$ and $f_\ell(z)$, respectively, are negative for a non-empty interval of $\mathbb{R}_+$. Furthermore, if the function $\theta(z) := rz + \lambda\sigma(z) - \alpha(z)$ is non-decreasing, $f_i(z)$ (for $i = u, \ell$) is convex, and zero is a non-attracting boundary, then this solution is unique.*

Proof. See Appendix C. □

[14]Details are identical to those in Appendix B and therefore omitted.

Corollary 2. *Whenever solutions to Eqs.* (11) *and* (19) *exist,* $x_d^* \geq x_a^*$ *for any process* X.

Proof. See Appendix D. □

The above corollary generalises the GBM based result $x_d^* \geq x_a^*$, found by Mauer and Sarkar (2005), to a wider class of diffusion processes. This result is consistent with economic intuition since given the extra cash-flow burden of the amount R, one expects rational equityholders to abandon the project sooner (at a higher output price) given the lower overall cash inflows.

Next, to value the debt we observe that the debtholders' periodic cash flow is equal to the coupon payment R, provided that the equityholders do not default. In the case of default, debtholders receive the value of the unlevered project less bankruptcy costs. Therefore, the debt value is given by

$$D(x) := \mathbb{E}_x^{\mathbb{Q}}\left[\int_0^{T_d^*} e^{-rt}Rdt + e^{-rT_d^*}(1-b)V_u(X_{T_d^*})\right] \tag{20}$$

where T_d^* denotes the equityholders' *optimal* default time. Note that this is no longer an optimisation problem since the debtholders do not have any direct influence on the time of default. Accordingly, the debt value can be shown to be (see Appendix E)

$$D(x) = \begin{cases} \frac{R}{r} + \left((1-b)V_u(x_d^*) - \frac{R}{r}\right)\frac{\phi(x)}{\phi(x_d^*)}, & \text{for } x \geq x_d^*, \\ (1-b)V_u(x), & \text{for } x < x_d^*. \end{cases} \tag{21}$$

Note that the assumption of a natural boundary at infinity for the process X guarantees that $\lim_{x\to\infty}\phi(x) = 0$ and hence Eq. (21) shows that for very large values of the output price, the value of debt approaches the value of a perpetuity, R/r, indicating that the probability of default also approaches zero.

Substituting Eqs. (18) and (21) into Eq. (13) and judicious rearranging provides the following, particularly insightful, representation of the value of the levered project (for $x \geq x_d^*$)[15]

$$V_\ell(x) = V_u(x) + \frac{\tau R}{r}\left(1 - \frac{\phi(x)}{\phi(x_d^*)}\right) - bV_u(x_d^*)\frac{\phi(x)}{\phi(x_d^*)}. \tag{22}$$

Therefore, we find that the value of the levered project can be expressed as the sum of three components. The value of the unlevered project, the expected additional benefit provided by debt in the form of a tax shield, and the expected cost of bankruptcy. This representation forms the basis for the trade-off theory of optimal capital structure (Kraus and Litzenberger, 1973).

2.4. Second-best investment policy

Next, the option to invest in the project, which we call the *firm value*, is considered.[16] We begin with the case of the *second-best* investment policy based on the equityholders' desire to maximise

[15]See Appendix F for the derivation of Eq. (22). Note that for $x < x_d^*$, we have $V_\ell(x) = D(x) = (1-b)V_u(x)$.

[16]Note that, like Mauer and Sarkar (2005), we assume the firm undertaking this investment decision has no other existing operations or debt. Hence the investment option value is equivalent to the *pure* firm value, since there are no additional operations from which to derive value.

equity/shareholder value (as opposed to *first-best* or total firm value). This policy provides the optimal time to undertake the investment from the equityholders' point of view. The second-best value of the investment option (firm), denoted as F_2, is defined as

$$F_2(x) := \sup_{T_2} \mathbb{E}_x^{\mathbb{Q}} \left[e^{-rT_2}(E(X_{T_2}) - (I - K)) \right] \tag{23}$$

since the equityholders will only outlay $I - K$ for the investment and they wish to maximise the total value of equity at the time of investment. It can be verified that the equity value also satisfies the necessary conditions for the optimality of a threshold strategy in Eq. (23) and hence $T_2^* = \inf\{t \geq 0 \,|\, X_t = x_2^*\}$ where x_2^* denotes the second-best trigger price at which investment becomes optimal. The second-best value can thus be calculated as

$$F_2(x) = \max_{x_2} \left\{ (E(x_2) - (I - K)) \mathbb{E}_x^{\mathbb{Q}} \left[e^{-rT_2} \right] \right\} = \max_{x_2} \left\{ (E(x_2) - (I - K)) \frac{\psi(x)}{\psi(x_2)} \right\}$$

yielding

$$F_2(x) = \begin{cases} (E(x_2^*) - (I - K)) \frac{\psi(x)}{\psi(x_2^*)}, & \text{for } x < x_2^*, \\ E(x) - (I - K), & \text{for } x \geq x_2^*, \end{cases} \tag{24}$$

where x_2^* solves the equation

$$\frac{E'(x_2^*)}{E(x_2^*) - (I - K)} = \frac{\psi'(x_2^*)}{\psi(x_2^*)}. \tag{25}$$

Note that Eqs. (24) and (25) provide the second-best firm value and trigger price *conditional* on the equityholders and debtholders agreeing on the periodic coupon payment R in exchange for an initial loan of amount K. However, recall that debtholders rationally anticipate that equityholders will maximise equity value and will therefore charge appropriately high interest payments. In other words, the debtholders have no control over the equityholders' default and investment decisions but they can determine, given the coupon payment R, how much debt will be provided upon investment. Consequently, the *fair* value of debt, denoted as K^* and representing the amount of debt provided at the time of investment, is equal to Eq. (21) evaluated at the second-best trigger price x_2^*, i.e. $K^* = D(x_2^*)$ which yields[17]

$$K^* = \frac{R}{r} + \left((1 - b) V_u(x_d^*) - \frac{R}{r} \right) \frac{\psi(x_2^*)}{\psi(x_d^*)}. \tag{26}$$

Eq. (26) governs the equilibrium relationship between the coupon payment R and the amount of debt provided.[18] Given this relationship we can now determine the second-best firm value and

[17] The rational debtholders could provide less debt for a given coupon payment R, however we assume that competition amongst debt providers will enforce the stated equality.

[18] Alternatively, and perhaps more intuitively, one could solve (implicitly) for R and determine the fair coupon payment debtholders would expect for a given amount of debt K promised to equityholders at time of investment.

trigger price *in equilibrium*. Substituting Eqs. (26) and (13) into Eqs. (24) and (25) we see that

$$F_2(x) = \begin{cases} \left(V_\ell(x_2^*) - I\right) \frac{\psi(x)}{\psi(x_2^*)}, & \text{for } x < x_2^*, \\ V_\ell(x) - I, & \text{for } x \geq x_2^*, \end{cases} \tag{27}$$

and furthermore

$$\frac{E'(x_2^*)}{V_\ell(x_2^*) - I} = \frac{\psi'(x_2^*)}{\psi(x_2^*)}. \tag{28}$$

Note that Eq. (27) represents the expected discounted value of the *levered* project less total investment cost.

2.5. First-best investment policy

The comparison of the general results derived from Eqs. (27) and (28) to the first-best firm value and investment trigger price allows for a quantitative analysis of agency costs in the presence of conflicting equityholder-debtholder interests. We derive the first-best firm value and investment policy based on the setting in which the overall firm value, as opposed to equity value, is maximised. In this case, and analogous to the second-best value, the first-best firm value is defined as

$$F_1(x) := \sup_{T_1} \mathbb{E}_x^{\mathbb{Q}} \left[e^{-rT_1} (V_\ell(X_{T_1}) - I) \right] = \max_{x_1} \left\{ (V_\ell(x_1) - I) \frac{\psi(x)}{\psi(x_1)} \right\}, \tag{29}$$

hence

$$F_1(x) = \begin{cases} \left(V_\ell(x_1^*) - I\right) \frac{\psi(x)}{\psi(x_1^*)}, & \text{for } x < x_1^*, \\ V_\ell(x) - I, & \text{for } x \geq x_1^*, \end{cases} \tag{30}$$

where x_1^* denotes the first-best trigger price and satisfies the equation[19]

$$\frac{V_\ell'(x_1^*)}{V_\ell(x_1^*) - I} = \frac{\psi'(x_1^*)}{\psi(x_1^*)}. \tag{31}$$

Our chosen solution methodology allows us to highlight the following important results when comparing the first- and second-best firm values, i.e. Eqs. (30) and (27) respectively.

Proposition 3. *For coupon payment R fixed, the second-best firm value $F_2(x)$ is always lower than (or equal to) the first-best value $F_1(x)$, i.e. $F_2(x) \leq F_1(x)$, for all x, hence agency conflicts always reduce total firm value.*

Proof. We note that the representation of the two firm values given by Eqs. (27) and (30) differ only by the critical level x_i^* employed in each. Since the value of $F_1(x)$ was determined by maximisation over such investment triggers x_i, it follows that this must imply the relationship $F_1(x) \geq F_2(x)$ for all x. □

[19]Proof of the existence and uniqueness of the first- and second-best trigger prices x_1^* and x_2^* would appear more difficult than for x_a^* and x_d^* (since it requires proof of the convexity of V_ℓ). However, numerical studies of the cases considered in Section 3 indicate that for a wide range of reasonable parameter values this is indeed the case.

Corollary 4. *For coupon payment R fixed, the second-best investment trigger price always lies below the first-best investment trigger price, i.e.* $x_2^* \leq x_1^*$*, resulting in earlier (or over) investment by levered firms.*

Proof. Since $F_2(x) \leq F_1(x)$ for all x and both F_1 and F_2 dominate $V_\ell - I$ the result is evident. See Appendix G for a more detailed proof. ☐

The above results confirm the overinvestment of equityholders in (at least partially) debt-financed investments for a much wider class of uncertainty processes. However, it is important to note that the above results hold true for a coupon R fixed across first- and second-best outcomes. However, equityholders are free to choose the financing strategy and hence the coupon payment R which maximises equity value. From the problem formulation it seems clear that an optimal coupon payment exists due to the tradeoff between the expected benefits of the tax shields and the expected costs of financial distress; see Fig. 6 and Eq. (22). We defined the optimal coupon payment which maximises both the first- and second-best firm value as $R_i^* = \text{argmax}_R F_i(x; R)$. The resulting effect of this additional flexibility of the equityholders on the value of the firm and hence the agency costs of overinvestment is presented in the following proposition.

Proposition 5. *The first-best firm value (using the first-best optimal coupon* R_1^**) dominates the second-best firm value (using the second-best optimal coupon* R_2^**), i.e.* $F_1(x; R_1^*) \geq F_2(x; R_2^*)$*. Furthermore, this differential in firm value is greater than or equal to the differential in firm value for a fixed coupon applied to both first- and second-best outcomes, i.e.* $F_1(x; R_1^*) \geq F_1(x; R_2^*) \geq F_2(x; R_2^*)$*.*

Proof. The proof is trivial since $F_1(x; R_1^*) \geq F_1(x; R_2^*) \geq F_2(x; R_2^*)$ where the first inequality must be true from the definition of R_1^* and the second inequality is due to the result of Proposition 3. ☐

To summarise, for a fixed coupon payment R, the overinvestment of equityholders (i.e. their optimal timing decisions) results in a decrease in firm value. However, with the inclusion of the effect of agency conflicts on the optimal financing policy (i.e. the choice of coupon R) the equityholders' optimal leverage decision results in an even further reduction in firm value.

At this stage it would be desirable to provide a result comparing the relative sizes of the first- and second-best investment trigger prices when the equityholders are also allowed to optimise over the coupon R. Unfortunately, such a result appears to be elusive in the present (very general) setting. Numerical computations in Section 4, however, reveal that $x_1^*(R_1^*) \geq x_2^*(R_2^*)$ *for all* parameter regimes considered, indicating that overinvestment is indeed maintained by equityholders when they are also allowed to optimally choose the level of debt financing.

2.6. Agency costs

To quantify the agency cost of overinvestment by equityholders we follow Mauer and Sarkar (2005) and define the agency cost as the difference between first- and second-best firm values (evaluated at their respective optimal coupon payments) in percent of the second-best firm value

$$AC := \frac{F_1(x; R_1^*) - F_2(x; R_2^*)}{F_2(x; R_2^*)}. \tag{32}$$

Note that given our expressions for the firm values $F_i(x)$ in Eqs. (27) and (30), it can be seen that the agency cost AC is independent of the price x, provided that $x < x_2^*(R_2^*)$. If $x_2^*(R_2^*) < x < x_1^*(R_1^*)$ then the agency cost becomes dependent on x and further if $x > x_1^*(R_1^*)$ then there are no agency costs since it is optimal to immediately invest under both first- and second-best outcomes. This point was not previously noted by Mauer and Sarkar (2005) and can be important when assessing the agency cost of projects for which immediate investment is optimal under either the first- or second-best case.

