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A Stochastic Approach to the Valuation of Barrier Options in Heston's Stochastic Volatility Model

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Abstract. In the valuation of continuous barrier options the distribution of the first hitting time plays a substantial role. In general, the derivation of a hitting time distribution poses a mathematically challenging problem for continuous but otherwise arbitrary boundary curves. When considering barrier options in the Heston model the non-linearity of the variance process leads to the problem of a non-linear hitting boundary. Here, we choose a stochastic approach to solve this problem in the reduced Heston framework, when the correlation is zero and foreign and domestic interest rates are equal. In this context one of our main findings involves the proof of the reflection principle for a driftless Itô process with a time-dependent variance. Combining the two results, we derive a closed-form formula for the value of continuous barrier options. Compared to an existing pricing formula, our solution provides further insight into how the barrier option value in the Heston model is constructed. Extending the results to the general Heston framework with arbitrary correlation and drift, we obtain approximations for the joint random variables of the Itô process and its maximum in a weak sense. As a consequence, an approximate formula for pricing barrier options is established. A numerical case study is also performed which illustrates the agreement in results of our developed formulas with standard finite difference methods.

Keywords: Heston model, barrier options, reflection principle

JEL classification: C15, G12

1 Introduction and Problem Description

In this paper, we introduce a method to derive valuation formulas for options with a continuous barrier level under stochastic volatility dynamics. To this end, we consider the stochastic volatility model of Heston (first introduced by Heston [10]), which is characterized by the following system of stochastic differential equations under the risk-neutral measure as

$$\begin{aligned} \frac{dS_t}{S_t} &= rdt + \sqrt{v_t}dW_t^S \\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t^v. \end{aligned} \tag{1}$$

The process $\{S_t\}_{t\geq 0}$ denotes the spot price and the process $\{v_t\}_{t\geq 0}$ represents the instantaneous variance of the logarithmic spot price with initial variance $v_0 > 0$. In the following discussions we will put the focus on the currency market, without loss of generality, and assume our underlying asset to be a foreign exchange rate. Therefore, the risk-neutral drift term r of the underlying price process is set to the difference between the domestic and foreign interest rates $r_d - r_f$. The two Wiener processes $\{W_t^S\}_{t\geq 0}$ and $\{W_t^v\}_{t\geq 0}$ in the model are correlated with a constant correlation rate ρ, i.e.

$$dW_t^S \, dW_t^v = \rho dt.$$

The model allows for correlation to take on values between -1 and 1, $-1 < \rho < 1$. The parameters of the Heston model, meaning, the long term variance θ, the rate of mean reversion κ and the volatility of variance σ, are assumed to be constant and positive throughout.

While there exists a closed-form formula for vanilla and a few other options in the presence of stochastic volatility, practitioners in the financial industry depend mainly on numerical methods such as finite difference methods and Monte Carlo simulation to price barrier options. Even though the method of Monte Carlo simulation in the Heston model is easy to implement, it is often found to be too slow due to the high number of sample paths and discretization steps required for an accurate outcome. Finite difference methods in the Heston model have been studied in a number of articles. In particular, Chiarella, Meyer and Kang [2] developed a method of lines approach to numerically evaluate barrier option prices and Foulon and Hout [11] outline an ADI finite difference method for option pricing in the Heston model with correlation. These studies serve us as a good foundation to set up benchmark methods for our concluding numerical analysis.

Our contribution in this paper is to develop a novel approach to obtain a (semi-)closed-form valuation formula for continuous barrier options in a reduced Heston framework and approximations for these types of options in the general Heston model. We demonstrate our derivation by using the example of an up-and-out call option. Its payoff is given by

$$(S_T - K)^+ \mathbb{1}_{\left\{\max\limits_{0\leq t\leq T} S_t \leq B\right\}}, \tag{2}$$

where K denotes the strike price and B the knock-out level of the option. The value of an option with payoff (2) is given by its discounted expected payoff function under the risk-neutral measure. Hence, when pricing this type of barrier option one is generally confronted with the problem of deriving the joint distribution of the random variables contained in the payoff function, S_T and $\max_{0\leq t\leq T} S_t$. For the case of discretely monitoring the underlying asset price, this problem can be solved in closed analytical form as shown in [8] by Griebsch and Wystup. In continuous time, the derivation of the first hitting time in the Heston model poses a mathematically challenging problem.
In the Black-Scholes model, the option pricing problem above can be reformulated in terms of knock-out probabilities of a Brownian motion with drift and its maximum with respect to constant boundaries. This reduces the central issue of the problem to the derivation of the joint density of a Brownian motion with linearly time-dependent drift and its maximum. Hence, the entire problem is analytically tractable due to the constant volatility assumptions of the Black-Scholes model as shown for example in Shreve [15].
However, in the context of Heston's stochastic volatility model, two difficulties arise in pricing continuous barrier options. Firstly, the underlying price is not only driven by a Brownian motion, but also by a random volatility process. Secondly, since the realized variance path is neither constant nor linear in time, the task is to compute knock-out probabilities with respect to non-linear boundaries. Major parts of the sequel deal with these two challenges, and while the first issue can be tackled by conditioning on the variance path, the determination of the knock-out probabilities require an approximation approach. Although boundary crossing problems arise in many fields of applied mathematics and different approximation methods have been developed and successfully applied (see for example Novikov, Frishling and Kordzakhia [13] for an application to option pricing), we use an approximation method that is tailored to the general approach pursued in this paper.
This study pursues the line of work by Lipton [12] in developing methods to derive closed-form valuation formulas for single barrier options. In his work, Lipton proposes (semi-)analytical solutions for double barrier options. By setting one of the two barriers in his formula equal to a practically improbable value (which would be close to zero or very large) one can therefore apply the result to obtain an approximation of a single barrier option value. For the purpose of pricing barrier options, Lipton formulates the problem as a set of partial differential equations and presents two different methods to derive solutions, the method of images and the Eigenfunction expansion method. Applying either of the two methods, an expression for the bounded Green's function is derived, which leads to the final pricing formula for a double barrier option given by an infinite sum of rather complex expressions. However, for both approaches the derivation of the solution is restricted to the case where the correlation and the interest rate difference in the model is equal to zero. In his thesis [7] Faulhaber shows that and why an extension of these techniques to the general Heston framework fails. Nevertheless, for this special case, we can directly compare the outcome of our approach with Lipton's formula.

For our approach, we employ probabilistic methods rather than techniques to solve differential equations and obtain an analytical solution for the reduced Heston framework as well. In comparison to Lipton's pricing formula, our solution is of a different format and provides insight into how the barrier option value in the Heston model is constructed. Additionally, the techniques used to derive our result, provide us with a good foundation to extend the pricing of barrier options to the general Heston model. Thereby, our plan of attack consists of two major parts. One is to distinguish the cases for

(A) $\rho = 0$ and $r_d = r_f$,

(B) $\rho = 0$ and arbitrary r_d and r_f,

(C) $0 \leq |\rho| < 1$ and arbitrary r_d and r_f,

where we build up our solution from case A to case C, having to solve a more complicated problem for each case and developing techniques to do so along the way.
The other part of our strategy is to condition on the information of the variance path generated up until maturity. Thereby, we are allowed to treat the variance in the logarithmic spot price process as a deterministic quantity and for this reason we can solve the pricing problem. The last step is to reverse this conditioning by accounting for the distribution of the remaining random variables coming from the process v. Our Ansatz is summarized as follows:

I. Solve case A by

   (i) Conditioning on the variance paths until maturity of the option.
   (ii) Under the knowledge of the variance paths, the logarithmic spot price in the Heston model is normally distributed and we can apply similar techniques, known from the Black-Scholes model, for the derivation of the joint density of the logarithmic spot price and its maximum process.
   (iii) With the joint density from the previous step we derive the valuation formula for up-and-out options, still under the conditioning subject to the variance paths.
   (iv) Finally, we resolve the conditioning.

II. Solve case B and C by using the result obtained in I and extend it to an approximation pricing formula.

The remainder of this paper is organized as follows: first we describe the conditioning step and its consequences for the pricing problem in section 2. It turns out that certain probabilistic tools need to be at hand in order to proceed with the derivation. Hence, in section 3, the proof of the reflection principle for a driftless Itô process with a time-dependent variance is presented. Then, in the next section, the joint density of the logarithmic spot price and its maximum process is derived for case A and approximations for case B and C. Section 5 shows how the valuation and approximation formulas are obtained for all cases under the condition of known variance paths. And finally, the conditioning is resolved in section 6, such that the joint distribution of the remaining random variables with respect to the variance process is determined. The paper concludes with numerical case studies illustrating the agreement between the results of our developed formulas with a finite difference method, and presents a summary of our findings.

2 First Step: Conditioning

In this section, we describe the logarithmic spot price using integral notation incorporating both of the two model definitions in (1) for S and v into one. This leads to a representation which mainly depends on variance values and an Itô integral with respect to a Brownian motion independent of W^S and W^v. On account of this, the conditioning on the information generated by the variance paths gives rise to a normally distributed logarithmic spot price with time-dependent variance. We analyze how that affects the pricing problem of barrier options in the Heston model.

Let x_t denote the logarithmic spot value, $\log S_t$, at an arbitrary time $0 < t \leq T$. Applying Itô's lemma, the logarithmic spot price at time t, given the values x_0 and v_0, can be written in integral form as

$$\begin{aligned} x_t &= x_0 + (r_d - r_f)t - \frac{1}{2}\int_0^t v_s ds + \int_0^t \sqrt{v_s} dW_s^S \\ &= x_0 + (r_d - r_f)t - \frac{1}{2}\int_0^t v_s ds + \rho \int_0^t \sqrt{v_s} dW_s^v + \rho_2 \int_0^t \sqrt{v_s} dW_s, \end{aligned} \tag{3}$$

where $\rho_2 = \sqrt{1-\rho^2}$ and $dW_t^S = \rho dW_t^v + \rho_2 dW_t$ is the Cholesky decomposition of the Brownian motion W^S into the sum of W^v and another independent Brownian motion W.
The value of the variance at time t is given by the integral equation

$$v_t = v_0 + \kappa\theta t - \kappa \int_0^t v_s ds + \sigma \int_0^t \sqrt{v_s} dW_s^v. \tag{4}$$

Solving equation (4) with respect to the integral $\int_0^t \sqrt{v_s} dW_s^v$ and insertion of the corresponding result into equation (3) yields

$$x_t = x_0 + (r_d - r_f)t - \frac{1}{2}\int_0^t v_s ds + \frac{\rho}{\sigma}\left(v_t - v_0 - \kappa\theta t + \kappa \int_0^t v_s ds\right) + \rho_2 \int_0^t \sqrt{v_s} dW_s.$$

In the following, we use the abbreviation

$$\alpha(t) = (r_d - r_f)t - \frac{1}{2}\int_0^t v_s\, ds + \frac{\rho}{\sigma}\Big(v_t - v_0 - \kappa\theta t + \kappa \int_0^t v_s\, ds\Big),$$

so that the logarithmic spot price at time t is the sum of its initial value, the time-dependent drift α and an Itô integral

$$x_t = x_0 + \alpha(t) + \rho_2 \int_0^t \sqrt{v_s} dW_s.$$

Since v_t cannot be negative, conditioning on the σ-algebra generated by the variance process up to time t, i.e. $\mathcal{G}_t^v = \sigma(\{v_s : 0 \leq s \leq t\})$, yields a random variable $x_t^v := X_t|\mathcal{G}_t^v$, where the only random contribution arises from the integrator dW_s of the Itô integral,

$$Y_t = \rho_2 \int_0^t \sqrt{v_s}\, dW_s. \tag{5}$$

In Shreve [15], it is shown that an Itô integral Y_t of the format above is normally distributed with expected value zero and variance $\rho_2^2\nu^2(t)$, where $\nu^2(t) = \int_0^t v_s ds$. Hence, the random variable x_t^v is normally distributed with mean $\mu(t) = x_0 + \alpha(t)$ and the same variance:

$$x_t^v \sim \mathcal{N}\left(\mu(t), \rho_2^2\nu^2(t)\right).$$

Note that, $\alpha(t)$ is a deterministic continuous function of t when conditioned on a particular realization of $\{v_s\}_{0\leq s\leq t}$, but is not differentiable unless $\rho = 0$.

