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ASSET PRICING UNDER KEEPING UP WITH THE JONESES AND HETEROGENEOUS BELIEFS

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Asset Pricing under Keeping up with the Joneses and Heterogeneous Beliefs

ABSTRACT. When agents agree to disagree about the expected growth rate of the aggregate endowment process, we study the asset price dynamics under "Keeping up with the Joneses" (KUJ) meaning that each agent maximizes the expected life-time CRRA utility of his relative consumption to the other agent in the economy. By solving the optimal consumption policies analytically, we obtain the market equilibrium under heterogeneous beliefs. We provide conditions for agents' long-run survival and show that the market price of risk, risk-free rate, price-dividend ratio in market equilibrium are the consumption share weighted averages of these variables under each agent's belief. We also show the cyclical behaviour of Sharpe ratio, risk-free rate, price and dividend ratio and stock volatility. Through Monte Carlo simulations, we find that, when the less risk averse agent is relatively optimistic, allowing a small amount of disagreement between agents can explain many market characterizes including excess volatility, a high equity premium and a low risk-free rate identified in financial markets.

1. Introduction

The standard consumption-based asset pricing models of representative agent[1] have encountered difficulty in explaining various market characteristics of asset prices, including variation and cyclical movement of asset prices, and asset pricing puzzles such as high equity premia and low risk-free rates identified in financial markets. As a result, habit models have been developed[2]. With external habit preferences and countercyclical variation in risk aversion, Campbell and Cochrane (1999) use a consumption-based model that explains the procyclical variation of stock prices, the long-horizon predictability of excess stock return, the countercyclical variations of stock market volatility and risk premium, and the equity premium puzzle. Chan and Kogan (2002) further show that such risk aversion variation can be endogenously generated through wealth redistribution of multiple agents with different risk aversion coefficients, also leading to countercyclical behaviour in risk-free rate. However, Xiouros and Zapatero (2010) develop a discrete-time model of heterogeneous agents similar to that of Chan and Kogan (2002) and show that the heterogeneous risk aversion alone can only have a marginal, almost negligible, effect.

To reconcile the findings in Campbell and Cochrane (1999) and Chan and Kogan (2002), we follow the recent literature of differences-in-opinion models and develop an equilibrium asset pricing model of two heterogeneous agents under "Keeping up with the Joneses" (KUJ) preferences. Different from the current literature, the KUJ in this paper is to characterize a boundedly rational feature of an agent to maximize the expected utility of his relative consumption to the other agent. When agents disagree about the expected growth rate of the aggregate endowment process, we solve the optimal consumption of heterogeneous agents under the KUJ preference analytically and derive the market price of risk, risk-free rate, and price-dividend (P/D) ratio in closed form. We provide conditions for agents' long-run survival, showing that agents with less accurate prediction may survive in long-run. By showing that the market equilibrium under heterogeneous beliefs is equivalent to the market equilibrium under a consensus belief, we find that the market price of risk, risk-free rate, price-dividend ratio in market equilibrium are the consumption share weighted averages of these variables under each agent's belief. Consequently, we derive the countercyclical behaviour of Sharpe ratio in general and risk-free rate under certain condition. When less risk averse agent is more optimistic, we also show the countercyclical behaviour in volatility and procyclical behaviour in P/D ratio. To quantify the implications of the model, we use Monte Carlo simulations and show that, when the less risk averse agent is relatively optimistic, allowing a small amount of disagreement between agents can explain many market characterizes including excess volatility, high equity premia and low risk-free rates identified in financial markets.

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[1]Such as Lucas (1978), Breeden (1979) and Hansen and Singleton (1983), and Mehra and Prescott (1985).

[2]See Sundaresan (1989), Constantinides (1990), Abel (1990), Campbell and Cochrane (1999), and Chan and Kogan (2002).

