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Heterogeneous Beliefs and the Cross Section of Asset Returns

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HETEROGENEOUS BELIEFS AND THE CROSS-SECTION OF ASSET RETURNS

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ABSTRACT. When agents have irrational beliefs which are rational on average, it has been shown that the effect of their trades does not cancel out in general and can lead to time variations in market price of risk and volatility. In this paper, we follow the differences-in-opinion approach and show that the impact of unbiased disagreement on market equilibrium is much stronger in a multi-asset market than in a single-asset market, in which the impact of small disagreement may be negligible. More importantly, we show that different type of disagreement contribute significantly to explain the cross-section of expected returns, volatility and covariance between asset returns. In particular, disagreement can lead to excess volatility, a positive (negative) excess covariance when optimism/pessimism are positively (negatively) correlated between assets and the level of disagreement is negatively (positively) related to expected future returns when the relatively optimistic agent has a larger (smaller) wealth share than the pessimistic agent.

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1. INTRODUCTION

Standard asset pricing models usually assume that investors correctly perceive the *objective* probabilities of different states of nature underlying different market conditions. However, the objective probabilities are likely to be *unknown* to the investors ex-ante, and can only be estimated after observing a sufficient amount of data. Therefore, there is likely a disagreement among the investors about the probabilities underlying each state of nature, hence their optimal portfolio/consumption policies may significantly differ from each other.

It has been argued that if investors' expectations *on average* coincide with the objective or rational expectation, then the disagreement among individual agents has little or no effect on equilibrium asset returns at the aggregate level; this is the aggregation argument[1]. If the argument holds, then the effect of disagreement is canceled out by aggregation and assets can be priced in a framework with a representative agent who perceive the objective probabilities (at least when risk preferences are identical). However, if the aggregation does not hold, then one should take the heterogeneity in beliefs into account in empirical tests of asset pricing models[2].

The aggregation argument has been challenged recently. Fama and French (2007) argue that heterogeneity in beliefs cancels out in aggregation if and only if the mis-informed investors on aggregate hold the market portfolio, so that the market portfolio remains to be the tangency portfolio. In a static model, Yan (2010) shows that investors' disagreements about expected return *cancel* out when they are on average unbiased, however, unbiased disagreements about variance reduce the expected returns (under the objective probabilities). This is due to the fact that investors' demand for a risky asset is linear in expected return but nonlinear in variance. Duchin and Levy (2010) show that tiny fluctuations in the disagreement about the variance lead to substantial price fluctuations. In a dynamic equilibrium setting, expected stock return is related to the expected growth in dividends, if investors on average have unbiased expectations about dividend growth, then they are also on average unbiased about the expected return. Yan (2010) shows in a two-period model that even when investors agrees about the variance and have heterogeneous but on average unbiased beliefs about the expected dividend growth rate, disagreements among agents do not cancel out over time. He finds that disagreement leads to price *overshooting* and *mean-reversion* in stock returns. Intuitively, the stock price is a wealth-share-weighted average of the stock prices that would prevail in economies in each of which there is only one investor. If dividend growth is high (above the true average), then the optimistic investors would have larger wealth shares compared to the pessimistic investors, hence the stock would be overpriced compared to the market in which investors have homogeneous

[1]See Yan (2010) and Jouini and Napp (2011) for related literature and discussion.

[2]For example, Anderson, Ghysels and Juergens (2005) estimate a structural model and show that analysts' forecast of earnings used as a proxy for investors' beliefs can be a pricing factor.

unbiased beliefs. Similarly, when dividend growth is low (below the true average), the stock price would overshoot in the opposite direction. When stock price is over(under)-priced, the expected return of next period would be lower (higher) compared to the market in which investors have homogeneous beliefs, which implies mean-reverting of the expected returns. In an infinite horizon continuous-time version of Yan's model, Jouini and Napp (2011) assume that all investors survive in the long, which requires the amount of *optimism* for one investor to exactly equal the amount of *pessimism* for another investor. They find "waves of pessimism and optimism that lead to countercyclical market prices of risk and procyclical risk-free rates".

Most of the dynamic disagreement models analyze an economy in which there is a single risky asset, which is a claim to the *aggregate* dividend or the aggregate endowment.[3] However, in reality, an investor's portfolio choice involves a risk-free asset and *multiple* risky assets, which are the claims to *different* dividend processes. The question is how disagreement affects stock return in multi-asset market differently compared to a single risky asset market. In a static mean-variance framework, by assuming heterogeneity in belief about the covariance matrix of asset returns, He and Shi (2012) show that the effect of disagreement on the risk-free rate and market risk premium can be very different in a multi-risky-asset market compared to a single-risky-asset market.

In this paper, we consider a market with one risk-free asset and multiple risky assets in continuous time. Investors agree about the expected growth rate of the aggregate endowment (as in Yan (2010)) but disagree about the expected dividend growth rates of the risky assets. By assuming a stationary economy[4], we show that the fluctuations in the distribution of wealth or consumption shares among investors depend on the distance between the subjective probability measures among investors, which increases with the overall disagreement in investors' beliefs. In a single risky asset market, although disagreement leads to time-variation in expected stock returns and excess volatility caused by price overshooting, the effect is *negligible* if the dispersion in beliefs is *small*, which is likely to be the case given a long time-series of data on the aggregate stock returns. In comparison, when there are multiple stocks and dividend processes, investors may have small disagreement about the expected dividend growth rate of each stock, however, when the number of stocks is relatively large (as in the real financial market), the overall disagreements in investors beliefs could be much higher compared to a market with single risky asset. Consequently, we show that there can be significantly large time-variations in expected stock returns and excess volatility even though the amount of disagreement for each stock is small. More importantly, we examine the impact of disagreement on the dynamics of

[3]In Yan (2010), the aggregate dividend is modeled separately from the aggregate endowment, however, investors are assumed to disagree about the expected growth rate of the aggregate dividend only.

[4]An economy is stationary if all agents survive in the long run, which require the *distance* between the subjective probability measure and the objective probability measure to be equal, see Yan (2008).

cross-sectional returns, including covariance of stock returns (which cannot be examined in a single risky asset market). We find that disagreement can lead to a positive (negative) *excess covariance* when optimism/pessimism are positive (negatively) correlated between two stocks, meaning that the investor who is relatively optimistic about the dividend growth for one stock is also relatively optimistic (pessimistic) about the dividend growth for another stock. This finding has strong implications for portfolio risk management. When investors' optimism/pessimism are on average positively (negatively) correlated across different stocks, the aggregate market portfolio becomes more (less) risky. In addition, a high disagreement can lead to low (high) expected return among otherwise similar assets when the relatively optimistic agents has a larger (smaller) wealth share than the pessimistic agents. Furthermore, when the number of risky assets increases, the impact becomes more significant.

This paper is related to the literature of survival and stationary economy. When agents are not fully rational, it has been argued that irrational investors would consistently lose money to the rational investors and eventually be driven out of the market, therefore their beliefs have no price impact in the long run and thus can be ignored. This is the so-called evolution argument or the market selection hypothesis, first proposed by Friedman (1953). However, it has been shown that market selection does not work in some cases. For example, Blume and Easley (2006) show that market selection may not work in an incomplete market, a rational investor may choose to opt out of the market because he cannot arbitrage against the irrational investors. Kogan, Ross, Wang and Westerfield (2006) show that, without intermediate consumption, investors with negligible wealth can have significant price impact. Yan (2008) consider a complete market and show that incorrect beliefs are always disadvantageous for an investor's survival, depending on an investor's survival index, which is defined by his belief, patience and risk aversion. An investor with incorrect belief may survive because of greater patience or a lower risk aversion. In this paper, we do not consider the survivability of investors and assume that all investors have the same survival indices and survive in the long run. To focus on the impact of disagreement, we assume a complete market and investors have the same risk aversion and patience, but different beliefs about the expected dividend growth rates. Note that beliefs can differ even though investors have the same survival indices. This is also the approach taken by Jouini and Napp (2011).

This paper belongs to a growing literature on equilibrium effects of heterogeneous beliefs started by Lintner (1969), Williams (1977) and Detemple and Murthy (1994). Recently, Li (2007) analyzes the joint effect of heterogeneities in beliefs and patience parameters. Bhamra and Uppal (2010) examines the joint pricing effect of heterogeneity in beliefs and risk aversions. Buraschi and Jiltsov (2006) model the open interest in options in an incomplete market with heterogeneous beliefs. David (2008) tries to explain the average level of equity premium with heterogeneous beliefs where there are recurrent jumps in the expected dividend and endowment growth rates. Dumas, Kurshev and Uppal (2009) identify the optimal portfolio strategy in a market with investors who are driven by sentiment and causes excess volatility in stock returns. All of the above models are either static in nature or dynamic but in a single-asset market. Different from this literature, this paper examines the effect of disagreement in a multi-asset dynamic equilibrium model. The framework used in this paper follows the differences-in-opinion approach, instead of asymmetric information approach[5]. This paper is closely related to recent literature on cross-sectional returns, see, for example, Varian (1985), Miller (1977), Bart and Masse (1981); in particular, Diether, Malloy and Scherbina (2002), Johnson (2004) and Ang, Hodrick, Xing and Zhang (2006). In examining the cross-sectional returns, Diether et al. (2002) provide empirical evidence that stocks with higher dispersion in analysts earnings forecasts earn lower future returns than otherwise similar stocks, in particular for small cap stocks and stocks that have performed poorly over the past year. Johnson (2004) offers a simple explanation for this phenomenon based on the interpretation of dispersion as a proxy for un-priced information risk arising when asset values are unobservable. Ang et al. (2006) examine the empirical relation between cross-sectional volatility and expected returns and find that stocks with high sensitivities to innovations in aggregate volatility have low average returns. This paper helps to provide a theoretical understanding of these empirical findings.

This paper is organized as follows. With two heterogeneous agents, the model is presented and equilibrium is obtained in Section 2; the cross-sectional asset return dynamics are then analyzed in Section 3. Section 4 extends the model to multiple heterogeneous agents. Section 5 concludes. All proofs are provided in Appendix A.

2. THE MODEL

We consider a continuous-time pure exchange economy with an infinite time horizon and two agents. indexed by $i = 1, 2$. The uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$ on which a multi-dimensional Wiener process $\mathbf{Z}(t) \equiv (Z_0(t), Z_1(t), \cdots, Z_K(t))^T$ is defined. $\mathcal{F}_t$ is the information generated by the Wiener processes $\mathbf{Z}(t)$, and the correlation matrix is denoted by $\boldsymbol{\rho} \equiv (\rho_{j,k})_{(K+1)\times(K+1)}$. The rate of aggregate endowment follows

$$de(t) = e(t)[\mu_e dt + \sigma_e dZ_0(t)],$$

[5]The asymmetric information approach assumes that informed investors receive private signals about future payoffs of the asset and uninformed investors attempt to extract the private information from the commonly observed asset prices. Such models include Grossman and Stiglitz (1980), Hellwig (1980), Admati (1985), Wang (1993), Watanabe (2008) and Biais, Bossaerts and Spatt (2010).

where μ_e and σ_e are constants. There are K dividend streams $D_k(t), k = 1, 2, \cdots, K$, governed by

$$dD_k(t) = D_k(t)[\mu_{D,k}dt + \sigma_{D,k}dZ_k(t)],$$

where $\mu_{D,k}$ and $\sigma_{D,k}$ are constants.