Now, an up-and-out call has the following payoff

$$V_T = \begin{cases} 0 & \text{if} \quad \max\limits_{0\leq t\leq T} S_t \geq B \\ (S_T - K)^+ & \text{if} \quad \max\limits_{0\leq t\leq T} S_t < B. \end{cases} \tag{6}$$

Since the value of the up-and-out call is given by its discounted expected payoff function, using the tower property results in

$$\begin{aligned} V_0 &= e^{-r_d T}\mathbb{E}\left[V_T \,|\, \mathcal{F}_0\right] \\ &= e^{-r_d T}\mathbb{E}\Big[\,\mathbb{E}\left[V_T \,|\, \mathcal{G}_T^v\right] \,|\, \mathcal{F}_0\Big]. \end{aligned}$$

Inserting the definition of the payoff in equation (6) yields

$$V_0 = e^{-r_d T}\mathbb{E}\Big[\underbrace{\mathbb{E}[(S_T - K)^+\mathbb{1}_{\{\max\limits_{0\leq t\leq T} S_t<B\}} \,|\, \mathcal{G}_T^v]}_{=:\mathcal{E}^v}\Big|\,\mathcal{F}_0\Big]. \tag{7}$$

We proceed by analyzing the inner expectation $\mathcal{E}^v$ and define the random variables $\hat{M}$ and $\hat{Y}$ as

$$\begin{aligned} \hat{Y}_t &= \alpha(t) + Y_t \\ \hat{M}_t &= \max\{\hat{Y}_s \,:\, 0 \leq s \leq t\}. \end{aligned}$$

Then the inner part of the pricing problem of an up-and-out barrier call option in the Heston model reads as

$$\begin{aligned} \mathcal{E}^v &= \mathbb{E}\left[(S_0 e^{\hat{Y}_T} - K)^+\mathbb{1}_{\{S_0 e^{\hat{M}_T}<B\}} \,|\, \mathcal{G}_T^v\right] \\ &= \mathbb{E}^v\left[(S_0 e^{\hat{Y}_T} - K)\mathbb{1}_{\{\hat{M}_T<b, \hat{Y}_T>k\}}\right], \end{aligned} \tag{8}$$

where $k = \ln(K/S_0)$ and $b = \ln(B/S_0)$. $\mathbb{P}^v$ is associated with the probability measure under the filtration $\mathcal{G}_T^v$.

We observe that computing the inner expectation $\mathcal{E}^v$ entails at least finding the unknown joint distribution of $(\hat{Y}, \hat{M})$ under the conditional probability measure $\mathbb{P}^v$. In the next section we derive an important mathematical tool that is required to solve the just described problem, the reflection principle for Y_t and $M_t = \max\{Y_s \,:\, 0 \leq s \leq t\}$.

3 Reflection Principle

In this section, we show that a driftless Itô process with time-dependent variance, as given in (5), satisfies the reflection principle. A general process $\{Y_t\}_{0\leq t\leq T}$ (for some finite T) of the form

$$Y_t = \int_0^t \beta(s)\, dW_s, \tag{9}$$

is considered, where $\beta(\cdot)$ is some deterministic continuous function bounded on $[0,T]$[1]. Note that Y is a Gaussian process with heterogeneous instantaneous variance, which distinguishes Y from a Brownian motion. Nevertheless, as shown in the appendix A.1, the first parts of the proof work analogously to their counterparts that deal with Brownian motions: first we prove in lemma 1 that the random variable $Y_t - Y_s$ is independent from $\mathcal{F}_s$ for all $0 \leq s < t < \infty$. Then we show that $M_t = \exp\left(iuY_t + \frac{1}{2}u^2\int_0^t \beta^2(s)\,ds\right)$, for $u \in \mathbb{R}$, is a martingale in lemma 2 and finally in lemma 3 we show that the process Y has the strong Markov property.
In the derivation of the reflection principle for Brownian motions B, usually a stronger statement than the strong Markov property is proven, i.e. that the increments $B_{\tau+t} - B_t$ are independent of $\mathcal{F}_\tau$, for a stopping time τ. This is not true for the process $\{Y_t\}_{t\geq 0}$, since the variance depends on time. Hence, different techniques must be applied to prove the reflection principle for Y. We denote the reflected process (with respect to Y) by the process

$$\tilde{Y}_t = Y_{\tau\wedge t} - (Y_t - Y_{\tau\wedge t}) \quad t \geq 0 \tag{10}$$

and prove the reflection principle for Y and its supremum.

Theorem 1 *Let $\{Y_t\}_{t\geq 0}$ be an Itô process of the form in equation (9) with deterministic function β and $M_t = \sup_{s\leq t} Y_s$ for $t \geq 0$. Then the reflection principle holds,*

$$\mathbb{P}(M_t \geq x, Y_t < y) = \mathbb{P}(Y_t > 2x - y) \quad \textit{for all } t \geq 0,\ x \geq y \vee 0.$$

Proof: The proof is carried out in two steps. First we prove that the process Y defined in (9) and the reflected process $\tilde{Y}$ defined in (10) have the same distribution, i.e. $Y \overset{\mathrm{d}}{=} \tilde{Y}$. Therefore, define the process $Y'_t = Y_{\tau+t} - Y_\tau$. Then Y can be expressed by

$$Y_t = Y_{\tau\wedge t} + Y'_{(t-\tau)^+} = Y_{\tau\wedge t} + \mathbb{1}_{\{t\geq\tau\}}Y'_{t-\tau} + \mathbb{1}_{\{t<\tau\}}Y'_0 = Y_{\tau\wedge t} + \mathbb{1}_{\{t\geq\tau\}}Y'_{t-\tau}$$

and $\tilde{Y}$ by

$$\tilde{Y}_t = Y_{\tau\wedge t} - Y'_{(t-\tau)^+} = Y_{\tau\wedge t} - \mathbb{1}_{\{t\geq\tau\}}Y'_{t-\tau} - \mathbb{1}_{\{t<\tau\}}Y'_0 = Y_{\tau\wedge t} - \mathbb{1}_{\{t\geq\tau\}}Y'_{t-\tau}.$$

[1] In this section, we distinguish between the stochastic process v_t and a deterministic function $\beta(t)$ as the integrand of Y. Although v_t is stochastic in the Heston model setup, we make use of the results established in this section for a process Y with a deterministic integrand by conditioning on the variance paths.

Note that, the random variables Y'_s are defined for $s \geq -\tau$. Since $Y'_{t-\tau} = Y_t - Y_\tau$, $\{\tau \leq t\} \in \mathcal{F}_\tau$ and

$$\exp\left(iu(Y_t - Y_\tau)\mathbb{1}_{\{\tau\leq t\}}\right) = 1 - \mathbb{1}_{\{\tau\leq t\}} + \mathbb{1}_{\{\tau\leq t\}} \exp\left(iu(Y_t - Y_\tau)\right),$$

we have almost surely that

$$\begin{aligned}\mathbb{E}\left[\exp\left(iuY'_{(t-\tau)^+}\right) \mid \mathcal{F}_\tau\right] &= 1 - \mathbb{1}_{\{\tau\leq t\}} + \mathbb{1}_{\{\tau\leq t\}}\mathbb{E}[\exp\left(iu(Y_t - Y_\tau)\right) \mid \mathcal{F}_\tau] \\ &= 1 - \mathbb{1}_{\{\tau\leq t\}} + \mathbb{1}_{\{\tau\leq t\}}\mathbb{E}[\exp\left(-iu(Y_t - Y_\tau)\right) \mid \mathcal{F}_\tau] \\ &= \mathbb{E}\left[\exp\left(iu(-Y'_{(t-\tau)^+})\right) \mid \mathcal{F}_\tau\right], \end{aligned} \tag{11}$$

where in the second equation we used the fact that $(Y_t - Y_\tau)$ conditioned on $\mathcal{F}_\tau$ is normally distributed with mean zero for $t > \tau$ (justified by lemma 3 and remark 2 in the appendix A.1).
Because the equality in (11) is true for all $t \geq 0$ we can conclude that

$$\begin{aligned}\mathbb{E}\left[\exp\left(iuY'_{(t-\tau)^+}\right)\right] &= \mathbb{E}\left[\mathbb{E}\left[\exp\left(iuY'_{(t-\tau)^+}\right) \mid \mathcal{F}_\tau\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\exp\left(iu(-Y'_{(t-\tau)^+})\right) \mid \mathcal{F}_\tau\right]\right] \\ &= \mathbb{E}\left[\exp\left(iu(-Y'_{(t-\tau)^+})\right)\right].\end{aligned}$$

The first part of the assertion follows thereafter.

Secondly, using the first part of the proof, $Y \overset{\mathrm{d}}{=} \tilde{Y}$, it immediately follows that

$$\mathbb{P}(M_t \geq x, Y_t \leq y) = \mathbb{P}(\tilde{Y}_t \geq 2x - y) = \mathbb{P}(Y_t \geq 2x - y).$$

□

In section 4, we make use of the above derived reflection principle in order to determine the distribution of the first hitting time of the process Y and in turn the joint distribution of (Y, M) under the conditional filtration $\mathcal{G}_T^v$ and the probability measure $\mathbb{P}^v$. Then, we tackle the problem of finding the joint distribution of the drift included process $(\alpha + Y, \hat{M})$.

4 Second Step: Derivation of the Joint Density

The valuation of continuous barrier options depends on the distribution of the first hitting time. In the Heston model, the derivation of this distribution under the knowledge of the variance paths up until maturity is as straightforward as in the Black-Scholes model if $r_d = r_f$ and $\rho = 0$. This case will be discussed in section 4.1 and we follow closely the theorems 3.7.1, 3.7.3 and 7.2.1 given in Shreve [15]. In section 4.2, we develop a (semi-)closed approximation formula for the case $\rho = 0$ but with arbitrary domestic and foreign interest rates. Finally, the general case will be considered in section 4.3. The following two propositions will be useful for all of the three cases.

Proposition 1 *For $m \neq 0$ the random variable $\tau_m = \inf\{t \geq 0 : Y_t \geq m\}$ has the following distribution*

$$\mathbb{P}^v(M_t \geq m) = \mathbb{P}^v(\tau_m \leq t) = \frac{2}{\sqrt{2\pi}} \int_{\frac{|m|}{\sqrt{\rho_2^2 \nu^2(t)}}}^{+\infty} e^{-\frac{1}{2}y^2} dy.$$

Proof: First consider the case $m > 0$. Setting $x = y = m$ and utilizing the reflection principle stated in theorem 1 and the equality $\{M_t \geq m\} = \{\tau_m \leq t\}$ we obtain

$$\mathbb{P}^v(\tau_m \leq t, Y_t \leq m) = \mathbb{P}^v(Y_t \geq m).$$

On the other hand, if $Y_t \geq m$, then it is $\tau_m \leq t$, since the process Y starts at 0. Hence,

$$\mathbb{P}^v(\tau_m \leq t, Y_t \geq m) = \mathbb{P}^v(Y_t \geq m).$$

Adding the two equations gives the cumulative distribution of τ_m:

$$\begin{aligned}
\mathbb{P}^v(\tau_m \leq t) &= \mathbb{P}^v(\tau_m \leq t, Y_t \leq m) + \mathbb{P}^v(\tau_m \leq t, Y_t \geq m) \\
&= 2\mathbb{P}^v(Y_t \geq m) \\
&= \frac{2}{\sqrt{2\pi}\rho_2\nu(t)} \int_m^{\infty} e^{-\frac{1}{2}\frac{x^2}{\rho_2^2\nu^2(t)}} dx.
\end{aligned}$$

If $m < 0$, then $\tau_m \overset{d}{=} \tau_{|m|}$. □

From proposition 1 and the reflection principle,

$$\mathbb{P}^v(M_t \geq m, Y_t \leq w) = \mathbb{P}^v(Y_t \geq 2m - w) \qquad \text{for } w \leq m, m > 0, \tag{12}$$

we obtain the joint distribution of M and Y under the measure $\mathbb{P}^v$ as follows.

Proposition 2 *For $t > 0$ the joint distribution of (M_t, Y_t) is given by*

$$f_{M,Y}(m, w) = \frac{2(2m - w)}{\sqrt{2\pi}\rho_2^3\nu^3(t)} \exp\left(-\frac{1}{2}\frac{(2m - w)^2}{\rho_2^2\nu^2(t)}\right) \qquad \text{for } w \leq m, m > 0.$$

Proof: By definition, it is

$$\mathbb{P}^v(M_t \geq m, Y_t \leq w) = \int_m^{\infty} \int_{-\infty}^{w} f_{Y,M}(u, s) du ds$$

and by theorem 1 we know that

$$\mathbb{P}^v(Y_t \geq 2m - w) = \frac{1}{\sqrt{2\pi}\rho_2\nu(t)} \int_{2m-w}^{\infty} e^{-\frac{1}{2}\frac{y^2}{\rho_2^2\nu^2(t)}} dy.$$

From equation (12) it follows that

$$\int_m^\infty \int_{-\infty}^w f_{Y,M}(u,s)duds = \frac{1}{\sqrt{2\pi}\rho_2\nu(t)} \int_{2m-w}^\infty e^{-\frac{1}{2}\frac{y^2}{\rho_2^2\nu^2(t)}} dy.$$

Differentiation first with respect to m and then with respect to w leads to

$$-f_{Y,M}(m,w) = -\frac{2(2m-w)}{\sqrt{2\pi}\rho_2^3\nu^3(t)} \exp\left(-\frac{1}{2}\frac{(2m-w)^2}{\rho_2^2\nu^2(t)}\right).$$

□

Proposition 2 states the joint density $f_{Y,M}$ of Y and M. In order to solve the barrier pricing problem, we established in section 2 that one important part of the solution is to find the unknown joint distribution $f_{\hat{Y},\hat{M}}$ of Y and M with drift under the conditional filtration $\mathcal{G}_T^v$ and the probability measure $\mathbb{P}^v$ (see equation (8)). This problem is treated differently for all three cases A, B, C, that were outlined in the introduction.