In our model, as in the differences-in-opinion models[3], agents disagree based on the same set of information; that is, agents agree to disagree[4]. When agents are not fully rational, the survivability of agents play an important role in aseet pricing literature. To examine the impact of irrational agents on markets and to explain high trading volume and excess volatility, asset pricing models under heterogeneous beliefs have been developed in the literature. By allowing intermediate consumption, Sandroni (2000) and Blume and Easley (2006) show that irrational traders do not survive in the long run and hence have no impact on the market price. However, in the absence of intermediate consumption, Kogan, Ross, Wang and Westerfield (2006) show that survival and price impact are two independent concepts; irrational traders can have a significant impact on asset prices even when their wealth becomes negligible. Even when the disagreements among agents cancel out on average, it is found that the effect of disagreement on asset pricing only "cancels out" under some very restrictive assumptions (see, for example, Fama and French (2007) and Yan (2010)). In general, disagreement may increase trading volume and market volatility[5], and have implications for pricing options [6], and also induce time-variation in market price of risk and interest rate[7]. Therefore, heterogeneous beliefs can be a pricing factor. Based on the disagreement among analysts about expected earnings, Anderson et al. (2005) develop an empirical measure to examine whether heterogeneity in beliefs is a priced factor. They find that the inclusion of this factor does improve the fit of the factor models, especially for small firms, however, on average, dispersion only captures 9 to 26 basis points of excess returns. Jouini and Napp (2006) show that pessimism and doubt at the individual level lead to pessimism and doubt at the aggregate level, which according to Abel (2002) increases equity premium and reduces the risk-free rate. However the explanation of pessimism and doubt is lack of empirical support[8]. The discussion above indicates that heterogeneity in beliefs alone seems not enough to explain the level of equity premium and risk-free rate observed in financial markets.

The KUJ preference introduced in this paper is closely related but significantly different from the current habit models. The difficulty in much of the consumption asset pricing literature lies

[3]See, for example, Detemple and Murthy (1994), Zapatero (1998), Anderson, Ghysels and Juergens (2005), Jouini and Napp (2006), Hong and Stein (2007), David (2008), Cao and Ou-Yang (2009), Xiong and Yan (2010), and Yan (2010).

[4]Different from this paper, the heterogeneous beliefs can also be characterized by asymmetric information, see Admati (1985), Watanabe (2008), and Biais, Bossaerts and Spatt (2010).

[5]See Zapatero (1998), Hong and Stein (2007), Berraday (2009), Dumas, Kurshev and Uppal (2009) and Duchin and Levy (2010).

[6]See Buraschi and Jiltsov (2006) and Cao and Ou-Yang (2009).

[7]See Detemple and Murthy (1994), Basak (2005), Jouini and Napp (2006, 2007, 2011), David (2008) and Xiong and Yan (2010).

[8]Based on survey data on aggregate consumption growth and GDP, Giordani and Söderlind (2006) find that investors are overconfident rather than doubtful. Moreover, although disagreement can lead to doubt, the amount of disagreement is too small to produce doubt that is statistically significant. By assuming that there are recurrent jumps in the expected growth rates of the aggregate endowment and the aggregate dividend between good and bad states, David (2008) explains half of the equity premium and reduce the risk-free rate slightly by one percent.

in the low correlation between equity returns and investors' marginal utility, characterized by the stochastic discount factor (SDF). To overcome this difficulty, some have argued for the importance of *habit formation* such that an agent's utility derived from his consumption stream depends on a benchmark process or *habit*. There are two types of habit, "internal" and "external" habits. Internal habit formation captures the notion that an agent benchmarks his consumption on his own past consumption pattern and he is happier when he consumes more than the past. External habit models assume that the benchmark is not the agent's own past consumption, but an exogenous process such as the aggregate consumption process. Abel (1990) calls the latter "relative consumption models" or "catching up with the Joneses"[9]. There is also an important difference between ratio models and difference models, the former (latter) assumes that utility is derived from the ratio (difference) between an investor's consumption and his habit. Abel (1990) prescribes a ratio model, but as pointed out by Campbell (2003), ratio models or catching up with the Joneses simply add a term to the risk-free rate that does not affect the equity premium, because investor's relative risk aversion is constant. Therefore, one still requires a high relative risk aversion to explain equity premium. For this reason, many have chosen difference models, however the problem with difference models is that utility can be undefined when consumption does not exceed the habit (for example, power utility). Campbell and Cochrane (1999) overcome this problem by modeling directly the log surplus consumption ratio as an external habit. They show that a slow varying, countercyclical risk premium in stock returns can be reproduced within a model of representative agent whose utility function exhibits countercyclical variation in risk aversion. Chan and Kogan (2002) show that such a variation in risk aversion of the representative agent can be the result of the endogenous cross-sectional distribution of wealth. However, Xiouros and Zapatero (2010) show by calibration that the effect of heterogeneity in risk aversion alone is almost negligible. This highlights the limitation of habit models in explaining the aggregate market behavior when beliefs are homogeneous. Different from catching up with the Joneses in Abel (1990) which uses the lagged value of the aggregate consumption as the external benchmark, this paper model "keeping up with the Joneses" (KUJ) preferences that an agent uses the consumption process of another agent as his benchmark process[10].