2.1. **Beliefs.** Agents receive the same information $\mathcal{F}_t$, but they agree to disagree about the expected dividend growth rates[6]. Under the subjective probability measure of agent i, the dividend processes follow

$$dD_k(t) = D_k(t)[\mu^{(i)}_{D,k}dt + \sigma_{D,k}dZ^{(i)}_k(t)], \qquad k = 1, \cdots, K, \qquad i = 1, 2,$$

where

$$dZ^{(i)}_k(t) \equiv dZ_k(t) - \theta_{i,k}dt \qquad \text{and} \qquad \theta_{i,k} \equiv \frac{\mu^{(i)}_{D,k} - \mu_{D,k}}{\sigma_{D,k}}.$$

We treat $\mathcal{P}$ as the *objective probability measure*. Then $\theta_{i,k}$ measures the amount of optimism in agent i's subjective belief about the expected growth rate of $D_k(t)$. To focus on the cross sectional returns, we follow Yan (2010) and assume that agents agree on the expected growth rate of the aggregate endowment process, that is $\theta_{i,0} = 0$ for $i = 1, 2$. Let $\boldsymbol{\theta}_i \equiv (\theta_{i,0}, \theta_{i,1}, \cdots, \theta_{i,K})^T$. By Girsanov's theorem, agent i's *subjective probability measure* $\mathcal{P}_i$ can be characterized by the positive density process

$$M_i(t) = \frac{d\mathcal{P}_i}{d\mathcal{P}} = \exp\left\{ -\frac{1}{2}\boldsymbol{\theta}_i^T \boldsymbol{\rho}^{-1}\boldsymbol{\theta}_i t + \boldsymbol{\theta}_i^T \boldsymbol{\rho}^{-1}\mathbf{Z}(t)\right\}, \tag{2.1}$$

which is a martingale with respect to the objective measure $\mathcal{P}$ satisfying

$$\frac{dM_i(t)}{M_i(t)} = \boldsymbol{\theta}_i^T \boldsymbol{\rho}^{-1} d\mathbf{Z}(t).$$

2.2. **Securities.** There is a risk-free bond with price $B(t)$ following

$$dB(t) = r(t)B(t)dt,$$

where $r(t)$ is the risk-free interest rate. There are $K+1$ risky assets indexed by $k = 0, 1, 2, \cdots,$ K. When $k = 0$, the corresponding asset is a claim to the aggregate endowment, and when $k \neq 0$, the corresponding asset is a claim to the k^{th} dividend process. The ex-dividend price for

[6]Following Merton (1980), high-frequency data allows agents to estimate volatilities to any desired precision. However, the growth rate of a process is much harder to estimate, especially for processes with large volatilities. Accurate estimation of the growth rates requires a long time-series of data. This means that agents in the economy may disagree about the growth rate of the dividend processes though they have the same information (observe the same processes) given by the filtration $\mathcal{F}_t$.

asset k is denoted by $S_k(t)$ and the instantaneous return of asset k is defined as

$$dR_k(t) \equiv \frac{dS_k(t) + D_k(t)dt}{S_k(t)}.$$

Let $d\mathbf{R}(t) \equiv (dR_0(t), dR_1(t), \cdots, dR_K(t))^T$ be the vector of instantaneous asset returns and $d\mathbf{Z}(t) = (dZ_0(t), dZ_1(t), \cdots, dZ_K(t))^T$. Then the vector of instantaneous returns can be written as

$$d\mathbf{R}(t) = \boldsymbol{\mu}(t)dt + \boldsymbol{\sigma}(t)d\mathbf{Z}(t),$$

where $\boldsymbol{\mu}(t) \equiv (\mu_0(t), \mu_1(t) \cdots, \mu_K(t))^T$ and $\boldsymbol{\sigma}(t) \equiv (\sigma_{j,k}(t))_{(K+1)\times(K+1)}$ are the vector of *expected returns* and the *volatility matrix* respectively. We assume that the volatility matrix is *invertible*.

2.3. **Portfolio Optimization.** To focus on the impact of disagreement, we assume that agents have a *logarithmic* preferences. Agent i maximizes his expected utility of life-time consumption under his subjective probability measure $\mathcal{P}_i$,

$$\max_{c_i(t)} \mathbb{E}\left[\int_0^\infty M_i(t)e^{-\beta t}\ln(c_i(t))dt\right],$$

subjected to the *dynamic budget constraint*,

$$\begin{aligned} dW_i(t) = W_i(t)\big[r(t) + \boldsymbol{\pi}_i(t)^T(\boldsymbol{\mu}(t) - r(t)\mathbf{1})\big]dt + (e_i(t) - c_i(t))dt \\ + W_i(t)\big[\boldsymbol{\pi}_i(t)^T\boldsymbol{\sigma}(t)d\mathbf{Z}(t)\big], \end{aligned} \tag{2.2}$$

where β is the subjective discount rate, $\boldsymbol{\pi}_i(t) \equiv (\pi_{i,0}(t), \pi_{i,1}(t), \cdots, \pi_{i,K}(t))^T$, and $\pi_{i,k}(t)$ is the proportion of agent i's wealth invested in asset k at time t. The market is dynamically complete in the sense that any contingent claims on the assets can be replicated. This implies the existence of a unique state price density (SPD) process $\xi(t)$ with $\xi(0) = 1$ and

$$\frac{d\xi(t)}{\xi(t)} = -r(t)dt - \boldsymbol{\kappa}(t)^T d\mathbf{Z}(t), \tag{2.3}$$

where

$$\boldsymbol{\kappa}(t) \equiv (\kappa_0(t), \kappa_1(t), \cdots, \kappa_K(t))^T = (\boldsymbol{\sigma}(t)\boldsymbol{\rho})^{-1}\ (\boldsymbol{\mu}(t) - r(t)\mathbf{1}) \tag{2.4}$$

is the vector of *market prices of risk*. Thus, the dynamic budget constraint can be written as a static one (see Cox and Huang (1989), Karatzas, Lehoczky and Shreve (1987) and Cvitanic and Zapatero (2004)),

$$\mathbb{E}\left[\int_0^\infty \xi(t)c_i(t)dt\right] \le \mathbb{E}\left[\int_0^\infty \xi(t)e_i(t)dt\right]. \tag{2.5}$$

We assume that each agent is endowed with half of the aggregate endowment, that is $e_i(t) = \frac{1}{2}e(t)$ for $i = 1, 2$. We first obtain the optimal consumptions and consumption shares of agents.

Lemma 2.1. *(Optimal Consumption) The optimal consumption of agent i is given by*

$$c_i(t) = \frac{M_i(t)e^{-\beta t}}{\eta\ \xi(t)}, \tag{2.6}$$

where η is agent i's Lagrange multiplier and $1/\eta = \frac{1}{2}\beta\mathbb{E}\big[\int_0^\infty \xi(t)e(t)dt\big]$, independent of i. Let $\lambda(t)$ and $(1-\lambda(t))$ be the consumption ratios of agents 1 and 2 respectively. Then

$$\lambda(t) \equiv \frac{c_1(t)}{c_1(t)+c_2(t)} = \frac{M_1(t)}{M_1(t)+M_2(t)}. \tag{2.7}$$

Lemma 2.1 shows that under a homogeneous logarithmic preference, agents' optimal consumptions only differ by their subjective beliefs $M_i(t)$. Furthermore, the consumption ratio depends on the ratio of agents' beliefs. This result is consistent with Jouini and Napp (2007, 2011).

2.4. **Consensus Belief and Market Equilibrium.** We now follow the approach in Jouini and Napp (2007) to construct a consensus belief and to determine the market equilibrium. Firstly, we present a modified version of the equilibrium definition in Basak (2005).

Definition 2.2. *An equilibrium is a price system $(\{S_k(t)\}, B(t))$ and consumption-portfolio strategies ($c_i(t), \boldsymbol{\pi}_i(t)$) such that*

(i) *agents choose their optimal consumption-portfolio strategies given their perceived dividend processes;*

(ii) *observed asset prices are consistent across agents, that is,*

$$S_k(t) = \mathbb{E}_{i,t}\left[\int_0^\infty e^{-\beta(s-t)}\frac{c_i(t)}{c_i(s)}D_k(s)\ ds\right]$$

for all $k = 0, 1, 2, \cdots, K$ and $i = 1, 2$; and

(iii) *goods and security markets clear,*

$$c_1(t)+c_2(t) = e(t), \qquad W_1(t)+W_2(t) = 0, \text{ and}$$
$$W_1(t)\boldsymbol{\pi}_1(t) + W_2(t)\boldsymbol{\pi}_2(t) = \mathbf{0}.$$

Secondly we follow Jouini and Napp (2007) to construct a consensus consumer with logarithmic preference and belief $M(t)$ who is endowed with the total endowment in the economy. The consensus belief in this case is given by

$$M(t) = \frac{1}{2}(M_1(t)+M_2(t)), \tag{2.8}$$

which satisfies

$$\frac{dM(t)}{M(t)} = \boldsymbol{\theta}(t)^T\boldsymbol{\rho}^{-1}d\mathbf{Z}(t) \qquad \text{with} \qquad \boldsymbol{\theta}(t) = \lambda(t)\boldsymbol{\theta}_1 + (1-\lambda(t))\boldsymbol{\theta}_2.$$

Here $\theta(t)$ measures the amount of optimism in the consensus belief about the expected dividend growth rates. The economy with a consensus consumer generates the same equilibrium prices as the economy populated by agents with heterogeneous beliefs. Therefore, the SPD process $\xi(t)$ can be determined using the fact that the consensus consumer must optimally consume the aggregate endowment and therefore

$$\xi(t) = \frac{M(t)e^{-\beta t}}{e(t)}. \tag{2.9}$$

Thus, we can write the equilibrium price for asset k as

$$S_k(t) = \mathbb{E}_t\left[\int_0^\infty \frac{\xi(s)}{\xi(t)} D_k(s)\ ds\right] = \mathbb{E}_t\left[\int_0^\infty e^{-\beta(s-t)}\frac{M(s)}{M(t)}\frac{e(t)}{e(s)} D_k(s)\ ds\right]. \tag{2.10}$$

With the consensus belief, the characterizes of the market equilibrium can be obtained.

