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Fractal Market Time

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Abstract

Ané and Geman (2000) observed that market returns appear to follow a conditional Gaussian distribution where the conditioning is a stochastic clock based on cumulative transaction count. The existence of long range dependence in the squared and absolute value of market returns is a 'stylized fact' and researchers have interpreted this to imply that the stochastic clock is self-similar, multi-fractal (Mandelbrot, Fisher and Calvet; 1997) or mono-fractal (Heyde; 1999). We model the market stochastic clock as the stochastic integrated intensity of a doubly stochastic Poisson (Cox) point process of the cumulative transaction count of stocks traded on the New York Stock Exchange (NYSE). A comparative empirical analysis of a self-normalized version of the stochastic integrated intensity is consistent with a mono-fractal market clock with a Hurst exponent of 0.75.

Keywords: Market Time Deformation, Long Range Dependent, Stochastic Clock, Fractal Activity Time, Doubly Stochastic Binomial Point Process

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1 Introduction

This paper models the intra-day stochastic market clock using the daily trade count $N(t)$ of stocks traded on the New York Stock Exchange (NYSE). A comparative empirical analysis of a self-normalized version of the daily trade count is consistent with a mono-fractal market clock with a Hurst exponent of 0.75. In particular, this result excludes increasing Lévy processes (subordinators) as models of the stochastic market clock.

Cumulative trade arrivals of stocks traded on the New York Stock Exchange (NYSE) are modeled as a Cox (doubly stochastic Poisson) point process. This allows the representation of the stochastic market clock as the stochastic integrated intensity $\Lambda(t)$ of the Cox process. This is superior to cumulative trade count as a model to the stochastic market clock because it excludes the 'Poisson noise' of cumulative trade count and has a natural interpretation as a stochastic clock through the point process *random time change* theorem [14].

To simplify notation and exposition we assume that the period $\tilde{t} \in [0, T]$ (09:30-16:00) of a single days trading on the New York Stock Exchange (NYSE) is the normalized $t = \tilde{t}/T$ to the unit interval $t \in [0, 1]$. The cumulative intra-day trade counts $N(t)$ of NYSE stocks with different final trade counts $N(1) = K$ is also normalized to the unit interval $[0, 1]$ by the simple expedient of self-normalizing with final trade count $R(t) = N(t)/N(1)$. The self-normalized trade count $R(t)$ and the associated self-normalized integrated intensity are both bridge processes that are conceptually and mathematically similar to a Brownian bridge.

This self-normalization has the useful econometric property that the finite dimensional distributions of $R(t)$ are simply and readily approximated using a 2 dimensional histogram (figure 2). The moments of the closely related self-normalized integrated intensity $\Lambda(t)/\Lambda(1)$ can also be extracted from this histogram [31].

The statistical innovation of this paper is that if the integrated intensity is assumed to be a time accelerated base-line integrated intensity $\Lambda_1(Kt)$ time-scaled by final trade count $N(1) = K$ then the variance of the self-normalized baseline integrated intensity has an approximate power scaling of $1/\sqrt{K}$.

$$\Lambda(t) \;=\; \Lambda_1(Kt)$$

$$\mathrm{Var}\left[\frac{\Lambda(t)}{\Lambda(1)}\right] \;=\; \mathrm{Var}\left[\frac{\Lambda_1(Kt)}{\Lambda_1(K)}\right] \;\propto K^{\beta}\,, \quad \beta = -0.5, \qquad t \in [0,1]$$

We then show that the power scaling β between final trade count K and the variance of the self-normalized integrated intensity is different for different mathematical models of stochastic market time. A variance power scaling of $\beta = -0.5$ is consistent with the monofractal model of stochastic market time 'Fractal Activity Time' (FAT) proposed by by Heyde [20] and Heyde and Liu [22]. Furthermore, we can estimate the Hurst exponent of the FAT stochastic market clock as $2H = \beta + 2$.

$$K_2^{\beta+2}\,\mathrm{Var}\big[\Lambda_1(K_1 t)\big] \;\approx\; K_1^{\beta+2}\,\mathrm{Var}\big[\Lambda_1(K_2 t)\big]\;, \qquad K_1, K_2 \gg 1$$

This method of estimating the Hurst exponent of an increasing process is novel.

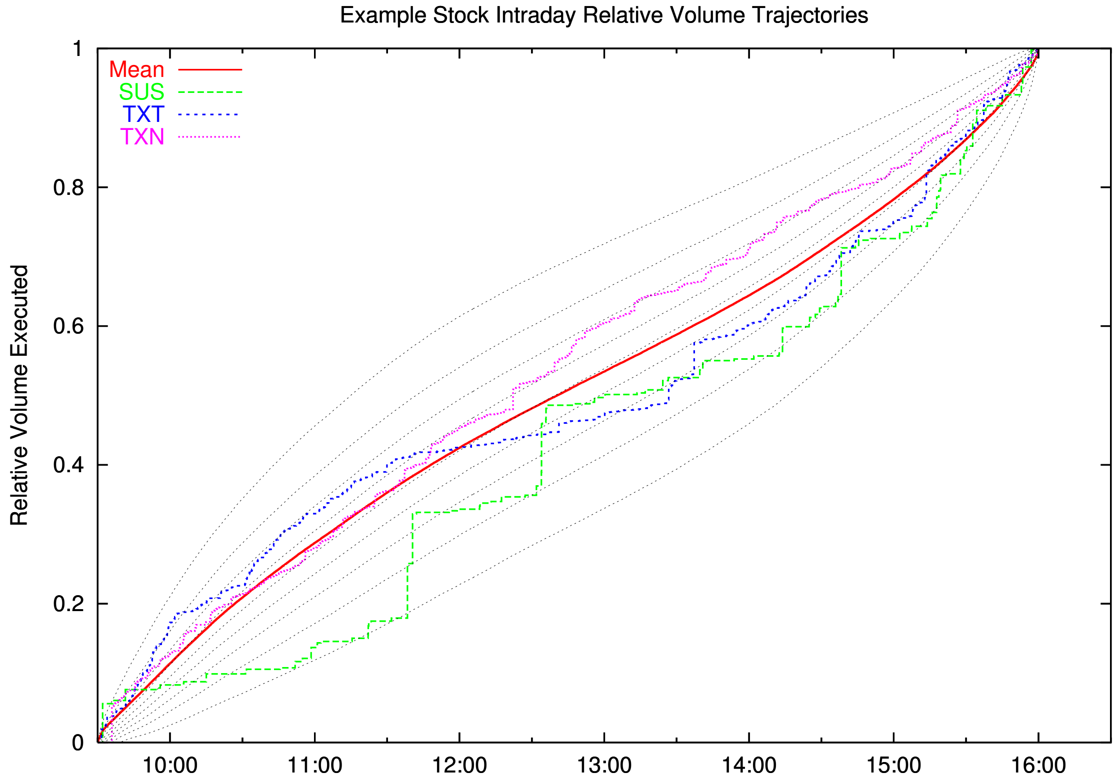

Figure 1: The graph shows examples of self-normalized trade count trajectories of 3 stocks with different daily turnovers on 2 July 2001 between 9:30 and 16:00. *SUS - Storage USA*, 101 trades. *TXT - Textron Incorporated*, 946 trades. *TXN - Texas Instruments*, 2183 trades. Also shown is the mean number of relative trades executed at different times during the trading day for all NYSE stocks trading more than 50 times per day. Trading is not constant during the day and the derivative of this curve is the 'U' shaped trading intensity commonly observed on equity markets.

1.1 Introduction to Empirical Results

Clark [12] observed that returns appear to follow a conditional Gaussian Distribution where the conditioning is taken on a latent stochastic information flow process. As a consequence, the unconditional returns $r(t)$ will be generated by a mixture where the returns are a Wiener process $W(.)$ subject to a time deformation or subordination process $\Lambda_1(t)$:

$$r(t) \;=\; W\big[\Lambda_1(t)\big]$$

A large number of different mathematical models have been put forward to account for the observed characteristics of the unconditional return process. The process of empirically discriminating between them has proved difficult – see Barndorff-Nielsen and Shephard [8] and the associated discussion. Martingale returns and the existence of long range dependence (LRD, Embrechts and Maejima [17]) in squared returns and absolute value of returns (Ding, Engle and Granger [15]) is a 'stylized fact'. A wide sense self-similar model of market activity time was introduced by Heyde [20] and Heyde and Liu [22] as consistent with this empirically observed market behaviour, which they termed 'Fractal Activity Time' (FAT).

Empirical researchers have examined the most suitable observable proxies for market time and standard activity variables such as the volume and the number of trades have been used – see LeFol and Mercier [24] for a discussion of different possible economic time scales. An obvious choice for an empirical proxy of market time is relative transactional activity or the cumulative trade count in a period of clock time.

Ané and Geman [2] show that the market unconditional return distribution is generated from conditioning an ordinary Brownian diffusion by a stochastic clock based on cumulative trade count $N(t)$. This empirical analysis uses intra-day cumulative trade counts from the New York Stock Exchange (NYSE) to explore the characteristics of the integrated intensity as the time deformation process by self-normalizing cumulative trade count $R(t)$ and modeling the self-normalized trade count as a doubly stochastic binomial point process [31]:

$$R(t) \;=\; \frac{N(t)}{N(1)}\,, \qquad t \in [0,1]$$

We then show that the scaling between final trade count K and the variance of the self-normalized integrated intensity $\Lambda_1(Kt)/\Lambda_1(K)$ is different for different mathematical models of stochastic market time $\Lambda_1(Kt)$.

1. If $\Lambda_1(t)$ is modeled as a *finite variance Lévy subordinator* then the variance of the self-normalized integrated intensity will vary approximately as the inverse of trade count $1/K$ (lemma 6.1). Prominent examples of finite variance Lévy subordinators used as models of market time in the mathematical finance literature include the Variance Gamma model [25], [26], the Normal Inverse Gaussian model [3], [7], [33], [6], [5], [4] and the Hyperbolic model [16]:

$$\mathrm{Var}\left[\frac{\Lambda_1(Kt)}{\Lambda_1(K)}\right] \propto \frac{1}{K}$$

2. If $\Lambda_1(t)$ is modeled as *Fractal Activity Time* (FAT) proposed by Heyde [20] and Heyde and Liu [22] then the variance of the self-normalized integrated intensity will vary approximately with trade count K as a power of the Hurst exponent H of the FAT:

$$\mathrm{Var}\left[\frac{\Lambda_1(Kt)}{\Lambda_1(K)}\right] \propto K^{2H-2}$$

3. If $\Lambda_1(t)$ is modeled as an *α-stable Lévy subordinator* then the resultant subordinated Wiener process $W[\Lambda(t)]$ is the stable Paretian process suggested as a model of returns by Mandelbrot [27], Fama [18] and Mandelbrot and Taylor [28]. The variance of the self-normalized integrated intensity will not vary with trade count K:

$$\mathrm{Var}\left[\frac{\Lambda_1(Kt)}{\Lambda_1(K)}\right] \propto 1$$

The variance of the normalized integrated intensity is found to scale proportionally to the inverse square root of final trade count $1/\sqrt{K}$. This is consistent with the FAT proposed by Heyde [20] and Heyde and Liu [22] and excludes the Lévy subordinator models (cases 1 & 3) examined above.