4.1 Case $r_d = r_f$ and $\rho = 0$:

In the following, under the assumption that $r_d = r_f$ and $\rho = 0$, the joint density for the process $\alpha(t) + Y_t$ and its maximum is derived. To this end, we recall that under $\mathbb{P}^v$, the process Y_t has no drift and the joint density of (Y, M) was derived under $\mathbb{P}^v$. By contrast, $\hat{Y}_t = \alpha(t) + Y_t$ has drift $\frac{d}{dt}\alpha(t)$ under $\mathbb{P}^v$. To use the propositions above, we need to find a new probability measure $\hat{\mathbb{P}}$ under which $\hat{Y}$ has a drift zero.

Proposition 3 *The joint density of $(\hat{M}_T, \hat{Y}_T)$ under $\mathbb{P}^v$ is given by*

$$f_{\hat{M},\hat{Y}}(m,w) = \exp\left(-\frac{1}{2}w - \frac{1}{8}\nu^2(T)\right) \frac{2(2m-w)}{\sqrt{2\pi}\nu^3(T)} e^{-\frac{1}{2}\frac{(2m-w)^2}{\nu^2(T)}} \qquad \text{for } w \le m, m > 0. \quad (13)$$

Proof: Since at time $t = 0$, the process $\hat{Y}_t$ takes on the value 0, we obviously have $\hat{M}_t \ge 0$ and $\hat{M}_t \ge \hat{Y}_t$. Hence, the pair of random variables $(\hat{M}_t, \hat{Y}_t)$ takes values in the set $\{(m,w) : w \le m, m \ge 0\}$.

For $\rho = 0$ the process Y satisfies

$$dY_t = \sqrt{v_t} dW_t,$$

where W_t is a Brownian motion under the probability measure $\mathbb{P}^v$ with zero drift. Since $\alpha(t) + Y_t$ where $\alpha(t) = -\frac{1}{2}\int_0^t v_s ds$, we have

$$d\hat{Y}_t = d\alpha(t) + \sqrt{v_t}dW_t = \sqrt{v_t}d\hat{W}_t,$$

where

$$d\hat{W}_t = dW_t + \frac{d\alpha(t)}{\sqrt{v_t}} = dW_t + \gamma(t)dt \quad \text{and} \quad \gamma(t) = -\frac{1}{2}\sqrt{v_t}.$$

Hence, $\hat{W}_t = \int_0^t \gamma(s)ds + W_t$ is a Brownian motion under $\mathbb{P}^v$ with drift $\gamma(t)$. We define the exponential martingale

$$\hat{H}_t = \exp\left(-\int_0^t \gamma(s)dW_s - \frac{1}{2}\int_0^t \gamma^2(s)ds\right) = \exp\left(-\int_0^t \gamma(s)d\hat{W}_s + \frac{1}{2}\int_0^t \gamma^2(s)ds\right)$$

and the new probability measure $\hat{\mathbb{P}}$ by

$$\hat{\mathbb{P}}(A) = \int_A \hat{H}_T d\mathbb{P}^v \qquad \text{for all } A \in \mathcal{G}_T^v.$$

Conditioned on the sigma-algebra generated by the variance paths, Novikov's condition,

$$\mathbb{E}\Big[\exp\Big\{\frac{1}{2}\int_0^T \Big(-\frac{1}{2}v_s\Big)^2 ds\Big\} \Bigm| \mathcal{G}_T^v\Big] < \infty,$$

is immediately satisfied, because $+\infty$ is a natural boundary, i.e. cannot be reached by the process v_t in finite time (cf. Borodin and Salminen, Chapter II). According to Girsanov's theorem, $\hat{W}_t$ is a Brownian motion with zero drift under $\hat{\mathbb{P}}$. Proposition 2 gives us now the joint density of $(\hat{M}, \hat{Y})$ under $\hat{\mathbb{P}}$, which is $\hat{f}_{\hat{M},\hat{Y}}$. To work out the density of $(\hat{M}, \hat{Y})$ under $\mathbb{P}^v$ we find that

$$\begin{aligned}
\mathbb{P}^v[\hat{M}_T \leq m, \hat{Y}_T \leq w] &= \mathbb{E}^v\left[\mathbb{1}_{\{\hat{M}_T \leq m, \hat{Y}_T \leq w\}}\right] \\
&= \hat{\mathbb{E}}\left[\frac{1}{\hat{H}_T}\mathbb{1}_{\{\hat{M}_T \leq m, \hat{Y}_T \leq w\}}\right] \\
&= \hat{\mathbb{E}}\left[\exp\left(\int_0^T \gamma(s)d\hat{W}_s - \frac{1}{2}\int_0^T \gamma^2(s)ds\right)\mathbb{1}_{\{\hat{M}_T \leq m, \hat{Y}_T \leq w\}}\right] \\
&= \hat{\mathbb{E}}\left[\exp\left(-\frac{1}{2}\int_0^T \sqrt{v_s}d\hat{W}_s - \frac{1}{8}\int_0^T v_s ds\right)\mathbb{1}_{\{\hat{M}_T \leq m, \hat{Y}_T \leq w\}}\right] \\
&= \int_{-\infty}^w \int_{-\infty}^m \exp\left(-\frac{1}{2}y - \frac{1}{8}\nu^2(T)\right)\hat{f}_{\hat{M}_T,\hat{Y}_T}(x,y)dxdy.
\end{aligned}$$

Therefore, the density of $(\hat{M}, \hat{Y})$ under $\mathbb{P}^v$ is

$$\frac{\partial^2}{\partial m \partial w}\mathbb{P}^v[\hat{M}_T \leq m, \hat{Y}_T \leq w] = \exp\left(-\frac{1}{2}w - \frac{1}{8}\nu^2(T)\right)\frac{2(2m-w)}{\sqrt{2\pi\nu^3(T)}}e^{-\frac{1}{2}\frac{(2m-w)^2}{\nu^2(T)}}.$$

When $w \leq m$ and $m \geq 0$, this is formula (13). For other values of m and w, we obtain zero because $\hat{f}_{\hat{M}_T,\hat{Y}_T}$ is zero. □

4.2 Case ρ=0 and $r_d \neq r_f$:

The same approach as in section 4.1 can be used up to proposition 3, since

$$\alpha(t) = (r_d - r_f)t - \frac{1}{2}\int_0^t v_s\, ds$$

is still differentiable. However, $\alpha(t)$ is different than in the section before, hence the drift of the new Brownian motion $\hat{W}$ must be newly defined as well,

$$\gamma(s) \;=\; \frac{(r_d - r_f)}{\sqrt{v_s}} - \frac{1}{2}\sqrt{v_s}.$$

Unfortunately, the term $\sqrt{v_s}$ in the denominator prevents us from applying exactly the same techniques as in the case of section 4.1. Proceeding as in proposition 3 yields

$$\begin{aligned}\mathbb{P}^v[\hat{M}_T \le m, \hat{Y}_T \le w] \;&=\; \hat{\mathbb{E}}\Big[\exp\Big(\int_0^T \gamma(s) d\hat{W}_s - \frac{1}{2}\int_0^T \gamma^2(s)ds\Big)\mathbb{1}_{\{\hat{M}_T\le m, \hat{Y}_T \le w\}}\Big] \\ &=\; \hat{\mathbb{E}}\Big[\exp\Big((r_d - r_f)\hat{X}_T - \frac{1}{2}\hat{Y}_T - \frac{1}{2}\int_0^T \gamma^2(s)ds\Big)\mathbb{1}_{\{\hat{M}_T\le m, \hat{Y}_T \le w\}}\Big], \quad (14)\end{aligned}$$

with

$$\hat{X}_T \;=\; \int_0^T \frac{1}{\sqrt{v_s}} d\hat{W}_s.$$

The expectation in (14) contains three random variables $\hat{X}_T, \hat{Y}_T$ and $\hat{M}_T$. Therefore, to compute the distribution function of $(\hat{M}_T, \hat{Y}_T)$ under $\mathbb{P}^v$ we need the joint density of $(\hat{X}_T, \hat{Y}_T, \hat{M}_T)$ under $\hat{\mathbb{P}}$,

$$\hat{f}_{\hat{X}_T, \hat{Y}_T, \hat{M}_T} \;=\; ?.$$

However, it seems difficult to derive this density by means of the reflection principle (as done for the joint density of $(\hat{Y}_T, \hat{M}_T)$ under $\hat{\mathbb{P}}$), and for that reason we decide to approximate the random variable $\hat{X}_T$ in terms of $\hat{Y}_T$. First, we outline the idea before going into details. Abbreviating

$$r = r_d - r_f, \qquad b_I(t) = \frac{1}{2}\int_0^t \gamma^2(s)ds = \frac{1}{2}\int_0^t \left(\frac{(r_d - r_f)^2}{v_s} - (r_d - r_f) + \frac{1}{4}v_s\right) ds$$

we can write

$$\mathbb{P}^v[\hat{M}_T \le m, \hat{Y}_T \le w] \;=\; \iint_{\mathbb{R}^2}\int_0^\infty e^{rx - \frac{1}{2}y - b_I(T)}\mathbb{1}_{\{z\le m, y\le w\}}\hat{f}_{\hat{X}_T, \hat{Y}_T, \hat{M}_T}(x,y,z)\, dz\, dy\, dx.$$

Using conditional densities and interchanging integrations yields

$$= \int_{\mathbb{R}}\int_0^\infty \exp\left(-\frac{1}{2}y - b_I(T)\right)\mathbb{1}_{\{z\le m, y\le w\}}\hat{f}_{\hat{Y}, \hat{M}}(y,z)\hat{\mathbb{E}}[e^{r\hat{X}_T} \,|\, \hat{Y}_T = y, \hat{M}_T = z]\, dz\, dy.$$

A simple approximation of $\hat{X}_T$ that results in a tractable expression of (14) employs the fact that $(\hat{X}_T, \hat{Y}_T)$ is jointly normally distributed with zero mean and a certain covariance matrix as shown in the appendix A.2. Then there exists a representation of $\hat{X}_T$ in terms of $\hat{Y}_T$ and some independent (with respect to $\hat{Y}_T$) standard normal random variable U: $\hat{X}_T = \kappa_1\hat{Y}_T + \kappa_2 U$ (this representation is discussed in the next paragraph). The inner expectation then reads as

$$\hat{\mathbb{E}}[\exp(r\hat{X}_T) \,|\, \hat{Y}_T = y, \hat{M}_T = z] \;=\; \exp(r\kappa_1 y)\hat{\mathbb{E}}[\exp(r\kappa_2 U) \,|\, \hat{M}_T = z].$$

Now, the remaining conditional expected value will be approximated by taking the unconditional expectation, i.e. discarding the information $\hat{M}_T = z$. Intuitively, the random variable $\hat{M}_T$ (which refers to the path of $\hat{Y}_T$ only) should not contribute too much information to the value of $\hat{X}_T$ in excess of the information the random variable $\hat{Y}_T$ is contributing to $\hat{X}_T$. In the following we develop this approach formally.

Define ε by the equation $\hat{X}_T = \kappa_0 + \kappa_1 \hat{Y}_T + \varepsilon$, then the goal is to determine κ_0 and κ_1 such that $\hat{\mathbb{E}}[\varepsilon] = 0$ and $\hat{\mathbb{E}}[\varepsilon^2]$ is minimal. Obviously, we have to set $\kappa_0 = 0$ in order to satisfy the first constraint. With respect to the second constraint, we note that

$$\hat{\mathbb{E}}[\varepsilon^2] \;=\; \mathrm{Var}(\hat{X}_T) + \mathrm{Var}(\kappa_1 \hat{Y}_T) - 2\mathrm{Cov}(\hat{X}_T, \kappa_1 \hat{Y}_T) \;=\; \sigma^2_{\hat{X}} + \kappa_1^2 \sigma^2_{\hat{Y}} - 2\kappa_1 \sigma_{\hat{X},\hat{Y}},$$

with $\sigma^2_{\hat{X}} = \int_0^T \frac{1}{v_s} ds$, $\sigma^2_{\hat{Y}} = \int_0^T v_s ds$ and $\sigma^2_{\hat{X},\hat{Y}} = T$ as shown in proposition 4 in the appendix A.2. A minimization with respect to κ_1 yields

$$\kappa_1 \;=\; \frac{\sigma_{\hat{X},\hat{Y}}}{\sigma^2_{\hat{Y}}} \;=\; \frac{T}{\nu^2(T)}.$$

In proposition 5 in the appendix A.2, we show that the random variable $(\hat{X}_T, \kappa_1 \hat{Y}_T)$ is normally distributed with zero mean and a certain covariance matrix, and that ε is normal with zero mean and variance $\sigma^2_\varepsilon = \int_0^T \frac{1}{v_s} ds - \frac{T^2}{\nu^2(T)}$. In the spirit of the definition for ν^2, we define

$$\nu^2_{\mathrm{inv}}(T) = \int_0^T \frac{1}{v_s} ds.$$

Since in the notation introduced above $\varepsilon = \kappa_2 U$, we have

$$\kappa_2 \;=\; \sigma_\varepsilon \;=\; \sqrt{\nu^2_{\mathrm{inv}}(T) - \frac{T^2}{\nu^2(T)}}.$$

It is easy to check that $\mathrm{Cov}(\hat{Y}_T, \varepsilon) = 0$, hence U is independent from $\hat{Y}_T$. Consequently, the just derived expression for $\hat{X}_T$ equates to its Cholesky decomposition with respect to $\frac{1}{\nu(T)} \hat{Y}_T$ and an independent standard normal random variable U.