Proposition 2.3. *(Market Equilibrium) Denote*

$$s_i = \boldsymbol{\theta}_i^T \boldsymbol{\rho}^{-1}\boldsymbol{\theta}_i, \qquad \boldsymbol{\epsilon} \equiv \boldsymbol{\theta}_1 - \boldsymbol{\theta}_2, \qquad \varphi \equiv \boldsymbol{\epsilon}\boldsymbol{\rho}^{-1}\boldsymbol{\epsilon} > 0, \qquad i = 1, 2.$$

Then

(i) *the market prices of risk are given by*

$$\boldsymbol{\kappa}(t) = (\kappa_o(t), \kappa_1(t), \cdots, \kappa_K(t))^T = \sigma_e(1, 0, \cdots, 0)^T + \boldsymbol{\rho}^{-1}\boldsymbol{\theta}(t). \tag{2.11}$$

(ii) *the equilibrium interest rate is a constant and given by*

$$r = \beta + \mu_e - \sigma_e^2. \tag{2.12}$$

(iii) *agent 1's share of aggregate consumption is given by*

$$\lambda(t) = \left[1 + \exp\left\{-\frac{1}{2}(s_2 - s_1)t - \boldsymbol{\epsilon}^T\boldsymbol{\rho}^{-1}\mathbf{Z}(t)\right\}\right]^{-1}, \tag{2.13}$$

which follows

$$d\lambda(t) = \lambda(t)(1-\lambda(t))\left[\left(\left(\frac{1}{2} - \lambda(t)\right)\varphi + \frac{1}{2}(s_2 - s_1)\right)dt + \boldsymbol{\epsilon}^T\boldsymbol{\rho}^{-1}d\mathbf{Z}(t)\right]. \tag{2.14}$$

(iv) *the equilibrium price for asset k is given by*

$$S_k(t) = \lambda(t)S_{1,k}(t) + (1-\lambda(t))S_{2,k}(t), \tag{2.15}$$

where

$$S_{i,k}(t) = \mathbb{E}_{i,t}\left[\int_0^\infty e^{-\beta(s-t)}\frac{e(t)}{e(s)}D_k(s)\ ds\right] = \frac{D_k(t)}{\delta_{i,k}}$$

is the equilibrium price of asset k if the economy is populated by agent i alone, and $\delta_{i,k} = r + \rho_{0k}\sigma_{D,k}\sigma_e - \mu_{D,k}^{(i)}$ assuming that $\mu_{D,k}^{(i)} < r + \rho_{0k}\sigma_{D,k}\sigma_e$ for $i = 1, 2$.

In Proposition 2.3, equation (2.11) shows that the impact of heterogeneity in beliefs on the market prices of risk is positively related to the amount of pessimism in the consensus belief augmented by the inverse of the correlation matrix of the dividend processes. Equation (2.12) shows that the risk-free rate in the economy is constant, which is the standard one under logarithmic utility. This is due to that agents agree on the expected aggregate endowment growth. As in Yan (2008), s_i defines the survival index of agent i for $i = 1, 2$. Equation (2.13) shows that the long-run consumption shares of agents depend on the difference $s_2 - s_1$. When $s_2 - s_1 > 0$, agent 1 is in aggregate making a smaller error based on the *Kullback-Leibler (KL) distance*[7] between his subjective probability measure $\mathcal{P}_1$ and the objective probability measure $\mathcal{P}$, thus agent 1 survives and agent 2 vanishes in long run. Similarly, if $s_2 - s_1 < 0$, then agent 2 survives and agent 1 vanishes. The market is stationary when both agents survive, which only occurs when $s_2 = s_1$. In this case, equation (2.13) shows that $\lambda(t)$ and $1 - \lambda(t)$ are equal in distribution, therefore the *unconditional means* of consumption shares of two agents under their objective probability measures are equal, $\mathbb{E}[\lambda(t)] = \mathbb{E}[1 - \lambda(t)] = \frac{1}{2}$. This is also pointed out in Jouini and Napp (2011). Note that in (2.14), when $s_2 = s_1$ and $\lambda(t) = \frac{1}{2}$, the expected change in agent 1's consumption share is zero. Moreover, the fluctuations in consumption shares are level dependent – fluctuations diminish when $\lambda(t)$ approaches zero or one. It also depend on the KL distance between agents' subjective probability measures $\varphi = D_{KL}[\mathcal{P}_1, \mathcal{P}_2]$. More precisely, $Var_t[d\lambda(t)] = \lambda(t)^2(1 - \lambda(t))^2\varphi dt$, hence the fluctuations in consumption shares are the greatest when $\lambda(t) = \frac{1}{2}$. The equilibrium price of asset k is a consumption-share-weighted average of the prices that would prevail in economies in each of which there is only one agent, see equation (2.15). In each economy with a single agent, the asset is evaluated under the *Dividend Discount Model* (DDM), where the expected return is $r + \rho\sigma_{D,k}\sigma_e$ and the expected dividend growth rate is $\mu_{D,k}^{(i)}$, and the price-dividend (PD) ratio is $\phi_{i,k} \equiv 1/\delta_{i,k}$. In equilibrium, the PD ratio of asset k is given by

$$\phi_k(t) = \lambda(t)\phi_{1,k} + (1 - \lambda(t))\phi_{2,k}. \tag{2.16}$$

Therefore, under heterogeneity in beliefs, PD ratios are no longer constant but stochastic due to the fluctuations in agents' consumption shares, and the instantaneous return of asset k can be in general decomposed into

$$dR_k(t) = \frac{d\phi_k(t)}{\phi_k(t)} + \frac{dD_k(t)}{D_k(t)} + \frac{d\phi_k(t)}{\phi_k(t)}\frac{dD_k(t)}{D_k(t)} + \frac{1}{\phi_k(t)}dt. \tag{2.17}$$

[7]The KL distance is a non-symmetric measure of the difference between two probability distributions. If both distributions are multivariate normal and have a common covariance matrix Σ, but different mean vectors $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$, then the KL distance between the two distributions is given by $(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)^T\Sigma^{-1}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)$.

Moreover, since there is no disagreement about the expected growth rate of the aggregate endowment process, the instantaneous return of asset 0 (claim on the aggregate endowment) is given by

$$dR_0(t) = (r + \sigma_e^2)dt + \sigma_e dZ_0(t). \tag{2.18}$$

Note that if agents disagree about the expected aggregate endowment growth, then the expected return and variance of asset 0 are no longer constant, and the risk-free rate would have interesting dynamics (see Xiong and Yan (2010) for a recent analysis of the impact of disagreement on the term structure of interest rates).

3. Cross Section of Asset Returns

Based on the market equilibrium results in the previous section, we examine the impact of disagreement on the dynamics of cross sectional returns in this section. First, from equation (2.15), we can derive the instantaneous return of asset k in a closed form.

Corollary 3.1. *(Instantaneous Return) The equilibrium return for asset k is given by*

$$dR_k(t) = \frac{d\phi_k(t)}{\phi_k(t)} + (r + \rho_{0k}\sigma_e\sigma_{D,k} - \sigma_{D,k}\theta_k(t))dt + \sigma_{D,k}dZ_k(t). \tag{3.1}$$

Corollary 3.1 shows that the instantaneous return of asset k depends on the growth in PD ratio and the amount of optimism in the consensus belief. When there is no disagreement about the expected dividend growth rate of asset k, the PD ratio is a constant and the consensus belief coincides with the objective belief, so that $\theta_k(t) = 0$. Consequently the instantaneous return of asset k under the objective belief, $d\hat{R}_k$, is given by

$$d\hat{R}_k(t) = (r + \rho_{0k}\sigma_e\sigma_{D,k})dt + \sigma_{D,k}dZ_k(t). \tag{3.2}$$

Therefore, the excess equilibrium return for asset k under disagreement (from the return under the objective belief) can be written as $dR_k(t) - d\hat{R}_k(t) = d\phi_k(t)/\phi_k(t) - \sigma_{D,k}\theta_k(t)dt$, which increases when the market is pessimistic about the expected dividend growth rate of asset k.

Secondly, we can characterize the excess volatility due to the disagreement.

Corollary 3.2. *(Excess Volatility) The variance of asset k's instantaneous return is given by*

$$Var_t[dR_k(t)] = \sigma_{D,k}^2[1 + \gamma_k(t)]dt, \tag{3.3}$$

where $\gamma_k(t)$ represents the proportional excess variance of asset return due to dispersion in beliefs defined by

$$\gamma_k(t) \equiv \epsilon_k^2\big[G_k^2(\lambda(t))\varphi + 2G_k(\lambda(t))\big] \geq 0 \quad \textit{with} \quad G_k(\lambda(t)) \equiv \frac{\lambda(t)(1-\lambda(t))}{\phi_k(t)\delta_k^{(1)}\delta_k^{(2)}} > 0. \tag{3.4}$$

Corollary 3.2 implies that heterogeneity in beliefs leads to excess volatility in the instantaneous returns, that is $Var_t[dR_k(t)] \geq \sigma^2_{D,k}dt$. Without disagreement about asset k's dividend, $\epsilon_k = 0$, the variance of instantaneous return is a constant and equals to the variance of dividend growth, that is $Var_t[dR_k(t)] = \sigma^2_{D,k}dt$. As pointed out in Yan (2010), the excess volatility is caused by a *double-kick* effect. When there is positive shock in dividend, that is $dZ_k(t) > 0$, the price of asset k increases not only by the increase in dividend $D_k(t)$, but also by the increase in the PD ratio caused by the transfer of consumption share from the agent who is relatively pessimistic about asset k to the agent who is relatively optimistic. Equation (3.4) shows that the excess volatility depends on the current distribution of the consumption shares among agents.

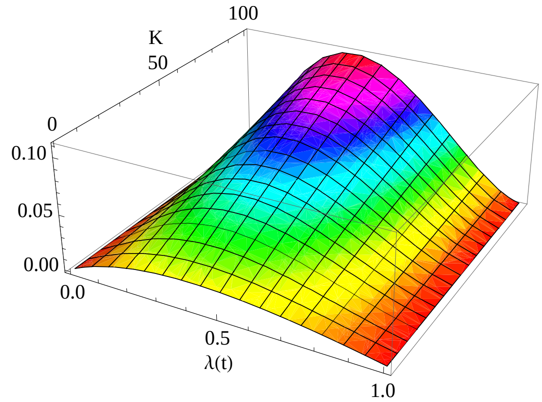


FIGURE 3.1. The impact of unbiased disagreement on the level of *excess volatility* $\bar{\sigma}_k(t)$ of asset k in a stationary market with respect to the consumption share $\lambda(t)$ of the agent 1 is and the number of the risky assets K. Here $\epsilon_k = 0.1$,

In the case of multiple risky assets, we show that the level of excess variance depends on not only the dispersion ϵ_k^2 in beliefs about asset k, but also depend on the overall dispersion $\varphi = \boldsymbol{\epsilon}\boldsymbol{\rho}^{-1}\boldsymbol{\epsilon}$ in beliefs of all the assets, which leads to greater fluctuations in the distribution of consumption shares. The overall dispersion φ in beliefs depends on the *number of risky assets* of which agents disagree about their expected dividend growth rates. To quantify the impact of the number of assets on the excess volatility, we conduct a numerical example. Let[8]

$$\beta = 0.02, \;\; \mu_e = 0.02, \;\; \sigma_e = 0.02, \;\; \mu_{D,k} = 0.02, \;\; \sigma_{D,k} = 0.15, \;\; \rho_{j,k} = 0(j \neq k), \qquad (3.5)$$

where $k, j = 1, 2, \cdots, K$, and K is the total number of risky assets of which agents disagree about their expected dividend growth rates.[9] We assume the *average squared disagreement*

[8]The same set of parameter values apply to all the numerical analysis unless specified otherwise.