Mauer and Sarkar (2005) also chose to decompose agency cost into two components. The first being the loss of pure operating value due to agency conflicts and the second the loss in the net benefit of debt financing. Given the above considerations we instead choose to decompose the agency cost differently. In order to pinpoint the agency costs due to differences in *timing* decisions (x_1^* vs. x_2^*) and those due to differences in *financing* decisions (R_1^* vs. R_2^*) we define

$$AC = \left[\frac{F_1(x;R_1^*) - F_1(x;R_2^*)}{F_2(x;R_2^*)}\right] + \left[\frac{F_1(x;R_2^*) - F_2(x;R_2^*)}{F_2(x;R_2^*)}\right] =: AC^{fin} + AC^{tim}. \tag{33}$$

This novel decomposition provides additional insights and highlights important results when comparing GBM and mean reversion in Section 3, the results of which are reported and analysed in detail in Section 4.

Finally, we note that the differing optimal coupon payments also affect the optimal leverage at investment, defined as $L_i^* = D(x_i^*;R_i^*)/V_\ell(x_i^*;R_i^*)$ for $i = 1, 2$. These leverage ratios will allow us to evaluate the impact of agency conflicts (and also mean reversion) on optimal capital structure decisions.

3. Mean-reverting commodity prices

In this section we apply the general model from Section 2 to a well known mean-reverting process. Applying the GBM process (with $\alpha(x) = \alpha x$ and $\sigma(x) = \sigma x$) to our general setup will result in a model similar to the one studied by Mauer and Sarkar (2005). Results for both GBM and MR will be provided in Section 4.

To incorporate mean reversion into the price dynamics we consider the following arithmetic mean-reverting (AMR) process

$$dX_t = \eta(\overline{x} - X_t)dt + \sigma X_t dW_t^{\mathrm{P}}, \tag{34}$$

also known as *inhomogeneous geometric Brownian motion* (IGBM)[20] due to the inhomogeneity of its expected return in the state variable X. Under the equivalent risk-neutral measure Q the price process becomes

$$dX_t = (\eta(\overline{x} - X_t) - \lambda\sigma X_t)\,dt + \sigma X_t dW_t^{\mathrm{Q}}. \tag{35}$$

[20] In the existing literature, this process has been called, amongst other things, ‘inhomogeneous geometric Brownian motion’ (IGBM) (see Abadie and Chamorro, 2008, Zhao, 2009), ‘Geometric Ornstein-Uhlenbeck’ (GOU) (see Insley, 2002) or ‘geometric Brownian motion with affine drift’ (see Linetsky, 2004). To be consistent with the more recent literature we refer to this process as IGBM.

Here η denotes the speed of mean reversion and determines the rate at which X_t returns to $\overline{x}$, the expected long-run price level. In comparison to Mauer and Sarkar (2005) the process in Eq. (34) is a stationary process as opposed to the non-stationary GBM process employed by those and many other authors. All other process elements are identical to the ones described in the context of Eq. (2).

In the real option literature, the use of the process (34) dates back to Bhattacharya (1978) and has been applied more recently by Insley (2002), Abadie and Chamorro (2008), Hong and Sarkar (2008) and Tsekrekos (2010) amongst others. It has also been used in other areas such as stochastic volatility (see Lewis, 2000) and interest rate modelling (see Brennan and Schwartz, 1980). The reasons for choosing this particular process are manyfold. First, like GBM, the IGBM mean-reverting model guarantees positive process values, consistent with our oil price application. Second, Zhao (2009) showed that IGBM has many nice closed form properties despite not being of the more tractable affine class. Third, due to the explicit inclusion of the risk-discounting effect in this paper, it is advantageous to have a process for which the volatility of returns is a constant instead of exhibiting the so-called *leverage effect*.[21] Finally, and perhaps most importantly, it can be seen that geometric Brownian motion (GBM) can be obtained as a special case of IGBM by setting $\eta = 0$ or $\overline{x} = 0$ and $\eta = -\alpha$, this allows for a direct comparison of both the IGBM and GBM processes. Given these considerations the application of IGBM in modelling positive commodity prices appear to be a natural fit.

Remark 1. We note that the IGBM process exhibits an entrance (not exit) boundary at zero in contrast to the natural barrier for the GBM process employed by Mauer and Sarkar (2005).[22] This indicates that the origin is inaccessible after the process has started and the only feasible case of a zero price level is given when the current price is zero, i.e. $X_0 = 0$. These considerations can have an important effect on the option value. In the case of the investment option discussed in Section 3.3, the entrance boundary guarantees a positive firm value at $x = 0$ for positive long-run mean-reverting levels ($\overline{x} > 0$). The intuition behind this is that, even if prices are currently zero, we can still expect future prices to revert back to a positive long-run mean level $\overline{x}$. Hence at $x = 0$ the investment option still holds time value and so is positive.

It might appear that these are merely technical considerations, however a proper understanding of the boundary behaviour of the chosen uncertainty process is crucial in the correct application of the appropriate boundary conditions. For example, Tsekrekos (2010), who also employs the IGBM process, attempted to apply an incorrect boundary condition at zero (Eq. (7) in Tsekrekos, 2010). The stated condition is applicable for GBM (and its natural boundary) but not for IGBM. In fact, the only solution to Eq. (6) in Tsekrekos (2010) satisfying the incorrect boundary condition is indeed the trivial solution $V_0 \equiv 0$. However, a careful scrutinizing of Tsekrekos (2010) reveals

[21]This is not to say that the leverage effect is unimportant, just that we choose to isolate the effects of risk-discounting from the leverage effect. The influence of the leverage effect on investment and financing decisions could be investigated easily using the CEV model but this is left for the subject of future research. For a step in this direction see Nunes (2009).

[22]This can be seen most clearly by the application of the well known *Feller tests* for boundary classifications. Zhao (2009) applied such tests and showed that, for the process (35), zero in an entrance boundary and infinity a natural boundary.

that an incorrect application of the incorrect boundary condition results, fortuitously, in the correct expression for the solution to the associated free-boundary problem.

3.1. Unlevered project value

The process in Eq. (35) can be rearranged to

$$dX_t = (\eta + \lambda\sigma)\left(\frac{\overline{x}}{1 + \lambda\sigma/\eta} - X_t\right)dt + \sigma X_t dW_t^{\mathrm{Q}}, \tag{36}$$

which is identified as an AMR process with a speed of mean reversion of $\eta + \lambda\sigma$ and a long-run mean level of $\overline{x}/(1 + \lambda\sigma/\eta)$. Hence it is well known that the expected value at time t is given by

$$\mathbb{E}_x^{\mathrm{Q}}[X_t] = \frac{\overline{x}}{1 + \lambda\sigma/\eta} + \left(x - \frac{\overline{x}}{1 + \lambda\sigma/\eta}\right)e^{-(\eta+\lambda\sigma)t}. \tag{37}$$

Therefore, by Eq. (6), the value of all future discounted expected profits is given by

$$f_u(x) = (1 - \tau)\left(\frac{x}{r + \eta + \lambda\sigma} + \frac{\eta\overline{x}}{r(r + \eta + \lambda\sigma)} - \frac{C}{r}\right). \tag{38}$$

Given Eq. (38) the abandonment trigger price x_a^* in Eq. (11) can be determined and the value of the unlevered project $V_u(x)$ in Eq. (10) calculated. To do so, the functions $\phi(x)$ and $\psi(x)$ associated with the IGBM process have to be determined.

Proposition 6. *The fundamental decreasing and increasing functions ϕ and ψ associated with the process* (35) *are given by*

$$\phi(x) = x^\gamma M\left(-\gamma, 2(1-\gamma) + \frac{2(\eta + \lambda\sigma)}{\sigma^2}; \frac{a}{x}\right), \tag{39}$$

$$\psi(x) = x^\gamma U\left(-\gamma, 2(1-\gamma) + \frac{2(\eta + \lambda\sigma)}{\sigma^2}; \frac{a}{x}\right), \tag{40}$$

where M and U are confluent hypergeometric functions, $a := 2\eta\overline{x}/\sigma^2$, and γ is the negative root of the quadratic $\frac{1}{2}\sigma^2\gamma(\gamma - 1) - (\eta + \lambda\sigma)\gamma - r = 0$.

Proof. The derivation of the solution to Eq. (9) in the case of IGBM relies on its reduction to the standard form of the so-called Kummer's equation[23] (see Appendix H). □

3.2. Levered project value

In the case of the levered project the calculation of the value of all future discounted expected profits f_ℓ requires the substitution of Eq. (37) into Eq. (16) (or alternatively by substitution of Eq. (38) into Eq. (17)) yielding

$$f_\ell(x) = (1 - \tau)\left(\frac{x}{r + \eta + \lambda\sigma} + \frac{\eta\overline{x}}{r(r + \eta + \lambda\sigma)} - \frac{C + R}{r}\right). \tag{41}$$

[23] See, for example, Abramowitz and Stegun (1972)

This equation allows us to solve for the default trigger price x_d^* in Eq. (19) and thus determine the optimal equity value of the levered project $E(x)$ in Eq. (18). Next, and following Eq. (21), the value function of debt $D(x)$ can be specified. Lastly, the value function of the levered project follows from applying the previous results to Eq. (13).

Inspection of Eq. (41) leads to the following important observation.

Proposition 7. *Considering the uncertainty dynamics of Eq.* (35)*, the optimal default trigger price x_d^* does not exist, and hence it is never optimal to default on the levered project, if the following condition is satisfied*

$$\frac{\overline{x}}{C+R} > 1 + \frac{r+\lambda\sigma}{\eta}. \tag{42}$$

Furthermore, if Eq. (42) *is satisfied, the project value $V_\ell(x)$ remains finite and is given by $f_\ell(x) + R/r$. The same results hold for the optimal abandonment trigger price x_a^* when $R = 0$.*

Proof. Noting that f_ℓ is linear (and hence convex), we recall from Proposition 1 that a unique optimal default (abandonment) trigger exists provided that f_ℓ (f_u) becomes negative in $\mathbb{R}_+$. Since f_ℓ is increasing it suffices to show that $f_\ell(0+) < 0$ for this condition to be satisfied. Evaluating Eq. (41) at $x = 0$ provides the above result. □

Corollary 8. *Under geometric Brownian motion with drift α, the optimal default and abandonment trigger prices x_d^* and x_a^*, respectively, do not exist (for $C + R > 0$) if $\alpha \geq r + \lambda\sigma$ (and hence default is never optimal). Furthermore, in this case, the project value becomes infinite.*

Proof. Setting $\overline{x} = 0$ and $\eta = -\alpha$ in Eq. (42) provides the required result. □

The above results provide clear qualitative differences in investors' behaviour between the GBM and IGBM case. Under the assumption of a GBM price process the condition $\alpha < r + \lambda\sigma$ is required to ensure that the project valuation is finite and investors will always optimally default on (abandon) the project in this case if subjected to positive costs. Otherwise, if $\alpha \geq r + \lambda\sigma$, the project has an infinite value and trivially should never be abandoned.[24] This condition effectively restricts the region of applicability of the GBM model to valuing projects in this (infinite horizon) case. Under the assumption of mean-reverting prices on the other hand, the project value f_ℓ can be seen to remain finite *for all* parameter regimes, even when it is optimal to never default on the project. Finally, we note that Eq. (42) indicates that if $\overline{x} < C + R$, hence the project is not profitable in the long-run, then it will always be optimal to default/abandon at sufficiently low prices, irrespective of other parameters.

3.3. *First- and second-best investment policy*

Following the steps of the general model, the firm value functions $F_i(x)$ ($i = 1, 2$) for the specific case of process (34) can be found by applying Eqs. (39) and (40), along with Eq. (22),

[24] This condition is reminiscent of the popular (Gordon) constant growth model for equity valuation (see Gordon, 1959) in which the equity cannot be valued if the expected future growth rate of dividends exceeds the risk-adjusted required rate of return.

in the general representations given by Eqs. (27) and (30). In addition, the first- and second-best trigger prices can be calculated numerically with standard root-finding algorithms applied to the specific cases of Eqs. (28) and (31). Other required inputs to these calculations are the abandonment and default trigger prices x_a^* and x_d^*, determined by the specific cases of Eqs. (11) and (19) respectively, as well as the integrated profit function $f_\ell(x)$ given in Eq. (41).