2 The Self-Normalized NYSE Trade Counts

New York Stock Exchange (NYSE) trade data from the TAQ database was used to collect relative trade count data of all stocks that traded from 1 June 2001 to 31 August 2001 (a total of 62 trading days[1]) for a total of 203,158 relative trade count sample paths for all stocks. The relative trade count data was collected in a 391×253 2-D histogram with time in minutes in the x-axis and relative volume in the y-axis.

The histogram size was chosen for the following reasons. In the time x-axis, the NYSE is open from 9:30 to 16:00, a total of 390 minutes. It is natural to collect sample path data each minute at precisely 9:31 to 16:00 giving 390 data points in the time axis. This 390 points is augmented with the initial state of the market at 9:30 when no trades have executed, giving a total of 391 data points in the time x-axis.

In the relative volume y axis (proportion of final trading completed) an examination of the properties of the sample path distributions indicated that approximately 250 bins (see page 22, Simonoff [38] for details) was appropriate. However if 250 bins were used then stocks with n final trades and k intra-day trades at some time during the day would have an intra-day executed proportional trade count of $\frac{k}{n}$. This fraction would fall on the bin boundary of a 250 bin grid if n and 250 have a common divisor. On exact boundaries, a problem arises caused by rounding a floating point y value (proportion executed) to an integer number of bins - the actual integer value generated (which bin) becomes uncertain and is dependent on the hardware/software implementation of the particular computer floating point algorithm. The solution was to use a prime number of bins (no common divisor), 251, so that the proportional trade count of $\frac{k}{n}$ never falls on exact bin boundaries. The total number of bins in the y axis is then augmented by including the two end conditions of 'no trades executed' and 'no further trades executed' to give a total of 253 y axis bins.

[1] 3 July 2001 (half day trading) and 8 June 2001 (NYSE computer malfunction delayed market opening) were excluded from the analysis.

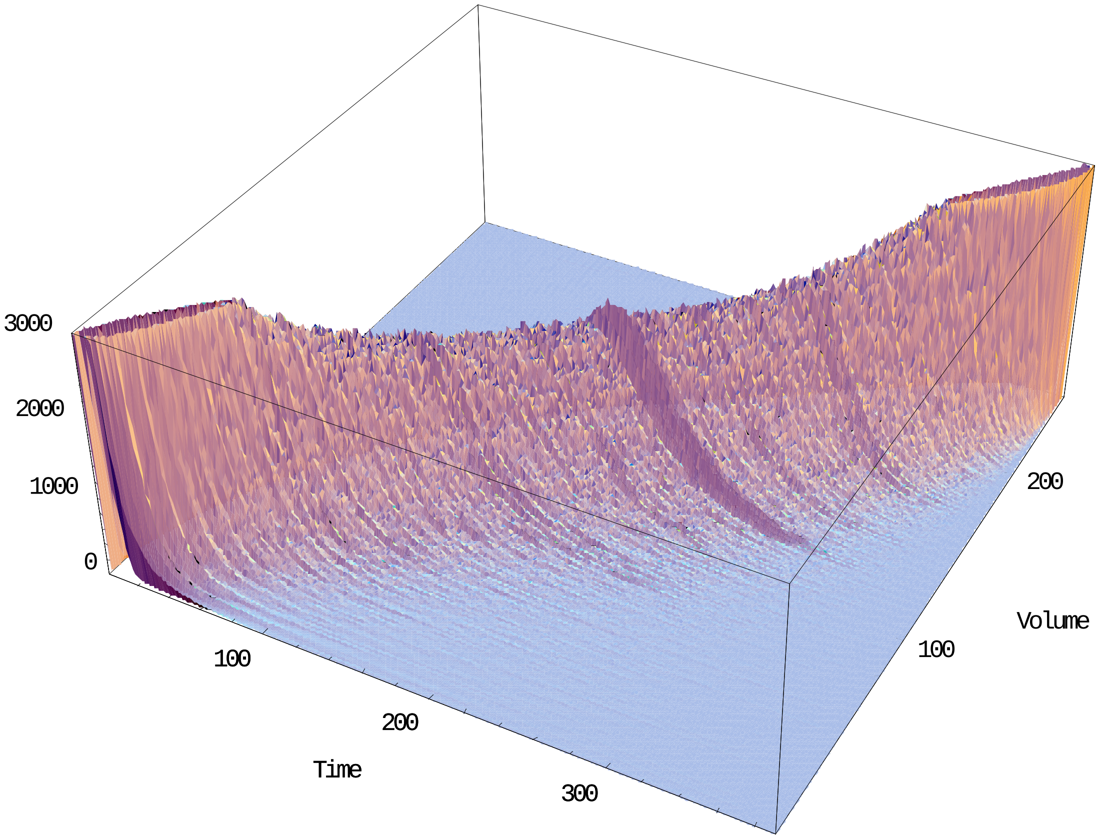

Figure 2: The fidi distributions of $R(t)$ are approximated by recording self-normalized trade count trajectories ($R(t) = N(t)/N(1), \forall t \in [0, 1]$) in a 253×391 histogram. The graph shows binned trajectories that were generated from stocks on the NYSE that traded at least 50 times daily on the 62 trading days from 1 June 2001 to 31 August 2001. The histogram is comprised of 203,158 self-normalized daily trade count trajectories over this period. The histogram is truncated at the beginning and end of the day. This histogram is used to calculate the moments of the self-normalized integrated intensity $\Lambda(t)/\Lambda(1)$ for different trade count bands.

3 A Point Process Model of Trade Arrival

Trade arrivals can be modeled as a point process with the cumulative trade count defined by the random counting measure $N(t)$. In particular, trade arrivals can be modeled as a Cox process, a simple (no co-occurring points) point process directed by a stochastic integrated intensity $\Lambda(t)$.

The following limit[2] defines the non-negative Lebesque integrable *intensity process* of the Cox process. Intuitively, the magnitude of intensity process describes the rate at which trades are occurring:

$$\lambda(t) = \lim_{\delta \to 0^+} \frac{Pr[N(t+\delta) - N(t) > 0 \,|\, \mathscr{F}_t]}{\delta}$$

The absolutely continuous integrated intensity $\Lambda(t)$ is the integral of the intensity process:

$$\Lambda(t) \;=\; \int_0^t \lambda(s)\, ds \tag{1}$$

Conditional on a realization of the random integrated intensity, the Cox process is a Poisson point process (Daley and Vere-Jones [14] definition 6.2.I):

$$Pr\{N(t) = k \,|\, \Lambda(t)\} \;=\; \frac{\Lambda(t)^k}{k!}\, e^{-\Lambda(t)}$$

3.1 The Integrated Intensity $\Lambda(t)$ as a Stochastic Clock

We follow Ané and Geman [2] and associate the market stochastic clock with cumulative trade count. However, instead of modeling the market clock directly on trade count $N(t)$ we assume *the stochastic integrated intensity* $\Lambda(t)$ *to be the market stochastic clock*. The integrated intensity has a natural interpretation as a stochastic clock through the important *random time change* theorem (Daley and Vere-Jones [14] theorem 7.4.I) which states that the cumulative trade count can be considered to be a simple Poisson point process of unit intensity with the time argument t re-scaled as $t \to \Lambda(t)$.

[2]Formally a Cox process may exist where $\Lambda(t)$ has no integral representation (no intensity process $\lambda(t)$ exists) – see Segall and Kalaith [37] for an example. We will assume the existence of the intensity process in this paper.

3.2 Trade Count Accelerated Integrated Intensity $\Lambda_1(Kt)$

The the difference between trade count $N(t)$ and integrated intensity $\Lambda(t)$ is a martingale (Brémaud [10]):

$$M(t) \;=\; N(t) \;-\; \Lambda(t)\;, \quad \mathbb{E}[M(t)] \;=\; 0$$

It is intuitive and true that $M(t)$ and $N(t)$ contain the inaccessible and unpredictable 'Poisson noise' of the cumulative trade count and the absence of this noise is an additional advantage in specifying integrated intensity $\Lambda(t)$ as the market clock. The martingale difference also implies that the expectation of integrated intensity $\Lambda(t)$ is equal to the expectation to the trade count $N(t)$. In particular, if a stock has K observed final cumulative trades $N(1) = K$ then $\mathbb{E}[\Lambda(1)] = K$.

We introduce the baseline integrated intensity $\Lambda_1(t)$ such that at the end of the trading day the expectation of baseline integrated intensity equals elapsed daily market time interval. With the daily time interval re-scaled to the unit interval $[0, 1]$:

$$\mathbb{E}[\Lambda_1(1)] \;=\; 1$$

We assume the baseline integrated intensity is equivalent to stochastic market time and is a stationary increment process.

Since stocks have different final daily trade counts $N(1) = K$ and $\mathbb{E}[\Lambda(1)] = K$, then we can model the observed integrated intensity $\Lambda(t)$ using the baseline integrated intensity (stochastic market time) $\Lambda_1(t)$ in two ways;

1. The integrated intensity for a stock is modeled as baseline integrated intensity multiplied by the stock final trade count K:

$$\Lambda(t) \;=\; K\Lambda_1(t)$$

2. The time parameter of the base-line integrated intensity is multiplied by K:

$$\Lambda(t) = \Lambda_1(Kt).$$

In the first case, the self-normalized integrated intensity is the same for all stocks irrespective of final trade count and therefore all moments of the self-normalized integrated intensity will be the same irrespective of trade count:

$$\frac{K\Lambda_1(t)}{K\Lambda_1(1)} = \frac{\Lambda_1(t)}{\Lambda_1(1)}$$

However, the variance for the self-normalized integrated intensity is different for different trade counts K and is proportional to $1/\sqrt{K}$ (see figure 4) and therefore the first case is rejected by the empirical evidence. This supports case 2, that the integrated intensity for a stock with final trade count K may be simply modeled as baseline integrated intensity where the time parameter is multiplied by K. Since the base-line integrated intensity process $\Lambda_1(t)$ is assumed to have stationary increments:

$$\mathbb{E}[\Lambda(1)] = \mathbb{E}[\Lambda_1(K)] = \sum_{i=1}^{K} \mathbb{E}\big[\Lambda_1(i) - \Lambda_1(i-1)\big] = K$$

Note that the assumption of stationary increments is an over-simplification of daily NYSE market dynamics. The base-line integrated intensity process $\Lambda_1(t) - t$ has an intra-day sinusoidal 'S' shaped expectation (figures 1 and 3) and the corresponding intensity process $\lambda_1(t)$ has an intra-day 'U' shaped expectation. However, we can introduce an exogenous deterministic intra-day periodicity function $\Delta(t)$ such that the inverse function $\Delta^{-1}(t)$ removes this periodicity and $\Lambda_1(\Delta^{-1}(t))$ can be considered a stationary increment process. See section 4.1 for further details.