[9]Note that asset 0 is a claim to the aggregate endowment process which agents agree about.

among agents for each asset is given by

$$\bar{\epsilon}^2 \equiv \frac{1}{K}\sum_{k=1}^{K}\epsilon_k^2 = 0.01 \qquad \text{for} \qquad k = 1, 2, \cdots, K,$$

which implies that on average $|\mu_{D,k}^{(1)} - \mu_{D,k}^{(2)}| = \bar{\epsilon}\sigma_{D,k} = 0.015$. Thus the overall effect of dispersion in beliefs on the fluctuation in the distribution of consumption shares is given by $\varphi = K\bar{\epsilon}^2$. We define *excess volatility* of asset k as

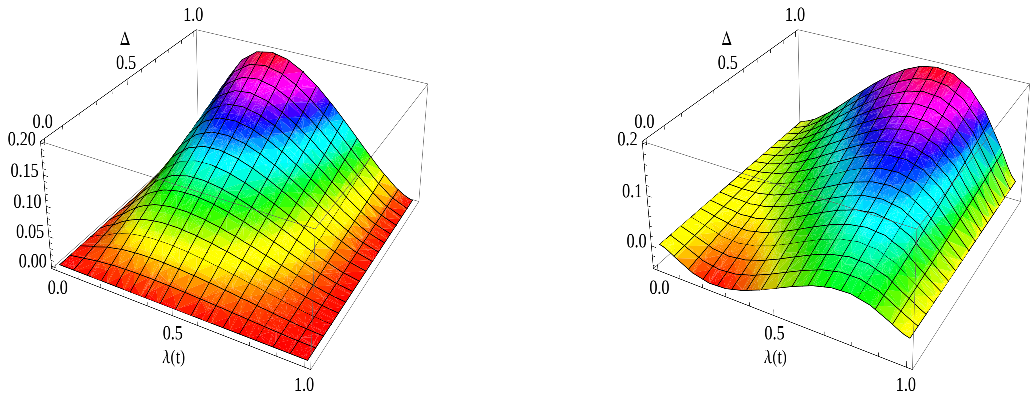
$$\bar{\sigma}_k(t) \equiv \sigma_{D,k}[\sqrt{(1+\gamma_k(t))} - 1].$$

Suppose $\epsilon_k = 0.1$, that is agent 1 is relatively more optimistic than agent 2 about asset k's dividend growth, Fig. 3.1 shows that the excess volatility for asset k increases with the number of assets K. In a market with a single risky asset $K = 1$, excess volatility can only reach a maximum of 1.99% p.a when $\lambda(t) = 0.4$. In comparison, when $K = 100$, the excess volatility can reach a maximum of 11.08% p.a when $\lambda(t) = 0.4$. Intuitively, even though agents may have only limited amount of disagreement about the expected dividend growth for each asset; but when the number of assets is large, the overall effect of disagreement can lead to substantial fluctuations in the distribution of consumption shares, which then causes excess volatility. Note that the excess volatility $\bar{\sigma}_k(t) = 0$ when $\lambda(t)$ approaches to either zero or one, because the fluctuations in the consumption shares are close to zero, see equation (2.14). Therefore, when the market conditions are *strongly* in favor of the agent who is relatively optimistic or pessimistic about asset k, the level of excess volatility is relatively low. However, the level of excess volatility *peaks* when market conditions are *slightly* in favor of the agent who is relatively pessimistic about asset k as indicated by Fig. 3.1.

To examine the *cross section* of excess volatilities when the level of disagreement differs between two assets, we assume that there is a greater disagreement among agents about the expected dividend growth for asset 2 than for asset 1, more precisely,

$$\epsilon_1^2 = \bar{\epsilon}^2(1-\Delta) \qquad \text{and} \qquad \epsilon_2^2 = \bar{\epsilon}^2(1+\Delta), \tag{3.6}$$

where $\Delta \geq 0$ measures the difference in the level of disagreement between assets 1 and 2. In general, Fig. 3.2 shows that there is a *positive* relationship between the level of disagreement and excess volatility, and the volatility is higher for the stock with higher disagreement (which is stock 2). When optimism/pessimism are positively correlated between assets 1 and 2, that is $\epsilon_1 > 0$ and $\epsilon_2 > 0$, Fig. 3.2 (a) shows that the difference in excess volatility between assets 1 and 2 is the highest at $\lambda(t) = 0.4$, which means that when market conditions are slightly favoring agent 2 who is relative pessimistic compared to agent 1 about the dividend growth of both assets. In comparison, when optimism/pessimism are negatively correlated between



(a) $\epsilon_1 > 0, \ \ \epsilon_2 > 0$ (b) $\epsilon_1 > 0, \ \ \epsilon_2 < 0$

FIGURE 3.2. The difference in *excess volatility* $\bar{\sigma}_2(t) - \bar{\sigma}_1(t)$ between assets 2 and 1 in a stationary market with respect to the consumption share $\lambda(t)$ of the agent 1 and the difference in the level of disagreement Δ. Here the number of assets $K = 100$.

the two assets, Fig. 3.2 (b) shows that the difference in excess volatility is the highest when $\lambda(t) = 0.6$, which means that when market conditions are slightly favoring agent 1 who is relatively optimistic about asset 1's dividend growth, and relative pessimistic about asset 2's dividend growth.

Next, we examine the impact on the excess covariance of any two risky assets about which agents disagree.

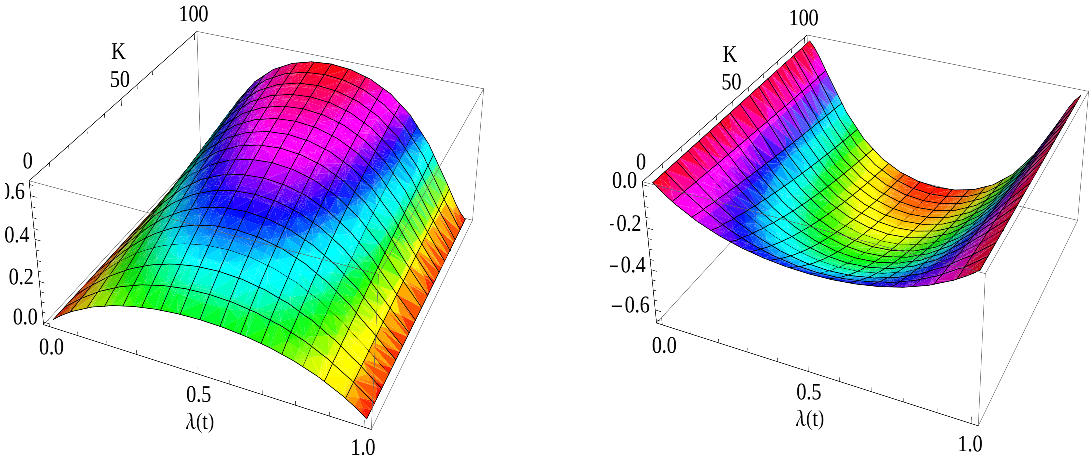
Corollary 3.3. *(Excess Covariance) The covariance between assets* 1 *and* 2 *is given by*

$$Cov_t[dR_1(t), dR_2(t)] = \sigma_{D,1}\sigma_{D,2}\big[\rho_{12} + \gamma_{1,2}(t)\big]dt,$$

where $\gamma_{1,2}(t)$ *represents the excess covariance given by*

$$\gamma_{1,2}(t) = \epsilon_1\epsilon_2\big[\varphi G_1(\lambda(t))G_2(\lambda(t)) + G_1(\lambda(t)) + G_2(\lambda(t))\big].$$

Corollary 3.3 shows that the excess covariance depends on the correlation between agents' optimism/pessimism about the two assets. When optimism/pessimsm are positively (negatively) correlated between assets 1 and 2, that is $\epsilon_1\epsilon_2 > (<)0$, there is a positive (negative) excess covariance between the instantaneous returns, that is $Cov_t[dR_1(t), dR_2(t)] > (<)\rho_{12}\sigma_{D,1}\sigma_{D,2}dt$. Intuitively, as already used to explain the excess variance, a positive shock in asset 1's dividend leads to a transfer of consumption shares from the agent who is relatively pessimistic about asset 1 to the agent who is relatively optimistic, which then leads to an increase in the PD ratio of asset 1. Now if the optimisim/pessimism is positively (negatively) correlated, the transfer of consumption share also contributes to an increase (decrease) in the PD ratio of asst 2. Therefore, although two assets may have independent dividend processes, however, with heterogeneity in beliefs, fluctuations in the distribution of consumption shares among agents can lead to correlated returns of two assets. Note that, if agents have homogeneous beliefs about the expected dividend growth rate for one asset, say for example, asset 1 ($\epsilon_1 = 0$), then optimism/pessimism is uncorrelated between assets and so the covariance between their returns is given by the covariance between their dividend processes. Hence the excess correlation is generated by the disagreements in both assets.



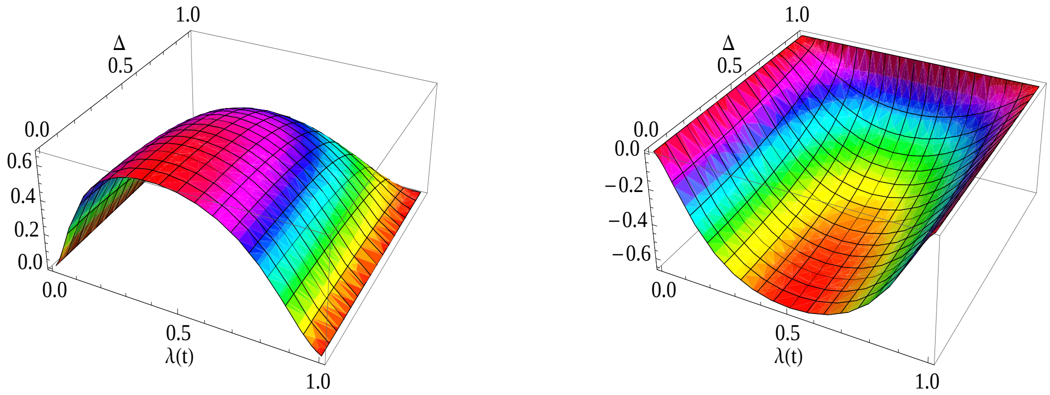
(a) $\epsilon_1 > 0, \;\; \epsilon_2 > 0$ (b) $\epsilon_1 > 0, \;\; \epsilon_2 < 0$

FIGURE 3.3. The impact of disagreement on the level of *excess correlation* $\bar{\rho}_{1,2}(t)$ between assets 1 and 2 in a stationary economy with $\epsilon_1^2 = \epsilon_2^2 = \bar{\epsilon}^2$ with respect to the consumption share $\lambda(t)$ of the agent who is relative optimistic about asset k and the total number K of assets to which agents disagree.