4. Results

In this section we derive numerical results based on the IGBM process described in Section 3, which, upon setting $\eta = 0$, allows for the comparison of our investment, financing and policy related results with those based on a standard GBM price process. We proceed as follows. After discussing our base case parameters we first focus on the effects of mean reversion on the two-layered optimal stopping problem to study how the default option affects investment timing and financing decisions. We then focus on quantifying the agency costs of debt financing (and its components) and perform extensive comparative statics analysis for all model parameters, focusing in particular on the speed of mean reversion η, which plays a crucial role for our model results.

4.1. Base case parameters

To illustrate the model results we consider the investment into an oil production facility (such as an oil rig). Oil continues to be a key energy resource in the 21st century and therefore has received much attention in the the real options literature (see, for example, Paddock, Siegel, and Smith, 1988). Furthermore, many studies indicate that oil price dynamics exhibit mean-reverting behaviour, at least over longer time periods (see Bessembinder et al., 1995), therefore oil would appear a natural choice as our illustrative example.

We estimated the parameters of the IGBM model using 12 years (January 2000–December 2011) of monthly West Texas Intermediate (WTI) oil price data (US Dollars per Barrel).[25] We employed the estimation method of Longstaff and Schwartz (1995), which has also been employed in many other papers since, including Insley (2002), Sarkar and Zapatero (2003) and Hong and Sarkar (2008). The estimation yielded the following base case uncertainty parameters: $\overline{x} = \$98.22$, $\eta = 0.1733$ and $\sigma = 0.274$.

The other (non-process specific) parameters are taken to be: $r = 0.04$, $\lambda = 0.32$, $\tau = 0.3$, $b = 0.35$, $C = \$60$, $R = \$13.03$, $I = \$180$ and $X_0 = \$100$.[26] The production cost per barrel of \$60 is set to be the average production cost for several oil production technologies (see International Energy Agency, 2008, p.218). Note that these costs (assumed to be constant) are less than the long-run price level $\overline{x}$ and so, in the absence of debt, the project is expected to make a profit in the long-run. The base case coupon payment R is derived as the optimal second-best coupon for the base case parameters (in which 65.77% of the project is financed by debt). This choice is

[25] The data was obtained from the *US Energy Information Administration*. We choose this period to respond to what appears to have been a structural change in the price of oil (to a much higher price regime) around the turn of the last century. It should also be noted that we estimated the process using real prices since it is the real price (not the nominal) that is assumed to mean revert. As such we converted all prices to December 2011 prices using the *Producers Price Index* (PPI).

[26] Note that all costs are in units of *per barrel* since the estimated price is in these units.

consistent with the procedure adopted in Mauer and Sarkar (2005) for choosing their base case. The effective tax rate τ of 30% and bankruptcy costs b of 35% also follow Mauer and Sarkar (2005). The investment cost I is assumed to be three times the yearly costs. We propose X_0 to be $100 which approximately reflects WTI oil prices during the first half of 2012. For the Sharpe ratio λ we assume a value of 0.32, taken from Henriques and Sadorsky (2008) who reported this to be the Sharpe ratio for oil prices over a similar sample period. Finally, the risk-free rate r is chosen to be 4%.

4.2. Project (inner option) and firm (outer option) values

Fig. 1 illustrates project values $V_i(x)$ for $i = u, \ell$, and the project abandonment and default trigger prices for the base case. The unlevered project is optimally abandoned at an oil price of $23.25, whereas the addition of leverage increases this trigger to $48.77, thereby confirming Corollary 2, i.e. $x_a^* \leq x_d^*$. We note that x_a^* is very low compared to the prices observed during the data sample period, indicating that this level would be highly unlikely to be reached if price dynamics continued as in the sample period. On the other hand, x_d^* is more than twice as large, indicating a much higher probability of default due to the effect of debt on the project cash flows. We also note that for the base case parameters $V_\ell(X_0) = \$181.53$ and $V_u(X_0) = \$172.79$, hence debt financing adds $8.74 (or 5.06%) to the total project value, reflecting the expected value of the tax shield in excess of bankruptcy costs.

*** Insert Figure 1 about here ***

*** Insert Figure 2 about here ***

Fig. 2 illustrates the first- and second-best firm values.[27] In accordance with Proposition 3, the first-best firm value $F_1(x)$ is greater than the second-best firm value $F_2(x)$ with the difference reflecting the agency cost of debt financing. Also, overinvestment is observed since $x_1^* \leq x_2^*$ which confirms Corollary 4. Note that the smooth pasting of $F_1(x)$ onto $V_\ell(x) - I$ at x_1^* indicates the optimality of the first-best trigger price to maximise the levered firm value. In contrast to this, the optimality of the second-best investment trigger dictates that the $F_2(x)$ pastes smoothly onto the equity value $E(x) - (I - K)$ (not depicted in Fig. 2), not total project value (depicted in Fig. 2), thus explaining the kink in the firm value at $F_2(x_2^*)$.

4.3. Abandonment and default trigger prices

Trigger prices of the inner and outer option are of utmost importance, governing the investor's optimal behaviour both before and after investment hence influencing the rational debtholders' behaviour and the magnitude of agency costs. Key drivers for these trigger prices are the parameters

[27]To better emphasise the difference between first- and second-best firm values graphically we deviate from the base case coupon payment and used $R = \$50$ for *this figure only*.

of the mean-reverting process employed ($\overline{x}$, η and σ). Comparative statics for the abandonment and default trigger levels, x_a^* and x_d^*, are presented in Fig. 3. In addition, comparative statics for the non-process dependent, discount parameters r and λ and cost parameters C and R are presented. These are crucial in our understanding of the effect of mean reversion on the entrepreneur's optimal timing, particularly in the presence of risk discounting and in light of the results of Proposition 7 and Corollary 8. As for the parameters b and τ, it can be seen from Eqs. (11) and (19) that the abandonment and default trigger prices are independent of both.

It is important to note that these comparative statics were produced for a fixed (base case) coupon R. However, the investment timing and financing decisions are intimately linked, therefore once we depart from the base case, the optimal equilibrium coupon payment changes, providing additional effects on the default and investment trigger prices. We begin by analysing the isolated effect of parameters on the optimal timing decisions by fixing R (and hence the financing decision). In Section 4.6 we will extend this analysis by considering the equilibrium coupon payment and hence the effect of the optimal financing decisions, presenting the general mechanics of this complex and highly nonlinear model.

*** Insert Figure 3 about here ***

Fig. 3 demonstrates that $x_a^* \leq x_d^*$ for all parameter values. Also, for higher $\overline{x}$, lower costs (either C or R) or a lower discount rate (due to a lower r or λ) abandonment and default occurs at a lower trigger price because the expected profitability of the project, in present value terms, increases in these cases. Project owners therefore tolerate much lower output prices in light of this increased expected profitability.

Also, from Fig. 3(b) we observe that a higher speed of mean reversion results in a lower abandonment or default trigger price. For low values of η there is very little effect on the default and abandonment trigger prices, whereas there is a more pronounced effect for higher levels of η. As noted previously, for the base case parameters the long-run profitability of the production facility is positive (i.e. $\overline{x} - C - R > 0$) and so higher levels of η indicate that price departures from $\overline{x}$ (and hence from a profitable region) are corrected more quickly through a stronger mean-reversion force. This reduces the price variance and the equityholders' are willing to tolerate lower output prices.

However, note that the η dependence of the trigger prices is intimately linked with the values of the long-run price level $\overline{x}$ and costs $C + R$. Specifically, it can be shown that when the project is not expected to be profitable in the long-run (i.e. $\overline{x} < C+R$) we find that an increase in the speed of mean reversion actually increases the abandonment and default trigger prices, and hence increases the probability of such default (which in turn would impact debt provision and the equilibrium outcome).[28] This result emphasises the importance of the long-run profitability on the models outcomes which will be discussed in more detail in Section 4.6.

[28] It can be shown that because $\frac{\partial x_d^*}{\partial \eta} = -\frac{(1-\tau)(\overline{x}-C-R)\phi'(x_d^*)}{r(r+\eta+\lambda\sigma)\phi''(x_d^*)f_\ell(x_d^*)}$, x_d^* is decreasing for $\overline{x} > C+R$ and increasing for $\overline{x} < C+R$.

In reference to Fig. 3(c), it is well understood that (in the absence of risk discounting) an increase in volatility σ would result in an increase in the value of the default and abandonment options, with an associated decrease in the default and abandonment trigger prices. However, the inclusion of the risk-discounting effect results in the impact of volatility on the required rate of return having an additional and competing effect on the default and abandonment trigger prices. An increase in σ results in a higher risk-adjusted discount rate and hence a lower option/project value and a higher trigger price. These two competing forces explain the observed σ comparative statics indicating that the risk-discounting effect dominates for low volatilities. We thus state our first result.

Result 1. *Under mean-reverting dynamics the inclusion of the risk-discounting effect results in non-monotonic behaviour of the abandonment and default trigger prices with changes in the volatility parameter* σ.

Fig. 3 also shows that for certain parameter regimes it is optimal to never abandon or default ($x_a^* = 0$ and $x_d^* = 0$ respectively), see Proposition 7. No-default regions occur for very profitable projects, when either $\overline{x}$ is very high or variable costs C are very low. No default or abandonment also becomes optimal for sufficiently low volatility σ or Sharpe ratio λ, and for sufficiently high speeds of mean reversion η since these scenarios describe an increased certainty in price, and hence profitability.

Critical parameter values which separate the default versus no-default regions can be determined by rearranging Eq. (42) for the required parameter. For example, the critical value of η above which the investor would never default on the project in Fig 3(b) is

$$\eta^* := \frac{(r + \lambda\sigma)(C + R)}{\overline{x} - (C + R)} = 0.37 \tag{43}$$

with the associated critical value for abandonment obtained by setting $R = 0$ to yield $\eta^* = 0.2$.

4.4. Investment trigger prices

Next, we investigate the optimal first- and second-best investment trigger prices. Fig. 4 plots the comparative statics of the investment trigger prices x_i^*, $i = 1, 2$, along with the abandonment and default trigger prices x_a^* and x_d^* for comparison. Again, we begin the analysis with a fixed coupon payment R.

*** Insert Figure 4 about here ***

The analysis of the critical investment trigger prices x_1^* and x_2^* as a function of the process and non-process model parameters demonstrates the overinvestment by equityholders, confirming Corollary 4. Fig. 4 also shows that the trigger prices decrease for higher long-run price levels $\overline{x}$ and a higher speed of mean reversion η. The opposite relationship holds when considering the optimal investment trigger prices and σ, r, λ, τ, C or I.

Furthermore, Fig. 4(c) indicates that investment trigger prices increase as uncertainty increases accompanied with an increase in overinvestment (since the gap between first- and second-best trigger prices also appears to widen). To help explain this increase in overinvestment, we note that the equityholders share the benefits of the higher prices (resulting from an increased volatility) but are still limited on the downside by their ability to default on the project and hand the project over to the debtholders. This asymmetric payoff thus results in increased incentives for equityholders to overinvest as volatility increases (see Mauer and Sarkar, 2005).

When considering the first- and second-best investment trigger prices as a function of the coupon R we observe that for an initial increase in R both the first- and second-best trigger prices are reduced. As R increases further this initial decrease reverses and the trigger prices start to increase.[29] We then observe that for the second-best outcome, there is a critical value of R above which the default trigger price is actually higher than the investment trigger price. For the base-case parameters this can be seen to be approximately \$133.82. This region corresponds to the case in which the equity value would be eroded to zero and therefore coupons above this value are not economically meaningful.

The economic insight behind the non-monotonicity of the investment trigger prices in the coupon payment R differs for the first- and second-best outcomes. For the first-best optimiser the investment cost is fixed at I as R increases and so the optimal behaviour is simply a result of the expected tax-shield/bankruptcy-cost trade off. For the second-best optimiser however, the investment cost $(I - K)$ is no longer fixed as the coupon R increases. For an increase in R, more cost is added to the levered project which reduces the equity value of the up-and-running firm. However, this additional cost finances an initial cash injection of K from the debtholders which reduces the cost of purchasing the project for the equityholders. For small coupon payments it can be seen that the reduction in investment cost faced by the equityholders is greater than the reduction in the equity value, resulting in a net *increase* in total equity value prior to investment, and hence producing a lower investment trigger price. As the coupon increases further, the rational debtholders become more reluctant to provide additional debt while the value of equity in the up-and-running firm continues to fall. The result being a net *reduction* in equity value prior to investment and hence a higher investment trigger price.