3.3 Self-Normalized Integrated Intensity

The problem with using the stochastic integrated intensity $\Lambda(t)$ of different stocks to determine the aggregate statistical properties of the market stochastic clock is that stocks trade at different rates. The solution is to re-scale the intra-day trade count to between 0 and 1 by the simple expedient of dividing the intra-day count ($N(t) = k$) by the final trade count ($N(1) = K$). This defines the self-normalized trade count process $R(t)$ which is formally named the *random relative counting measure*:

$$R(t) \;=\; \frac{N(t)}{N(1)} \;=\; \frac{k}{K} \;=\; a, \quad a \in \{0, \frac{1}{K} \ldots, \frac{K-1}{K}, 1\}$$

The great utility of the random relative counting measure $R(t)$ is that the finite dimensional (fidi) distributions of this measure are simply and readily approximated by recording self-normalized intra-day stock trade count trajectories in a 2-d histogram (see section 2).

It is unsurprising that the random relative counting measure $R(t)$ is described by a binomial point process directed by the self-normalized integrated intensity. This point process is related to a binomial point process in a way directly analogous to the relationship between a Cox point process and the Poisson point process. Formally, the random relative counting measure $R(t)$, conditioned on the final value of the integrated intensity $\Lambda(1)$, is a binomial point process directed by the stochastic self-normalized integrated intensity of the related Cox process (McCulloch [31]):

$$\begin{aligned} Pr\{\, R(t) = a \,|\, \Lambda(1)\} \;&=\; Pr\{\, N(t) \;=\; aK \,|\, N(1) \;=\; K, \Lambda(1)\} \\ &=\; \binom{K}{aK} \left[\frac{\Lambda(t)}{\Lambda(1)}\right]^{aK} \left[1 - \frac{\Lambda(t)}{\Lambda(1)}\right]^{(1-a)K} \qquad (2) \\ a \in \{0, \frac{1}{K}, \ldots, \frac{K-1}{K}, 1\} \end{aligned}$$

Conceptually the point process has two sources of randomness: uninteresting binomial 'noise' and the randomness of the self-normalized integrated intensity. We can calculate the moments of the self-normalized intensity by examining stock trade count trajectories in the empirical 2-d histogram (figure 2).

4 Empirical Self-Normalized Integrated Intensity Moments

The time indexed empirical expectation and variance of the self-normalized integrated intensity are graphed and analyzed below. See C and McCulloch [31] for details on how the moments are calculated from the 2-Histogram in section 2.

4.1 Intra-day Trading Periodicity and the Self-Normalized Integrated Intensity Expectation

The mean of the self-normalized integrated intensity is closely related to market intra-day trading periodicity. Figure 3 examines the mean trading intensity for stocks split into 4 different trade count bands. The mean trading intensity is reduced on market open for high trade count stocks because these stocks only experience 1 trade during the NYSE market opening period whereas these stocks would expect to trade several times during the same period (NYSE market open is approximately 4-5 minutes).

4.2 The Self-Normalized Integrated Intensity Variance

The variance of the self-normalized integrated intensity is dependent on trade count with higher trade count stocks having lower variance (see figure 4). Given the assumption that the integrated intensity $\Lambda(t)$ is wide-sense self-similar, this scaling can be shown (section 5 below) to be related to the Hurst exponent of the integrated intensity. *The self-similar and long-range dependent properties of the stochastic market clock (the integrated intensity)* $\Lambda(t)$ *can be determined by examining the scaling of the variance of the self-normalized integrated intensity* $\Lambda(t)/\Lambda(1)$ *for different trade counts.* Note that the variance is zero at the beginning of trading and the end of trading. This is because the self-normalized integrated intensity is a 'bridge' process, conceptually similar to a Brownian bridge.

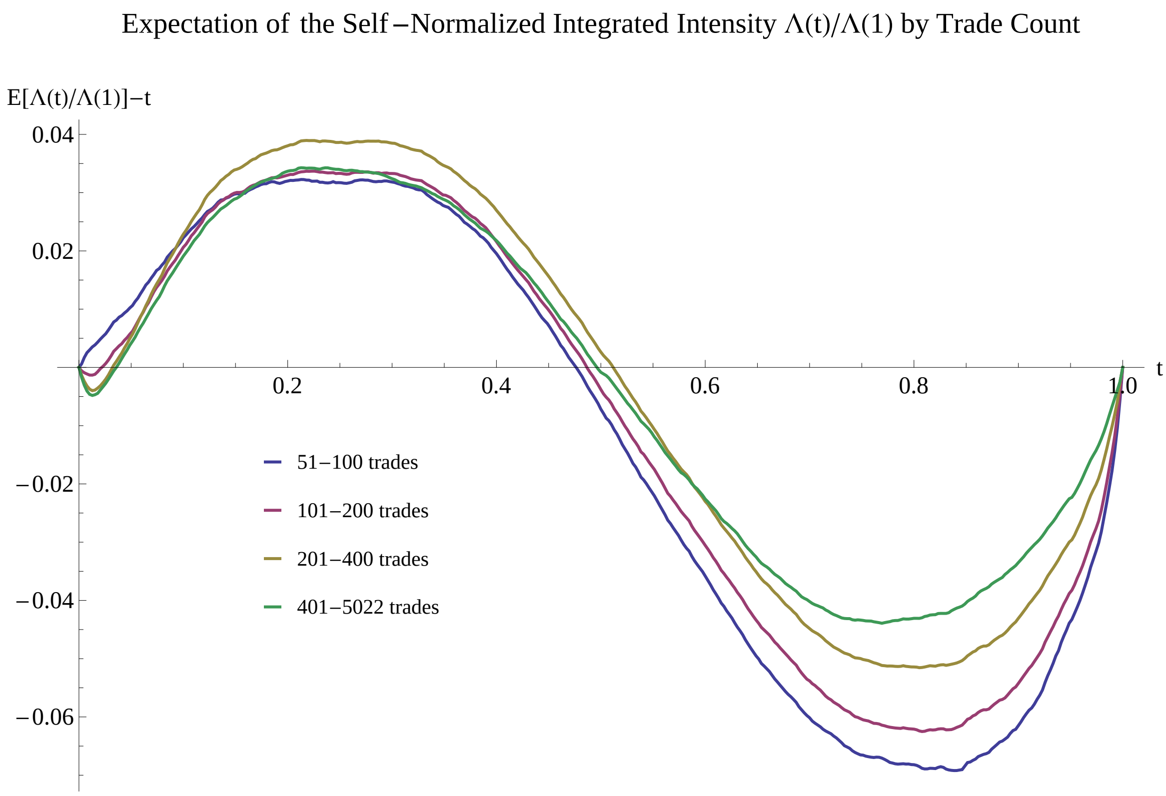

Figure 3: The mean of the self-normalized integrated intensity for NYSE stocks with 50 or more trades per day within different trade count bands. *This is closely related intra-day trading periodicity.* To better illustrate the differences between the self-normalized integrated intensity mean curves, the constant time component has been subtracted, $\mathbb{E}\left[\Lambda(t)/\Lambda(1)\right] - t$ (all of these expectations are actually monotonically increasing from 0 to 1). The derivative $d\mathbb{E}\left[\Lambda(t)/\Lambda(1)\right]/dt$ of the curves is closely related to the classic 'U' shaped trading intensity variation found on equity markets.

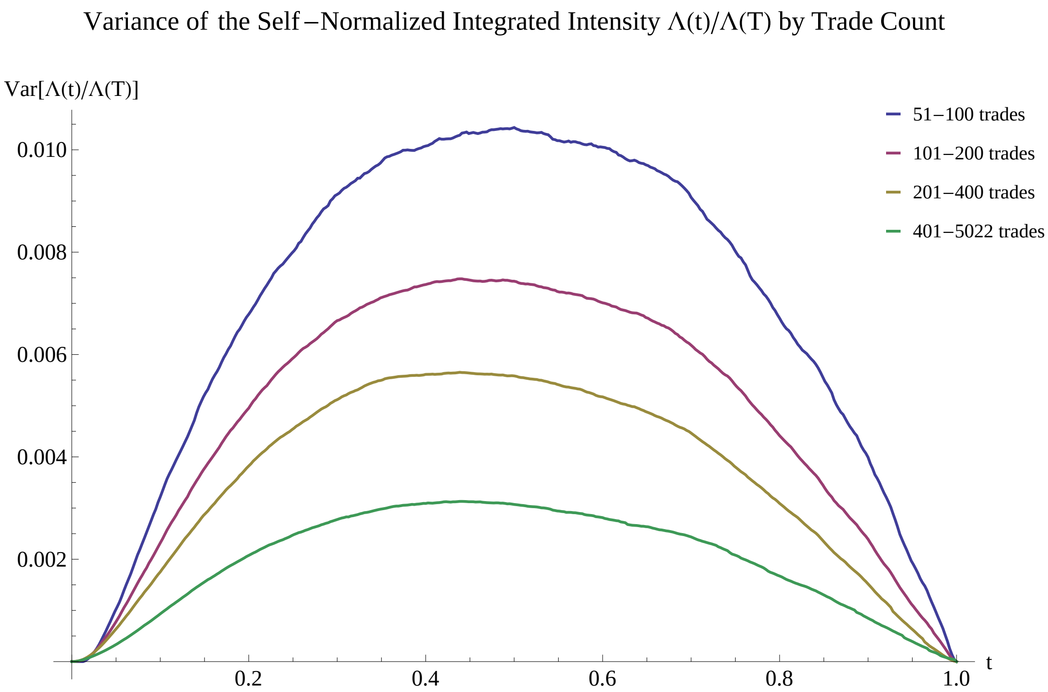

Figure 4: The variance of the self-normalized integrated intensity $\mathrm{Var}\left[\Lambda(t)/\Lambda(1)\right]$ for NYSE stocks within 4 different trade count bands. Note that final trade count K has a significant effect on variance and the variances scale as the inverse square root of final trade count $1/\sqrt{K}$. This is related to the Hurst exponent of the the integrated intensity.

5 Fractal Activity Time

A stochastic process $\mathcal{T}$ is called wide-sense self-similar (Sato[3] [35]) if, for each $c > 0$, there are a positive number a and a function $b(t)$ such that $\mathcal{T}(ct) \overset{d}{=} a\mathcal{T}(t) + b(t)$ have common finite-dimensional distributions. A wide sense self-similar *stationary increment* model of market activity time was introduced by Heyde [20] and Heyde and Liu [22] as consistent with empirically observed market behaviour, which they termed 'Fractal Activity Time' (FAT). Heyde and Leonenko [21] developed a FAT with an inverse gamma marginal distribution implying Student-t distributed returns and Finlay and Seneta [19] have defined a FAT with gamma marginal distribution implying variance-gamma distributed returns:

$$\mathcal{T}(t) - t \overset{d}{=} t^H\big(\mathcal{T}(1) - 1\big)\,, \qquad \frac{1}{2} \leq H < 1$$

$$\mathbb{E}[\mathcal{T}(t)] = t + t^H\big(\mathbb{E}[\mathcal{T}(1)] - 1\big) = t\,, \qquad t \in [0,1]$$

5.1 Self-Normalized Fractal Activity Time Moments

If integrated intensity is assumed equivalent to the fractal activity time process with a rate parameter K, then a Taylor series approximation of the ratio of random variables (see A for details) can be used to formulate the expectation and variance of the self-normalized integrated intensity with respect to time t and rate K.