The contribution of this paper is to model equilibrium asset price dynamic under KUJ by incorporating agents' heterogeneity in beliefs and preferences. We model a pure-exchange economy in continuous-time with two agents who disagree about the expected growth rate of the aggregate endowment (consumption) process.[11] Following Kogan et al. (2006), Yan (2008),

[9]In Abel (1990), the external habit is modeled as the lagged aggregate consumption, therefore it is referred to as "catching up with the Joneses" rather than "keeping up with the Joneses" introduced in this paper.

[10]According to Wikipedia, *keeping up with the Joneses is referring to the comparison to one's neighbor as a benchmark for social caste or the accumulation of material goods. To fail to "keep up with the Joneses" is perceived as demonstrating socio-economic or cultural inferiority.*

[11]This is also the framework adopted by Lintner (1969), Williams (1977), Varian (1989), Harris and Raviv (1993) and others in modeling heterogenous beliefs.

and Jouini and Napp (2011), we assume that disagreement between agents is constant over time in order to focus on the impact of disagreement (instead of learning) under KUJ preferences. The external habit or benchmark process in our model for one agent is the optimal consumption process of the other agent, which is endogenously determined in equilibrium. In other words, one agent's optimal consumption decision affects the other agent's benchmark or habit which then feeds back to his own optimal consumption process. As a result, we solve for both agents' optimal consumption process simultaneously. On survivability of the agents, we find that the an agent who consistently makes more accurate predications about the aggregate endowment process may not always survive in the long-run, in particular, when the sum of investors' relative risk aversions is less than one. We then use the aggregation method developed in Jouini and Napp (2007) and construct a consensus consumer to determine the consumption shares of the agents, market price of risk, interest rate, and price-dividend ratio in equilibrium in closed-form. We study various cyclical behaviour of market equilibrium and use Monte Carlo simulation to analyze the time-series average of the equity premium and risk free rate.

The paper is structured as follows. In Section 2, we solve the optimal consumption under KUJ preferences and heterogeneous beliefs explicitly and examine the impact of risk aversion and KUJ preference on agent's consumption behaviour. In Section 3, conditions for agents' long-run survivability and speed of market selection are provided. In Section 4, by constructing a consensus consumer in market equilibrium, we derive in closed-form, the equilibrium market price of risk, risk-free interest rate, and price-dividend ratio, and examine the cyclical behaviour of market equilibrium. Section 5 examines stock return volatility, equity premium and risk-free rate, and presents a Monte Carlo simulation analysis on the average equity premium, risk-free rate and stock volatility. Section 6 concludes. All proofs can be found in Appendix A.

2. THE MODEL

2.1. **The Economy.** The setup of the economy is standard. Consider a pure exchange, continuous time competitive economy over an infinite time interval $[0, \infty)$ with complete financial markets. There is only one source of uncertainty and agents trade in securities to share risk. The uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$ on which an one-dimensional Brownian motion ω_t is defined, where $\{\mathcal{F}_t^\omega\}$ is the information generated by ω_t. The aggregate endowment process D_t follows a geometric Brownian motion

$$\frac{dD_t}{D_t} = \mu_D dt + \sigma_D d\omega_t, \tag{2.1}$$

in which μ_D and σ_D are constants.

2.2. **Heterogeneous Beliefs.** There are two agents in the economy, indexed by $i = 1, 2$, and they agree to disagree on the expected growth rate of the aggregate endowment. Agent i perceives the expected growth rate of the aggregate endowment to be $\mu_{i,D}$. Hence, agent i lives in a filtered probability space[12] $(\Omega, \mathcal{F}_i, \{\mathcal{F}_{i,t}\}, \mathcal{P}_i)$ in which the endowment process follows,

$$\frac{dD_t}{D_t} = \mu_{i,D}dt + \sigma_D d\omega_{i,t} \tag{2.2}$$

with constant $\mu_{i,D}$[13]. Define

$$\theta_i \equiv \frac{\mu_{i,D} - \mu_D}{\sigma_D} \qquad \text{and} \qquad d\omega_{i,t} \equiv d\omega_t - \theta_i dt. \tag{2.3}$$