Now, substituting $\kappa_1 \hat{Y}_T + \kappa_2 U$ for $\hat{X}_T$ in (14) yields

$$\begin{aligned}\mathbb{P}^v[\hat{M}_T \le m, \hat{Y}_T \le w] \;&=\; \hat{\mathbb{E}}\Big[\exp\Big(r\hat{X}_T - \frac{1}{2}\hat{Y}_T - b_I(T)\Big)\mathbb{1}_{\{\hat{M}_T \le m, \hat{Y}_T \le w\}}\Big] \\ &=\; \hat{\mathbb{E}}\Big[\exp\Big(r\left(\kappa_1 \hat{Y}_T + \kappa_2 U\right) - \frac{1}{2}\hat{Y}_T - b_I(T)\Big)\mathbb{1}_{\{\hat{M}_T \le m, \hat{Y}_T \le w\}}\Big]\end{aligned}$$

Due to the independence of $\hat{Y}_T$ and U, but ignoring the dependence of $\hat{M}_T$ and U, we get

$$\begin{aligned}\mathbb{P}^v[\hat{M}_T \le m, \hat{Y}_T \le w] \;&\approx\; \hat{\mathbb{E}}[\exp(r\kappa_2 U)] \times \hat{\mathbb{E}}\Big[\exp\Big(\Big(r\kappa_1 - \frac{1}{2}\Big)\hat{Y}_T - b_I(T)\Big)\mathbb{1}_{\{\hat{M}_T \le m, \hat{Y}_T \le w\}}\Big] \\ &=\; e^{\frac{1}{2}\kappa_2^2 r^2} \int_{-\infty}^{w} \int_{-\infty}^{m} \exp\Big(\Big(r\kappa_1 - \frac{1}{2}\Big)y - b_I(T)\Big)\, \hat{f}_{\hat{M}_T, \hat{Y}_T}(x, y)\, dx\, dy.\end{aligned}$$

Therefore, the density of $(\hat{M}, \hat{Y})$ under $\mathbb{P}^v$ is given by

$$\frac{\partial^2}{\partial m \partial w}\mathbb{P}^v[\hat{M}_T \leq m, \hat{Y}_T \leq w] \quad \approx \quad \frac{2(2m-w)}{\sqrt{2\pi}\nu^3(T)} e^{\left(r\kappa_1 - \frac{1}{2}\right)w - b_1(T) - \frac{1}{2}\frac{(2m-w)^2}{\nu^2(T)}},$$

where $b_1(T) = \frac{1}{2}\left[(r_d - r_f)^2 \frac{T^2}{\nu^2(T)} - (r_d - r_f)T + \frac{1}{4}\nu^2(T)\right]$.

Remark 1 *Another possible approach is to define the Radon-Nikodym derivative as in section 4.2, i.e.*

$$\frac{d\mathbb{P}}{d\hat{\mathbb{P}}} \quad = \quad \exp\left(-\frac{1}{2}\int_0^T \sqrt{v_s}d\hat{W}_s - \frac{1}{8}\int_0^T v_s^2 ds\right).$$

As a consequence the maximum process $\hat{M}_t$ refers to a process $\hat{Y}_t$ with non-zero drift $\frac{(r_d - r_f)}{\sqrt{v_t}}$. This drift is even non-linear, and to date no closed formula has been found for barrier hitting times of Brownian motions with non-linear drift. A huge amount of literature deals with this problem, since it has important applications in other areas than financial mathematics (e.g. sequential tests in mathematical statistics) and therefore different approximation methods were developed for the calculation of the hitting time distribution (see [5] for an overview). However, these methods require at least several values v_s for times $s \in [0, T]$, and therefore cause an equally large number of nested integrals when resolving the conditioning on the volatility path.

4.3 Case $r_d \neq r_f$ and $\rho \neq 0$

Allowing for $\rho \neq 0$ complicates the pricing problem even further compared to the cases before. Again we are looking for a Girsanov's transformation in order to eliminate the drift $\alpha(t)$ of $\hat{Y}$ and to proceed the pricing procedure as in the Black-Scholes case. Though $\alpha(t)$ is still deterministic under $\mathbb{P}^v$ in this general case, $d\alpha(t)$ is not of order dt anymore, as the term v_t appears in the drift which is a realization of the Heston variance process[2].

However, we decide to follow up on the approximation approach taken in the section before and additionally assume a sufficiently smooth variance around v_t such that the derivative with respect to t can be taken. The following theorem shows that any differentiable approximation of v_t can be chosen to define the probability measure $\hat{\mathbb{P}}$ and using this measure to derive the joint density $\hat{f}_{\hat{M},\hat{Y}}$ will always lead to the same result as long as the approximation coincides at time $t = 0$ and $t = T$. The necessity of resolving the conditioning on the variance path in the end suggests to additionally require that the approximation is equal to the original variance path at time 0 and the option's maturity T. Then the characteristic function for the random variable $(v_T, \nu^2(T))$ allows a fast calculation of the corresponding density function. At the end of this section, we furthermore justify this particular choice of

[2] One possible approach is to apply Itô's formula instead of the standard derivative to $\alpha(t)$ and then substitute the arising dv_t term by the stochastic differential equation from the model definition in (1). But in that case neither of the introduced Brownian motions, W_t^v nor W_t^S (see also equations (3) and (4)), are any longer independent from the variance path, which rules out our technique of conditioning on the variance path.

probability measure by showing that the exercise probability of a call option is valued correctly under this approach.

Theorem 2 *Consider the general case with r_d, r_f and ρ arbitrary. Let $\bar{v}_t$ be any differentiable approximation for v_t (i.e. for which $d\bar{v}_t/dt$ exists), satisfying $\bar{v}_0 = v_0$ and $\bar{v}_T = v_T$. Then the density of $(\hat{M}, \hat{Y})$ under $\mathbb{P}^v$ is given by*

$$\hat{f}_{\hat{M},\hat{Y}}(m,w) \approx \frac{2(2m-w)}{\sqrt{2\pi}\rho_2^3\nu^3(T)} \exp\left(-\frac{1}{2}\frac{(2m-w)^2}{\rho_2^2\nu^2(T)}\right) \exp\left(\frac{c_1\kappa_1 + c_2 + c_3\kappa_3}{\rho_2^2} w - b_2(T)\right)$$

with $b_2(T) = \frac{1}{2\rho_2^2}\left[c_1^2\frac{T^2}{\nu^2(T)} + c_3^2\frac{(v_T-v_0)^2}{\nu^2(T)} + 2c_1c_3\frac{(v_T-v_0)T}{\nu^2(T)} + c_2^2\nu^2(T) + 2c_1c_2T + 2c_2c_3(v_T-v_0)\right]$ *and* c_1, c_2 *and* c_3 *given by*

$$c_1 = \left((r_d - r_f) - \frac{\kappa\theta\rho}{\sigma}\right), \quad c_2 = \left(\frac{\kappa\rho}{\sigma} - \frac{1}{2}\right), \quad \text{and } c_3 = \frac{\rho}{\sigma}. \tag{15}$$

Proof: Recall the logarithmic spot to be $x_t^\nu = x_0 + \alpha(t) + Y_t$ with the deterministic drift

$$\alpha(t) = (r_d - r_f)t - \frac{1}{2}\int_0^t v_s\,ds + \frac{\rho}{\sigma}\left(v_t - v_0 - \kappa\theta t + \kappa\int_0^t v_s\,ds\right).$$

In order to get a representation $\frac{d\alpha(t)}{\rho_2\sqrt{v_t}} = \gamma(t)dt$, we substitute $\bar{v}_t'$ for dv_t/dt, and define $\gamma(t)$ by

$$\begin{aligned}\gamma(t) &= \frac{1}{\rho_2\sqrt{v_t}}\left\{(r_d - r_f) - \frac{1}{2}v_t + \frac{\rho}{\sigma}\left(\bar{v}_t' - \kappa\theta + \kappa v_t\right)\right\} \\ &= \frac{1}{\rho_2}\left((r_d - r_f) - \frac{\kappa\theta\rho}{\sigma}\right)\frac{1}{\sqrt{v_t}} + \frac{1}{\rho_2}\left(\frac{\kappa\rho}{\sigma} - \frac{1}{2}\right)\sqrt{v_t} + \frac{1}{\rho_2}\frac{\rho}{\sigma}\frac{\bar{v}_t'}{\sqrt{v_t}}.\end{aligned}$$

We see that γ is composed out of three terms, the term with $\sqrt{v_t}$, the term with $1/\sqrt{v_t}$ and the one with $\frac{\bar{v}_t'}{\sqrt{v_t}}$. A gamma function involving terms of the first two types was treated in the cases 4.1 and 4.2 before. The term with respect to $\frac{\bar{v}_t'}{\sqrt{v_t}}$ in the gamma function is new and we have to determine how it changes the approach taken before.
First, we repeat the definitions for $\hat{Y}$ and $\hat{X}$ and note that we need to take the factor ρ_2 into account of our calculations for this case. For case A and B, the variable ρ_2 disappears due to ρ being equal to zero.

$$\begin{aligned}\hat{Y}_t &= \rho_2\int_0^t \sqrt{v_s}d\hat{W}_s, \\ \hat{X}_t &= \rho_2\int_0^t \frac{1}{\sqrt{v_s}}d\hat{W}_s.\end{aligned}$$

Next we define

$$\hat{Z}_t = \rho_2\int_0^t \frac{\bar{v}_s'}{\sqrt{v_s}}d\hat{W}_s,$$

and in the spirit of the definitions of $\nu(T)$ and $\nu_{\text{inv}}(T)$, we further define

$$\nu_I = \sqrt{\int_0^T \frac{\bar{v}_s'^2}{v_s} ds}, \quad \nu_{II} = \sqrt{\int_0^T \frac{\bar{v}_s'}{v_s} ds}, \quad \nu' = \sqrt{\int_0^T \bar{v}_s' ds},$$

Accordingly, we assume that these integrals exist. Then the distribution of the random variable $(\hat{X}_T, \hat{Y}_T, \hat{Z}_T)$ is a standard normal one with zero mean and a covariance matrix given in proposition 5 in the appendix A.2.

Again, we find constants κ_1 and κ_3 for the Cholesky decomposition such that

$$\hat{X}_T = \kappa_1 \hat{Y}_T + \varepsilon^X \quad \text{and} \quad \hat{Z}_T = \kappa_3 \hat{Y}_T + \varepsilon^Z,$$

with $\mathbb{E}[\varepsilon^X] = \mathbb{E}[\varepsilon^Z] = 0$ and each $\mathbb{E}[(\varepsilon^X)^2]$ and $\mathbb{E}[(\varepsilon^Z)^2]$ is minimal. A straightforward solution found as in section 4.2 yields

$$\kappa_1 = \frac{T}{\nu^2(T)} \quad \text{and} \quad \kappa_3 = \frac{(\nu'(T))^2}{\nu^2(T)} = \frac{v_T - v_0}{\nu^2(T)}.$$

The random variable $(\varepsilon^X, \varepsilon^Z)$ is normally distributed with zero mean and covariance matrix as given in proposition 5. Furthermore, it is independent of $\hat{Y}_T$.

Finally, we approximate the distribution function $\mathbb{P}^v[\hat{M}_T \le m, \hat{Y}_T \le w]$. A Girsanov transformation that eliminates the drift $\alpha(T)$ of $\hat{Y}_T$ gives the cumulative distribution function (CDF) for the random variable $(\hat{M}_T, \hat{Y}_T)$ under the $\mathbb{P}^v$-measure,

$$\mathbb{P}^v[\hat{M}_T \le m, \hat{Y}_T \le w] \quad = \quad \hat{\mathbb{E}}\Big[\exp\Big\{-\frac{1}{2}\int_0^T \gamma^2(s)ds + \int_0^T \gamma(s)d\hat{W}_s\Big\}\mathbb{1}_{\{\hat{M}_T \le m, \hat{Y}_T \le w\}}\Big].$$

Abbreviating c_1, c_2 and c_3 by the terms given in (15), we proceed by computing the two integrals in the exponential function of the CDF. For the deterministic integral the expression becomes

$$\begin{aligned} b_I(T) &= -\frac{1}{2\rho_2^2}\Big[c_1^2\nu_{\text{inv}}^2(T) + c_2^2\nu^2(T) + c_3^2\int_0^T \frac{(\bar{v}_s')^2}{v_s}ds + 2c_1c_2T + 2c_1c_3\int_0^T \frac{\bar{v}_s'}{v_s}ds + 2c_2c_3\int_0^T \bar{v}_s' ds\Big] \\ &= -\frac{1}{2\rho_2^2}\Big[c_1^2\nu_{\text{inv}}^2(T) + c_2^2\nu^2(T) + c_3^2\nu_I^2(T) + 2c_1c_2T + 2c_1c_3\nu_{II}^2(T) + 2c_2c_3(v_T - v_0)\Big]. \end{aligned}$$

Now, the stochastic integral yields

$$\frac{c_1}{\rho_2^2}\int_0^T \rho_2\frac{1}{\sqrt{v_s}}d\hat{W}_s + \frac{c_2}{\rho_2^2}\int_0^T \rho_2\sqrt{v_s}d\hat{W}_s + \frac{c_3}{\rho_2^2}\int_0^T \rho_2\frac{\bar{v}_s'}{\sqrt{v_s}}d\hat{W}_s \quad = \quad \frac{c_1}{\rho_2^2}\hat{X}_T + \frac{c_2}{\rho_2^2}\hat{Y}_T + \frac{c_3}{\rho_2^2}\hat{Z}_T.$$