5.1.1 Self-Normalized Fractal Activity Time Expectation

The Taylor series approximation of the expectation of the self-normalized integrated intensity (eqn 3) has a sinusoidal non-linear component that is functionally similar to the sinusoidal component of the empirical data shown in figure 3 and has an amplitude proportional to trade count as a power of the Hurst exponent K^{2H-2} (see also figure 5):

[3]The author also describes these processes as "broad-sense" self similar in his book, Sato [36]. We adopt the earlier term.

$$\mathbb{E}\left[\frac{\mathcal{T}(Kt)}{\mathcal{T}(K)}\right] \approx t + \left(t - \frac{t^{2H} + 1 - (1-t)^{2H}}{2}\right) K^{2H-2}\,\mathrm{Var}[\mathcal{T}(1)] \qquad (3)$$

For example, the amplitude of the non-linear component is scaled by the inverse square root of trade count $1/\sqrt{K}$ for an integrated intensity with a Hurst exponent of $H = 0.75$ or by inverse trade count $1/K$ if the integrated intensity process Hurst exponent is $H = 0.5$. This is not what is observed in the empirical data shown in figure 3 where there is only slight variation in the sinusoidal non-linear intra-day variation across trade count bands.

In addition, for the empirical value (see below) of the nominal variance, $\mathrm{Var}[\mathcal{T}(1)] = 0.875$, the amplitude of the eqn 3 is an order of magnitude smaller than the observed sinusoidal amplitude (figures 3 and 5). This is because we have assumed that the integrated intensity process expectation is a linear function of time and that $\Lambda_1(t) = \mathcal{T}(t)$. However, it is a 'stylized fact' that equity markets are active on market open, less active mid session and active on market close. This is the classic 'U' shape in intra-day trading intensity found in all major equity markets. For further discussion and explanations of the causes of the 'U' shaped intra-day trading rate periodicity see Brock and Kleidon [11], Admati and Pfleiderer [1] and Coppejans, Domowitz and Madhavan [13]. If the average *rate* of trading is 'U' shaped then the *integrated rate* is market time and this will be a monotonically increasing linear rate with a superimposed 'S' shaped sinusoid – see figures 1 and 3.

We arbitrarily model the exogenous 'S' shaped non-linear variation in market time as a deterministic function with the same functional form as the expectation of the FAT model of the self-normalized integrated intensity (eqn 3). Thus market time as integrated intensity is formulated as $\Lambda_1(t) = \mathcal{T}(\Delta(t))$ where $\Delta(t)$ is the deterministic function defined below with constant a D that determines the magnitude of the 'S' shaped non-linear variation with $\Delta(0) = 0$, $\Delta(0.5) = 0.5$ and $\Delta(1) = 1$:

$$\Delta(t) = t + \left(t - \frac{t^{2H} + 1 - (1-t)^{2H}}{2}\right) D\,, \qquad t \in [0,1] \qquad (4)$$

If the baseline intensity/stochastic clock is defined as $\Lambda_1(t) = \mathcal{T}(\Delta(t))$ then it is obvious that a stationary increment version of the baseline intensity/stochastic is $\Lambda_1(\Delta^{-1}(t)) = \mathcal{T}(\Delta^{-1}(\Delta(t))) = \mathcal{T}(t)$ where $\Delta^{-1}(t)$ is the inverse function of $\Delta(t)$. For a stock with K observed final trades the integrated intensity is modeled using the FAT as:

$$\Lambda(t) = \Lambda_1(Kt) = \mathcal{T}(K\Delta(t))$$

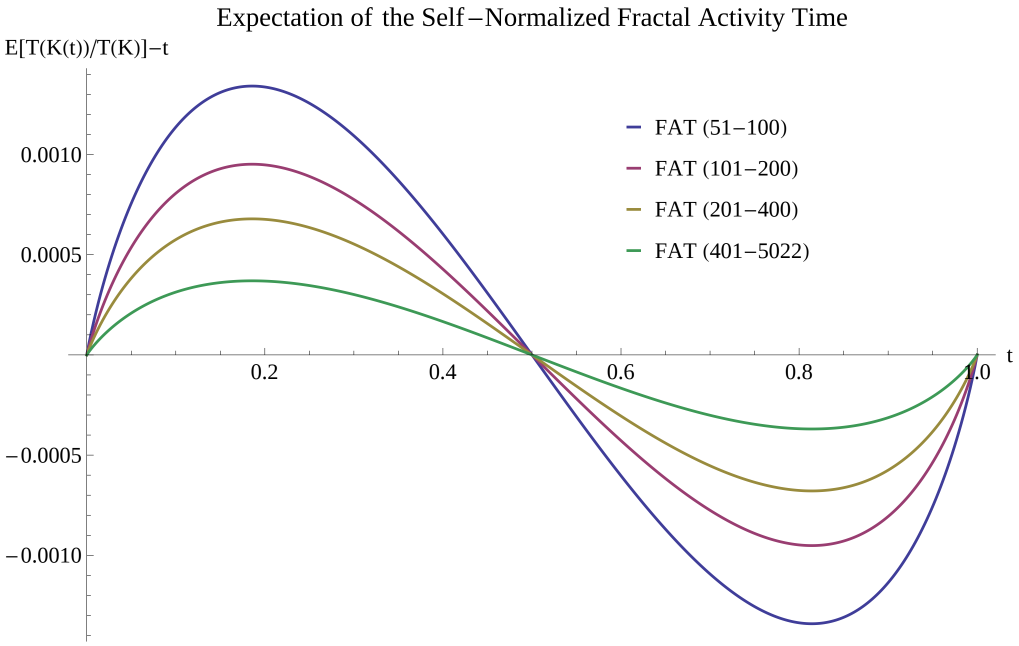

Figure 5: The expectation of the self-normalized FAT without intra-day periodicity $\mathbb{E}[\mathcal{T}(Kt)/\mathcal{T}(K)] - t$ (eqn 3 with the linear component t removed) for different trade count bands K. The Hurst exponent is $H = 0.75$ and the nominal variance $\text{Var}[\Lambda_1(1)] = \text{Var}[\mathcal{T}(1)] = 0.875$. The functional form of the intra-day expectation is similar to the empirical intra-day variation but an order of magnitude smaller in amplitude. See figure 3 and the commentary in this section. Note that the plots 'FAT (51-100)' etc are single trade counts representing the weighted average trade counts for stocks trading in the trade count bands, see table 1.

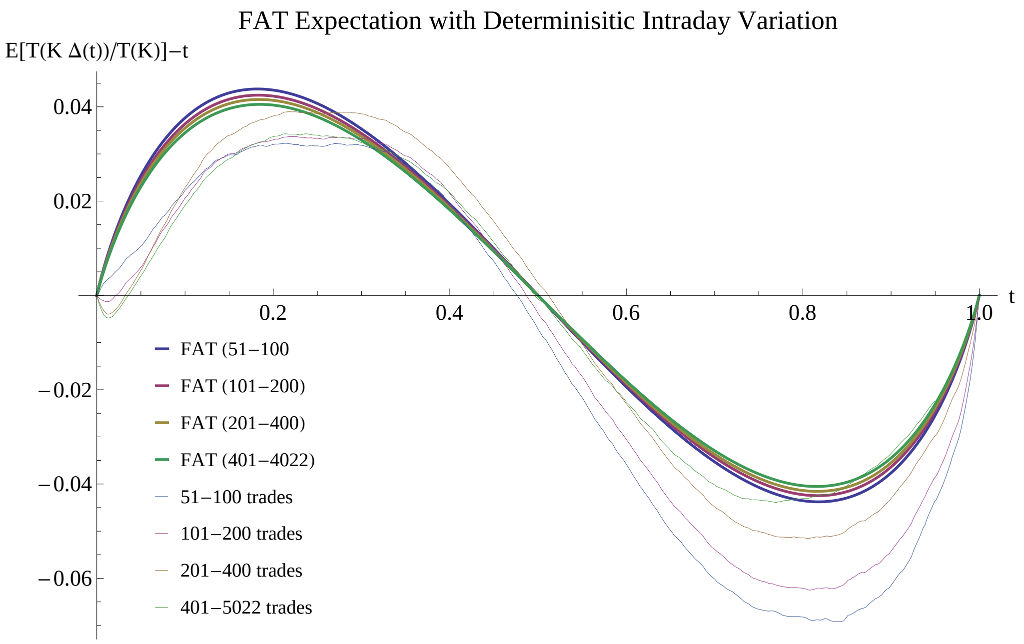

Figure 6: The expectation of the self-normalized FAT with intra-day periodicity $\mathbb{E}[\mathcal{T}(K\Delta(t))/\mathcal{T}(K)]-t$ (linear trend removed).We model the exogenous 'S' shaped non-linear intra-day periodicity in market time as a deterministic function $\Delta(t)$ (eqn 4) where $D = 3$. For comparison the empirical expected intra-day variation is also displayed as thin plot lines. The empirical expected intra-day variation exhibits an asymmetry between the morning and afternoon variations that are not captured by the formal FAT model. The slight difference in intra-day variation amplitude between trade counts in the formal FAT model is due to the deterministic function $\Delta(t)$ plus the functional form of eqn 3 (see figure 5).

5.1.2 Self-Normalized Fractal Activity Time Variance

The Taylor series approximation of the variance of the self-normalized integrated intensity (see appendix A for details) has terms that scale with trade count as both K^{2H-2} and K^{4H-4}. However, with a nominal variance of $\mathrm{Var}[\mathcal{T}(1)] = 0.875$ and Hurst exponent of $H = 0.75$ the K^{4H-4} term is small relative to the K^{2H-2} term:

$$\mathrm{Var}\left[\frac{\mathcal{T}(Kt)}{\mathcal{T}(K)}\right] \approx \Big(t^2 - t\,(t^{2H} + 1 - (1-t)^{2H}) + t^{2H}\Big) K^{2H-2}\mathrm{Var}\big[\mathcal{T}(1)\big] - \left(t - \frac{t^{2H} + 1 - (1-t)^{2H}}{2}\right)^2 K^{4H-4}\left(\mathrm{Var}[\mathcal{T}(1)]\right)^2 \tag{5}$$