We demonstrate the result numerically in Fig. 3.3, in which we assume the same level of disagreement for both assets, more precisely, $\epsilon_1^2 = \epsilon_2^2 = \bar{\epsilon}^2$. We define the excess correlation between the returns of assets 1 and 2 as

$$\bar{\rho}_{1,2}(t) \equiv \frac{\gamma_{1,2}(t)}{\sqrt{(1+\gamma_1(t))(1+\gamma_2(t))}}.$$

In Fig. 3.3 (a), we show that although the dividends between assets 1 and 2 are uncorrelated, when $\epsilon_1 > 0, \;\; \epsilon_2 > 0$, the correlation $\bar{\rho}_{1,2}(t)$ between the instantaneous returns of assets 1 and 2 is *strictly positive* for all value of $\lambda(t) \in (0, 1)$. In Fig. 3.3 (b), we show that when $\epsilon_1 > 0, \;\; \epsilon_2 < 0$, $\bar{\rho}_{1,2}(t)$ is strictly *negative* for $\lambda(t) \in (0, 1)$. This demonstrates that there is a positive (negative) excess covariance between the instantaneous returns when optimism/pessimsm are positively (negatively) correlated between assets 1 and 2. Moreover, the absolute magnitude of the correlation increases with the number of assets K in the market. In a market with *two* risky assets, the absolute correlation $|\bar{\rho}_{1,2}(t)|$ can reach a maximum of 0.223 (when $\lambda(t) = 0.5$). In comparison, in a market with $K = 100$ assets, the absolute correlation can reach a maximum of 0.653 (when $\lambda(t) = 0.5$).



(a) $\epsilon_1 > 0, \ \ \epsilon_2 > 0$ (b) $\epsilon_1 > 0, \ \ \epsilon_2 < 0$

FIGURE 3.4. The *excess correlation* between assets 1 and 2, $\bar{\rho}_{1,2}(t)$, with respect to the consumption share $\lambda(t)$ of agent 1 and the difference in disagreement level Δ. Here $K = 100$.

When the level of disagreement differs between assets, we show in Fig. 3.4 that the absolute magnitude of excess correlation *decreases* in the difference of disagreement level (measured by Δ in equation (3.6)). The intuition is that as Δ increases, there is more disagreement about asset 2's dividend growth. However the beliefs about asset 1's dividend growth are becoming more homogeneous. When $\Delta = 1$, beliefs are homogeneous about asset 1's dividend growth, that is $\epsilon_1 = 0$, and according to Corollary 3.3, in this case the excess covariance between assets is $\gamma_{1,2} = 0$. When the dividend processes are uncorrelated, this means the excess correlation between assets $\bar{\rho}_{1,2}(t) = 0$.

Finally, we examine the impact of disagreement on the cross section of expected excess returns.

Corollary 3.4. *(Expected Excess Return) Assume $s_1 = s_2$, then the expected excess return of asset k is given by*

$$\mathbb{E}_t[dR_t - rdt] = \sigma_{D,k}\big[\rho_{0,k}\sigma_e - \theta_k(t) + \epsilon_k G_k(\lambda(t))[1/2 - \lambda(t)]\varphi\big]dt.$$

In Corollary 3.4, the assumption of $s_1 = s_2$ implies that both agents survive in the long-run and thus heterogeneity in beliefs can persist. The result shows that when there is no disagreement about asset k's dividend growth rate, that is $\epsilon_k = 0$, the expected excess return is equal to the covariance between the aggregate endowment process and asset k's dividend process. However, with heterogeneity in beliefs, there is a negative (positive) relation between dispersion in beliefs and expected excess return when the *relatively optimistic agent* has a consumption share

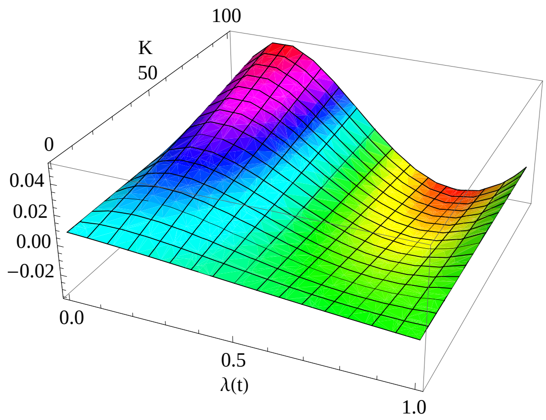
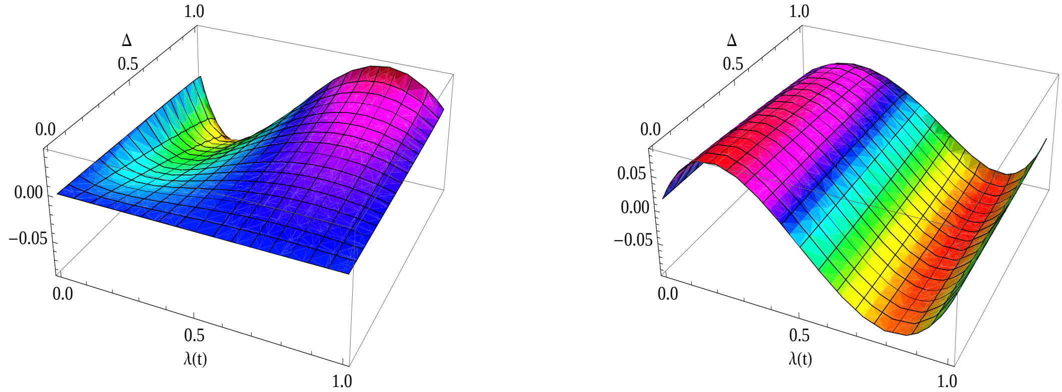


FIGURE 3.5. The impact of disagreement on the level of *expected excess return* $\mu_k(t) - r$ of asset k in a stationary market with respect to the consumption share $\lambda(t)$ of the agent who is relative optimistic about asset k and the total number of assets K to which agents disagree. Here $\epsilon_k = 0.1$.

less (greater) than a half. The level of the expected excess return is greater than the standard one ($\rho_{0,k}\sigma_{D,k}\sigma_e$) when $\epsilon_k G_k(\lambda(t))(\frac{1}{2} - \lambda(t))\varphi > \theta_k(t)$. When $\lambda(t) = \frac{1}{2}$, the expected excess return is given by the standard one in a market with homogeneous beliefs.

To quantify the effect of disagreement, in Fig. 3.5, we assume $\epsilon_k = 0.1$, and $|\theta_{i,k}| = \epsilon_k/2$ for $i = 1, 2$, so that when consumption shares are equal; agents on average perceive the objective expected dividend growth for asset k, that is $\theta_k(t) = 0$ when $\lambda(t) = \frac{1}{2}$. Fig. 3.5 shows that the absolute magnitude of the expected excess return $|\mu_k(t) - r|$ increases with the number of assets K. In a market with a single risky asset, the expected excess return $\mu_k(t) - r$ reaches a maximum of 0.75% p.a (when $\lambda(t) = 0$) and a minimum of -0.75% p.a (when $\lambda(t) = 1$). In comparison, in a market with $K = 100$ risky assets, $\mu_k(t) - r$ can reach a maximum of 5.22% p.a (when $\lambda(t) = 0.2$) and a minimum of -3.44% p.a (when $\lambda(t) = 0.8$). Intuitively, when the dividend processes are uncorrelated, the expected excess return or *equity premium* is a product of the MPR and stock volatility, that is $\mu_k(t) - r = \sigma_k(t)\kappa_k(t)$, where $\sigma_k(t) = \sigma_{D,k} + \bar{\sigma}_k(t)$. Therefore, the equity premium is large and positive when the MPR and excess volatility are both positive and reasonably large, which occurs when the market conditions are *moderately* in favor of the agent who is pessimistic about asset k (as shown in Fig. 3.5), and the magnitude increases with the total number of assets in the market. Note that equity premium does not *monotonically* decrease with $\lambda(t)$. This is because the stock volatility is a *concave* function of $\lambda(t)$, although the MPR increases monotonically as $\lambda(t)$ decrease.

Next, we examine the relationship between the level of disagreement and expected returns. We consider two assets, same as in Figs 3.2 and 3.4, agents have a higher level of disagreement about asset 2 than asset 1, and the difference in disagreement levels is measured Δ. Fig. 3.6 (a)



(a) $\epsilon_1 > 0, \;\; \epsilon_2 > 0$ (b) $\epsilon_1 > 0, \;\; \epsilon_2 < 0$

FIGURE 3.6. The impact of difference in disagreement level (Δ) on the difference in *expected returns* between assets 1 and 2, $\mu_1(t) - \mu_2(t)$, where $\lambda(t)$ is the consumption share of agent 1. Assume the number of assets $K = 100$.

shows that when optimism/pessimism are positively correlated between the assets, asset 1 has a higher expected return than asset 2 so that $\mu_1(t) > \mu_2(t)$ when $\lambda(t) > 0.5$; and the difference in expected returns $\mu_1(t) - \mu_2(t)$ increases with larger difference in disagreement levels Δ. However, the opposite is true, that is $\mu_1(t) < \mu_2(t)$ when $\lambda(t) > 0.5$ and $\mu_1(t) - \mu_2(t)$ decreases in Δ. Intuitively, when $\lambda(t) > 0.5$, both assets are over-priced compared to the market under homogeneous beliefs. However, asset 2 is more over-price because of there is a greater level of disagreement compared to asset 1. Hence the expected future return of asset 2 is lower than that of asset 1. Diether et al. (2002) find a negative relationship between dispersion in analysts' forecasts and expected stock returns. If dispersion in analysts' forecasts is a good proxy for disagreement,[10] then our result is consistent with the finding in Diether et al. (2002) when $\lambda(t) > 0.5$, that is when the relatively optimistic agent has a larger consumption/wealth share than the pessimistic agent.

Note that since excess volatility is always increasing in disagreement level. In the case of $\lambda(t) > 0.5$, there is a *negative* relationship between excess volatility and expected returns. Now we explain how is our result related to the empirical finding in Ang et al. (2006). Suppose that the average disagreement level $\bar{\epsilon}$, which is positively correlated with aggregate market volatility is *time-varying*, intuitively, the expected return of assets with a relatively high disagreement level are more sensitive to changes in $\bar{\epsilon}$ compare to assets with a low disagreement level.[11] And

[10] Johnson (2004) suggests dispersion in analysts' forecast is a measure of uncertainty in signals about the firm's value. By separating the effect of uncertainty from the effect of disagreement, Doukas, Kim and Pantzalis (2006) find a positive relationship between disagreement and expected stock returns.

[11] For example, when $\epsilon_k = 0$, the expected return of asset k is a constant and thus independent to the changes in the average disagreement level $\bar{\epsilon}$.

if $\lambda(t) > 0.5$, which is case consistent with the finding of Diether et al. (2002), high disagreement assets which are more sensitive to changes in aggregate market volatility is expected to have lower future return compare to low disagreement assets. Thus, in this case, we are able to reconcile the findings of Diether et al. (2002) and Ang et al. (2006). Furthermore, when optimism/pessimism are positively correlated, there should also be a *positive* excess correlation between asset returns, which have not been tested empirically.