By Girsanov's theorem, $\omega_{i,t}$ is a Brownian motion with respect to agent i's probability measure $\mathcal{P}_i$ and agent i's subjective belief $M_{i,t}$ can be characterized by the positive density process of $\mathcal{P}_i$ with respect to $\mathcal{P}$,

$$M_{i,t} \equiv d\mathcal{P}_i/d\mathcal{P} = \exp\left\{ -\frac{1}{2}\theta_i^2 t + \theta_i \omega_t \right\}, \tag{2.4}$$

which satisfies

$$\frac{dM_{i,t}}{M_{i,t}} = \theta_i d\omega_t. \tag{2.5}$$

When $\theta_i > (<) 0$, agent i is over-optimistic (over-pessimistic) about the aggregate endowment growth. The difference between agents' subjective beliefs is measured by

$$\delta \equiv \theta_1 - \theta_2 = \frac{\mu_{1,D} - \mu_{2,D}}{\sigma_D}. \tag{2.6}$$

When $\delta > (<) 0$, agent 1 is more optimistic (pessimistic) than agent 2.

2.3. **Securities Market.** There are two traded securities in the market, a riskless bond and a risky asset or stock. The riskless bond is in zero supply and there is one share of the stock endowment equally shared between the two agents. The bond price B_t follows

$$\frac{dB_t}{B_t} = r_t dt \tag{2.7}$$

with $B_0 = 1$. The stock is a claim on the aggregate endowment. Its price S_t and return R_t dynamics satisfy

$$\begin{aligned} dR_t \equiv \frac{dS_t + D_t dt}{S_t} &= \mu_t dt + \sigma_t d\omega_t \\ &= \mu_{i,t} dt + \sigma_t d\omega_{i,t} \qquad \text{for agent} \quad i = 1, 2. \end{aligned} \tag{2.8}$$

The risk-free rate r_t, expected return of the stock μ_t and the volatility σ_t are endogenously determined in equilibrium based on agents' beliefs and preferences. Assume σ_t to be $\mathcal{F}_{i,t}$ adapted and μ_t to be $\mathcal{F}_t$ adapted. Agents observe and agree on the asset prices and the volatility of stock returns, but do not observe the expected return μ_t, so they use their own inferences $\mu_{i,t}$. This

[12]Agents have the same information since $\mathcal{F}_{i,t} := \mathcal{F}_t^{\omega_i} = \mathcal{F}^{\omega_t}$, therefore they agree to disagree.

[13]In general, agents may update their beliefs about the expected growth rate of the aggregate endowment as new information becomes available. However, in order to focus on the joint impact of *Keeping up with the Joneses* and disagreement rather than the impact of learning, we assume that agents have constant disagreement with the objective belief, thus they also have constant disagreement between themselves.

implies the following "consistency" relation (see Basak (2005) and David (2008)):

$$\frac{\mu_{1,t} - \mu_{2,t}}{\sigma_t} = \delta. \tag{2.9}$$

The market is dynamically complete in the sense that any contingent claims can be replicated. This implies that there exists a unique state price density (SPD) process ξ_t that follows

$$\frac{d\xi_t}{\xi_t} = -r_t dt - \kappa_t d\omega_t, \tag{2.10}$$

where $\xi_0 = 1$. Denote

$$\kappa_t \equiv \frac{\mu_t - r_t}{\sigma_t}, \qquad \kappa_{i,t} \equiv \frac{\mu_{i,t} - r_t}{\sigma_t}. \tag{2.11}$$

Then κ_t defines the market price of risk (MPR) or the Sharp ratio for the stock and $\kappa_{i,t}$ is the perceived Sharp ratio of agent i. Following equation (2.9), the difference in perceived Sharp ratio is related to the disagreement of agents

$$\kappa_{1,t} - \kappa_{2,t} = \delta.$$

2.4. **Keeping Up with the Joneses (KUJ) Preferences and Optimal Consumption.** Different from the Keeping Up with the Joneses in habit literature, we assume that agent's utility is a power function of the ratio of his consumption to the other agent's consumption. Let $c_{i,t}$ be the consumption of agent i at time t. Under KUJ, the utility of agent i is defined by

$$u_{i,t}(c_{i,t}, c_{k,t}) := \frac{e^{-\beta t}}{1-\alpha_i}\left(\frac{c_{i,t}}{c_{k,t}}\right)^{1-\alpha_i} \qquad i \neq k; i,k = 1,2. \tag{2.12}$$