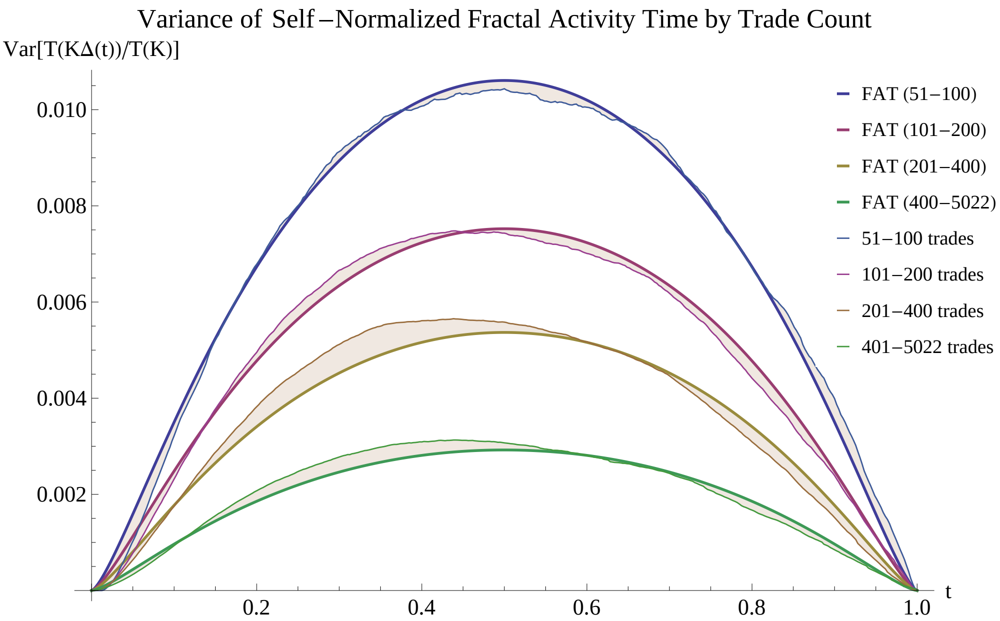

Figure 7: The variance of the FAT model of self-normalized integrated intensity $\mathrm{Var}[\mathcal{T}(K\Delta(t))/\mathcal{T}(K)]$ (eqn 5) for different trade count bands K. The Hurst exponent is $H = 0.75$ and nominal variance is $\mathrm{Var}[\mathcal{T}(1)] = 0.875$. For convenient comparison, the empirical variance $\mathrm{Var}[\Lambda(t)/\Lambda(1)]$ is also plotted as thin lines and the difference between the two is shaded. The difference between the empirical variance of the self-normalized integrated intensity and the FAT model is largely due to the symmetry of the functional form of the deterministic intra-day variation $\Delta(t)$ (eqn 4) compared to the asymmetry of the empirical intra-day variation, see figure 6 and related commentary.

6 Lévy Subordinators

Lévy subordinators are non-decreasing Lévy processes (Sato [36]). There has been considerable research proposing the use of Lévy subordinators as models of stochastic market time. A number of different mixtures have been put forward to account for the observed characteristics of the unconditional return process and prominent examples of subordinated Wiener processes include the Variance Gamma model of [25], [26] and the Normal Inverse Gaussian model, [3], [7], [33], [6], [5], [4], the Hyperbolic model of [16] and the α-stable model of [27] [18] [28]. An example of a subordinated Lévy process is the α-stable Gamma model of [30], [29]. Indeed a large array of different mixture models could in principle be analysed and the process of empirically discriminating between them is likely to prove difficult – see [8] and the associated discussion.

Much of the impetus of this modeling has come from the need to more accurately evaluate options and derivatives given the inadequacy of the price process assumption of Geometric Brownian Motion with constant volatility. The standard European option (Black and Scholes [9]) contract has a definite and fixed expiry/payoff interval and this research has concentrated on modeling the marginal distributions of the unconditional return process rather than the sample path properties of the returns process. It is intuitive and true that subordinated Lévy processes are also Lévy processes and retain the property of independent increments. However the square and absolute value of empirical returns exhibit clustering, strong autocorrelation and long range dependence [15] and this is inconsistent with a Lévy returns processes.

6.1 Finite Variance Subordinators

Lemma 6.1. *Any time-accelerated self-normalized Lévy subordinator $\Gamma(Kt)$ with a finite variance scales approximately as a function of $1/K$ for values of $K \gg 1$.*

$$Var\left[\frac{\Gamma(Kt)}{\Gamma(K)}\right] \propto \frac{1}{K}\,, \qquad K \gg 1, \quad t \in [0,1]$$

Proof. See B □

We examine the closely related case where the random activity time is assumed to be an independent increment additive process (a time changed Lévy subordinator, Sato [36]). Using the results in James, Lijoi and Prünster [23] the variance of self-normalized increasing additive processes can be explicitly formulated. As an example, the variance of the self-normalized Gamma process and self-normalized Inverse Gaussian process are formulated in B.1 and are shown to scale approximately, as expected, as a function of $1/K$. These are then compared to the scaling of the empirical self-normalized stochastic clock and self-normalized Fractal Activity Time (FAT) process and the results are graphed in figure 8. It is immediately clear from this graph that only the FAT process with Hurst exponent $H = 0.75$ scales as $1/\sqrt{K}$ as required by the empirical variances of figures 4 and 7.

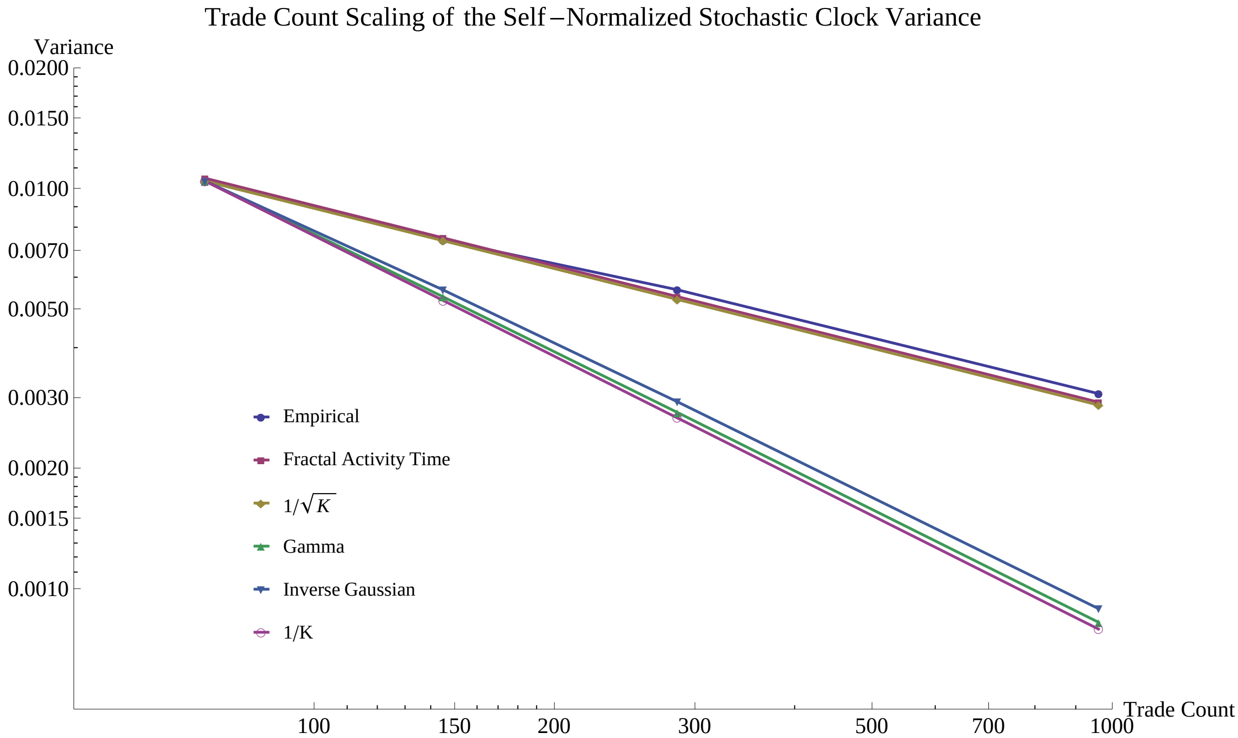

Figure 8: The variance scaling of the empirical self-normalized stochastic clock $\Lambda(0.5)/\Lambda(1)$ for different trade count bands K compared to the variance scaling of self-normalized versions the Fractal Activity Time (FAT) process and Lévy subordinators. It is clear from this graph that the empirical stochastic clock and FAT ($H = 0.75$) scale close to $1/\sqrt{K}$. Conversely the Gamma and Inverse Gaussian subordinators scale close to $1/K$ and are misspecified.

6.2 α-Stable Subordinators

Another class of Lévy subordinators are α-stable processes Γ^α with $0 < \alpha < 1$. These processes have no defined moments (all moments are infinite) and are self-similar with $\Gamma^\alpha(t) \stackrel{d}{=} t^{1/\alpha}\Gamma^\alpha(1)$ corresponding to a Hurst exponent $H = 1/\alpha$.

Mandelbrot [27], Fama [18] and Mandelbrot and Taylor [28] introduced stable Paretian processes as models of financial market returns. These are infinite variance symmetric distributions with $1 \leq \alpha < 2$ ($\alpha = 2$ is the Gaussian distribution). It is well known (Samorodnitsky and Taqqu [34]) that a standard Wiener process $W(t)$ subordinated to an α-stable Lévy subordinator with $0.5 \leq \alpha < 1$ is distributed as a symmetric stable Paretian process with index 2α:

$$\Gamma^{2\alpha}(t) \stackrel{d}{=} W(\Gamma^\alpha(t)) \,, \qquad 0.5 \leq \alpha < 1$$

Although α-stable processes with $0 < \alpha < 1$ have no defined moments the variance of the corresponding self-normalized process exists and James, Lijoi and Prünster [23] show that the variance of the self-normalized time transformed α-stable subordinator is:

$$\mathrm{Var}\left[\frac{\Gamma^\alpha(K\Delta(t))}{\Gamma^\alpha(K)}\right] = \Delta(t)\big(1 - \Delta(t)\big)(1-\alpha) \,, \qquad 0 < \alpha < 1$$

Therefore a self-normalized α-stable Lévy subordinator does not scale with trade count. However, the empirical variance of the self-normalized market clock displays $1/\sqrt{K}$ scaling (figure 4) and the α-stable Lévy subordinator model is not consistent with this evidence.

7 Conclusion and Summary

This paper assumes that the market stochastic clock can be empirically modeled by observing market transaction count $N(t)$. Modeling cumulative transaction count as a Cox point process allows the representation of the stochastic clock as the closely related stochastic integrated intensity, $\Lambda(t)$. This is superior to cumulative trade count as a model to the stochastic market clock because it excludes the 'Poisson noise' of cumulative trade count and has a natural interpretation as a stochastic clock through the point process *random time change* theorem.

The fundamental innovation in this research is self-normalizing the cumulative trade count $R(t) = N(t)/N(1)$. The resultant process termed the *random relative counting measure* has the powerful property that the finite dimensional distributions of $R(t)$ are simply and readily approximated using a 2 dimensional histogram. In this paper the finite-dimensional properties of $R(t)$ are approximated by recording approximately 200,000 self-normalized daily trade count trajectories from the New York Stock Exchange (NYSE) into a 2D histogram.

If the cumulative trade count $N(t)$ is a Cox process then the *random relative counting measure* $R(t)$ generated by normalizing cumulative trade count by final trade count is shown to be a binomial point process directed by a self-normalized version of the stochastic integrated intensity:

$$\frac{\Lambda(t)}{\Lambda(1)} \in [0,1], \qquad t \in [0,1]$$

The moments of the finite dimensional distributions of the self-normalized stochastic integrated intensity are readily extracted from the empirical 2D histogram approximation of the finite dimensional distributions of the *random relative counting measure* $R(t)$ [31].