Therefore,

$$\begin{aligned} \mathbb{P}^v[\hat{M}_T \le m, \hat{Y}_T \le w] &= e^{b_I(T)}\hat{\mathbb{E}}\left[\exp\left(\frac{c_1}{\rho_2^2}\hat{X}_T + \frac{c_2}{\rho_2^2}\hat{Y}_T + \frac{c_3}{\rho_2^2}\hat{Z}_T\right)\mathbb{1}_{\{\hat{M}_T \le m, \hat{Y}_T \le w\}}\right] \\ &= e^{b_I(T)}\hat{\mathbb{E}}\left[e^{\left(c_1(\kappa_1\hat{Y}_T + \varepsilon^X) + c_2\hat{Y}_T + c_3(\kappa_3\hat{Y}_T + \varepsilon^Z)\right)/\rho_2^2}\mathbb{1}_{\{\hat{M}_T \le m, \hat{Y}_T \le w\}}\right]. \end{aligned}$$

Due to the independence of $(\varepsilon^X, \varepsilon^Z)$ from $\hat{Y}_T$, we obtain

$$\begin{aligned}\mathbb{P}^v[\hat{M}_T \leq m, \hat{Y}_T \leq w] &\approx \hat{\mathbb{E}}\left[\exp\left((c_1\varepsilon^X + c_3\varepsilon^Z)/\rho_2^2\right)\right] \\ &\quad \times e^{b_I(T)}\hat{\mathbb{E}}\left[\exp\left(\hat{Y}_T\left(c_1\kappa_1 + c_2 + c_3\kappa_3\right)/\rho_2^2\right)\mathbb{1}_{\{\hat{M}_T \leq m, \hat{Y}_T \leq w\}}\right] \\ &= e^{b_I(T)+b_{II}(T)}\int_{-\infty}^{w}\int_{-\infty}^{m} e^{y(c_1\kappa_1+c_2+c_3\kappa_3)/\rho_2^2}\hat{f}_{\hat{M}_T,\hat{Y}_T}(x,y)dxdy.\end{aligned}$$

with $b_{II}(T) = \frac{1}{2\rho_2^2}\left[c_1^2\left(\nu_{\text{inv}}^2(T) - \frac{T^2}{\nu^2(T)}\right) + c_3^2\left(\nu_I^2(T) - \frac{(v_T - v_0)^2}{\nu^2(T)}\right) + 2c_1c_3\left(\nu_{II}^2(T) - \frac{(v_T - v_0)T}{\nu^2(T)}\right)\right]$.

Now, the density of $(\hat{M}, \hat{Y})$ under $\mathbb{P}^v$ is given by

$$\begin{aligned}\frac{\partial^2}{\partial m \partial w}\mathbb{P}^v[\hat{M}_T \leq m, \hat{Y}_T \leq w] &\approx \frac{2(2m-w)}{\sqrt{2\pi}\rho_2^3\nu^3(T)}\exp\left(-\frac{1}{2}\frac{(2m-w)^2}{\rho_2^2\nu^2(T)}\right) \\ &\quad \times \exp\left(\frac{c_1\kappa_1 + c_2 + c_3\kappa_3}{\rho_2^2}w - b_2(T)\right)\end{aligned}$$

where $-b_2(T) = b_I(T) + b_{II}(T)$. □

We conclude this section by arguing that the above introduced measure change according to γ is an appropriate choice, since it provides us with the potential to eliminate the drift $\alpha(t)$ of $\hat{Y}$, which allows us to make use of the analytical format of $f_{Y,M}$. Moreover, it features the pleasing property that the exercise probability $\mathbb{P}[S_T > K]$ of a call (which is inherent in the up-and-out barrier call) is priced correctly under this approach. The exact value of $\mathbb{P}[S_T > K]$ under $\mathcal{G}^v$ is equal to

$$\begin{aligned}\mathbb{P}^v[S_T > K] &= \mathbb{P}^v[x_T^v > \ln(K)] \\ &= \mathbb{P}^v\left[x_0 + \alpha(T) + \rho_2\int_0^T \sqrt{v_s}dW_s > \ln(K)\right] \\ &= 1 - \mathbb{P}^v[\hat{Y}_T < \ln(K) - x_0] \\ &= 1 - N\left(\frac{\ln(K/S) - \alpha(T)}{\rho_2\nu(T)}\right),\end{aligned}$$

since $\hat{Y}_T$ is normally distributed with mean $\alpha(T)$ and variance $\rho_2^2\nu^2(T)$ under the measure $\mathbb{P}^v$. Therefore,

$$\mathbb{P}^v[S_T > K] = 1 - N\left(\frac{\ln(K/S) - (r_d - r_f)T - \left(\frac{\kappa\rho}{\sigma} - \frac{1}{2}\right)\nu^2 - \frac{\rho}{\sigma}(v_T - v_0) + \frac{\kappa\theta\rho}{\sigma}T}{\rho_2\nu(T)}\right). \quad (16)$$

Now, with the approach developed in theorem 2, we obtain the following

$$\begin{aligned}\mathbb{P}^v[S_T > K] &= 1 - \mathbb{P}^v[\hat{Y}_T < \ln(K) - x_0] \\ &= 1 - \mathbb{E}^v\left[\mathbb{1}_{\{\hat{Y}_T < \ln(K/S)\}}\right] \\ &= 1 - \hat{\mathbb{E}}\left[\exp\left(-\frac{1}{2}\int_0^T \gamma^2(s)ds + \int_0^T \gamma(s)d\hat{W}\right)\mathbb{1}_{\{\hat{Y}_T < \ln(K/S)\}}\right].\end{aligned}$$

Under $\hat{\mathbb{P}}$, the Itô-integral $\hat{Y}_T$ is normally distributed with zero mean and variance equal to $\rho_2^2\nu^2(T)$. Thus,

$$\mathbb{P}^v[S_T > K] \quad = \quad 1 - e^{b_I(T)+b_{II}(T)} \int_{-\infty}^{\ln(K/S)} \exp\left(yq/(\rho_2^2\nu^2(T))\right) \hat{f}_{\hat{Y}}(y)dy$$

with density $\hat{f}_{\hat{Y}}$ equal to the normal probability distribution function and

$$\begin{aligned} q &= c_1T + c_2\nu^2(T) + c_3(v_T - v_0) \\ &= (r_d - r_f)T + \left(\frac{\kappa\rho}{\sigma} - \frac{1}{2}\right)\nu^2(T) + \frac{\rho}{\sigma}(v_T - v_0) - \frac{\kappa\theta\rho}{\sigma}T. \end{aligned}$$

Some simple algebra and integration by substitution leads to the correct result given in (16). Note that, in this case the complication with respect to the entanglement of the processes $\hat{M}$ and $\hat{X}$ do not arise. Hence, we are able to derive closed-form and exact expressions for the exercise probability $\mathbb{P}^v[S_T > K]$ of the call.

5 Third Step: Up-and-out Call Formula

In this section, we solve the inner expectation of the pricing problem for the continuous barrier option as stated in equation (8) as

$$\mathcal{E}^v \quad = \quad \mathbb{E}\left[(S_0e^{\hat{Y}_T} - K)\mathbb{1}_{\{\hat{M}_T<b,\hat{Y}_T>k\}}\,|\,\mathcal{G}_T^v\right].$$

Therefore, we use the joint density of the random variables, $(\hat{Y}, \hat{M})$, that appears in the expectation under the conditional probability measure $\mathbb{P}^v$ derived in the previous section 4. We express the joint density in a unified format for the three cases A, B and C:

$$f_{\hat{M},\hat{Y}}(m,w) \cong \frac{2(2m-w)}{\sqrt{2\pi}\rho_2^3\nu^3(T)} e^{-\frac{1}{2}\frac{(2m-w)^2}{\rho_2^2\nu^2(T)}+Fw+G},$$

where

$$F = \begin{cases} -\frac{1}{2} & \text{for case A,} \\ r\kappa_1 - \frac{1}{2} & \text{for case B,} \\ (c_1\kappa_1 + c_2 + c_3\kappa_3)/\rho_2^2 & \text{for case C.} \end{cases} \quad \text{and} \quad G = \begin{cases} -\frac{1}{8}\nu^2(T) & \text{for case A,} \\ -b_1(T) & \text{for case B,} \\ -b_2(T) & \text{for case C.} \end{cases}$$

The quantities r, c_1, c_2, c_3 are dependent on the model parameters only, whereas the κ_1, κ_2, κ_3 and b_1, b_2 are additionally functions of time-integrated variance: ν and v_T. For case A the exact equality applies.

For all three cases, we find the integration limits of the inner expectation, apply integration by substitution, and express the remaining integrals in terms of the cumulative normal distribution function.

5.1 Solving the Inner Expectation

The integration limits of the inner expectation in (8) are given by $\{k \leq w \leq b, \max(w, 0) \leq m \leq b\}$ provided that $0 < S_0 < B$. It follows that

$$\begin{aligned}
\mathcal{E}^v &= \int_k^b \int_{w^+}^b (S_0 e^w - K) f_{\hat{M},\hat{Y}}(m, w)\, dm\, dw \\
&= \int_k^b \int_{w^+}^b (S_0 e^w - K) \frac{2(2m-w)}{\sqrt{2\pi}\rho_2^3 \nu^3(T)} \exp\left(Fw + G - \frac{1}{2}\frac{(2m-w)^2}{\rho_2^2\nu^2(T)}\right) dm\, dw.
\end{aligned}$$

Because

$$\frac{\partial}{\partial m} \exp\left(-\frac{(2m-w)^2}{2\rho_2^2\nu^2(T)}\right) = -\frac{2(2m-w)}{\rho_2^2\nu^2(T)} \exp\left(-\frac{(2m-w)^2}{2\rho_2^2\nu^2(T)}\right)$$

we can apply integration by substitution as follows

$$\begin{aligned}
&\frac{1}{\rho_2\nu(T)\sqrt{2\pi}} \int_{w^+}^b \frac{2(2m-w)}{\rho_2^2\nu^2(T)} \exp\left(-\frac{(2m-w)^2}{2\rho_2^2\nu^2(T)}\right) dm \\
&= -\frac{1}{\rho_2\nu(T)\sqrt{2\pi}} \int_{-\frac{(2w^+-w)^2}{2\rho_2^2\nu^2(T)}}^{-\frac{(2b-w)^2}{2\rho_2^2\nu^2(T)}} e^x dx = -\frac{1}{\rho_2\nu(T)\sqrt{2\pi}} \exp\left(-\frac{(2m-w)^2}{2\rho_2^2\nu^2(T)}\right)\Big|_{m=w^+}^{m=b}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{E}^v &= -\frac{1}{\sqrt{2\pi}\rho_2\nu(T)} \int_k^b (S_0 e^w - K) \exp\left(Fw + G - \frac{1}{2}\frac{(2m-w)^2}{\rho_2^2\nu^2(T)}\right)\Big|_{m=w^+}^{m=b} dw \\
&= \frac{1}{\sqrt{2\pi}\rho_2\nu(T)} \int_k^b (S_0 e^w - K) \exp\left(Fw + G - \frac{1}{2}\frac{w^2}{\rho_2^2\nu^2(T)}\right) dw \\
&\quad -\frac{1}{\sqrt{2\pi}\rho_2\nu(T)} \int_k^b (S_0 e^w - K) \exp\left(Fw + G - \frac{1}{2}\frac{(2b-w)^2}{\rho_2^2\nu^2(T)}\right) dw.
\end{aligned}$$