Finally, for some parameter regimes (high $\overline{x}$ and η, as well as low σ, λ and C) the first- and second-best investment trigger prices converge, resulting in identical first- and second-best outcomes. This would indicate that the agency cost is negligible in these regimes. Inspection of Eqs. (28) and (31), reveals that trigger prices x_1^* and x_2^* are equal if $V_\ell'(x) = E'(x)$. This is the case when $D'(x) = 0$ since $V_\ell'(x) := E'(x) + D'(x)$. Further, it can be seen that this will be the case for parameter regimes in which it is optimal for the equityholders to never default on the project. Inspection of Eq. (21) reveals that in this case the fair value of debt reduces to R/r, indicating that debtholders are not concerned with default. The debt value therefore becomes insensitive to the output price, i.e. $D'(x) = 0$, yielding $V_\ell'(x) = E'(x)$ and hence $x_1^* = x_2^*$. Note that this feature is unique to the model under mean-reverting dynamics since it is always optimal to default under the

[29] Footnote 21 of Mauer and Sarkar (2005) states that, for their base case parameters, x_2^* is monotonically decreasing in R. Our result differs from this (even for the GBM case) indicating that the monotonic behaviour is parameter dependent and not a general result of the model.

GBM assumption.

Result 2. *When it is never optimal for equityholders to default on the levered project, the first- and second-best investment trigger prices coincide, and the agency-costs are zero.*

We also note that there are in fact two mechanisms through which agency costs can disappear in our model. The first occurs if, for a given parameter regime, immediate investment becomes optimal for both first- and second-best outcomes. This mechanism is independent of the process chosen. The second mechanism occurs when we are in a parameter regime for which it is never optimal to default on the levered project (as characterised by Proposition 7). Therefore, mean reversion provides additional mechanisms for the reduction of agency cost.

4.5. Equilibrium debt provision

Next we investigate the equilibrium provision of debt in the presence of agency conflicts and mean-reverting prices. Fig. 5 represents the equilibrium amount of debt financing for a given coupon payment R.

*** Insert Figure 5 about here ***

Whilst it is not surprising that more debt is provided as the coupon payment R increases, the concavity of the relationship reveals the impact of the increased credit risk to the debt providers as R, and therefore the amount of debt, increases. Higher coupon payments put the firm in a worse financial position, burdened with higher financing cost (R), which increases the probability of equityholders' default. It is important to note that in equilibrium the debtholders are very reluctant to provide debt in excess of the investment amount I (i.e. $K^* > I$). This result differs from the results of Mauer and Sarkar (2005), since for their base case a particularly high value of debt financing in equilibrium is observed, equal to an amount exceeding 2.75 times the investment cost.[30] Our model generates perhaps more realistic equilibrium debt levels for economically reasonable annual coupon payments, where the first- and second-best optimal coupons, $R_1^* = \$15.82$ and $R_2^* = \$13.03$ correspond to an equilibrium debt financing of 74.13% and 65.77% of the project cost, respectively. Only extremely high coupon payments (which are suboptimal for the equityholders) result in debt provision of more than I. One comparison to make with Mauer and Sarkar (2005) is to consider our model with $\eta = 0$ to evaluate the equilibrium debt financing fraction for (zero drift) GBM. In this case $R_1^* = \$44.01$ and $R_2^* = 25.09$ corresponding to $K^*(R_1^*)/I = 128.36\%$ and $K^*(R_2^*)/I = 78.51\%$. Thus, clearly more debt is provided in equilibrium under GBM but at the cost of much higher coupon payments. Furthermore, these values indicate that the even higher debt financing fraction in Mauer and Sarkar (2005)—275%, a number not reported by the authors—is perhaps related to the positive drift of the GBM process employed.

[30]Mauer and Sarkar (2005) suggest that excess debt is paid as a dividend to equityholders at time of investment as mentioned by the authors in Footnote 8. Hence, in their base case debtholders agree to provide equityholders with a relatively large dividend; a practice not typically observed in actual investments.

Result 3. *Under mean-reverting dynamics, debtholders are very reluctant to provide more funding than the purchase price of the project.*

*** Insert Figure 6 about here ***

Fig. 6 shows the equilibrium first- and second-best firm values as a function of the coupon payment R. We observe well-defined unique maximum first- and second-best firm values (cf. Modigliani and Miller, 1959, Baxter, 1967). Furthermore, the optimal first-best coupon, R_1^*, is higher than the second-best outcome, R_2^*. Therefore, equityholders maximising equity value would not only invest *sooner* but would also pick a *lower* coupon than a manager maximising total firm value, resulting in increased agency costs. This behaviour would also result in a lower second-best leverage ratio at time of investment due to the lower coupon payment. Economically, the incentive for the first-best optimiser to take on more debt is a result of the substantial benefits of the tax shield in increasing total firm value.

4.6. Equilibrium comparative statics and the agency cost of debt

Drawing on our previous results, we now provide an investigation of the influence of mean reversion on the *true* equilibrium outcomes, i.e. when the additional flexibility of equityholders to select the optimal coupon payment R^* is taken into account as model parameters are varied. Focusing in particular on the speed of mean reversion parameter—since $\eta = 0$ corresponds to GBM dynamics—we investigate its effect on the level of the optimal coupon itself, the investment and abandonment trigger prices, optimal leverage and the agency costs of debt.

*** Insert Table 1 about here ***

Table 1 reports the comparative statics analysis of the optimal coupons, optimal trigger prices (both default and investment), firm value, optimal leverage (at investment), credit spreads,[31] and total agency cost as the model parameters are varied. First, we reiterate that $x_1^*(R_1^*) > x_2^*(R_2^*)$ for all parameter regimes, demonstrating overinvestment for the case when the coupon payment is also independently optimised.

Focusing on the agency costs we observe that our comparative statics results for the financing parameters (b and τ) and the discount parameters (r and λ) are consistent with Mauer and Sarkar (2005). However, under mean-reverting dynamics, we previously observed that a key driver for the qualitative behaviour of the model's output was the long-run profitability of the project $\overline{x} - C - R$, which increases when either $\overline{x}$ increases or costs C decrease. Accordingly, inspection of Table 1 provides the following important result.

Result 4. *As the long-run profitability of the project increases, agency costs are reduced.*

[31] Defined as $CS_i = R_i^*/D(x_i^*(R_i^*); R_i^*) - r$ for $i = 1, 2$ as in Mauer and Sarkar (2005).

Fig. 7 shows the true comparative statics (allowing for the change in optimal coupon) of several equilibrium outputs as η is varied. Figs. 7(a) and 7(b) show the optimal outcomes for the equityholders financing and timing decisions. Fig. 7(c) shows the credit spreads, a proxy for the willingness of the debtholders to provide debt in equilibrium, and Fig. 7(d) depicts the optimal leverage ratio of the firm upon investment.

*** Insert Figure 7 about here ***

Note that for the base case parameters we have $\overline{x} - C = \$38.22$, hence the project is profitable in the long-run in the absence of any debt. Consequently, the firm can take on a coupon payment of up to \$38.22 and still remain profitable in the long-run. Inspection of Fig. 7(a) indicates that the optimal coupon payment for the second-best outcome is indeed always below this maximum value.[32]

Inspection of Fig. 7 also indicated the existence of two distinct regimes of behaviour as the parameter η is varied. The first regime, for $\eta < \eta_c$, exhibits relatively high optimal coupon payments which are decreasing as η increases, whereas both the investment and default trigger prices and the optimal leverage ratios remain relatively insensitive to changes in η. Credit spreads are also high in this regime. In the second regime, for $\eta > \eta_c$, the optimal coupon payment is increasing in η (with an associated increase in optimal leverage), the optimal investment trigger price is decreasing in η, and the optimal default trigger price is at a much lower level but is still fairly insensitive to changes in η. Credit spreads are very low in this regime. Note that for the base case parameters $\eta_c \simeq 0.184$.

Result 5. *There are two distinct regimes of equilibrium behaviour (both financing and investing) as the speed of mean reversion η varies, separated by a critical value η_c.*

The non-monotonicity of the model outcomes with respect to η is a feature of the strategic interaction between equity- and debtholders, resulting in highly nonlinear behaviour. An economic interpretation of the equilibrium outcomes exhibited in Fig. 7 is as follows. As η initially increases above zero the expected profits f_ℓ of the levered project increase due to the reduction in variance around the long-run (positive) profit level. This increase in profitability results in an initial decrease in the (still relatively high) equilibrium coupon payment as the debtholders reduce their credit spreads due to a lower perceived credit risk. We note that at high levels of the optimal coupon, the sensitivity of the default and investment trigger prices to η is very low,[33] resulting in negligible feedback effects on optimal financing (coupon) from the optimal timing decisions. Hence, the financing decisions are driven mainly by the change in the expected profit function f_ℓ (as η varies). However as η increases further, the associated decrease in optimal coupon payment

[32]The first-best outcome does optimally make the the project loss-making in the long run for very low speeds of mean reversion. However, note that the mean-reverting level $\overline{x}$ effectively disappears as $\eta \to 0$.

[33]This insensitively is evidenced in Fig. 4(b) and can also be seen by evaluating the critical value η^* defined in Eq. (43); a higher η^* corresponds to a lower sensitivity (for low values of η).

results in an increased sensitivity of the trigger prices to changes in η (since η^* decreases). The highly sensitive investment trigger prices link the optimal timing decisions and the optimal coupon choice, therefore putting additional downward pressure on R^*. We interpret this behaviour as equityholders attempting to retain the majority of the firm's (now very certain) operating profits by optimally choosing to take on minimal debt.

Note that the optimal coupon payment does not decrease to zero however, since there always remains some tax-shield benefits to the equityholders, resulting in a positive minimum optimal coupon payment. The surprising implication is that it is *always* optimal for equityholders to default on the project in equilibrium, even for very high values of η. As η increases to the region in which it would be optimal to never default (for a fixed coupon payment; see Fig. 3(b)), the equityholders dramatically reduce their coupon payment so that default still remains optimal at sufficiently low prices. Economically this implies that there is always some incentive for the equityholders to transfer wealth from the debtholders.

At the start of the high η regime, the optimal coupons are very low and the profitability of the project is very certain (due to both a low variance and low coupon payments). This results in a very low default trigger price and an ever decreasing probability of default as η increases further. This causes credit spreads to plummet, expressing a decrease in debtholders' concern about project default. However, since debt is now very affordable for equityholders to employ, and given the very low expected bankruptcy costs, the optimal coupon payment starts to increase in η, causing expected operating profits to increase further. Hence in the high η regime more debt is employed to maximise the expected benefits from the tax shield as η increases further.

*** Insert Figure 8 about here ***

*** Insert Table 2 about here ***

Fig. 8 shows the total agency cost and its decomposition into the two components (as defined by Eq. (33)) as a function of the speed of mean reversion η.[34] In addition, Table 2 also shows the remaining agency cost comparative statics for other model parameters.

Result 6. *The agency costs decrease (approximately linearly) as the speed of mean reversion η increases and they become extremely small above a critical value of η ($\simeq 0.184$ for our base case).*

Furthermore, in regards to Fig. 8 we see that, consistent with Proposition 5, the agency cost due to financing and timing decisions are both positive and for very low η the total agency cost

[34]In addition, we further decomposed the agency costs into the loss in pure operating value and the loss in net benefits of debt financing, in the spirit of Mauer and Sarkar (2005). However, details are omitted in the interests of brevity but the results are available from the authors upon request.

is evenly split between the financing and timing components. However, as η increases the impact of (suboptimal) timing decisions becomes more important relative to (suboptimal) financing decisions.

Finally, in order to investigate the effect of the growth prospects of the debt financed project we also choose to perform comparative statics for the GBM case with drift $\alpha \neq 0$, obtained by setting $\overline{x} = 0$ and $\eta = -\alpha$. Results can be found at the bottom of Table 2. We conclude from this our final result.

Result 7. *The agency cost due to (suboptimal) financing decisions* increases *as the growth rate α increases whereas the agency cost due to (suboptimal) timing* decreases*, resulting in a fairly constant total agency cost.*

Furthermore, the total amount of the agency cost is found to be approximately 8-9%, which is consistent with the values reported in Mauer and Sarkar (2005). The implication of the above result is that the relatively large agency costs reported in Mauer and Sarkar (2005) appear to be due to the non-stationarity of the output price dynamics.

5. Conclusions

In this paper, we have shown that the choice of the uncertainty process used to model (leveraged) investment project cash flows can have a significant impact on investment timing and related project financing decisions. The application of a mean-reverting (MR) process to our proposed model has revealed important equilibrium results with respect to the investment, default, and financing strategies of equityholders, as well as the optimal debt provision of rational debtholders.