Extracting the variance of the self-normalized stochastic integrated intensity $\mathrm{Var}[\Lambda(t)/\Lambda(1)]$ for different trade count bands K allows the observation that these variances are scaled by the inverse square root of trade count $1/\sqrt{K}$ (figure 4). Stochastic market time $\Lambda(t)$ is modeled as Fractal Activity Time (FAT) introduced by Heyde [20] and Heyde and Liu [22] with a deterministic intra-day periodicity $\mathcal{T}(K\Delta(t))$. This model is consistent with the empirically observed behaviour of the market stochastic clock (figures 6, 7 and 8). It is then shown that the $1/\sqrt{K}$ scaling is related to the Hurst exponent H of the FAT model:

$$\mathrm{Var}\left[\frac{\Lambda(t)}{\Lambda(1)}\right] \approx \mathrm{Var}\left[\frac{\mathcal{T}(K\Delta(t))}{\mathcal{T}(K)}\right] \propto K^{2H-2}\,, \qquad t \in [0,1]$$

Thus by simply measuring the variance scaling of different trade count bands, the Hurst exponent the stochastic market clock is shown to be $H = 0.75$. This is consistent with the strong autocorrelation and the long range dependence of the square and absolute value of returns while preserving the no arbitrage requirement of martingale returns.

This is also inconsistent with the independent increment property of subordinated Lévy process models of returns and we formally show that the $1/\sqrt{K}$ scaling is inconsistent with Variance Gamma, Normal Inverse Gaussian and α-stable subordinated models of returns.

References

[1] A. Admati and P. Pfleiderer, *A Theory of Intraday Patterns: Volume and Price Variability*, Review of Financial Studies **1** (1988), 3–40.

[2] T. Ané and H. Geman, *Order flow, transaction clock, and normality of asset returns.*, The Journal of Finance. **55** (2000), no. 5, 2259–2284.

[3] O.E. Barndorff-Nielsen, *Normal inverse gaussian distributions and stochastic volatility modelling*, Scandinavian Journal of Statistics **24** (1996), 1–13.

[4] ______, *Normal inverse gaussian distributions and stochastic volatility modelling*, Scand. J. Statist. **24** (1996), 1–13.

[5] ______, *Proceses of the normal inverse gaussian type*, Finance and Stochastics **2** (1998), 41–68.

[6] ______, *Processes of normal inverse gaussian type*, Finance and Stochastics **2** (1998), 41–68.

[7] O.E. Barndorff-Nielsen and S.Z. Levendorskii, *Feller processes of the normal inverse gaussian type*, Quantitative Finance **1** (2001), no. 3, 318–331.

[8] O.E. Barndorff-Nielsen and N. Shephard, *Non-gaussian ornstein-uhlenbeck-based models and some of their uses in financial economics (with discussion)*, Journal of the Royal Statistical Society **series B 63** (2001), 167–241.

[9] F. Black and M. Scholes, *The pricing of options and corporate liabilities*, Journal of Political Economy **81** (1973), 637–654.

[10] P. Brémaud, *Point Processes and Queues:Martingale Dynamics*, Springer-Verlag, New York, 1981.

[11] William Brock and Allan Kleidon, *Periodic Market Closure and Trading Volume: A Model of Intraday Bids and Asks*, Journal of Economic Dynamics and Control **16** (1992), 451–490.

[12] P.K. Clark, *A subordinated stochastic process model with finite variance for speculative prices*, Econometrica **41** (1973), no. 1, 135–155.

[13] M. Coppejans, I. Domowitz, and A. Madhavan, *Liquidity in an Automated Auction*, Working Paper. March 2001 version.

[14] D. Daley and D. Vere-Jones, *An Introduction to the Theory of Point Processes, Volume 1: Elementary Theory and Methods*, Springer-Verlag, New York, 2002.

[15] Z. Ding, R. Engle, and C. Granger, *A long memory property of stock market returns and a new model*, Journal of Empirical Finance **1** (1993), 83–106.

[16] E. Eberlin and U. Keller., *Hyberbolic distributions in finance.*, Bernoulli **1** (1995), 281–299.

[17] P. Embrechts and M. Maejima, *Selfsimilar Processes*, Princeton University Press, 2002.

[18] E. F. Fama, *Mandelbrot and the Stable Paretian Hypothesis*, Journal of Business of the University of Chicago **36** (1963), 420–429.

[19] R. Finlay and E. Seneta, *Stationay-Increment Student and Variance-Gamma Processes*, Journal Of Applied Probability **43** (2006), 441–453.

[20] C. C. Heyde, *A Risky Asset Model with Strong Dependence Through Fractal Activity Time*, Journal of Applied Probability **36** (1999), 1234–1239.

[21] C. C. Heyde and N. Leonenko, *Student processes*, Advances in Applied Probability **37** (2005), 342–365.

[22] C. C. Heyde and S. Liu, *Empirical realities for a minimal description risky asset model. The need for fractal features.*, Journal of the Korean Mathematical Society **38** (2001), no. 5, 1047–1059.

[23] L. F. James, A. Lijoi, and I. Prünster, *Conjugacy as a Distinctive Feature of the Dirichlet Process*, Scandinavian Journal of Statistics **33** (2006), no. 1, 105–120.

[24] G. LeFol and L. Mercier., *Time deformation: Definition and comparsions*, Working Paper, Laboratoire de Finance-Assurance and Laboratoire de Statistiques, Crest and CEME-Paris University and Crest and Dauphine-Paris IX University, April 1998.

[25] D. Madan, P. Carr, and E. Chang, *The variance gamma process and option pricing*, European Finance Review. **2** (1998), 79–105.

[26] D. Madan and E. Seneta., *The variance gamma (VG) model for share market returns*, Journal of Business **63** (1990), 511–524.

[27] B. Mandelbrot, *The Variation of Certain Speculative Prices*, Journal of Business of the University of Chicago **36** (1963), 394–411.

[28] B. Mandelbrot and H. M. Taylor, *On the Distribution of Stock Price Differences*, Operations Research **15** (1967), 1057–1062.

[29] C. Marinelli, T. Rachev, and R. Roll, *Subordinated exchange rate models: Evidence for heavy tailed distributions and long-range dependance*, Mathematical and Computer Modelling. **34** (2001), 955–1001.

[30] C. Marinelli, T. Rachev, R. Roll, and H. Goppl., *Subordinated stock price models: Heavy tails and long-range dependance in the high-frequency deutche bank price record.*, Data Mining and Computational Finance (G. Bol, ed.), Springer, 1999.

[31] J. McCulloch, *Relative Volume as a Doubly Stochastic Binomial Point Process*, Quantitative Finance **7** (2007), 55–62.

[32] J. A. Rice, *Mathematical Statistics and Data Analysis - 2nd Ed.*, Duxbury Press, 1996.

[33] T. H. Rydberg, *The normal inverse gaussian levy process: Simulation and approximation.*, Communications in Statistics: Stochastic Models **13** (1997), 887–910.

[34] G. Samorodnitsky and M. Taqqu, *Stable Non-Gaussian Random Processes*, Chapman and Hall, 1994.

[35] K. Sato, *Self-similar processes with independent increments*, Probability Theory and Related Fields **89** (1991), no. 3, 285–300.

[36] ______, *Lévy Processes and Infinitely Indivisible Distributions*, Cambridge University Press, 2002.

[37] A. Segall and T. Kailath, *The Modeling of Randomly Modulated Jump Processes*, IEEE Transactions on Information Theory **IT-21, 2** (1975), 135–143.

[38] J. Simonoff, *Smoothing Methods in Statistics*, Springer-Verlag, New York., 1996.

A Moments of the Self-Normalized Monofractal Activity Time

The Taylor series of a ratio of random variables Y and X at the expectation of each random variable:

$$\frac{Y}{X} = \left(\frac{\mathbb{E}[Y]}{\mathbb{E}[X]} + \frac{(Y - \mathbb{E}[Y])}{\mathbb{E}[X]}\right) \sum_{n=0}^{\infty} (-1)^n \frac{(X - \mathbb{E}[X])^n}{\mathbb{E}[X]^n} \tag{6}$$

The Taylor series expansion is valid if all moments of the law of X are defined (or zero). For example the log-normal distribution has all moments defined. Before applying the Taylor series expansion to a self similar model of stochastic activity time we introduce a useful lemma on scaling the cross moments of a self-similar process. This lemma can be seen as analogous to the cross moments of a stationary process being dependent on the time difference of the random variables and not the absolute time of the random variables.

Lemma A.1. *Let process X_t be self similar where $X_t \overset{d}{=} t^H X_1$ for any $t > 0$ and $0 < H < 1$. Then for any integer $n \geq 1$ the nth cross moments of X are scaled with c^{nH} for any $c > 0$ and any $t_1, \ldots, t_n > 0$ where:*

$$c^{nH}\, \mathbb{E}\big[\, X_{t_1} X_{t_2} \ldots X_{t_n} \big] \;=\; \mathbb{E}\big[\, X_{ct_1} X_{ct_2} \ldots X_{ct_n} \big]$$

Proof. Define the vectors $\mathbf{X} = \{X_{t_1}, \ldots, X_{t_n}\}$, $\mathbf{Y} = \{X_{ct_1}, \ldots, X_{ct_n}\}$ and $\mathbf{z} = \{z_1, \ldots, z_n\}$. By the self similarity of process X, $\mathbf{Y} \overset{d}{=} c^H \mathbf{X}$. Examining the characteristic functions of the random vectors with the change of variable $\tilde{\mathbf{z}} = c^H \mathbf{z}$:

$$\varphi(\mathbf{z}; \mathbf{Y}) \;=\; \sum_{k=0}^{\infty} \frac{i^k c^{kH} \mathbb{E}\big[\langle \mathbf{z}, \mathbf{X}\rangle^k\big]}{k!} \;=\; \varphi(c^H \mathbf{z}; \mathbf{X}) \;=\; \varphi(\tilde{\mathbf{z}}; \mathbf{X})$$

Using the chain rule:

$$\frac{\partial^n \varphi(\mathbf{z};\mathbf{Y})}{\partial z_1 \ldots \partial z_n} \;=\; \frac{\partial^n \varphi(\tilde{\mathbf{z}};\mathbf{X})}{\partial z_1 \ldots \partial z_n} \frac{\partial \tilde{\mathbf{z}}}{\partial z_1} \cdots \frac{\partial \tilde{\mathbf{z}}}{\partial z_n} \;=\; c^{nH} \frac{\partial^n \varphi(\tilde{\mathbf{z}};\mathbf{X})}{\partial z_1 \ldots \partial z_n}$$