Alternatively, when optimism/pessimism are negatively correlated between assets, that is $\epsilon_1 > 0,\ \epsilon_2 < 0$, which is examined in Fig. 3.6 (b). In this case, the difference in expected return $\mu_1(t) - \mu_2(t)$ between assets 1 and 2 is positive when $\lambda(t) < 0.5$; that is when the consumption share of the agent who is relatively optimistic (pessimistic) about the dividend growth of asset 1 (asset 2) is less than a half. Intuitively, in this case, since market conditions are in favor of agent 2 who is pessimistic (optimistic) about asset 1 (asset 2)'s dividend growth, asset 1 is under-priced while asset 2 is over-priced compared to the market under homogeneous beliefs. Therefore the expected return is higher for asset 1 than for asset 2. However, Fig. 3.6 (b) shows that the difference in expected returns $\mu_1(t)-\mu_2(t)$ is actually *decreasing* in Δ, which however *does not* support the finding of Diether et al. (2002) that a greater level of disagreement leads to a lower future return.

In summary, we have shown that the disagreement can have significant impact on market characteristics of cross sectional returns.

4. MULTIPLE AGENTS

In this section, we extend the analysis in the previous sections for two agents to multiple agents in the economy. We assume that there are I agents in the economy indexed by $i = 1, 2, \cdots, I$ with equal endowments, that is $e_i(t) = e(t)/I$ for $i = 1, 2, \cdots, I$. Agent i's subjective probability measure is given by equation (2.1). Moreover, we assume that the KL distances among agent's subjective probability measure and the objective probability measure are the same, that is $s_i = s$ for all i, so that for any $i, h \in I$,

$$\frac{M_h(t)}{M_i(t)} = \frac{\lambda_h(t)}{\lambda_i(t)} = \exp\left\{\boldsymbol{\epsilon}_{hi}^T \boldsymbol{\rho}^{-1}\mathbf{Z}(t)\right\},$$

where $\boldsymbol{\epsilon}_{hi} \equiv \boldsymbol{\theta}_h - \boldsymbol{\theta}_i$ measures the disagreement between agents h and i about the expected dividend growth rates and $\lambda_i(t)$ is the consumption share of agent i. It can be shown that the consensus belief with multiple agents is given by

$$M(t) = \sum_{h=1}^{I} \lambda_h(t) M_h(t) \qquad \text{and} \qquad \frac{dM(t)}{M(t)} = \boldsymbol{\theta}(t)^T \boldsymbol{\rho}^{-1} d\mathbf{Z}(t),$$

where $\boldsymbol{\theta}(t) = \sum_{h=1}^{I} \lambda_h(t)\boldsymbol{\theta}_h$ measures the amount of *optimism* in the consensus belief. Note that

$$\bar{\boldsymbol{\epsilon}}_i(t) \equiv \sum_{h=1}^{I} \lambda_h(t)\boldsymbol{\epsilon}_{hi} = \boldsymbol{\theta}(t) - \boldsymbol{\theta}_i \tag{4.1}$$

measures the disagreement in the expected dividend growth rates between agent i's subjective belief and the consensus belief. Using the fact that

$$\lambda_i(t) = \frac{M_i(t)}{\sum_{h=1}^{I} M_h(t)} = \left[\sum_{h=1}^{I} \frac{M_h(t)}{M_i(t)}\right]^{-1} = \left[\sum_{h=1}^{I} \exp\left\{\boldsymbol{\epsilon}_{hi}^T \boldsymbol{\rho}^{-1}\mathbf{Z}(t)\right\}\right]^{-1}, \tag{4.2}$$

we have the stochastic differential equation for the consumption share $\lambda_i(t)$ of agent i,

$$\frac{d\lambda_i(t)}{\lambda_i(t)} = \left[\bar{\varphi}_i(t) - \frac{1}{2}\sum_{h=1}^{I} \lambda_h(t)\varphi_{hi}\right]dt - \bar{\boldsymbol{\epsilon}}_i(t)^T \boldsymbol{\rho}^{-1} d\mathbf{Z}(t), \tag{4.3}$$

where $\varphi_{hi} \equiv \boldsymbol{\epsilon}_{hi}^T \boldsymbol{\rho}^{-1}\boldsymbol{\epsilon}_{hi}$ measures the KL distance between the subjective probability measures of agents h and i, and $\bar{\varphi}_i(t) \equiv \bar{\boldsymbol{\epsilon}}_i(t)^T \boldsymbol{\rho}^{-1}\bar{\boldsymbol{\epsilon}}_i(t)$ measures the KL distance between agent i's subjective probability measure with the consensus probability measure at time t, which also determines the amount of fluctuations in agent i's consumption share. Intuitively, agent i's *relative changes in consumption share* fluctuates more when his subjective belief deviates further from the consensus belief in the market. Furthermore, equation (4.3) also shows that one can expect a positive (negative) change in agent i's consumption share when $\bar{\varphi}_i(t) > (<)\frac{1}{2}\sum_{h=1}^{I} \lambda_h(t)\varphi_{hi}$, that is when the distance between probability measures of agent i and the consensus consumer at time t is greater (less) than *a half* of the average distance between the probability measures of agent i with the rest the agents in the economy.

In the case of multiple agents, Equation (4.2) shows that, even if all agents survive in the long run, their consumption shares may not be identically distributed. Therefore, in general the unconditional means of their consumptions shares are not the same. They are only the same, that is $\mathbb{E}[\lambda_i(t)] = \frac{1}{I}$ for $i = 1, 2, \cdots, I$ when φ_{hi} is independent of h and i, meaning that the KL distances between agents' subjective probability measures are all equal.

Similar, we can show that the equilibrium asset prices is a consumption-share-weighted average of the prices which would prevail in economies in each of which there is only one agent,

$$S_k(t) = \phi_k(t)D_k(t) = \left[\sum_{i=1}^{I} \frac{\lambda_i(t)}{\delta_{i,k}}\right] D_k(t)$$

for $k = 1, \cdots, K$ where $\delta_{i,k} = r + \rho_{0k}\sigma_e\sigma_{D,k} - \mu_{D,k}^{(i)} > 0$ and $r = \beta + \mu_e - \sigma_e^2$ is the equilibrium interest rate (since agents agree on the expected growth rate of the aggregate endowment process). Furthermore, we derive the equilibrium asset return in the following corollary.

Corollary 4.1. *(Instantaneous asset return) The equilibrium return for asset k is given by*

$$dR_k(t) = \frac{d\phi_k(t)}{\phi_k(t)} + (r + \rho_{0k}\sigma_e\sigma_{D,k} - \sigma_{D,k}\theta_k(t))dt + \sigma_{D,k}dZ_k(t). \tag{4.4}$$

The result in Corollary 4.1 is a general version of Corollary 3.4. The only *difference* is that asset k's PD ratio $\phi_k(t)$ and the level of optimism in the consensus belief $\theta_k(t)$ now depend on the distribution of consumption shares among I agents.

Corollary 4.2. *(Expected excess return) Define for each asset k a conditional expectation operator $\mathbb{E}_{k,t}(X) = \sum_{i=1}^{I} \omega_{i,k}(t)x_i$ of $X = (x_1, \cdots, x_I)$ based on the "price impact" of each agent on asset k,*

$$\{\omega_{i,k}(t)\} = \left\{\lambda_i(t)\frac{1/\delta_{i,k}}{\phi_k(t)}\right\} = \left\{\frac{\lambda_i(t)/\delta_{i,k}}{\sum_{h=1}^{I}\lambda_h(t)/\delta_{h,k}}\right\}. \tag{4.5}$$

Then the expected excess instantaneous return of asset k is given by

$$\mathbb{E}_t[dR_k(t) - rdt] = \big(\rho_{0k}\sigma_e\sigma_{D,k} - \sigma_{D,k}\theta_k(t) + \mathbb{E}_{k,t}[\mu_\lambda]\big)dt, \tag{4.6}$$

where $\mu_\lambda^{(i)}(t) \equiv \mathbb{E}_t[d\lambda_i(t)/\lambda_i(t)]$ is the expected instantaneous growth in agent i's consumption share under the objective probability measure.

Corollary 4.2 shows that when there are multiple agents in the economy, the importance of agent i's subjective belief on asset k is measured by his *price impact* $\omega_i(t)$ (which is strictly positive and sum up to one across all agents). Equation (4.6) shows that the expected excess instantaneous return of asset k is positively related to the *price-impact-weighted average* of the expected growth in agents' consumption shares measured by $\mathbb{E}_{k,t}[\mu_\lambda]$. Note that $\mathbb{E}_{k,t}[\mu_\lambda] = 0$ when agents have homogeneous beliefs about asset k's expected dividend growth rate. In this case, the expected excess return is given by the standard term $\rho\sigma_{D,k}\sigma_e$. Agent i's has a larger price impact on asset k if he is more optimistic about asset k relative to the other agents or he has larger consumption share. Suppose agent i has a large consumption share, then according to equation (4.3), the expected growth of his consumption share is relatively small compared to an agent with a small consumption share, which means that agent i's belief is actually less important in determining $\mathbb{E}_{k,t}[\mu_\lambda]$. Therefore, to obtain a higher level of expected excess return for asset k, there should be a negative correlation between agents' current level of consumption shares and their level of relative optimism of asset k. In other words, this is the case when the agents who are relatively optimistic about asset k's dividend growth have a smaller consumption share than agents who are relatively pessimistic about asset k.

Corollary 4.3. *(Variance and Covariance)*

(i). *The variance of asset k is given by*

$$Var_t[dR_k(t)] = [\sigma_{D,k}^2 + \gamma_k(t)]dt, \tag{4.7}$$

where the excess variance is given by

$$\gamma_k(t) = \mathbb{E}_{k,t}[\bar{\boldsymbol{\epsilon}}]^T \boldsymbol{\rho}^{-1} \mathbb{E}_{k,t}[\bar{\boldsymbol{\epsilon}}] - 2\sigma_{D,k} \mathbb{E}_{k,t}[\bar{\epsilon}_k] > 0.$$

(ii). *Consider assets* 1 *and* 2*, the covariance of their returns is given by*

$$Cov_t[dR_1(t), dR_2(t)] = [\rho_{12}\sigma_{D,1}\sigma_{D,2} + \gamma_{1,2}(t)]dt, \tag{4.8}$$

where the excess covariance is given by

$$\gamma_{1,2}(t) = \mathbb{E}_{1,t}[\bar{\boldsymbol{\epsilon}}]^T \boldsymbol{\rho}^{-1} \mathbb{E}_{2,t}[\bar{\boldsymbol{\epsilon}}] - \sigma_{D,1} \mathbb{E}_{2,t}[\bar{\epsilon}_1] - \sigma_{D,2} \mathbb{E}_{1,t}[\bar{\epsilon}_2].$$

In Corollary 4.3, equation (4.7) shows that the level of excess variance of asset k's instantaneous return consists of two components. The first component is the variance of growth in asset k's PD ratio, that is $Var_t[d\phi_k(t)/\phi_k(t)] = \mathbb{E}_{k,t}[\bar{\boldsymbol{\epsilon}}]^T \boldsymbol{\rho}^{-1} \mathbb{E}_{k,t}[\bar{\boldsymbol{\epsilon}}]$, which increases with the price-impact-weighted disagreement between agent i and the consensus consumer about the expected dividend growth of asset k. The second component is the covariance between the growth in asset k's PD ratio and its dividend process, that is $Cov_t[d\phi_k(t)/\phi_k(t), dD_k(t)/D_k(t)] = -\sigma_{D,k}\mathbb{E}_{k,t}[\bar{\epsilon}_k]$, which is positive (negative) if the price-impact-weighted average subjective belief about asset k's dividend growth is relatively optimistic compared to the consensus belief. Intuitively, this is likely to be the case since agent who are relatively optimistic about asset k has a larger price-impact, which implies that the excess variance is positive.