Under MR dynamics debtholders are very reluctant to provide more funding than the purchase price of the project, a result more consistent with observed investment practice than the existing geometric Brownian motion (GBM) based results. Furthermore, we observed two distinct regimes of equilibrium behaviour, dependent on the level of a key MR process parameter, the speed of mean reversion; demonstrating the increased complexity of the equilibrium financing and investment outcomes in the presence of MR dynamics.

In regards to the reduction in firm value due to agency conflicts (the agency costs) our results indicate that total agency costs are lower for a higher speed of mean reversion and also for a higher long-run profitability of the debt-financed project. Moreover, due to a novel agency cost decomposition, we have shown that under low-growth cash flows (modelled using GBM with zero or negative drifts) agency costs are driven mainly by equityholders' (investment and default) *timing* decisions rather than due to their financing decisions. On the other hand, for high-growth projects (modelled using GBM with positive drifts) it is the equityholders' *financing* decisions that contribute the greatest to agency costs. Assuming a desire to decrease agency costs in an economy, the above information about the underlying components and drivers of such agency costs would be valuable to both policy makers and regulators alike.

Future work in this area could include the extension of the current analysis to firms that have existing operations financed with pre-existing debt, therefore analysing the effect of mean reversion on possible *under*investment and the related *debt overhang* problem (see Moyen, 2007). The inclusion of some information asymmetry between equityholders and debtholders could also be another direction for future research. Technical extensions may include accounting for jumps in the underlying price dynamics.

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Appendix A. Derivation of risk-neutral price dynamics: Eq. (2)

To transform process (1) into one under the equivalent risk-neutral measure Q we follow Dixit and Pindyck (1994) and first divide Eq. (1) by X to obtain the rate of change of X

$$\frac{dX_t}{X_t} = \frac{\alpha(X_t)}{X_t}dt + \frac{\sigma(X_t)}{X_t}dW_t^{\mathrm{P}}.$$

Taking expectations yields the expected percentage change of X (or expected capital growth rate) denoted by $a(X)$[35]

$$a(X_t)dt := \mathbb{E}^{\mathrm{P}}\left[\frac{dX_t}{X_t}\right] = \frac{\alpha(X_t)}{X_t}dt.$$

Next, it is well known that under the risk-neutral measure the process must have the following dynamics

$$dX_t = (r - \delta(X_t))\, X_t dt + \sigma(X_t) dW_t^{\mathrm{Q}}$$

where r denotes the risk-free rate of return and $\delta(X)$ the (explicit or implicit) dividend or convenience yield. In the case of commodity prices such a convenience yield is not directly observable and therefore must be implied. To do so we note that the total expected rate of return on a commodity, denoted $\mu(X)$, must be equal to the expected capital appreciation $a(X)$, plus the implied dividend; in other words $\mu(X) = a(X) + \delta(X)$, from which we *imply* that $\delta(X) = \mu(X) - a(X)$. Substituting for $\delta(X)$ and $a(X)$ yields

$$dX_t = (\alpha(X_t) - (\mu(X) - r)X_t)dt + \sigma(X_t)dW_t^{\mathrm{Q}}. \tag{A.1}$$

To determine the total expected return on the commodity we appeal to equilibrium pricing arguments. Furthermore, since it would appear that there are more risk factors involved in commodity investments than simply market risk we choose to employ a multi-factor model.[36] In this case the expected total return (in excess of the risk-free rate) is linearly proportional to the expected excess returns of N risk factors and is given by the following pricing relationship

$$\mu(X) - r = \sum_{i=1}^{N} \beta_i(X)(\mu_i - r) \tag{A.2}$$

where μ_i is the expected return of the ith risk factor F_i and $\beta_i(X)$ is the sensitivity of the commodity X to this factor. Note that we allow this sensitivity to be dependent on X. The above equation can be modified by noting that, by definition, $\beta_i(X) = \mathrm{cov}(dX/X, dF_i/F_i)/\mathrm{var}(dF_i/F_i) = \sigma(X)\rho_{Xi}/X\sigma_i$ where ρ_{Xi} and σ_i denote the correlation of the commodity X and the ith risk factor, and the volatility of this factor, respectively. Using the above relationship we can perform the following manipulations

$$\mu(X) - r = \sum_{i}^{N} \frac{\sigma(X)\rho_{Xi}}{X\sigma_i}(\mu_i - r) = \frac{\sigma(X)}{X}\sum_{i}^{N}\rho_{Xi}\left(\frac{\mu_i - r}{\sigma_i}\right) = \frac{\sigma(X)}{X}\sum_{i}^{N}\rho_{Xi}\kappa_i = \frac{\sigma(X)}{X}\lambda \tag{A.3}$$

[35] Note that, technically, we are required to justify that the expectation of the Itô integral is zero, i.e. it is a true martingale and not a *strict local martingale*. For all cases considered in this paper this is indeed the case since $\sigma(X)/X$ will become a constant.

[36] A multi-factor model allows for more flexibility in modelling the risk-premium and its economic sources. For example, it is well know that, historically, the correlation of commodity prices (oil in particular) with the equity market are very low, hence there is limited equity beta in commodities. Under the assumption of a single factor model such as the CAPM, the resulting risk premium would be close to zero (or even negative).

where κ_i denotes the market price of risk for the ith risk factor and we have defined the parameter $\lambda := \rho_{X1}\kappa_1 + \rho_{X2}\kappa_2 + \ldots$, to be interpreted as the (theoretically justified) *Sharpe ratio* of the commodity X. Note that in the case when $\sigma(X) = \sigma X$ (the case considered in Section 3) we see that the risk-premium does not depend on the price level X. Finally, substitution of Eq. (A.3) into Eq. (A.1) yields the required result

$$dX_t = (\alpha(X_t) - \lambda\sigma(X_t))dt + \sigma(X_t)dW_t^{\mathbb{Q}},$$

and completes the derivation.

Appendix B. Reformulation of Eq. (3) into Eq. (5)

Assuming $\mathbb{E}_x^{\mathbb{Q}} \int_0^\infty e^{-rt}|\pi_u(X_t)|dt < \infty$ we have the following manipulations

$$\begin{aligned}
V_u(x) &= \sup_{T_a} \mathbb{E}_x^{\mathbb{Q}} \int_0^{T_a} e^{-rt}\pi_u(X_t)dt = \sup_{T_a} \mathbb{E}_x^{\mathbb{Q}} \left[\int_0^\infty e^{-rt}\pi_u(X_t)dt - \int_{T_a}^\infty e^{-rt}\pi_u(X_t)dt\right] \\
&= \mathbb{E}_x^{\mathbb{Q}} \int_0^\infty e^{-rt}\pi_u(X_t)dt + \sup_{T_a} \mathbb{E}_x^{\mathbb{Q}} \left[-\int_{T_a}^\infty e^{-rt}\pi_u(X_t)dt\right] \\
&=: f_u(x) + \sup_{T_a} \mathbb{E}_x^{\mathbb{Q}} \left[-\int_0^\infty e^{-r(s+T_a)}\pi_u(X_{s+T_a})ds\right] && \text{(setting } t = s + T_a\text{)} \\
&= f_u(x) + \sup_{T_a} \mathbb{E}_x^{\mathbb{Q}} \left[\mathbb{E}_x\left[-\int_0^\infty e^{-r(s+T_a)}\pi_u(X_{s+T_a})ds|\mathcal{F}_{T_a}^X\right]\right] && \text{(tower law property)} \\
&= f_u(x) + \sup_{T_a} \mathbb{E}_x^{\mathbb{Q}} \left[-e^{-rT_a}\int_0^\infty e^{-rs}\mathbb{E}_x\left[\pi_u(X_{s+T_a})|\mathcal{F}_{T_a}^X\right]ds\right] && \text{(Fubini's theorem)} \\
&= f_u(x) + \sup_{T_a} \mathbb{E}_x^{\mathbb{Q}} \left[-e^{-rT_a}\int_0^\infty e^{-rs}\mathbb{E}_{X_{T_a}}\left[\pi_u(X_s)\right]ds\right] && \text{(Markovian shift)} \\
&= f_u(x) + \sup_{T_a} \mathbb{E}_x^{\mathbb{Q}} \left[-e^{-rT_a}\mathbb{E}_{X_{T_a}}\left[\int_0^\infty e^{-rs}\pi_u(X_s)ds\right]\right] && \text{(Fubini's theorem)} \\
&= f_u(x) + \sup_{T_a} \mathbb{E}_x^{\mathbb{Q}} \left[e^{-rT_a}\left(-f_u(X_{T_a})\right)\right].
\end{aligned}$$

Appendix C. Proof of Proposition 1

Proof. To prove the existence and uniqueness of x_a^* and x_d^* we need only consider the levered equation

$$\frac{f_\ell'(z)}{f_\ell(z)} = \frac{\phi'(z)}{\phi(z)} \tag{C.1}$$

since the solution for the unlevered project is simply a special case of the above when $R = 0$. A cursory inspection of Eq. (C.1) indicates that since $\phi(z)$ is positive and decreasing and $f_\ell(z)$ is increasing, that any solution to this equation must occur in the region where $f_\ell(z) < 0$ or $z \in$

$(0, f_\ell^{-1}(0))$. To analyse the equation further we rearrange the above as follows (with the aim of producing better behaved left and right hand side functions)

$$\frac{f_\ell(z)}{f_\ell'(z)} - z = \frac{\phi(z)}{\phi'(z)} - z.$$

We thus define $g_{lhs}(z) := \frac{f_\ell(z)-zf_\ell'(z)}{f_\ell'(z)}$ and $g_{rhs}(z) := \frac{\phi(z)-z\phi'(z)}{\phi'(z)}$. If it can be shown that one of these functions is non-decreasing and the other non-increasing then we have uniqueness of the root of Eq. (C.1). For existence, we need to consider the limiting behaviour of these functions as $z \to 0$ and ∞.

Consider first g_{rhs}. Differentiation of this function yields

$$g_{rhs}'(z) = \frac{\phi'(z)}{\phi'(z)} + \phi(z)\left(\frac{1}{\phi(z)}\right)' - 1 = -\frac{\phi(z)\phi''(z)}{[\phi'(z)]^2}.$$

Theorem 1 in Alvarez (2003) states that, provided infinity is a natural boundary for the process X (a standing assumption in the present paper) then the fundamental solutions ϕ and ψ are convex if and only if the auxiliary function $\theta(z) := rz+\lambda\sigma(z)-\alpha(z)$ is non-decreasing, i.e. when $r+\lambda\sigma'(z)-\alpha'(z) \geq 0$. For all cases considered in Section 3 this condition will be satisfied, hence it can be seen that $g_{rhs}(z)$ is non-increasing.

Next we wish to consider the limits of $g_{rhs}(z)$ as $z \to 0$ and $z \to \infty$. The following limits of the fundamental solution ϕ are well known[37]

$$\lim_{z\downarrow 0} \phi(z) = \infty, \quad \lim_{z\uparrow\infty} \phi(z) = 0, \quad \lim_{z\uparrow\infty} \phi'(z) = 0.$$

Furthermore, it can be shown that if zero is *non-attracting* in the sense that $\lim_{z\downarrow 0} S(z) = \infty$, where $S(z)$ denotes the *scale function* of the diffusion X, then $\lim_{z\downarrow 0} \phi'(z) = -\infty$. The above limits thus indicate that both limits of $g_{rhs}(z)$ are of *indeterminate form*, therefore in order to compute them we must apply l'Hôpital's rule as follows

$$\lim_{z\to L} g_{rhs}(z) = \lim_{z\to L}\left(\frac{(\phi(z) - z\phi'(z))'}{\phi(z)''}\right) = \lim_{z\to L}(-z) = -L$$

where $L = 0, \infty$. Hence we have shown that $g_{rhs}(z)$ is non-increasing with limits $g_{rhs}(0+) = 0$ and $g_{rhs}(\infty-) = -\infty$.

Considering g_{lhs} we can calculate its derivative to be

$$g_{lhs}'(z) = -\frac{f_\ell(z) f_\ell''(z)}{f_\ell'(z)}$$

from which we see that, provided $f_\ell(z)$ is convex, $g_{lhs}(z)$ will be non-decreasing in the region where

[37] For more details see Borodin and Salminen (2002). The first limit is guaranteed by the natural or entrance (not-exit) boundary at zero, the second limit is a result of the natural boundary at infinity and the third limit is true for all diffusions since ϕ is a positive monotone decreasing function.