The nth cross-moment is the nth partial derivative of the characteristic function evaluated at $\mathbf{z} = \mathbf{0}$ and hence $\tilde{\mathbf{z}} = \mathbf{0}$:

$$\begin{aligned} \mathbb{E}\big[\, X_{ct_1}\, X_{ct_2} \ldots\, X_{ct_n} \,\big] \;&=\; i^{-n} \, \frac{\partial^n \varphi(\mathbf{z};\mathbf{Y})}{\partial z_1 \ldots \partial z_n} \Bigg]_{\tilde{\mathbf{z}}=0} \\ &=\; c^{nH}\, i^{-n} \, \frac{\partial^n \varphi(\tilde{\mathbf{z}};\mathbf{X})}{\partial z_1 \ldots \partial z_n} \Bigg]_{\tilde{\mathbf{z}}=0} \;=\; c^{nH} \mathbb{E}\big[\, X_{t_1}\, X_{t_2} \ldots\, X_{t_n} \,\big] \end{aligned}$$

□

The Heyde and Liu [22] model of the fractal activity time process with a rate parameter K and assuming, without loss of generality, that $t \in [0, 1]$:

$$\mathcal{T}(Kt) \;\overset{d}{=}\; Kt \;+\; \big(Kt\big)^H \,\big(\mathcal{T}(1) \;-\; 1\big)$$

Substituting into the ratio Taylor series (eqn 6) and:

$$\begin{aligned} \frac{\mathcal{T}(Kt)}{\mathcal{T}(K)} \;&=\; \left(t \;+\; \frac{\mathcal{T}(Kt) - Kt}{K}\right) \sum_{n=0}^{\infty} (-1)^n \, \frac{(\mathcal{T}(K) - K)^n}{K^n} \\ &=\; \sum_{n=0}^{\infty} (-1)^n \, \frac{t\,(\mathcal{T}(K) - K)^n}{K^n} \;+\; (-1)^n \, \frac{(\mathcal{T}(Kt) - Kt)\,(\mathcal{T}(K) - K)^n}{K^{n+1}} \end{aligned} \tag{7}$$

A.1 Expectation of the Ratio Bridge as a Taylor Series

Taking the expectation of eqn 7 and noting that $\mathbb{E}[\mathcal{T}(Kt) - Kt] = 0$ then:

$$\mathbb{E}\left[\frac{\mathcal{T}(Kt)}{\mathcal{T}(K)}\right] = t + \sum_{n=2}^{\infty} (-1)^n K^{-n} \Big(t\,\mathbb{E}\big[(\mathcal{T}(K) - K)^n\big] - \mathbb{E}\big[(\mathcal{T}(Kt) - Kt)(\mathcal{T}(K) - K)^{n-1}\big] \Big)$$

Invoking lemma A.1 on the the cross moment term:

$$\mathbb{E}\left[\frac{\mathcal{T}(Kt)}{\mathcal{T}(K)}\right] = t + \sum_{n=2}^{\infty} (-1)^n K^{n(H-1)} \Big(t\,\mathbb{E}\big[(\mathcal{T}(1) - 1)^n\big] - \mathbb{E}\big[(\mathcal{T}(t) - t)(\mathcal{T}(1) - 1)^{n-1}\big] \Big)$$

For notational convenience define:

$$\mathbb{E}\big[\mathcal{N}(n)\big] = (-1)^n K^{n(H-1)} \Big(t\,\mathbb{E}\big[(\mathcal{T}(1) - 1)^n\big] - \mathbb{E}\big[(\mathcal{T}(t) - t)\,(\mathcal{T}(1) - 1)^{n-1}\big] \Big)$$

Examining the terms $\mathbb{E}[\mathcal{N}(n)]$ for all n it is easily seen that $\mathbb{E}[\mathcal{N}(n)] = 0$ at $t = 0$ since, by definition, $\mathcal{T}_0 = 0$. Also $\mathbb{E}[\mathcal{N}(n)] = 0$ at $t = 1$ because the nth cross moment term is equal to the magnitude of the nth central moment at $t = 1$.

Expanding the quadratic terms (n = 2) using the autocovariance formula for a H-sssi self-similar process (Embrechts and Maejima [17]), the expectation of the Taylor series of the ratio bridge of process is:

$$\mathbb{E}\left[\frac{\mathcal{T}(Kt)}{\mathcal{T}(K)}\right] = t + \Big(t - \frac{t^{2H} + 1 - (1-t)^{2H}}{2} \Big) K^{2H-2}\,\mathrm{Var}[\mathcal{T}(1)] + \sum_{n=3}^{\infty} \mathbb{E}\big[\mathcal{N}(n)\big]$$

Finally we assume that $\sum_{n=3}^{\infty} \mathbb{E}\big[\mathcal{N}(n)\big]$ is small and make the standard approximation (e.g. Rice [32]):

$$\mathbb{E}\left[\frac{\mathcal{T}(Kt)}{\mathcal{T}(K)}\right] \approx t + \Big(t - \frac{t^{2H} + 1 - (1-t)^{2H}}{2} \Big) K^{2H-2}\,\mathrm{Var}[\mathcal{T}(1)]$$

A.2 Ratio Bridge Variance using Taylor Series

A truncated Taylor series expansion is used to approximate the variance of a ratio of random variables. This approximation is a standard result (Rice [32]). We extend it slightly and adapt it to the functional form of the fractal activity time process. Let Y and X be random variables then the Taylor series of the ratio of the squared random variables is as follows:

$$\left(\frac{Y}{X}\right)^2 = \frac{Y^2}{\mathbb{E}[X]^2} \sum_{n=0}^{\infty} (-1)^n (n+1) \frac{(X - \mathbb{E}[X])^n}{\mathbb{E}[X]^n}$$

$$= \frac{\mathbb{E}[Y]^2 + 2\mathbb{E}[Y](Y - \mathbb{E}[Y]) + (Y - \mathbb{E}[Y])^2}{\mathbb{E}[X]^2} \sum_{n=0}^{\infty} (-1)^n (n+1) \frac{(X - \mathbb{E}[X])^n}{\mathbb{E}[X]^n}$$

For notational convenience:

$$\mathbb{E}[J(n)] = (-1)^n (n+1) \frac{\mathbb{E}\big[(X - \mathbb{E}[X])^n\big]}{\mathbb{E}[X]^n}$$

$$\mathbb{E}[K(n)] = (-1)^n (n+1) \frac{\mathbb{E}\big[(Y - \mathbb{E}[Y])(X - \mathbb{E}[X])^n\big]}{\mathbb{E}[X]^n}$$

$$\mathbb{E}[L(n)] = (-1)^n (n+1) \frac{\mathbb{E}\big[(Y - \mathbb{E}[Y])^2(X - \mathbb{E}[X])^n\big]}{\mathbb{E}[X]^n}$$

Then the expectation of the squared ratio is:

$$\mathbb{E}\left[\frac{Y^2}{X^2}\right] = \frac{\mathbb{E}[Y]^2}{\mathbb{E}[X]^2} + 3\frac{\mathbb{E}[Y]^2}{\mathbb{E}[X]^4}\mathrm{Var}\big[X\big] - 4\frac{\mathbb{E}[Y]}{\mathbb{E}[X]^3}\mathrm{Cov}\big[Y, X\big] + \frac{1}{\mathbb{E}[X]^2}\mathrm{Var}\big[Y\big]$$
$$+ \frac{\mathbb{E}[Y]^2}{\mathbb{E}[X]^2}\sum_{n=3}^{\infty}\mathbb{E}[J(n)] + 2\frac{\mathbb{E}[Y]}{\mathbb{E}[X]^2}\sum_{n=2}^{\infty}\mathbb{E}[K(n)] + \frac{1}{\mathbb{E}[X]^2}\sum_{n=1}^{\infty}\mathbb{E}[L(n)] \qquad (8)$$

The fractal activity time process:

$$\mathcal{T}(Kt) \;=\; Kt \;+\; \big(Kt\big)^{H}\,\big(\mathcal{T}(1)\,-\,1\big)$$

For notational convenience and using lemma A.1 on the the cross moment terms:

$$\mathbb{E}[\mathcal{J}(n)] \;=\; (-1)^n\,(n+1)\,K^{(n-1)H}\,\mathbb{E}\big[(\mathcal{T}(1)-1)^n\big]$$

$$\mathbb{E}[\mathcal{K}(n)] \;=\; (-1)^n\,(n+1)\,K^{nH}\,\mathbb{E}\big[(\mathcal{T}(t)-t)(\mathcal{T}(1)-1)^n\big]$$

$$\mathbb{E}[\mathcal{L}(n)] \;=\; (-1)^n\,(n+1)\,K^{(n+1)H}\,\mathbb{E}\big[(\mathcal{T}(t)-t)^2(\mathcal{T}(1)-1)^n\big]$$

Substituting the above into equation 8 and using the autocovariance formula for H-sssi processes gives the expectation of the squared ratio:

$$\mathbb{E}\left[\left(\frac{\mathcal{T}Kt}{\mathcal{T}K}\right)^2\right] \;=\; t^2 \;+\; \left(3t^2 \;-\; 4t\,\frac{t^{2H} \;+\; 1 \;-\; (1-t)^{2H}}{2} \;+\; t^{2H}\right)K^{2H-2}\,\mathrm{Var}\big[\mathcal{T}(1)\big]$$

$$+\; t^2\sum_{n=3}^{\infty}\mathbb{E}[\mathcal{J}(n)] \;+\; 2\frac{t}{K}\sum_{n=2}^{\infty}\mathbb{E}[\mathcal{K}(n)] \;+\; \frac{1}{K^2}\sum_{n=1}^{\infty}\mathbb{E}[\mathcal{L}(n)] \tag{9}$$

The squared expectation of the ratio is:

$$\mathbb{E}\left[\frac{\mathcal{T}_t^K}{\mathcal{T}_1^K}\right]^2 \;=\; t^2 \;+\; 2t\left(t \;-\; \frac{t^{2H} \;+\; 1 \;-\; (1-t)^{2H}}{2}\right)K^{2H-2}\,\mathrm{Var}[\mathcal{T}(1)]$$

$$+\; K^{4H-4}\left(t \;-\; \frac{t^{2H} \;+\; 1 \;-\; (1-t)^{2H}}{2}\right)^2\mathrm{Var}[\mathcal{T}(1)]^2 \;+\; 2t\sum_{n=3}^{\infty}\mathbb{E}\big[\mathcal{N}(n)\big]$$

$$+\; 2K^{2H-2}\left(t \;-\; \frac{t^{2H} \;+\; 1 \;-\; (1-t)^{2H}}{2}\right)\mathrm{Var}[\mathcal{T}(1)]\sum_{n=3}^{\infty}\mathbb{E}\big[\mathcal{N}(n)\big] \;+\; \left(\sum_{n=3}^{\infty}\mathbb{E}\big[\mathcal{N}(n)\big]\right)^2$$