We set

$$\begin{aligned}
I_{1,x} &= \frac{1}{\sqrt{2\pi}\rho_2\nu(T)} e^G \int_k^b \exp\left(-\frac{1}{2}\left(-2(F+x)w + \frac{w^2}{\rho_2^2\nu^2(T)}\right)\right) dw \\
I_{2,x} &= \frac{1}{\sqrt{2\pi}\rho_2\nu(T)} e^{G - \frac{2b^2}{\rho_2^2\nu^2(T)}} \int_k^b \exp\left(-\frac{1}{2}\left(\left(-2(F+x) - \frac{4b}{\rho_2^2\nu^2(T)}\right) w + \frac{w^2}{\rho_2^2\nu^2(T)}\right)\right) dw,
\end{aligned}$$

with x taking on values in $\{0, 1\}$ and such that

$$\mathcal{E}^v = S_0 I_{1,1} - K I_{1,0} - S_0 I_{2,1} + K I_{2,0}.$$

It remains to solve four integrals of the form

$$\begin{aligned}\int_a^b e^{-\frac{1}{2}(cw+fw^2)}dw &= e^{\frac{1}{8}\frac{c^2}{f}}\int_a^b e^{-\frac{1}{2}f\left(w+\frac{1}{2}\frac{c}{f}\right)^2}dw\\ &= e^{\frac{1}{8}\frac{c^2}{f}}\frac{1}{\sqrt{f}}\int_{\sqrt{f}\left(a+\frac{1}{2}\frac{c}{f}\right)}^{\sqrt{f}\left(b+\frac{1}{2}\frac{c}{f}\right)} e^{-\frac{1}{2}y^2}dy\\ &= e^{\frac{1}{8}\frac{c^2}{f}}\frac{\sqrt{2\pi}}{\sqrt{f}}\left[N\left(\sqrt{f}\left(-a-\frac{1}{2}\frac{c}{f}\right)\right)-N\left(\sqrt{f}\left(-b-\frac{1}{2}\frac{c}{f}\right)\right)\right],\end{aligned}$$

for arbitrary constants a, b, c, f and using $N(z) = 1 - N(-z)$. We obtain

$$\begin{aligned}I_{1,x} &= \exp\left(\frac{1}{2}(F+x)^2\rho_2^2\nu^2(T)+G\right)\times\\ &\quad\left[N\left(\frac{\ln\left(\frac{S_0}{K}\right)+(F+x)\rho_2^2\nu^2(T)}{\rho_2\nu(T)}\right)-N\left(\frac{\ln\left(\frac{S_0}{B}\right)+(F+x)\rho_2^2\nu^2(T)}{\rho_2\nu(T)}\right)\right]\\ I_{2,x} &= \exp\left(\frac{1}{2}(F+x)^2\rho_2^2\nu^2(T)+G+2b(F+x)\right)\times\\ &\quad\left[N\left(\frac{\ln\left(\frac{B^2}{S_0K}\right)+(F+x)\rho_2^2\nu^2(T)}{\rho_2\nu(T)}\right)-N\left(\frac{\ln\left(\frac{B}{S_0}\right)+(F+x)\rho_2^2\nu^2(T)}{\rho_2\nu(T)}\right)\right].\end{aligned}$$

Finally, the value of an up-and-out call option in the Heston model at time $t = 0$ is given by

$$V_0 \cong e^{-r_dT}\mathbb{E}\left[S_0I_{1,1}-KI_{1,0}-S_0I_{2,1}+KI_{2,0}\right]. \tag{17}$$

For case A this is an exact equality, for the cases B and C it is an approximation.
Note that, both expressions, $I_{1,\cdot}$ and $I_{2,\cdot}$, remain to be functions of only two types of random variables when resolving the condition on the variance paths. For case A, B and C it is the time-integrated variance $\nu^2(T) = \int_0^T v_s ds$ and for case C it is additionally v_T.

Case A: $r_d = r_f$ and $\rho = 0$

$$\begin{aligned}I_{1,1} &= N\left(\frac{\ln\left(\frac{S_0}{K}\right)+\frac{1}{2}\nu^2(T)}{\nu(T)}\right)-N\left(\frac{\ln\left(\frac{S_0}{B}\right)+\frac{1}{2}\nu^2(T)}{\nu(T)}\right)\\ I_{1,0} &= N\left(\frac{\ln\left(\frac{S_0}{K}\right)-\frac{1}{2}\nu^2(T)}{\nu(T)}\right)-N\left(\frac{\ln\left(\frac{S_0}{B}\right)-\frac{1}{2}\nu^2(T)}{\nu(T)}\right)\\ I_{2,1} &= \frac{B}{S_0}N\left(\frac{\ln\left(\frac{B^2}{S_0K}\right)+\frac{1}{2}\nu^2(T)}{\nu(T)}\right)-\frac{B}{S_0}N\left(\frac{\ln\left(\frac{B}{S_0}\right)+\frac{1}{2}\nu^2(T)}{\nu(T)}\right)\\ I_{2,0} &= \frac{S_0}{B}N\left(\frac{\ln\left(\frac{B^2}{S_0K}\right)-\frac{1}{2}\nu^2(T)}{\nu(T)}\right)-\frac{S_0}{B}N\left(\frac{\ln\left(\frac{B}{S_0}\right)-\frac{1}{2}\nu^2(T)}{\nu(T)}\right).\end{aligned} \tag{18}$$

Case B: r_d, r_f arbitrary and $\rho = 0$

$$\begin{aligned}
I_{1,1} &= \exp\left((r_d - r_f)T\right) \times \\
&\left[N\left(\frac{\ln\left(\frac{S_0}{K}\right) + (r_d - r_f)T + \frac{1}{2}\nu^2(T)}{\nu(T)}\right) - N\left(\frac{\ln\left(\frac{S_0}{B}\right) + (r_d - r_f)T + \frac{1}{2}\nu^2(T)}{\nu(T)}\right)\right] \\
I_{1,0} &= N\left(\frac{\ln\left(\frac{S_0}{K}\right) + (r_d - r_f)T - \frac{1}{2}\nu^2(T)}{\nu(T)}\right) - N\left(\frac{\ln\left(\frac{S_0}{B}\right) + (r_d - r_f)T - \frac{1}{2}\nu^2(T)}{\nu(T)}\right) \\
I_{2,1} &= \exp\left(\left(2/\nu^2(T)\ln(B/S_0) + 1\right)(r_d - r_f)T\right) \times \\
\frac{B}{S_0} &\left[N\left(\frac{\ln\left(\frac{B^2}{S_0 K}\right) + (r_d - r_f)T + \frac{1}{2}\nu^2(T)}{\nu(T)}\right) - N\left(\frac{\ln\left(\frac{B}{S_0}\right) + (r_d - r_f)T + \frac{1}{2}\nu^2(T)}{\nu(T)}\right)\right] \\
I_{2,0} &= \exp\left(2/\nu^2(T)\ln(B/S_0)(r_d - r_f)T\right) \times \\
\frac{S_0}{B} &\left[N\left(\frac{\ln\left(\frac{B^2}{S_0 K}\right) + (r_d - r_f)T - \frac{1}{2}\nu^2(T)}{\nu(T)}\right) - N\left(\frac{\ln\left(\frac{B}{S_0}\right) + (r_d - r_f)T - \frac{1}{2}\nu^2(T)}{\nu(T)}\right)\right].
\end{aligned} \tag{19}$$

Case C: r_d, r_f arbitrary and ρ arbitrary

$$\begin{aligned}
I_{1,1} &= \exp\left(\frac{1}{2}\rho_2^2\nu^2(T) + q\right)\left[N\left(\frac{\ln\left(\frac{S_0}{K}\right) + q + \rho_2^2\nu^2(T)}{\rho_2\nu(T)}\right) - N\left(\frac{\ln\left(\frac{S_0}{B}\right) + q + \rho_2^2\nu^2(T)}{\rho_2\nu(T)}\right)\right] \\
I_{1,0} &= N\left(\frac{\ln\left(\frac{S_0}{K}\right) + q}{\rho_2\nu(T)}\right) - N\left(\frac{\ln\left(\frac{S_0}{B}\right) + q}{\rho_2\nu(T)}\right) \\
I_{2,1} &= \exp\left(\frac{1}{2}\rho_2^2\nu^2(T) + q\right)\left(\frac{B}{S_0}\right)^{\frac{2q}{\rho_2^2\nu^2(T)}+2} \\
&\times\left[N\left(\frac{\ln\left(\frac{B^2}{S_0 K}\right) + q + \rho_2^2\nu^2(T)}{\rho_2\nu(T)}\right) - N\left(\frac{\ln\left(\frac{B}{S_0}\right) + q + \rho_2^2\nu^2(T)}{\rho_2\nu(T)}\right)\right] \\
I_{2,0} &= \left(\frac{B}{S_0}\right)^{\frac{2q}{\rho_2^2\nu^2(T)}}\left[N\left(\frac{\ln\left(\frac{B^2}{S_0 K}\right) + q}{\rho_2\nu(T)}\right) - N\left(\frac{\ln\left(\frac{B}{S_0}\right) + q}{\rho_2\nu(T)}\right)\right].
\end{aligned} \tag{20}$$

with $q = c_1 T + c_2\nu^2(T) + c_3(v_T - v_0)$.

Hence, the last step in section 6, which deals with resolving the conditioning on the variance paths, pays attention to finding the distributions of the remaining random variables.

6 Fourth Step: Resolve Conditioning

After analytically solving the inner expectation of the pricing problem (7) for barrier options, the last step is to resolve the conditioning with respect to the information given by the variance paths up until maturity. For case A, we found an exact solution for the joint density as well as the inner expectation and therefore obtain an exact valuation formula for barrier options in this step. The final result is compared with an existing pricing formula for double barrier options. For case B, we can use the same approach as for case A resulting in an approximating formula. For case C, we extend the approach that is used in the other two cases.

6.1 Case $\rho = 0$ and $r_d = r_f$

The value of an up-and-out call option in the Heston model at time $t = 0$ in the special case where $\rho = 0$ and $r_d = r_f$ is given by

$$V_0 \quad = \quad e^{-r_d T}\mathbb{E}\left[S_0 I_{1,1} - K I_{1,0} - S_0 I_{2,1} + K I_{2,0}\right].$$

The terms $I_{1,\cdot}$ and $I_{2,\cdot}$ contain the random variable $\nu^2(T) = \int_0^T v_s ds$. In order to solve the outer expectation, the distribution of $\nu^2(T)$ must be determined. As shown in the appendix A.3, we can derive the characteristic function of $\nu^2(T)$ as

$$\phi_{\nu^2(T)}(u) \quad = \quad \mathbb{E}\left[e^{iu\nu^2(T)}\right] = \exp\left[A(u)v_0 + B(u)\right],$$

for functions A and B, with $d = \sqrt{\kappa^2 - 2\sigma^2 iu}$ and $e^{\pm} = 1 \pm \exp(-dT)$,

$$\begin{aligned} A(u) &= \frac{2iue^-}{de^+ + \kappa e^-}, \\ B(u) &= \frac{\kappa\theta}{\sigma^2}(\kappa - d)T + \frac{2\kappa\theta}{\sigma^2}\ln\left(\frac{2d}{de^+ + \kappa e^-}\right). \end{aligned}$$

Therefore, the density of $\nu^2(T)$ is given by Fourier inversion

$$d_{\nu^2(T)}(x) \quad = \quad \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ixu}\phi_{\nu^2(T)}(u)du = \frac{1}{\pi}\int_{\mathbb{R}+} \Re\left(e^{-ixu}\phi_{\nu^2(T)}(u)\right)du. \tag{21}$$

Hence, the value of an up-and-out call can be computed by dealing with $I_{j,\cdot}$, $j = 1, 2$, as functions of $\nu^2(T)$,

$$V_0 \quad = \quad e^{-r_d T}\int_{\mathbb{R}_+}\left[S_0 I_{1,1}(x) - K I_{1,0}(x) - S_0 I_{2,1}(x) + K I_{2,0}(x)\right] \times d_{\nu^2(T)}(x)\,dx, \tag{22}$$

where $I_{j,\cdot}$, for $j = 1, 2$, are defined in (18). In summary, the value of an up-and-out call can be interpreted as its Black-Scholes value with time-integrated variance.

In comparison, Lipton and Faulhaber state in [7] and in [12] the following formula for double barrier call options with lower barrier L and upper barrier U

$$V_0^D = e^{-r_dT} 2\sqrt{SK} \sum_{n=1}^{\infty} \sin\left(k_n \ln \frac{S}{L}\right) e^{A(k_n)v_0 + B(k_n)} \frac{(-1)^{n+1} k_n \left(\sqrt{\frac{U}{K}} - \sqrt{\frac{K}{U}}\right) + \sin\left(k_n \ln \frac{L}{K}\right)}{\left(k_n^2 + \frac{1}{4}\right) \ln(U/L)} \tag{23}$$

with $e^{\pm}(k) = 1 \pm e^{-\zeta(k)T}$, $k_n = \pi n / \ln(U/L)$, $\zeta(k) = \sqrt{\kappa^2 + \sigma^2 \left(k^2 + \frac{1}{4}\right)}$ and

$$\begin{aligned} A(k) &= \frac{-\left(k^2 + \frac{1}{4}\right) e^{-}(k)}{\zeta(k) e^{+}(k) + \kappa e^{-}(k)} \\ B(k) &= \frac{\kappa\theta}{\sigma^2}(\kappa - \zeta(k))T + \frac{2\kappa\theta}{\sigma^2} \ln\left(\frac{2\zeta(k)}{\zeta(k)e^{+}(k) - \kappa e^{-}(k)}\right) \end{aligned}$$

Lipton states in [12] that this series becomes more accurate as time to maturity becomes large or the barriers move closer to each other. Using this formula (23) for a barrier A with a value close to zero, we can approximate up-an-out call values. In section 7, we show the agreement in results for both formulas (22) and (23).

6.2 Case $r_d \neq r_f$ and $\rho = 0$

The second case, where the interest rate spread is arbitrary and the correlation is equal to zero, works similarly and results in an approximation formula for the up-and-out call option,

$$V_0 \approx e^{-r_dT} \int_{\mathbb{R}_+} \left[S_0 I_{1,1}(x) - K I_{1,0}(x) - S_0 I_{2,1}(x) + K I_{2,0}(x)\right] \times d_{\nu^2(T)}(x)\, dx, \tag{24}$$

where $I_{j,\cdot}$, for $j = 1, 2$, are defined in (20).