Furthermore, equation (4.8) shows that the excess covariance between assets 1 and 2 consists of three components. The first component is the covariance between the growth in PD ratios of the two assets, $Cov_t[d\phi_1(t)/\phi_1(t), d\phi_2(t)/\phi_2(t)] = \mathbb{E}_{1,t}[\bar{\boldsymbol{\epsilon}}]^T \boldsymbol{\rho}^{-1} \mathbb{E}_{2,t}[\bar{\boldsymbol{\epsilon}}]$, which is positive (negative) when agents' price-impact-weighted optimism/pessimism relative to the consensus belief is positively (negatively) correlated between assets 1 and 2. In other words, the growth in PD ratios are positively (negative) correlated if subjective probability measures of agents who have larger price impact on asset 1 deviate more from that of the consensus consumer, and those same agents who also have a larger (smaller) price impact on asset 2. The second and third components are the covariance between the growth of PD ratio of one asset and the growth of dividend of the other agent. These two components are positive (negative) if the agents with large price impact on one asset is relatively optimistic (pessimistic) about the other asset compared to the consensus belief.

5. CONCLUSION

In this paper, we analyze the impact of disagreement on cross section of asset returns in a market with *multiple* dividend processes and risky assets. In general, the market with multiple risky assets can be very different from a market with a single risky asset, although they share some common features. For example, in a market with a single risky asset, the equity premium is time-varying with the distribution of consumption shares among agents and stock volatility is stochastic and in excess of dividend volatility. However, in a market with multiple risky assets, fluctuations in the distribution of consumption shares among agents increase proportionally with the number of assets (to which agents disagree). As a result, equity premium varies significantly with the fluctuations in the distribution of consumption shares even when the disagreements among agents are relatively small about each asset, whereas the variation in the equity premium is *negligible* in a market with a single risky asset. Moreover, the level of excess volatility in stock return increases with the number of assets to which agents disagree, and greatly *exceeds* the level of excess volatility in a market with a single risky asset that is negligible when the disagreement is relatively small. Lastly and most importantly, we model the impact of disagreement on the *covariance* between the instantaneous returns of two different assets. Intuitively, we find that when optimism/pessimism are positively (negatively) correlated between the two assets, there is a positive (negative) excess covariance between their returns. When optimism/pessimism are uncorrelated, the covariance is equal to the covariance between assets' dividends. We show a positive relation between the level of disagreement and excess volatility. We also provide conditions for a negative relation between disagreement level and the expected return among similar stocks. The results obtained in this paper provide an explanation to time variation of cross sectional returns. They also provide some interesting implications to empirical literature in cross-sectional returns.

Appendix A. Proofs

A.1. **Proof of Lemma 2.1.** The agent i's first-order condition, $M_i(t)/c_i(t) = \eta_i e^{-\beta t}\xi(t)$ leads to equation (2.6). Substituting (2.6) into the budget constraint in equation (2.5) obtains

$$\frac{1}{\eta_i}\mathbb{E}[\int_0^\infty e^{-\beta t} M_i(t)dt] = \frac{1}{\eta_i}\int_0^\infty e^{-\beta t}\mathbb{E}[M_i(t)]dt = \frac{1}{2}\mathbb{E}[\int_0^\infty \xi(t)e(t)dt].$$

Since $\mathbb{E}[M_i(t)] = M_i(0) = 1$, we obtain that $\frac{1}{\eta_i}\int_0^\infty e^{-\beta t}dt = \frac{1}{2}\mathbb{E}[\int_0^\infty \xi(t)e(t)dt]$, which leads to the explicit expression for the Lagrange multiplier η_i. The share of aggregate consumption of agent 1 is then given by

$$\lambda(t) = \frac{c_1(t)}{c_1(t)+c_2(t)} = \frac{\eta_1 M_1(t)}{\eta_1 M_1(t) + \eta_2 M_2(t)}.$$

Note that $\eta_1 = \eta_2 = \eta$. Hence we obtain equation (2.7). □

A.2. **Proof of Proposition 2.3.** To prove (i) and (ii), given that the state price density process in equation (2.9), we obtain

$$\frac{d\xi(t)}{\xi(t)} = -\left[\beta + \mu_e - \sigma_e^2 - \sigma_e\theta_0(t)\right]dt - \left[\sigma_e dZ_0(t) + \boldsymbol{\theta}(t)^T\boldsymbol{\rho}^{-1}d\mathbf{Z}(t)\right].$$

However, the SDE of the state price density process must also satisfy the form in equation (2.3). Equating the diffusion coefficients leads the market prices of risk in (2.11), and equating the drift coefficient leads to the equilibrium interest rate in equation (2.12), given the assumption that $\theta_{i,0} = 0$ for all i, which implies that $\theta_0(t) = 0$.

To prove (iii), the aggregate consumption share of agent 1 in (2.13) is given by first re-writing equation (2.7) as

$$\lambda(t) = (1 + M_2(t)/M_1(t))^{-1}$$

and then substituting the expression for agents' subjective beliefs in equation (2.1) into the equation above. The SDE for $\lambda(t)$ can be obtained by applying Itô's lemma.

To prove (iv), the consensus belief in (2.8) satisfies

$$M(s)/M(t) = \lambda(t)M_1(s)/M_1(t) + (1-\lambda(t))M_2(s)/M_2(t) \text{ for } s > t.$$

Therefore, the equilibrium price of asset k in (2.10) can be written as in (2.15), where

$$\begin{aligned} 1/\delta_{i,k} &= \mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)}\frac{M_i(s)}{M_i(t)}\frac{e(t)}{e(s)}\frac{D_k(s)}{D_k(t)}\,ds\right] \\ &= \int_t^\infty \exp\{-(\beta + \mu_e - \sigma_e^2 + \rho_{0k}\sigma_e\sigma_{D,k} - \mu_{D,k}^{(i)}\}(s-t)ds, \end{aligned}$$

which gives $\delta_{i,k} = r + \rho_{0k}\sigma_e\sigma_{D,k} - \mu_{D,k}^{(i)}$ assuming that $\mu_{D,k}^{(i)} < r + \rho_{0k}\sigma_e\sigma_{D,k}$. □

A.3. **Proof of Corollary 3.1.** The equilibrium return of asset k is given by equation (2.17), while the covariance between the PD ratio and dividend are given by

$$\frac{d\phi_k(t)}{\phi_k(t)}\,\frac{dD_k(t)}{D_k(t)} = \frac{\sigma_{D,k}}{\phi_k(t)}\left[\frac{1}{\delta_{1,k}} - \frac{1}{\delta_{2,k}}\right]d\lambda(t)dZ_k(t) = \frac{\sigma_{D,k}}{\phi_k(t)}\frac{\sigma_{D,k}\epsilon_k}{\delta_{1,k}\delta_{2,k}}d\lambda(t)dZ_k(t).$$

From equation (2.14), we have

$$d\lambda(t)dZ_k(t) = \lambda(t)(1-\lambda(t))\boldsymbol{\epsilon}^T\boldsymbol{\rho}^{-1}d\mathbf{Z}(t)dZ_k(t) = \lambda(t)(1-\lambda(t))\epsilon_k dt.$$

Furthermore, from the expression of the equilibrium PD ratio in equation (2.16), we can write

$$\lambda(t)(1-\lambda(t)) = \frac{(\delta_{1,k}\delta_{2,k}\phi_k(t) - \delta_{1,k})(\delta_{2,k} - \delta_{1,k}\delta_{2,k}\phi_k(t))}{(\delta_{2,k}-\delta_{1,k})^2}.$$

After simplifying and rearranging, we obtain

$$\frac{d\phi_k(t)}{\phi_k(t)}\,\frac{dD_k(t)}{D_k(t)} = [r + \rho_{0k}\sigma_e\sigma_{D,k} + \sigma_{D,k}\theta_k(t) - (\mu_{D,k} + 1/\phi_k(t))]dt.$$

Substituting this expression into equation (2.17), we obtain equation (3.1). □

A.4. **Proof of Corollary 3.2.** Based on equation (3.1), we have the variance of the instantaneous return of asset k given by

$$Var_t[dR_k(t)] = \sigma_{D,k}^2 dt + Var_t\left[\frac{d\phi_k(t)}{\phi_k(t)}\right] + 2Cov_t\left[\frac{d\phi_k(t)}{\phi_k(t)}, \sigma_{D,k}dZ_k(t)\right]. \tag{A.1}$$

Following equation (2.14) and the fact that

$$d\phi_k(t) = \frac{\delta_{2,k} - \delta_{1,k}}{\delta_{1,k}\delta_{2,k}}d\lambda(t) = \frac{\epsilon_k}{\delta_{1,k}\delta_{2,k}}d\lambda(t),$$

the variance of the PD ratio becomes

$$Var_t\left[\frac{d\phi_k(t)}{\phi_k(t)}\right] = \left[\frac{\lambda(t)(1-\lambda(t))}{\delta_{1,k}\delta_{2,k}\phi_k(t)}\right]^2 \varphi\ \sigma_{D,k}^2\epsilon_k^2\ dt, \tag{A.2}$$

and the covariance between the PD ratio and asset's dividend is then given by

$$Cov_t\left[\frac{d\phi_k(t)}{\phi_k(t)}, \sigma_{D,k}dZ_k(t)\right] = \left[\frac{\lambda(t)(1-\lambda(t))}{\phi_k(t)\ \delta_{1,k}\delta_{2,k}}\right]\sigma_{D,k}^2\epsilon_k^2\ dt. \tag{A.3}$$

Substituting equations (A.2) and (A.3) into (A.1) leads to the volatility of asset k in (3.3) and the excess volatility of asset k in (3.4). □

A.5. **Proof of Corollary 3.4.** The expected excess return of asset k from equation (3.1) is given by

$$\mathbb{E}_t[dR_k(t) - rdt] = \rho_{0,k}\sigma_{D,k}\sigma_e + \mathbb{E}_t[d\lambda(t)](1/\delta_{1,k} - 1/\delta_{2,k})/\phi_k(t). \tag{A.4}$$