Therefore the variance of the ratio is:

$$\mathrm{Var}\left[\frac{\mathcal{T}(Kt)}{\mathcal{T}(K)}\right] = \mathbb{E}\left[\left(\frac{\mathcal{T}(Kt)}{\mathcal{T}(K)}\right)^2\right] - \mathbb{E}\left[\frac{\mathcal{T}(Kt)}{\mathcal{T}(K)}\right]^2$$

$$= \Big(t^2 - t\,(\,t^{2H} + 1 - (1-t)^{2H}) + t^{2H}\Big)K^{2H-2}\mathrm{Var}\big[\mathcal{T}(1)\big]$$

$$- \Big(t - \frac{t^{2H} + 1 - (1-t)^{2H}}{2}\Big)^2 K^{4H-4}\,(\mathrm{Var}[\mathcal{T}(1)])^2$$

$$- \sum_{n=3}^{\infty}\mathbb{E}\big[\mathcal{N}(n)]\left(2t + 2K^{2H-2}\Big(t - \frac{t^{2H} + 1 - (1-t)^{2H}}{2}\Big)\mathrm{Var}[\mathcal{T}(1)] + \sum_{n=3}^{\infty}\mathbb{E}\big[\mathcal{N}(n)]\right)$$

$$+ t^2\sum_{n=3}^{\infty}\mathbb{E}[\mathcal{J}(n)] + 2\frac{t}{K}\sum_{n=2}^{\infty}\mathbb{E}[\mathcal{K}(n)] + \frac{1}{K^2}\sum_{n=1}^{\infty}\mathbb{E}[\mathcal{L}(n)]$$

Finally, the Taylor series is truncated to approximate the variance of the ratio:

$$\mathrm{Var}\left[\frac{\mathcal{T}(Kt)}{\mathcal{T}(K)}\right] \approx \Big(t^2 - t\,(\,t^{2H} + 1 - (1-t)^{2H}) + t^{2H}\Big)K^{2H-2}\mathrm{Var}\big[\mathcal{T}(1)\big]$$

$$- \Big(t - \frac{t^{2H} + 1 - (1-t)^{2H}}{2}\Big)^2 K^{4H-4}\,(\mathrm{Var}[\mathcal{T}(1)])^2$$

B Variance Scaling of Time-Accelerated Self-Normalized Finite Variance Lévy Subordinators

Lemma B.1. *The following properties of finite variance Lévy subordinators are proved by examining the time dependent structure of the first two moments of a Lévy process and any self-normalized Lévy subordinator* $\Gamma(Kt)$ *with a finite variance scales approximately as a function of* $1/K$ *for values of* $K \gg 1$.

$$Var\left[\frac{\Gamma(Kt)}{\Gamma(K)}\right] \propto \frac{1}{K}\,, \qquad K \gg 1, \quad t \in [0,1]$$

Proof. For any Lévy process with a finite nth moment, the independence of disjoint increments property and the binomial theorem gives the following identity:

$$\begin{aligned}
\mathbb{E}\big[\big(\Gamma(s+t)\big)^n\big] &= \mathbb{E}\big[\big(\Gamma(s) + (\Gamma(s+t) - \Gamma(s))\big)^n\big] \\
&= \mathbb{E}\Big[\sum_{k=0}^{n} \binom{n}{k} \Gamma(s)^{n-k} \big(\Gamma(s+t) - \Gamma(s)\big)^k\Big] \\
&= \sum_{k=0}^{n} \binom{n}{k} \mathbb{E}\big[\Gamma(s)^{n-k}\big]\, \mathbb{E}\big[\Gamma(t)^k\big]
\end{aligned}$$

Using an induction argument and the identity above, it is straightforward to show that both variance and expectation scale linearly in K for any Lévy process with finite variance:

$$\mathbb{E}\big[\Gamma(2t)\big] = 2\mathbb{E}\big[\Gamma(t)\big]\,, \qquad \mathbb{E}\big[\Gamma(Kt)\big] = \mathbb{E}\big[\Gamma(t)\big] + \mathbb{E}\big[\Gamma((K-1)t)\big]$$

$$\mathrm{Var}\big[\Gamma(2t)\big] = 2\mathrm{Var}\big[\Gamma(t)\big]\,, \qquad \mathrm{Var}\big[\Gamma(Kt)\big] = \mathrm{Var}\big[\Gamma(t)\big] + \mathrm{Var}\big[\Gamma((K-1)t)\big]$$

$$\text{Hence:} \quad \mathbb{E}\big[\Gamma(Kt)\big] \;=\; K\mathbb{E}\big[\Gamma(t)\big]\,, \quad \mathrm{Var}\big[\Gamma(Kt)\big] \;=\; K\mathrm{Var}\big[\Gamma(t)\big]$$

The independent increments of a Lévy process also implies that the process auto-covariance is linear. We assume that $t > s$ then:

$$\begin{aligned}
\mathbb{E}\big[\Gamma(t)\Gamma(s)\big] - \mathbb{E}\big[\Gamma(t)\big]\mathbb{E}\big[\Gamma(s)\big] \;&=\; \mathbb{E}[\Gamma(s)^2] \,+\, \mathbb{E}\big[\Gamma(s)]\mathbb{E}\big[\Gamma(t) - \Gamma(s)\big] - \mathbb{E}\big[\Gamma(t)\big]\mathbb{E}\big[\Gamma(s)\big] \\
&=\; \mathrm{Var}\big[\Gamma(s)\big]
\end{aligned}$$

The argument for $s > t$ is identical and therefore:

$$\mathrm{Cov}\big[\Gamma(s), \Gamma(t)\big] \;=\; \mathrm{Var}\big[\Gamma(\min[s, t])\big]$$

A Taylor series expansion is used to approximate the variance of a ratio of random variables. This approximation is a standard result (Rice [32]):

$$\mathrm{Var}\left[\frac{X}{Y}\right] \approx \; \frac{1}{\mathbb{E}[Y]^2}\mathrm{Var}[X] - \frac{\mathbb{E}[X]^2}{\mathbb{E}[Y]^4}\mathrm{Var}[Y] \;-\; 2\frac{\mathbb{E}[X]}{\mathbb{E}[Y]^3}\mathrm{Cov}[X, Y]$$

Assigning $X = \Gamma(Kt)$ and $Y = \Gamma(K)$ and using the Lévy process moment relationships above, the Taylor series approximation shows the scaling relationship to be approximately inverse K and statement 2. is proved:

$$\begin{aligned}
\mathrm{Var}\left[\frac{\Gamma(Kt)}{\Gamma(KT)}\right] \;&\approx\; \frac{1}{K}\left(\frac{1}{\mathbb{E}[\Gamma(1)]^2}\mathrm{Var}[\Gamma(t)] - \frac{\mathbb{E}[\Gamma(t)]^2}{\mathbb{E}[\Gamma(1)]^4}\mathrm{Var}[\Gamma(1)] - 2\frac{\mathbb{E}[\Gamma(t)]}{\mathbb{E}[\Gamma(1)]^3}\mathrm{Cov}[\Gamma(t), \Gamma(1)]\right) \\
&\approx \frac{1}{K}\mathrm{Var}\left[\frac{\Gamma(t)}{\Gamma(1)}\right]
\end{aligned}$$

$\square$

B.1 The Variance Scaling of Gamma and Inverse Gaussian Subordinators

Using the results in James, Lijoi and Prünster [23] the variance of the self-normalized Gamma subordinator and self-normalized Inverse Gaussian subordinator are formulated explicitly.

Assuming subordinator $\Gamma(t)$ is a Gamma process, c is constant for all trade counts and $\Delta(t)$ is the deterministic intra-day periodicity (eqn 4), then the variance of the self-normalized Gamma process for a stock with K trades is:

$$\mathrm{Var}\left[\frac{\Gamma(K\Delta(t))}{\Gamma(K)}\right] = \Delta(t)\big(1 - \Delta(t)\big)\frac{1}{Kc+1}$$

Clearly for the self-normalized Gamma process the term $1/(Kc+1)$ approximates $1/K$ scaling for $Kc \gg 1$. Next we assume the subordinator $\Gamma(K\Delta(t))$ is an Inverse Gaussian process and c is constant for all trade counts, then the variance[4] of the self-normalized inverse Gaussian process is:

$$\mathrm{Var}\left[\frac{\Gamma(K\Delta(t))}{\Gamma(K)}\right] = \Delta(t)\big(1 - \Delta(t)\big)\,(Kc)^2\,e^{Kc}\int_{Kc}^{\infty}\frac{e^{-u}}{u^{-3}}\,du$$

The trade count term for Inverse Gaussian is less transparent than the Gamma case above but can be readily shown (figure 8) to approximate $1/K$ scaling for $K \gg 1$.

[4] The integral term is the upper incomplete gamma function $\mathrm{U}\Gamma(-2, Kc)$.

C The Empirical Moments of the Self-Normalized Integrated Intensity

For the convenience of the reader, below is a brief summary of how the time indexed moments of the self-normalized integrated intensity are extracted from the time indexed self-normalized trade count data. Details are available in McCulloch [31].

Fubini's theorem on product probability spaces implies that the unconditional probability distribution of the doubly stochastic binomial point process is a mixture distribution of the distribution of the self-normalized integrated intensity at time t, $\Psi_t(s)$ (the mixing distribution), and the binomial distribution of the probability of k trades given K final trades, $\mathcal{B}(k|K,\, s)$. Thus if $\Theta_t(k|K)$ is the probability of k trades at time t given K trades at time 1, then this distribution is the following mixture distribution:

$$\mathcal{B}(k|K,\, s) \;=\; \binom{K}{k}\, s^k \left(1-s\right)^{K-k}, \quad \forall s \in [0,1]$$

$$\Phi_t(s) \;=\; Pr\left(\frac{\Lambda(t)}{\Lambda(1)} \leq s\right), \quad \Psi_t(s) \;=\; \frac{d\Phi_t(s)}{ds}$$

$$\Theta_t(k|K) \;=\; \int_0^1 \mathcal{B}(k|K,\, s)\, \Psi_t(s)\, ds$$

Since the distribution of relative intra-day trade count is a binomial mixture distribution, it is easy to extract the moments of the self-normalized integrated intensity:

$$\sum_{k=0}^{K} \left(\frac{k}{K}\right)^n \Theta_t(k|K) \;=\; \frac{1}{K^n} \int_0^1 \left[\, \sum_{k=0}^{K} k^n\, \mathcal{B}(k|K,\, s) \right] \Psi_t(s)\, ds \qquad (10)$$

In this way the mean of the observed binned data is shown to equal the mean of the self-normalized integrated intensity.

$$\sum_{k=0}^{K} \frac{k}{K}\,\Theta_t(k|K) \;=\; \frac{1}{K}\int_0^1 \left[\,\sum_{k=0}^{K} k\,\mathcal{B}(k|K,\,s)\right]\,\Psi_t(s)\,ds \;=\; \int_0^1 s\;\Psi_t(s)\,ds$$

$$\mathbb{E}\left[\frac{N(t)}{N(1)}\right] \;=\; \mathbb{E}[R(t)] \;=\; \mathbb{E}\left[\frac{\Lambda(t)}{\Lambda(1)}\right]$$

An identical argument is used to calculate the second moment of the self-normalized integrated intensity in terms of the first two moments of the observed binned data.

Finally, a scaling can be introduced to allow moments to be readily extracted from a collections of stocks with different trade counts. This scaling is the weighted average of the trade count distribution of stocks being analyzed within each trade count band.

Trade Count Band	Weighted Av. Trade Count
51 - 100	72.96
101 - 200	145.02
201 - 400	284.94
401 - 5022	960.76

Table 1: Weighted average trade count for each trade count band. These averages can be used to represent the 'average' moment of a trade count band [31].