$f_\ell(z) \leq 0$. Furthermore, since $f_\ell(z)$ is increasing for all z we can deduce that $g_{lhs}(z) < 0$ whenever $f_\ell(z) < 0$. In sum, $g_{lhs}(z)$ is non-decreasing for all z and negative over the interval $z \in (0, f_\ell^{-1}(0))$. Coupled with our knowledge of $g_{rhs}(z)$ this proves the existence and uniqueness of a solution to Eq. (C.1) provided that $f_\ell(z)$ is convex and negative for at least some z in the state space of the process, that the auxiliary function $\theta(z)$, defined above, is non-decreasing, and that zero is a non-attracting boundary for the process. □

Appendix D. Proof of Corollary 2

Proof. The result follows from the fact that $g_{lhs}(z; R) = \frac{f_\ell(z)}{f'_\ell(z)} - z = \frac{f_u(z) - (1-\tau)R/r}{f'_u(z)} - z \leq g_{lhs}(z; 0)$, hence $g_{lhs}(z)$ for the levered project is dominated by $g_{lhs}(z)$ for the unlevered project. Recalling also that $g_{rhs}(z)$ defined in Appendix C is non-increasing it follows directly that $x_d^* \leq x_a^*$ and the proof is complete. □

Appendix E. Derivation of Eq. (21)

The debt value (for $x \geq x_d^*$) can be calculated using the following (trivial) manipulations

$$\begin{aligned}
D(x) &= \mathbb{E}_x^{\mathrm{Q}}\left[\int_0^{T_d^*} e^{-rt} R dt + e^{-rT_d^*}(1-b)V_u(X_{T_d^*})\right] \\
&= R\mathbb{E}_x^{\mathrm{Q}} \int_0^{T_d^*} e^{-rt} dt + (1-b)V_u(x_d^*)\mathbb{E}_x^{\mathrm{Q}}\left[e^{-rT_d^*}\right] \\
&= \frac{R}{r}\left(1 - \mathbb{E}_x^{\mathrm{Q}}\left[e^{-rT_d^*}\right]\right) + (1-b)V_u(x_d^*)\mathbb{E}_x^{\mathrm{Q}}\left[e^{-rT_d^*}\right] \\
&= \frac{R}{r} + \left((1-b)V_u(x_d^*) - \frac{R}{r}\right)\mathbb{E}_x^{\mathrm{Q}}\left[e^{-rT_d^*}\right] \\
&= \frac{R}{r} + \left((1-b)V_u(x_d^*) - \frac{R}{r}\right)\frac{\phi(x)}{\phi(x_d^*)}.
\end{aligned}$$

Appendix F. Derivation of Eq. (22)

To derive Eq. (22) we first substitute Eqs. (18) and (21) into Eq. (13) and rearrange to yield

$$\begin{aligned}
V_\ell(x) &= E(x) + D(x) \\
&= f_\ell(x) - f_\ell(x_d^*)\frac{\phi(x)}{\phi(x_d^*)} + \frac{R}{r} + \left((1-b)V_u(x_d^*) - \frac{R}{r}\right)\frac{\phi(x)}{\phi(x_d^*)} \\
&= -\underbrace{bV_u(x_d^*)\frac{\phi(x)}{\phi(x_d^*)}}_{\text{PV of bankruptcy cost}} + f_\ell(x) - f_\ell(x_d^*)\frac{\phi(x)}{\phi(x_d^*)} + \frac{R}{r} + \left(V_u(x_d^*) - \frac{R}{r}\right)\frac{\phi(x)}{\phi(x_d^*)}.
\end{aligned}$$

Furthermore, using $f_\ell(x) = f_u(x) - R(1-\tau)/r$ we see that

$$V_\ell(x) = -bV_u(x_d^*)\frac{\phi(x)}{\phi(x_d^*)} + f_u(x) - f_u(x_d^*)\frac{\phi(x)}{\phi(x_d^*)} + \underbrace{\frac{\tau R}{r}\left(1 - \frac{\phi(x)}{\phi(x_d^*)}\right)}_{\text{PV of tax shield}} + V_u(x_d^*)\frac{\phi(x)}{\phi(x_d^*)}. \tag{F.1}$$

Finally we recall from Eq. (10) that

$$V_u(x_d^*) = f_u(x_d^*) - f_u(x_a^*)\frac{\phi(x_d^*)}{\phi(x_a^*)}$$

and substituting in to (F.1) yields

$$\begin{aligned} V_\ell(x) &= f_u(x) - f_u(x_a^*)\frac{\phi(x)}{\phi(x_a^*)} + \frac{\tau R}{r}\left(1 - \frac{\phi(x)}{\phi(x_d^*)}\right) - bV_u(x_d^*)\frac{\phi(x)}{\phi(x_d^*)} \\ &= V_u(x) + \frac{\tau R}{r}\left(1 - \frac{\phi(x)}{\phi(x_d^*)}\right) - bV_u(x_d^*)\frac{\phi(x)}{\phi(x_d^*)} \end{aligned}$$

as stated.

Appendix G. Proof of Corollary 4

Proof. Defining the function

$$g(z) := (V_\ell(z) - I)\,\frac{\psi'(z)}{\psi(z)}$$

we see that x_1^* and x_2^* solve $V_\ell'(x_1^*) = g(x_1^*)$ and $E'(x_2^*) = g(x_2^*)$ respectively. Furthermore it is a straightforward matter to show that $g'(x_1^*) \geq 0$ and $g'(x_2^*) \geq 0$ using the following arguments. We have

$$g'(z) = V_\ell'(z)\frac{\psi'(z)}{\psi(z)} + (V_\ell(z) - I)\,\frac{\psi'(z)}{\psi(z)}\left[\frac{\psi''(z)}{\psi'(z)} - \frac{\psi'(z)}{\psi(z)}\right]$$

and thus at $z = x_1^*$, after identifying that $\left(V_\ell(x_1^*) - I\right)\frac{\psi'(x_1^*)}{\psi(x_1^*)} = V_\ell'(x_1^*)$ from the first order condition, we have

$$g'(x_1^*) = V_\ell'(x_1^*)\frac{\psi''(x_1^*)}{\psi'(x_1^*)} \geq 0$$

since we have seen (from the proof of Proposition 1) that ψ is convex under the condition that $\theta(z) = rz - \alpha(z) + \lambda\sigma(z)$ is non-decreasing. Applying the same procedure at $z = x_2^*$ yields

$$g'(x_2^*) = D'(x_2^*)\frac{\psi'(x_2^*)}{\psi(x_2^*)} + E'(x_2^*)\frac{\psi''(x_2^*)}{\psi'(x_2^*)} \geq 0$$

since $D'(z) \geq 0$ and $E'(z) \geq 0$ for all z. Finally it is also clear that

$$V'(z) = E'(z) + D'(z) \geq E'(z), \quad \forall z.$$

To summarise, we have that the roots x_1^* and x_2^* both occur when $V'_\ell(z)$ and $E'(z)$, respectively, cross the function $g(z)$ with a positive slope and that $V'_\ell(z)$ dominates $E'(z)$. Thus is is clear that the root x_1^* must be greater that x_2^*. ☐

Appendix H. Proof of Proposition 6

Proof. To calculate the fundamental decreasing and increasing solutions, ϕ and ψ we substitute $\alpha(X_t) = \eta(\overline{x} - X_t)$ and $\sigma(X_t) = \sigma X_t$ from Eq. (34) into the general ODE in Eq. (9) which, after rearranging, yields

$$\frac{1}{2}\sigma^2 x^2 u''(x) + (\eta\overline{x} - (\eta + \lambda\sigma)x)\, u'(x) - ru(x) = 0. \tag{H.1}$$

In order to solve this equation we transform it to the standard form of the so-called Kummer's equation for which the solutions are well understood (Abramowitz and Stegun, 1972, Chapter 13). To do this we first let $u(x) = x^\gamma v(x)$, where γ is to be determined, and then let $z = 2\eta\overline{x}/\sigma^2 x$ to yield

$$zv''(z) + \left(2 - 2\gamma + \frac{2(\eta + \lambda\sigma)}{\sigma^2} - z\right) v'(z) + \left(\gamma + \frac{1}{z}\left((\gamma - 1)\gamma - \frac{2\gamma(\eta + \lambda\sigma) - 2r}{\sigma^2}\right)\right) v(z) = 0.$$

The next step is to choose γ such that

$$\frac{1}{2}\sigma^2\gamma(\gamma - 1) - \gamma(\eta + \lambda\sigma) - r = 0 \tag{H.2}$$

to obtain

$$zv''(z) + (n - z)v'(z) - mv(z) = 0, \tag{H.3}$$

which we identify as Kummer's equation with $n = 2 - 2\gamma + \frac{2(\eta+\lambda\sigma)}{\sigma^2}$ and $m = -\gamma$. It is well known that Eq. (H.3) has two independent solutions $v(z) = U(m, n; z)$ and $v(z) = M(m, n; z)$ which are called, respectively, Kummer's and Tricomi's confluent hypergeometric functions. Furthermore, it can be shown that $U(m, n; z)$ is strictly decreasing and $M(m, n; z)$ is strictly increasing for $m > 0$. To ensure this is the case we are required to take γ to be the *negative* root of Eq. (H.2). It can also be verified that ϕ and ψ are strictly decreasing and increasing respectively. ☐

Figures

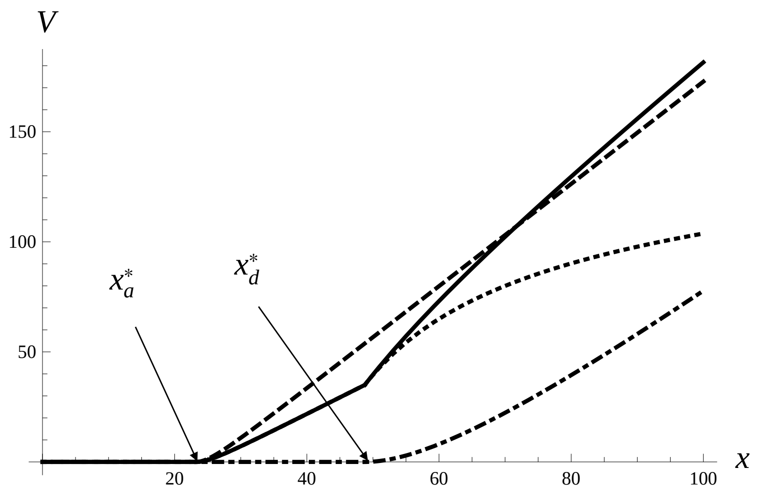

Figure 1: Unlevered and levered project values $V_u(x)$ and $V_\ell(x)$ as a function of the initial output price x (solid line = $V_\ell(x)$, dashed line = $V_u(x)$, dotted line = $D(x)$, dot-dashed line = $E(x)$; for base case parameters).

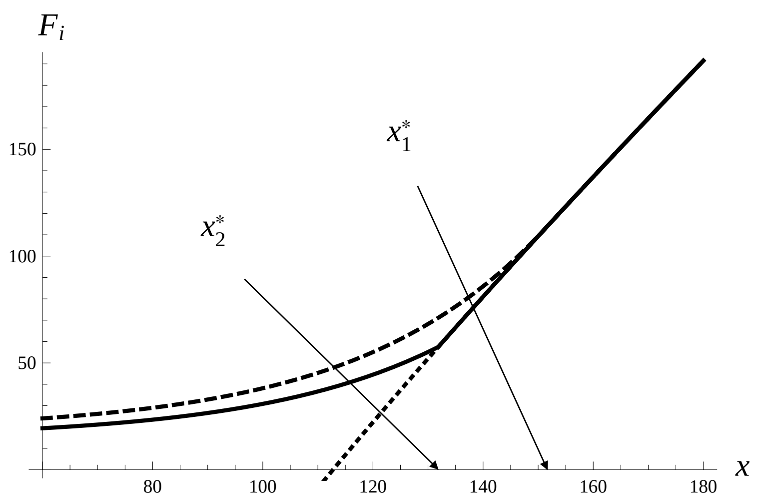

Figure 2: Value of the first- and second-best investment option (F_1 and F_2) as a function of the initial output price x (solid line = second-best option F_2, dashed line = first-best option F_1, dotted line is the value of the levered firm $V_\ell(x)$ less investment cost I; for base case parameters – except we use $R = \$50$ for emphasis). Note the smooth pasting of the first-best outcome but not the second best.