Substitute the expression for $\mathbb{E}_t[d\lambda(t)]$ from equation (2.14) into (A.4) leads to result. □

A.6. **Proof of Corollary 3.3.** The covariance between the instantaneous returns of assets 1 and 2 can be written as

$$Cov_t[dR_1(t), dR_2(t)] = \rho_{1,2}\sigma_{D,1}\sigma_{D,2}dt + Cov_t\left[\frac{d\phi_1(t)}{\phi_1(t)}, \frac{d\phi_2(t)}{\phi_2(t)}\right] + Cov_t\left[\frac{d\phi_1(t)}{\phi_1(t)}, \frac{dD_2(t)}{D_2(t)}\right]$$
$$+ Cov_t\left[\frac{d\phi_2(t)}{\phi_2(t)}, \frac{dD_1(t)}{D_1(t)}\right]. \tag{A.5}$$

Note that

$$Cov_t\left[\frac{d\phi_1(t)}{\phi_1(t)}, \frac{d\phi_2(t)}{\phi_2(t)}\right] = \left[\frac{\lambda(t)^2(1-\lambda(t))^2}{\phi_1(t)\phi_2(t)\delta_{1,1}\delta_{2,1}\delta_{1,2}\delta_{2,2}}\right]\varphi\sigma_{D,1}\sigma_{D,2}\epsilon_1\epsilon_2 dt,$$
$$Cov_t\left[\frac{d\phi_1(t)}{\phi_1(t)}, \frac{dD_2(t)}{D_2(t)}\right] = \left[\frac{\lambda(t)(1-\lambda(t))}{\phi_1(t)\delta_{1,1}\delta_{2,1}}\right]\sigma_{D,1}\sigma_{D,2}\epsilon_1\epsilon_2 dt,$$
$$\text{and } Cov_t\left[\frac{d\phi_2(t)}{\phi_2(t)}, \frac{dD_1(t)}{D_1(t)}\right] = \left[\frac{\lambda(t)(1-\lambda(t))}{\phi_2(t)\delta_{1,2}\delta_{2,2}}\right]\sigma_{D,1}\sigma_{D,2}\epsilon_1\epsilon_2 dt.$$

Substituting these expressions into equation (A.5) leads to the result. □

A.7. **Proof of Corollary 4.1.** Since the PD ratio of asset k follows

$$d\phi_k(t) = \sum_{i=1}^{I} \frac{d\lambda_i(t)}{\delta_{i,k}}, \tag{A.6}$$

the covariance between asset k's PD ratio and its dividend process is given by

$$\frac{d\phi_k(t)}{\phi_k(t)}\frac{dD_k(t)}{D_k(t)} = \frac{\sigma_{D,k}}{\phi_k(t)}\left[\sum_{i=1}^{I}\frac{\lambda_i(t)}{\delta_{i,k}}\bar{\epsilon}_{i,k}(t)\right]dt,$$

which (using the definition in equation (4.1)) can be rewritten as

$$\frac{d\phi_k(t)}{\phi_k(t)}\frac{dD_k(t)}{D_k(t)} = \frac{1}{\phi_k(t)}\left[\sum_{i=1}^{I}\frac{\lambda_i(t)}{\delta_{i,k}}\mu_{D,k}^{(i)}\right] - \mu_{D,k}^{(m)}(t),$$

where $\mu_{D,k}^{(m)}(t) \equiv \mu_{D,k} + \sigma_{D,k}\theta_k(t)$. Therefore, the term

$$\begin{aligned}
&\frac{d\phi_k(t)}{\phi_k(t)}\frac{dD_k(t)}{D_k(t)} + \Big[\mu_{D,k} + \frac{1}{\phi_k(t)}\Big]dt \\
&\qquad = \Big[(\mu_{D,k} - \mu_{D,k}^{(m)}(t)) + \frac{1}{\phi_k(t)}\Big[\sum_{h=1}^{I}\frac{\lambda_i(t)}{\delta_{i,k}}\mu_{D,k}^{(i)}\Big] + \frac{1}{\phi_k(t)}\Big]dt \\
&\qquad = \Big[-\sigma_{D,k}\theta_k(t) + \frac{1}{\phi_k(t)}\Big[\sum_{h=1}^{I}\frac{\lambda_i(t)}{\delta_{i,k}}\mu_{D,k}^{(i)}\Big] + \frac{1}{\phi_k(t)}\Big]dt.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\frac{1}{\phi_k(t)}\Big[\sum_{i=1}^{I}\frac{\lambda_i(t)}{\delta_{i,k}}\mu_{D,k}^{(i)}\Big] + \frac{1}{\phi_k(t)} = \frac{\sum_{i=1}^{I}\lambda_i(t)\mu_{D,k}^{(i)}/\delta_{i,k}}{\sum_{i=1}^{I}\lambda_i(t)/\delta_{i,k}} + \frac{1}{\sum_{i=1}^{I}\lambda_i(t)/\delta_{i,k}} \\
&= \frac{\sum_{i=1}^{I}\lambda_i(t)\mu_{D,k}^{(i)}\prod_{h\neq i}^{I}\delta_{h,k}}{\sum_{i=1}^{I}\lambda_i(t)\prod_{h\neq i}^{I}\delta_{h,k}} + \frac{\prod_{h=1}^{I}\delta_{h,k}}{\sum_{i=1}^{I}\lambda_i(t)\prod_{h\neq i}^{I}\delta_{h,k}}.
\end{aligned}$$

Using the fact that $\prod_{h=1}^{I}\delta_{h,k} = \sum_{i=1}^{I}\lambda_i(t)\delta_{i,k}\prod_{h\neq i}^{I}\delta_{h,k}$, we can obtain

$$\begin{aligned}
\frac{1}{\phi_k(t)}\Big[\sum_{i=1}^{I}\frac{\lambda_i(t)}{\delta_{i,k}}\mu_{D,k}^{(i)}\Big] + \frac{1}{\phi_k(t)} &= \frac{\sum_{i=1}^{I}\lambda_i(t)(\mu_{D,k}^{(i)} + \delta_{i,k})\prod_{h\neq i}^{I}\delta_{h,k}}{\sum_{i=1}^{I}\lambda_i(t)\prod_{h\neq i}^{I}\delta_{h,k}} \\
&= r + \rho_{0k}\sigma_e\sigma_{D,k},
\end{aligned}$$

since $\mu_{D,k}^{(i)} + \delta_{i,k} = r + \rho_{0k}\sigma_e\sigma_{D,k}$ for $i = 1, 2, \cdots, I$. Therefore,

$$\frac{d\phi_k(t)}{\phi_k(t)}\frac{dD_k(t)}{D_k(t)} + \Big[\mu_{D,k} + \frac{1}{\phi_k(t)}\Big]dt = \big[-\sigma_{D,k}\theta_k(t) + (r + \rho_{0k}\sigma_e\sigma_{D,k})\big]dt.$$

Substitute the above expression into equation (2.17) leads to the result in (4.4). □

A.8. **Proof of Corollary 4.2.** In the case of multiple agents, the expected excess return of asset k follows from equation (3.1) that

$$\mathbb{E}_t[dR_k(t) - rdt] = \rho_{0,k}\sigma_{D,k}\sigma_e - \sigma_{D,k}\theta_k(t) + \sum_{i=1}^{I}\frac{\lambda_i(t)}{\delta_{i,k}\phi_k(t)}\mathbb{E}_t\Big[\frac{d\lambda_i(t)}{\lambda_i(t)}\Big], \tag{A.7}$$

which leads to the result given the definition of $\mathbb{E}_{k,t}$ with probability weights defined in equation (4.5). □

A.9. **Proof of Corollary 4.3.** The variance of the growth in asset k's PD ratio is given by

$$Var_t\Big[\frac{d\phi_k(t)}{\phi_k(t)}\Big] = \mathbb{E}_{k,t}[\bar{\boldsymbol{\epsilon}}]^T\boldsymbol{\rho}^{-1}\mathbb{E}_{k,t}[\bar{\boldsymbol{\epsilon}}] \geq 0, \tag{A.8}$$

and the covariance between the growth in asset k's PD ratio and its dividend growth is given by

$$Cov_t\Big[\frac{d\phi_k(t)}{\phi_k(t)}, \frac{dD_k(t)}{D_k(t)}\Big] = -\sigma_{D,k}\mathbb{E}_{k,t}[\bar{\epsilon}_k]. \tag{A.9}$$

Substituting equations (A.8) and (A.9) into equation (A.1) leads to equation (4.7).

To show that the excess variance of asset k's instantaneous return is non-negative, it is sufficient to show from equation (A.9) that $\mathbb{E}_{k,t}[\bar{\epsilon}_k] \leq 0$. Using the definition of the expectation operator $\mathbb{E}_{k,t}$ in (4.5),

$$\mathbb{E}_{k,t}[\bar{\epsilon}_k] = \sum_{i=1}^{I} \lambda_i(t)\theta_{i,k} - \sum_{i=1}^{I} \omega_{i,k}(t)\theta_{i,k}.$$

We define an expectation operator $\mathbb{E}$ based on the weights $\{1/I\}_{I\times 1}$ across agents. Using the fact that $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] + Cov[X,Y]$, we obtain that

$$\frac{\sum_{i=1}^{I} \omega_{i,k}(t)\theta_{i,k}}{\sum_{i=1}^{I} \lambda_i(t)\theta_{i,k}} = \frac{\mathbb{E}[\omega_k(t)\theta_k]}{\mathbb{E}[\lambda(t)\theta_k]} = \frac{\mathbb{E}[\theta_k] + Cov[\omega_k(t), \theta_k]}{\mathbb{E}[\theta_k] + Cov[\lambda(t), \theta_k]} \geq 1.$$

Since $\mathbb{E}[\omega_k(t)] = \mathbb{E}[\lambda(t)] = 1$ by definition, $Cov[\lambda(t), \theta_k] \leq Cov[\omega_k(t), \theta_k]$ because $1/\delta_{i,k}$ is positively related to $\theta_{i,k}$.

The covariance between returns of asset 1 and asset 2 is given by equation (A.5). Using the definition in (4.5), we have

$$Cov_t\left[\frac{d\phi_1(t)}{\phi_1(t)}, \frac{d\phi_2(t)}{\phi_2(t)}\right] = \mathbb{E}_{1,t}[\bar{\boldsymbol{\epsilon}}]^T \boldsymbol{\rho}^{-1} \mathbb{E}_{2,t}[\bar{\boldsymbol{\epsilon}}]dt,$$

$$Cov_t\left[\frac{d\phi_1(t)}{\phi_1(t)}, \frac{dD_2(t)}{D_2(t)}\right] = -\sigma_{D,2}\mathbb{E}_{1,t}[\bar{\epsilon}_2]dt,$$

$$Cov_t\left[\frac{d\phi_2(t)}{\phi_2(t)}, \frac{dD_1(t)}{D_1(t)}\right] = -\sigma_{D,1}\mathbb{E}_{2,t}[\bar{\epsilon}_1]dt,$$

which then lead to the result in equation (4.8). □

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