6.3 Case $r_d \neq r_f$ and $\rho \neq 0$

The value of an up-and-out call option in the Heston model in the case where the correlation and the interest rate difference can be chosen arbitrary is approximated by

$$V_0 \approx e^{-r_dT} \int_{\mathbb{R}_+^2} \left[S_0 I_{1,1}(x,y) - K I_{1,0}(x,y) - S_0 I_{2,1}(x,y) + K I_{2,0}(x,y)\right] \times d_{\nu^2(T), v_T}(x,y)\, dx\, dy, \tag{25}$$

where $I_{j,\cdot}$, for $j = 1, 2$, are defined in (19). The functions $I_{1,\cdot}$ and $I_{2,\cdot}$ contain the random variables $\nu^2(T)$ and v_T. The joint distribution can be determined by the bivariate characteristic function given in the appendix A.3 and again the joint density is obtained by Fourier inversion. We simplify this representation further to a joint density involving only a single Fourier inversion

$$\begin{aligned} d_{v_T,\nu^2(T)}(x,y) &= \frac{1}{\pi}\exp\left(L\kappa\tau + (v_0-y)\frac{\kappa}{\sigma^2}\right) \\ &\times \int_{\mathbb{R}_+} \Re\left(\frac{2d}{\sigma^2 e^-} e^{-iux - Ld\tau - (v_0+y)\frac{de^+}{\sigma^2 e^-}} \left(\frac{ye^{d\tau}}{v_0}\right)^{L-\frac{1}{2}} I_{2L-1}\left(\frac{4d}{\sigma^2 e^-}\sqrt{yv_0 e^{-d\tau}}\right)\right) du. \end{aligned} \tag{26}$$

The derivation is given in the appendix A.3.

7 Numerical Analysis

In this section, we present some numerical results. We compare up-and-out call prices computed with three different methods: the here derived formulas for all cases, a finite difference scheme and for the special case of zero correlation and equal interest rates the pricing formula developed by Lipton. All computations were carried out using Matlab code.

Beforehand, we summarize the price of an up-and-out call for all the three cases in one formula as

$$V_0 \approx e^{-r_d T}\int_{\mathbb{R}_+^2} [S_0 I_{1,1}(x,y) - K I_{1,0}(x,y) - S_0 I_{2,1}(x,y) + K I_{2,0}(x,y)] \times d_{\nu^2(T),v_T}(x,y)\,dx\,dy. \tag{27}$$

with

$$\begin{aligned} I_{1,1} &= \exp\left(\frac{1}{2}\rho_2^2\nu^2(T)+q\right)\left[N\left(\frac{\ln\left(\frac{S_0}{K}\right)+q+\rho_2^2\nu^2(T)}{\rho_2\nu(T)}\right) - N\left(\frac{\ln\left(\frac{S_0}{B}\right)+q+\rho_2^2\nu^2(T)}{\rho_2\nu(T)}\right)\right] \\ I_{1,0} &= N\left(\frac{\ln\left(\frac{S_0}{K}\right)+q}{\rho_2\nu(T)}\right) - N\left(\frac{\ln\left(\frac{S_0}{B}\right)+q}{\rho_2\nu(T)}\right) \\ I_{2,1} &= \exp\left(\frac{1}{2}\rho_2^2\nu^2(T)+q\right)\left(\frac{B}{S_0}\right)^{\frac{2q}{\rho_2^2\nu^2(T)}+2} \\ &\times\left[N\left(\frac{\ln\left(\frac{B^2}{S_0K}\right)+q+\rho_2^2\nu^2(T)}{\rho_2\nu(T)}\right) - N\left(\frac{\ln\left(\frac{B}{S_0}\right)+q+\rho_2^2\nu^2(T)}{\rho_2\nu(T)}\right)\right] \\ I_{2,0} &= \left(\frac{B}{S_0}\right)^{\frac{2q}{\rho_2^2\nu^2(T)}}\left[N\left(\frac{\ln\left(\frac{B^2}{S_0K}\right)+q}{\rho_2\nu(T)}\right) - N\left(\frac{\ln\left(\frac{B}{S_0}\right)+q}{\rho_2\nu(T)}\right)\right]. \end{aligned}$$

and $q = (r_d - r_f)T - \frac{1}{2}\nu^2(T) + \frac{\rho}{\sigma}(v_T - v_0 - \kappa\theta T + \kappa\nu^2(T))$.
For case C, the density $d_{\nu^2(T),v_T}$ is specified in equation (26). For the cases A and B, the integration in (27) reduces to a single integral with respect to the univariate density of $\nu^2(T)$ given in (21). For case A, formula (27) is an exact valuation.

To demonstrate the performance of the formula (27), we analyze the results for a given set of model parameters, contract data and market data as shown in table 1. For the case B , we replace the

Model parameters	κ	θ	σ	ρ	v_0
	2	0.04	0.25	0	0.04
Contract data	Strike	Barrier	T		
	80-100	105-145	1 year		
Market data	S_0	r_d	r_f		
	100	0.03	0.03		
Finite Differences	# grid points in S		# grid points in v	# grid points in t	
	100		100	50	
Lipton Formula	Lower barrier		Summation accuracy		
	0.0001		10e-9		

Table 1: Parameters for case A.

interest rates by values such that their difference is non-zero and similarly, for case C, we change both the correlation and the interest rate difference to be non-zero. The model parameters chosen in table 1 define a Heston model set-up, which has been standard in many papers undertaking numerical studies of stochastic volatility models. Here, the parameters are chosen such that the Feller condition is fulfilled. This is a necessary condition for the application of the finite difference scheme we are using to benchmark our results in the following.

For case A, we compute prices for single barrier options with barrier levels between 105 and 145 using formula (22). We benchmark the results against the outcome of an ADI finite difference method (FD) as described in [11] and given parameters as in table 1 on the one hand and against the pricing formula for double barrier options by Lipton [12] using a lower barrier equal to an improbable value on the other hand. All methods work very efficiently and produce stable results. The results are given in terms of absolute errors on the left hand side of figure 1 and in terms of relative price differences on the right hand side. From figure 1, we see that the barrier prices generated with our derived formula are close to those produced with the other two numerical methods. In fact, the prices from both our formula and the one of Lipton match up to a scale of 10^{-7} in absolute terms. The approximation with the FD scheme is not as close, but nevertheless as accurate as at least 0.1% of a unit currency. The results of relative pricing errors in figure 1 indicate that the greatest deviations in prices between the finite difference method and the other two formulas arise for high strike prices and low barrier levels, which we need to take into account for the analysis of the other two cases. The computational times per price in Matlab are around 7.5 seconds for the FD scheme, between 0.35 and 1.1 seconds for formula (22) and around 0.01 seconds for Lipton's formula.

For case B, we compute prices for single barrier options for a Heston model with different interest rates.

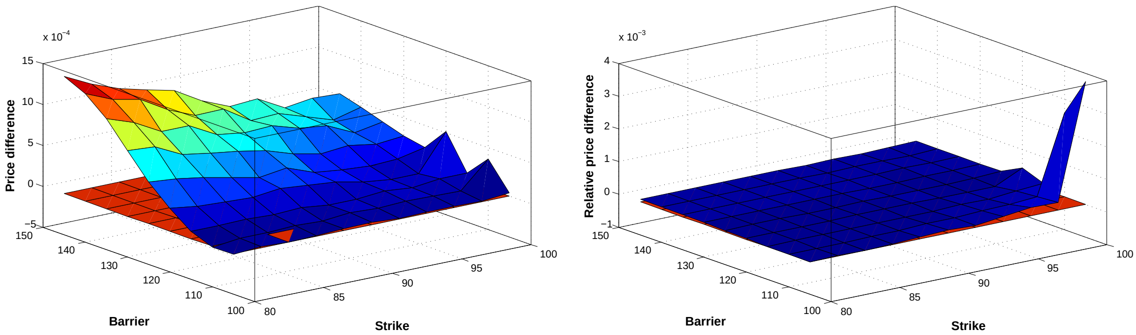

Figure 1: Case A. Left: Up-and-out call option price differences between the Lipton formula and our approach (red) and between prices obtained by the finite difference method and our approach (coloured). Right: Relative price differences between the Lipton formula and our approach (green) and between prices obtained by the finite difference method and our approach (coloured).

We choose $r_d = 0.05$ and $r_f = 0.02$ and compare absolute and relative errors between the results of formula (24) and the FD scheme. The absolute differences between the two methods are at most in the scale of 3% of a unit currency. The relative differences are at most 2.5%. Generally, we identify these errors as a result of discarding the information of the maximum process $\hat{M}$ on the independent part U of $\hat{Y}$, in the derivation of the joint distribution of $\hat{M}$ and $\hat{Y}$. The prices for our example set-up are reported in figure 2. The computational times for both methods remain the same as for case A.

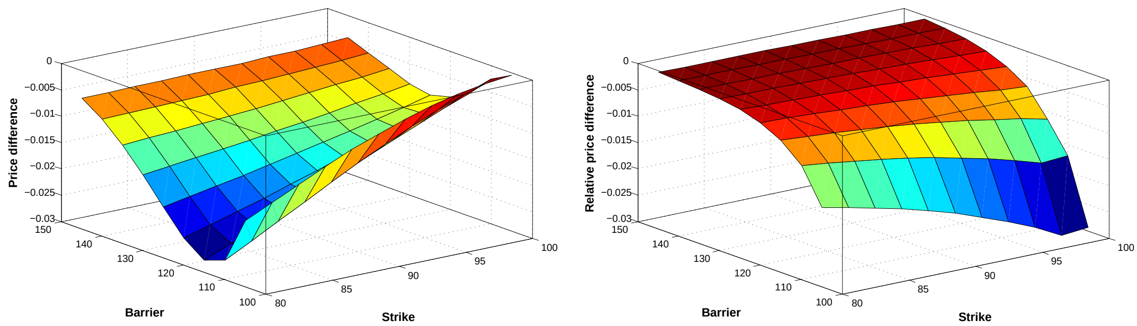

Figure 2: Case B. Differences of absolute and relative up-and-out call option prices between our approach and a finite differences scheme.

For case C, we compute prices for single barrier options for a model with different interest rates and non-zero correlation[3]. Here, we choose $\rho = -0.5$. All other parameters remain as chosen in case

[3]When pricing barrier options within a Heston model for which $2\kappa\theta/\sigma^2 \in [0, 1]$, the density in (26) exhibits a near-singular behavior in the variance direction (i.e. v_T). This behavior is described in [6] for the one-dimensional density of v_T and a logarithmic transformation of the variance domain is proposed as a remedy. We find that this suggestion is effective in our set-up as well.

B. The prices for up-and-out call options for different strikes and barrier levels are shown in figure 3. Two methods were used to compute these prices, the method developed in this paper and a finite difference method. We observe that both price surfaces of the two methods lie close together.

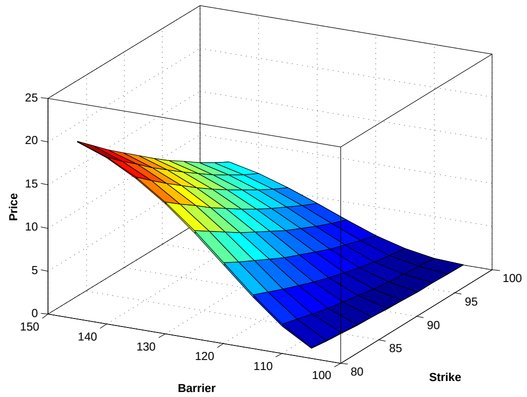

Figure 3: Case C. Up-and-out call option prices computed using our approach and a finite differences scheme.

The quality of the approximation is shown for the chosen example in figure 4. The absolute differences in this case lie between 0 and 27%, where the peak of the absolute differences arises for barrier options with strikes of 80 and barrier level around 115-120. In terms of relative errors this area exhibits only differences of around 3%, whereas up-and-out calls with barrier level 105, close to the initial spot price of 100, show the highest relative price differences of 1% up to 8%. However, from figure 3 we see that these options are practically zero in price due to the high probability of being knocked out. Hence, we can conclude that the effect of including a correlation factor to the logarithmic spot price in the model is incorporated numerically in our approximation formula (25) to a certain extent. Generally, deviations from the true price are contributed to the following aspects: the approximation of dv_t by a differentiable function and the related measure change, and the error of type "case B" arising from discarding the conditioning on $\hat{M}$. The computational time in this case mainly depends on the computation of the bivariate density $d_{\nu^2(T),v_T}(x,z)$. Since the density does not depend on initial spot price, strike price or barrier level, it can be pre-computed on a given discrete grid and cached. Using the numerical integration tools provided by Matlab we observe computational times between 1 and 8 seconds.