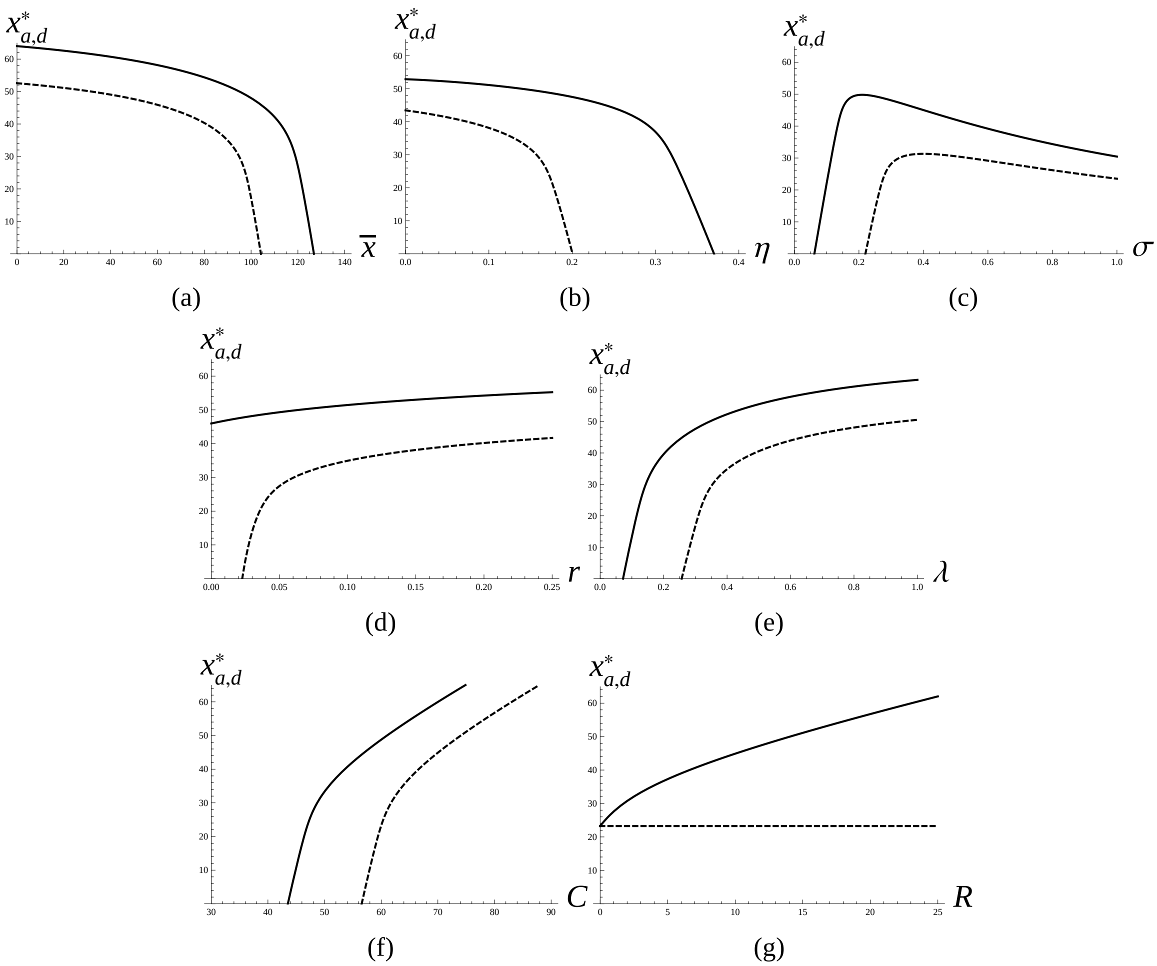

Figure 3: Default and abandonment trigger prices x_d^* and x_a^* as a function of (a) long-run mean price level $\overline{x}$, (b) speed of mean reversion η, (c) process volatility σ, (d) risk-free (real) interest rate r, (e) Sharpe ratio of oil λ, (f) variable costs C, and (g) debt coupon payment R (solid line = x_d^*, dashed line = x_a^*; for base case parameters: $\overline{x}$ = \$98.22, $\eta = 0.1733$, $\sigma = 0.274$, $r = 0.04$, $\lambda = 0.32$, C = \$60, R = \$13.03, $\tau = 0.3$ and $b = 0.35$). Note that these comparative statics are produced for fixed coupon R.

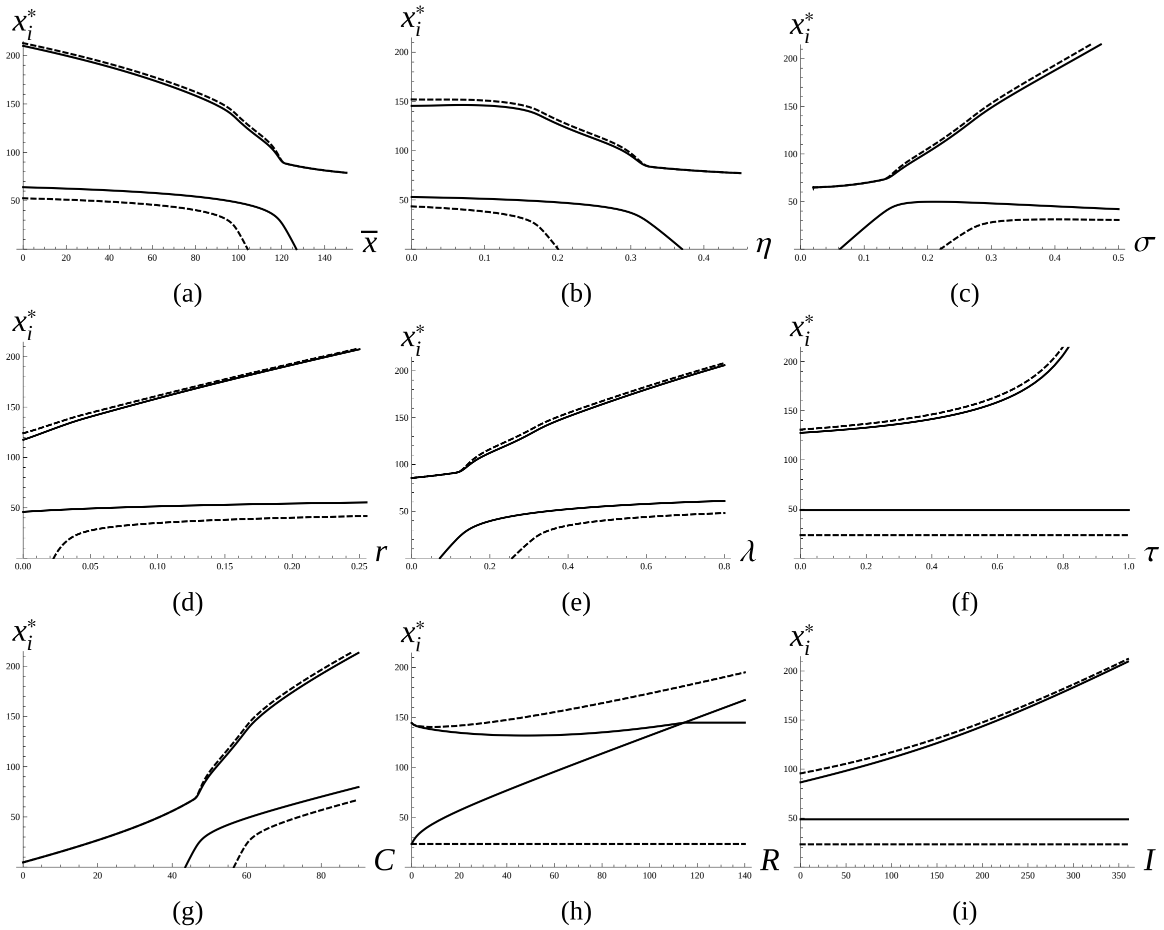

Figure 4: The first- and second-best investment trigger prices as function of (a) long-run mean price level $\overline{x}$, (b) speed of mean reversion η, (c) process volatility σ, (d) risk-free (real) interest rate r, (e) Sharpe ratio of oil λ, (f) effective tax rate τ, (g) variable costs C, (h) debt coupon payment R, and finally (i) investment cost I (upper solid line = x_2^*, upper dashed line = x_1^*, lower solid line = x_d^*, lower dashed line = x_a^*; for base case parameters: $\eta = 0.1733$, $\overline{x} = \$98.22$, $\sigma = 0.274$, $\lambda = 0.32$, $r = 0.04$, $\tau = 0.3$, $b = 0.35$, $C = \$60$, $R = \$13.03$ and $I = \$180$). Note that these comparative statics are produced for fixed coupon R.

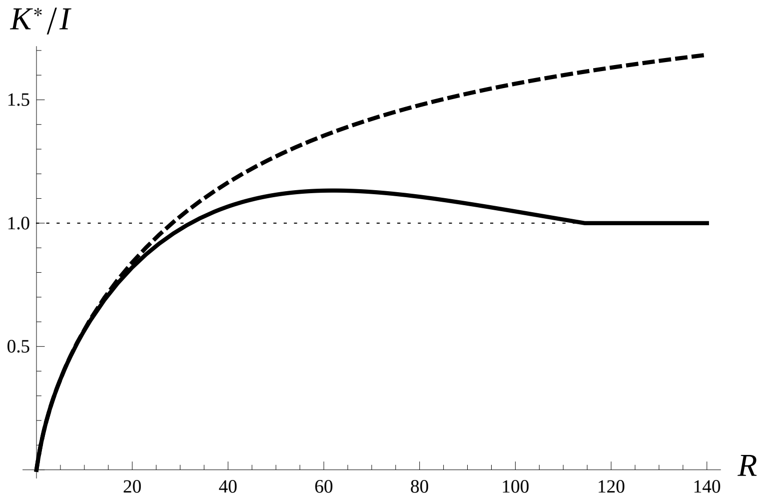

Figure 5: The equilibrium debt financing ratio K^*/I as a function of the annual debt coupon payment R (solid line = second-best outcome, dashed line = first-best outcome; for base case parameters). Note the debtholders are very reluctant to give any debt over the required investment for the project (dotted line: $K^* = I$).

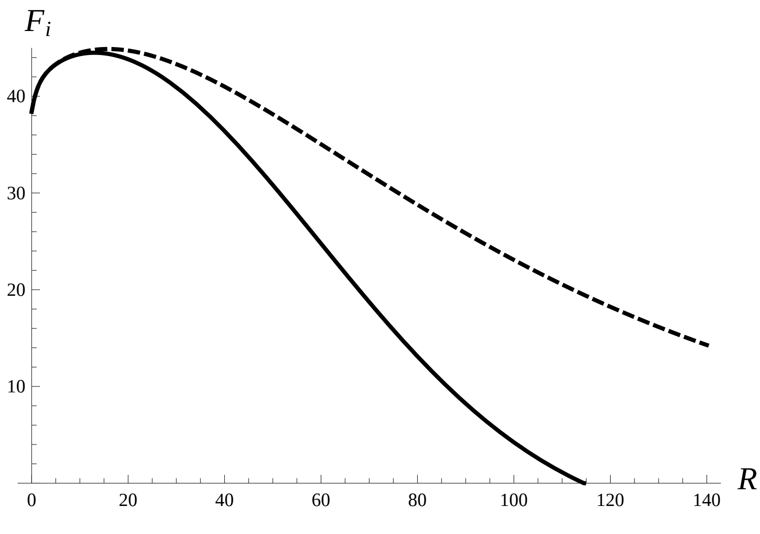

Figure 6: The first-best (dashed line) and second-best (solid line) firm value, as a function of the coupon payment R (for base case parameters). Note the first- and second-best firm value maximising coupon payments.

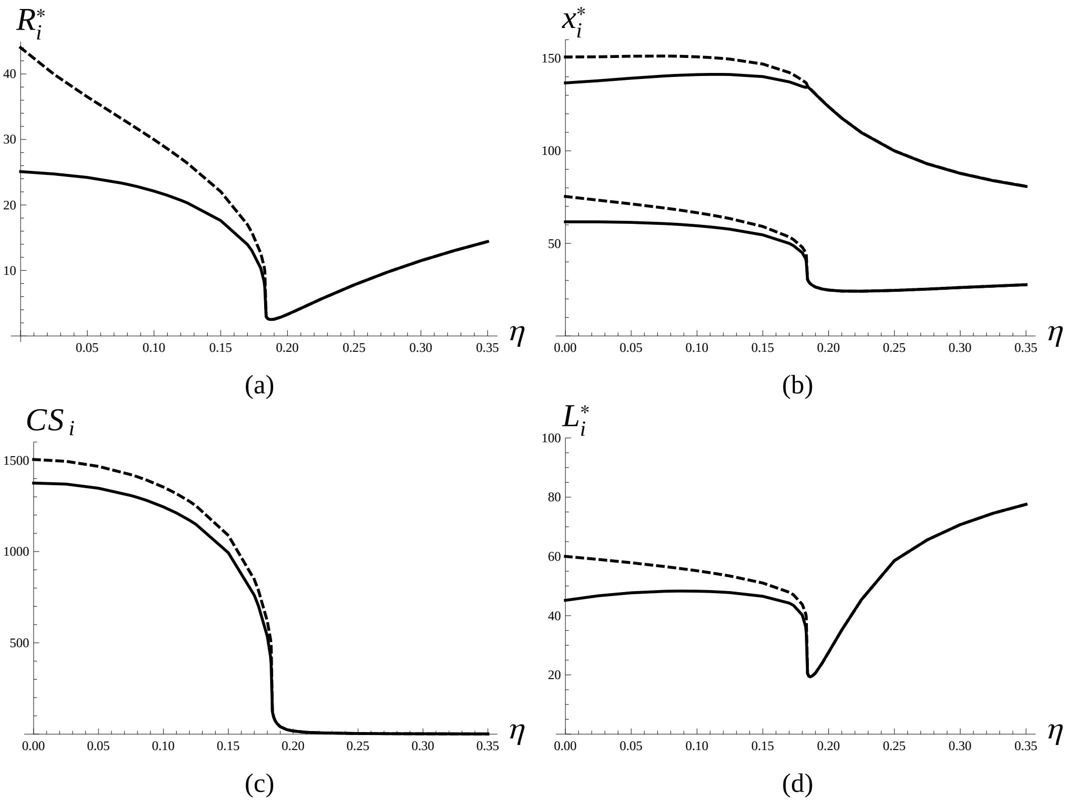

Figure 7: Comparative statics (in equilibrium) with η of (a) optimal coupon payment, (b) optimal investment and default trigger prices, (c) equilibrium credit spreads, and (d) optimal leverage ratio at time of investment. (solid line = second-best outcome, dashed line = first-best outcome; for base case parameters).

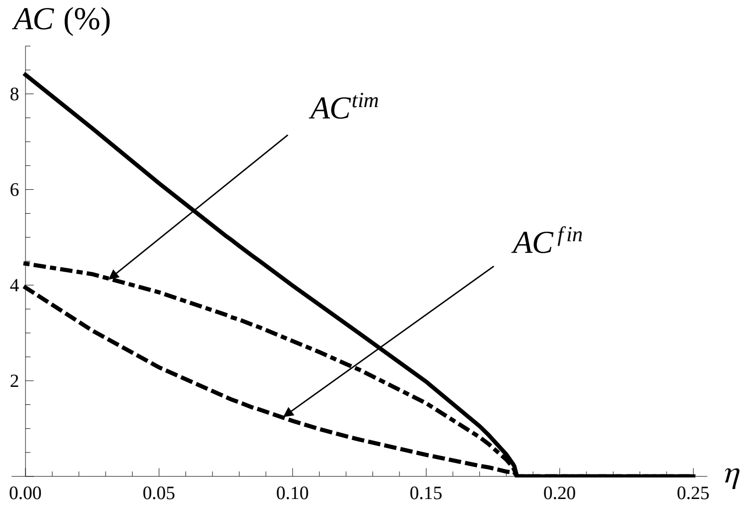

Figure 8: Comparative statics (in equilibrium) of total agency cost (solid line) with η and its decomposition into AC^{fin} (dashed line) and AC^{tim} (dot-dashed line)—see Eq. (33); for base case parameters.

Table 1: Comparative statics of the first-best and second-best optimal financing and investment decisions in equilibrium.

	Optimal coupon		Investment trigger		Firm value		Agency cost (%)	Optimal leverage		Default trigger		Credit spread (basis points)	
	R_1^*	R_2^*	x_1^*	x_2^*	$F_1(X_0)$	$F_2(X_0)$	AC	L_1^*	L_2^*	$x_d^*(R_1^*)$	$x_d^*(R_2^*)$	CS_1	CS_2
Base Case	15.82	13.03	141.10	136.49	44.88	44.50	0.87	47.03	43.49	52.07	48.77	785.60	701.01
Base Case (η = 0)	44.01	25.09	150.70	136.69	46.29	42.70	8.40	60.01	45.18	75.35	61.65	1,504.51	1,375.29
$\overline{x}$ =60	35.90	28.57	175.89	167.80	3.22	3.05	5.57	56.19	50.67	78.89	72.30	2,200.65	2,088.83
$\overline{x}$ =80	28.92	23.13	161.74	154.22	11.86	11.42	3.81	54.00	49.01	69.68	64.21	1,682.69	1,577.95
$\overline{x}$ =120	12.49	12.49	90.46	90.33	303.28	303.28	0.00	64.16	64.16	23.78	23.78	2.80	2.80
$\overline{x}$ =140	24.35	24.35	73.42	73.00	593.75	593.75	0.00	78.40	78.40	26.36	26.36	1.42	1.42
η =0.1	29.97	22.11	150.78	141.17	37.91	36.46	3.99	55.16	48.26	66.58	59.53	1,353.22	1,244.71
η =0.15	22.03	17.61	146.89	140.10	39.70	38.93	1.98	51.03	46.54	59.13	54.59	1,088.48	993.74
η =0.2	3.29	3.29	123.87	123.78	69.43	69.43	0.00	27.63	27.65	24.79	24.79	17.18	17.18
η =0.25	7.80	7.80	100.02	99.96	150.01	150.01	0.00	58.58	58.59	24.60	24.60	3.40	3.40
σ =0.20	6.14	6.14	96.24	96.11	139.38	139.38	0.00	46.95	46.95	30.50	30.50	9.50	9.50
σ =0.25	2.88	2.88	125.62	125.37	58.08	58.07	0.00	21.94	21.98	29.88	29.88	70.05	70.09
σ =0.30	21.84	17.48	153.75	146.95	39.24	38.62	1.59	49.69	45.36	57.50	52.96	1,117.53	1,011.46
σ =0.35	30.58	23.36	173.98	163.84	36.36	35.49	2.44	51.52	45.98	63.36	56.68	1,550.10	1,409.53
r =0.01	3.48	3.48	107.67	107.66	380.37	380.37	0.00	59.94	59.95	17.02	17.02	0.42	0.42
r =0.03	1.72	1.72	134.27	134.11	66.29	66.29	0.00	15.96	15.98	26.25	26.25	61.18	61.19
r =0.05	19.33	16.02	144.85	139.43	33.42	32.97	1.35	49.54	45.96	56.41	52.76	928.61	841.00
r =0.07	24.47	20.41	151.36	145.12	20.69	20.27	2.07	52.51	48.87	62.46	58.28	1,089.72	997.86
λ =0.20	7.19	7.19	106.39	106.27	194.32	194.32	0.00	45.31	45.35	22.69	22.69	7.10	7.10
λ =0.30	11.67	9.42	136.02	132.88	57.33	57.13	0.35	42.41	38.57	45.90	42.79	539.06	456.40
λ =0.40	23.06	18.95	155.11	148.94	18.04	17.63	2.32	51.18	47.15	62.88	58.67	1,352.55	1,255.95
λ =0.50	28.15	23.39	168.58	162.11	5.81	5.61	3.50	53.02	48.96	70.58	65.95	1,821.75	1,721.21

Table 1: (*continued*).

	Optimal coupon		Investment trigger		Firm value		Agency cost (%)	Optimal leverage		Default trigger		Credit spread (basis points)	
	R_1^*	R_2^*	x_1^*	x_2^*	$F_1(X_0)$	$F_2(X_0)$	AC	L_1^*	L_2^*	$x_d^*(R_1^*)$	$x_d^*(R_2^*)$	CS_1	CS_2
Base Case	15.82	13.03	141.10	136.49	44.88	44.50	0.87	47.03	43.49	52.07	48.77	785.60	701.01
Base Case (η = 0)	44.01	25.09	150.70	136.69	46.29	42.70	8.40	60.01	45.18	75.35	61.65	1,504.51	1,375.29
τ =0.15	2.22	2.20	132.67	132.35	65.74	65.73	0.00	12.48	12.43	31.35	31.30	183.37	182.21
τ =0.25	10.32	8.94	138.19	135.50	50.82	50.68	0.28	36.36	33.98	45.32	43.45	580.11	530.71
τ =0.35	20.60	16.43	144.05	137.56	39.63	38.92	1.82	54.66	50.49	57.38	52.77	954.37	838.84
τ =0.45	28.82	21.96	150.38	140.18	30.56	29.11	4.99	65.35	60.74	65.91	58.84	1,242.86	1,070.99
C =50	9.38	9.38	82.49	82.39	235.78	235.78	0.00	55.95	55.95	19.75	19.75	3.24	3.24
C =55	4.69	4.69	107.74	107.66	114.36	114.36	0.00	36.90	36.92	21.57	21.57	9.46	9.46
C =65	24.01	19.32	159.19	152.12	27.03	26.43	2.28	51.49	47.03	66.11	61.32	1,279.79	1,180.70
C =70	28.74	22.72	172.35	164.10	18.69	18.11	3.21	53.33	48.21	75.66	69.80	1,563.13	1,454.89
b =0.15	27.61	19.78	138.81	130.90	50.15	48.67	3.04	65.30	58.08	64.69	56.50	1,059.34	867.00
b =0.25	21.32	16.49	140.11	133.88	47.16	46.37	1.71	56.32	51.13	58.16	52.84	921.74	792.29
b =0.45	10.95	9.54	141.76	138.65	43.22	43.06	0.37	37.36	35.17	46.15	44.27	641.56	590.41
b =0.55	6.81	6.30	142.10	140.30	42.09	42.04	0.12	27.50	26.54	40.31	39.50	485.63	462.23
I =100	10.58	8.66	116.70	112.95	91.56	91.12	0.48	44.58	41.11	45.66	43.05	661.23	582.44
I =140	13.00	10.66	128.22	124.05	65.12	64.70	0.66	45.87	42.33	48.72	45.77	724.95	642.28
I =220	18.99	15.76	155.25	150.16	30.20	29.87	1.10	48.02	44.54	55.63	52.00	840.90	756.35
I =260	22.44	18.77	170.47	164.88	20.02	19.76	1.32	48.82	45.44	59.35	55.39	889.52	806.20
α =-0.08	44.42	31.75	176.85	166.34	3.67	3.39	8.18	59.26	50.35	85.96	75.53	2,352.50	2,223.84
α =-0.04	43.38	28.34	163.62	151.73	13.29	12.27	8.31	59.61	48.31	81.06	69.27	1,947.23	1,816.68
α =0.04	49.49	22.85	139.30	121.51	163.21	150.49	8.45	60.48	40.53	68.47	51.81	1,022.25	901.47
α =0.08	75.58	25.48	132.96	106.27	680.95	627.37	8.54	61.24	34.75	60.15	37.92	537.50	433.43

Table 2: Comparative statics of agency costs and its timing and financing components.

	AC	AC^{fin}	AC^{tim}
Base Case	0.87	0.18	0.68
Base Case ($\eta = 0$)	8.40	3.95	4.45
$\bar{x}$ =60	5.57	1.24	4.32
$\bar{x}$ =80	3.81	0.84	2.97
$\bar{x}$ =120	0.00	0.00	0.00
$\bar{x}$ =140	0.00	0.00	0.00
η =0.10	3.99	1.16	2.83
η =0.15	1.98	0.45	1.53
η =0.20	0.00	0.00	0.00
η =0.25	0.00	0.00	0.00
σ =0.20	0.00	0.00	0.00
σ =0.25	0.00	0.00	0.00
σ =0.30	1.59	0.36	1.23
σ =0.35	2.44	0.63	1.81
r =0.01	0.00	0.00	0.00
r =0.03	0.00	0.00	0.00
r =0.05	1.35	0.28	1.08
r =0.07	2.07	0.42	1.64
λ =0.2	0.00	0.00	0.00
λ =0.3	0.35	0.09	0.26
λ =0.4	2.32	0.46	1.86
λ =0.5	3.50	0.65	2.85

	AC	AC^{fin}	AC^{tim}
Base Case	0.87	0.18	0.68
Base Case ($\eta = 0$)	8.40	3.95	4.45
τ =0.15	0.00	0.00	0.00
τ =0.25	0.28	0.05	0.23
τ =0.35	1.82	0.43	1.38
τ =0.45	4.99	1.35	3.63
C =50	0.00	0.00	0.00
C =55	0.00	0.00	0.00
C =65	2.28	0.50	1.78
C =70	3.21	0.74	2.47
b =0.15	3.04	1.01	2.03
b =0.25	1.71	0.46	1.25
b =0.45	0.37	0.06	0.31
b =0.55	0.12	0.01	0.11
I =100	0.48	0.11	0.37
I =140	0.66	0.14	0.51
I =220	1.10	0.22	0.88
I =260	1.32	0.26	1.07
α =-0.08	8.18	2.55	5.62
α =-0.04	8.31	3.15	5.15
α =0.04	8.45	4.98	3.48
α =0.08	8.54	6.24	2.30