8 Conclusion and Future Work

In this paper, we have studied the pricing of continuously monitored barrier options under the stochastic volatility dynamics of Heston's model. We have derived a semi-analytical solution for barrier options within this model under the assumption of zero correlation and zero interest rate spread. This exact pricing approach was extended to the development of an approximation formula for this type of

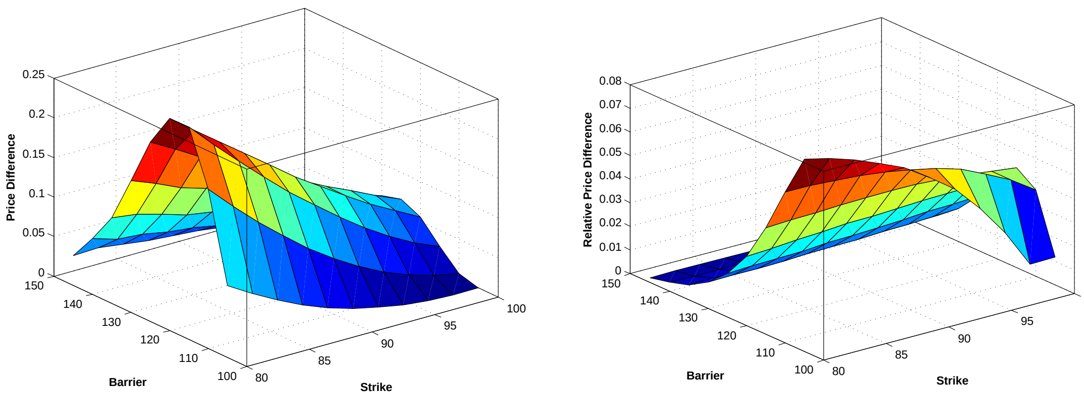

Figure 4: Case C. Differences of absolute and relative up-and-out call option prices between our approach and a finite differences scheme.

option for a Heston model with arbitrary interest rate spreads and zero correlation in a first step. In a second step, we have established approximation formulas for barrier options within the general Heston model with arbitrary interest rates and correlation parameter. All the derivations were carried out using the example of an up-and-out call option. Numerical examples have demonstrated that the developed pricing method leads to fairly accurate results compared to other conceptually different numerical pricing techniques such as finite difference methods and a formula developed by Lipton [12] for the case of reduced Heston framework. Generally, the remaining inaccuracies of our developed approximation pricing formula are contributed to the following aspects: the approximation of dv_t by a differentiable function and the related measure change, and the error of type "case B" arising from discarding the conditioning on $\hat{M}$. This last issue could possibly be further improved by regressing the random variable $\hat{X}_T$ not only on $\hat{Y}_T$, but additionally on a factor $\hat{M}_T$. We pursue this idea in depth in a subsequent study.

In future research, the pricing of other path-dependent options under stochastic volatility dynamics should be further investigated. Moreover, other stochastic volatility models and an extension of the presented method to stochastic volatility models with jumps should be considered.

A Appendix

A.1 Appendix for the Reflection Principle

Lemma 1 *Let Y be defined in (9). Then for all $0 \leq s < t < \infty$ the random variable $Y_t - Y_s$ is independent from $\mathcal{F}_s$.*

Lemma 2 *Let Y be defined in (9) and $u \in \mathbb{R}$. Define*

$$M_t = \exp\Big\{iuY_t + \frac{1}{2}u^2 \int_0^t \beta^2(s)\, ds\Big\}. \tag{28}$$

Then M_t is a martingale.

Lemma 3 *The process Y defined in (9) is strong Markov.*

The proofs are established in the same way as for Brownian motions (see for example Karatzas and Shreve [14], Theorem 6.15 and Lemma 6.14, Chapter 2.

Remark 2 *Since a conditional characteristic function determines a conditional distribution, the proof also shows that the distribution of $Y_{\tau+t}$ conditioned on $\mathcal{F}_\tau$ is normal with mean value Y_τ and variance $\int_\tau^{\tau+t} \beta^2(s)\, ds$.*

Remark 3 *A second possibility to show the strong Markov property is by assuming $\beta(t)$ to be continuous in t (which is fulfilled in our application to the Heston model). Then $\{Y_t\}_{t\geq 0}$ is a solution of the stochastic differential equation $dY_t = \beta(t)\, dW_t$ and since $\beta(t)$ does not depend on Y_t it is also twice continuously differentiable in the space variable. Now, theorems which state the strong Markov property of such solutions can be applied (see for example Da Prato [4], Theorem 8.2).*

A.2 Appendix for the Second Step

Proposition 4 *For the case $r_d = r_f$ and $\rho = 0$. The random variable $(\hat{X}_T, \hat{Y}_T)$ is normally distributed with zero mean and covariance matrix*

$$\Sigma = \begin{pmatrix} \nu^2_{\text{inv}}(T) & T \\ T & \nu^2(T) \end{pmatrix}.$$

Proof: The characteristic function of $(\hat{X}_T, \hat{Y}_T)$ is given by

$$\begin{aligned} \mathbb{E}^v\left[\exp\left\{iu_1\hat{X}_T + iu_2\hat{Y}_T\right\}\right] &= \mathbb{E}^v\left[\exp\left\{i\int_0^T \left(u_1\frac{1}{\sqrt{v_s}} + u_2\sqrt{v_s}\right) d\hat{W}_s\right\}\right] \\ &= \exp\left\{-\frac{1}{2}(u_1^2\int_0^T \frac{1}{v_s}ds + 2u_1u_2T + u_2^2\int_0^T v_s ds)\right\}. \end{aligned}$$

□

Proposition 5 *The following assertions are proved as in proposition 4 and are derived from the theory of normal distributions.*

- *For the case r_d, r_f arbitrary and $\rho = 0$. The random variable $(\hat{X}_T, \kappa_1 \hat{Y}_T)$ is normally distributed with zero mean and covariance matrix*

$$\Sigma = \begin{pmatrix} \nu^2_{\text{inv}}(T) & \frac{T^2}{\nu^2(T)} \\ \frac{T^2}{\nu^2(T)} & \frac{T^2}{\nu^2(T)} \end{pmatrix},$$

  hence ε is normal with zero mean and variance $\sigma^2_\varepsilon = \nu^2_{\text{inv}}(T) - \frac{T^2}{\nu^2(T)}$.

- *For the case r_d, r_f arbitrary and arbitrary ρ. The random variable $(\hat{X}_T, \hat{Y}_T, \hat{Z}_T)$ is normally distributed with zero mean and covariance matrix*

$$\Sigma = \begin{pmatrix} \rho_2^2 \nu^2_{\text{inv}}(T) & \rho_2^2 T & \rho_2^2 \nu^2_{II}(T) \\ \rho_2^2 T & \rho_2^2 \nu^2(T) & \rho_2^2 (\nu'(T))^2 \\ \rho_2^2 \nu^2_{II}(T) & \rho_2^2 (\nu'(T))^2 & \rho_2^2 \nu^2_I(T) \end{pmatrix}.$$

- *For the case r_d, r_f arbitrary and arbitrary ρ. The random variable $(\varepsilon^X_T, \varepsilon^Z_T)$ is normally distributed with zero mean and covariance matrix*

$$\Sigma = \begin{pmatrix} \rho_2^2 \nu^2_{\text{inv}}(T) - \kappa_1 T \rho_2^2 & \rho_2^2 \nu^2_{II}(T) - \kappa_3 T \rho_2^2 \\ \nu^2_{II}(T) \rho_2^2 - \kappa_3 T \rho_2^2 & \rho_2^2 \nu^2_I(T) - \kappa_3 \rho_2^2 (\nu'(T))^2 \end{pmatrix},$$

  and is independent from $\hat{Y}_T$.

A.3 Appendix for the Fourth Step

The bivariate characteristic function of v_T and $\int_0^T v_t dt$ is defined by

$$\varphi(u, w) = \mathbb{E}\left[\exp\left(iu \int_0^T v_t dt + iwv_T\right)\right].$$

The expectation is solvable by applying the Feynman-Kac formula and solving the resulting partial differential equation. Then the above expectation has the following solution

$$\mathbb{E}\left[\exp\left(iwv_T + iu \int_0^T v_t dt\right)\right] = \exp\left[A(T, u, w) v_0 + B(T, u, w)\right] \quad (29)$$

with $d = \sqrt{\kappa^2 - 2\sigma^2 iu}$, $\gamma = 2d \exp(-d\tau) + (\kappa + d - \sigma^2 iw)(1 - \exp(-d\tau))$ and

$$A(\tau, u, w) = \frac{-(1 - \exp(-d\tau)(-2iu + \kappa iw) + diw(1 + \exp(-d\tau))}{\gamma}$$

$$B(\tau, u, w) = \frac{\kappa\theta}{\sigma^2}(\kappa - d)\tau + \frac{2\kappa\theta}{\sigma^2} \ln \frac{2d}{\gamma}.$$

The joint density of v_T and $\nu^2(T) = \int_0^T v_s ds$ is given by the double Fourier inversion of the characteristic function in equation (29)

$$\varphi_{v_T,\nu^2(T)}(u,w) = \mathbb{E}\left[\exp\left(iwv_T + iu\nu^2(T)\right)\right] = \exp\left[A(T,u,w)v_0 + B(T,u,w)\right]$$

as

$$d_{v_T,\nu^2(T)}(x,y) = \frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} e^{-iux-iwy}\varphi(u,w)dwdu.$$

In [3], using PDE methods Chiarella and Ziogas show that the joint density of v_T and $\nu^2(T)$ has a closed-form solution with respect to v_T. Here, we follow the approach described in [9] using integration techniques to derive this representation of the joint density.

Inserting the definitions for the functions A and B and some simple algebra results in

$$\begin{aligned} d_{v_T,\nu^2(T)}(x,y) &= \frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} e^{-iux-iwy}\exp\left[A(T,u,w)v_0 + B(T,u,w)\right]dwdu \\ &= \frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} e^{-iux-iwy}e^{\frac{2iue^- - \kappa iwe^- + diwe^+}{\gamma(w)}v_0 + \frac{\kappa\theta}{\sigma^2}(\kappa-d)\tau - \frac{2\kappa\theta}{\sigma^2}\ln\frac{\gamma(w)}{2d}}dwdu \\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}} e^{-iux}e^{\frac{\kappa\theta}{\sigma^2}(\kappa-d)\tau}\int_{\mathbb{R}} e^{-iwy}e^{\frac{2iue^- - \kappa iwe^- + diwe^+}{\gamma(w)}v_0}\left(\frac{\gamma(w)}{2d}\right)^{-\frac{2\kappa\theta}{\sigma^2}}dw\,du \end{aligned}$$

where $e^{\pm} = 1 \pm \exp(-d\tau)$. The aim of the following calculations is to solve the inner integral with respect to w. Substituting for

$$z = \frac{\gamma(w)}{2d} = \frac{de^+ + \kappa e^-}{2d} - iw\frac{\sigma^2 e^-}{2d} \qquad =: m - iwn$$

leads to

$$\begin{aligned} &\frac{1}{2\pi}\int_{\mathbb{R}} e^{-iwy}\exp\left(\frac{2iue^- - \kappa iwe^- + diwe^+}{\gamma(w)}v_0\right)\left(\frac{\gamma(w)}{2d}\right)^{-\frac{2\kappa\theta}{\sigma^2}}dw \\ = &-\frac{1}{2\pi in}e^{-\frac{y}{n}m}e^{\frac{\kappa e^- - de^+}{2dn}v_0}\int_{m-in\infty}^{m+in\infty} e^{cz}z^{-2L}e^{kz^{-1}}dz, \end{aligned}$$

with $L = \frac{\kappa\theta}{\sigma^2}$, $c = \frac{y}{n}$ and $k = \frac{2iune^- - m(\kappa e^- - de^+)}{2dn}v_0$. The previous calculations reduce the problem to solving the following inverse Laplace transform

$$\frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} e^{cz}z^{-2L}e^{kz^{-1}}dz = \mathcal{L}_c^{-1}\left(z^{-2L}e^{kz^{-1}}\right),$$

for some $\xi \in \mathbb{R}$. Generally, we know the inverse Laplace transform of a function $f(s) = e^{ks^{-1}}s^{-\mu}$ is given by

$$\mathcal{L}_t^{-1}\left(f(s)\right) = \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} e^{ts}f(s)ds = \left(\frac{t}{k}\right)^{0.5\mu-0.5} I_{\mu-1}\left(2\sqrt{kt}\right),$$

where I denotes the modified Bessel function for complex arguments. Hence, after some simple algebra, the inner Fourier inversion is resolved to

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\mathbb{R}} e^{-iwy}\exp\left(\frac{2iue^- - \kappa iwe^- + diwe^+}{\gamma(w)}v_0\right)\left(\frac{\gamma(w)}{2d}\right)^{-\frac{2\kappa\theta}{\sigma^2}} dw \\
= \; & \frac{2d}{\sigma^2 e^-} e^{-y\frac{de^+ + \kappa e^-}{\sigma^2 e^-}} e^{v_0\frac{\kappa e^- - de^+}{\sigma^2 e^-}} \left(\frac{y}{v_0}e^{d\tau}\right)^{L-\frac{1}{2}} I_{2L-1}\left(\frac{4d}{\sigma^2 e^-}\sqrt{yv_0 e^{-d\tau}}\right)
\end{aligned}$$

In summary, we have

$$\begin{aligned}
d_{v_T,\nu^2(T)}(x,y) \; = \; & \frac{1}{2\pi}\exp\left(L\kappa\tau + (v_0 - y)\frac{\kappa}{\sigma^2}\right) \\
& \times \int_{\mathbb{R}} \frac{2d}{\sigma^2 e^-} e^{-iux - Ld\tau - (v_0+y)\frac{de^+}{\sigma^2 e^-}} \left(\frac{ye^{d\tau}}{v_0}\right)^{L-\frac{1}{2}} I_{2L-1}\left(\frac{4d}{\sigma^2 e^-}\sqrt{yv_0 e^{-d\tau}}\right) du.
\end{aligned}$$

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