Here $\beta > 0$ is the subjective discount rate[14] for *future relative consumption* and α_i is the relative risk aversion coefficient of agent i. In equation (2.12), the consumption of agent k, $c_{k,t}$, serves as the time-varying habit level of agent i, which agent i takes as exogenously given. Agent i's objective is to

$$\max_{c_{i,t}} \mathbb{E}\left[\int_0^\infty M_{i,t} u_{i,t}(c_{i,t}, c_{k,t}) dt | c_{k,t}\right], \tag{2.13}$$

subjected to the budget constraint,

$$\mathbb{E}\left[\int_0^\infty \xi_t c_{i,t} dt\right] \leq \frac{1}{2} S_0. \tag{2.14}$$

We refer to utility sepcification in (2.12) as *Keeping Up with the Joneses* (KUJ) under heterogeneous beliefs, since agent i behaves to keep up with the consumption of agent k. In equilibrium, agent i's habit level is also affected by his optimal consumption plan since agent k also maximizes his expected utility of relative consumption to agent i. Therefore, there is a *feedback effect*: agent's habit level affects his consumption, which then feeds back to his habit level. This feedback effect has not been studied in the habit formation literature.

[14]In this paper, we focus on the impact of disagreement, instead of different discount rate. However, the analysis also applies when agents have different discount rate.

Without loss of generality, we assume that agent 1 is less risk averse than agent 2, that is $\alpha_1 \leq \alpha_2$ and denote the *difference in risk aversion* and the *average risk aversion* respectively as

$$\Delta\alpha \equiv \alpha_2 - \alpha_1 (\geq 0), \qquad \bar{\alpha} \equiv \frac{\alpha_1 + \alpha_2}{2}. \tag{2.15}$$

Assume $\bar{\alpha} \neq \frac{1}{2}$. For each agent i, define $\{p_i, q_i\}$ as

$$p_1 = \frac{\alpha_2}{2\bar{\alpha} - 1}, \qquad q_1 = \frac{\alpha_1 - 1}{2\bar{\alpha} - 1} = 1 - p_1$$

and

$$p_2 = \frac{\alpha_2 - 1}{2\bar{\alpha} - 1}, \qquad q_2 = \frac{\alpha_1}{2\bar{\alpha} - 1} = 1 - p_2.$$

Let

$$\bar{M}_{i,t} \equiv M_{1,t}^{p_i} M_{2,t}^{q_i}, \qquad i = 1, 2.$$

Then

$$\frac{d\bar{M}_{i,t}}{\bar{M}_{i,t}} = -\frac{1}{2} p_i q_i \delta^2 dt + \bar{\theta}_i d\omega_t, \tag{2.16}$$

where $\bar{\theta}_i \equiv p_i\theta_1 + q_i\theta_2$. Also assume $\bar{\beta}_i > 0$, where $\bar{\beta}_i \equiv \beta + \frac{1}{2}p_i q_i \delta^2$ for $i = 1, 2$. The optimal consumption and the aggregate consumption share of agents can be characterized as follows[15].

Lemma 2.1. *(Optimal Consumption and the Aggregate Consumption Share)*

- *The optimal consumption of agent i is given by*

$$c_{i,t} = \lambda_{i,t} D_t \qquad \textit{for} \qquad i = 1, 2, \tag{2.17}$$

  where agents' share of aggregate consumption are given by

$$\lambda_{1,t} = \frac{\bar{\beta}_1 \bar{M}_{1,t}}{\bar{\beta}_1 \bar{M}_{1,t} + \bar{\beta}_2 \bar{M}_{2,t}} \qquad \textit{and} \qquad \lambda_{2,t} = 1 - \lambda_{1,t}. \tag{2.18}$$

- *The aggregate consumption share of agent i satisfies*

$$d\lambda_{1,t} = \lambda_{1,t}\lambda_{2,t}\Big(\big[\bar{\beta}_2 - \bar{\beta}_1 - \bar{\theta}_1(\bar{\theta}_1 - \bar{\theta}_2) + \lambda_{2,t}(\bar{\theta}_2 - \bar{\theta}_1)^2\big] dt + (\bar{\theta}_1 - \bar{\theta}_2) d\omega_t \Big). \tag{2.19}$$

Lemma 2.1 provides solutions to the optimal consumption of agents under KUJ preference with the following interesting observations. First, Equation (2.18) shows that the aggregate consumption share of agent i depends on $\bar{M}_{i,t}$, instead of the subjective belief $M_{i,t}$ of agent i. This is because that agent i is concerned about his level of consumption relative to the other agent rather than his absolute level of consumption. We can interpret $\bar{M}_{i,t}$ as the *equivalent belief* of agent i under KUJ. In fact, suppose that agent i maximizes logarithmic utility of his *absolute consumption* based on his equivalent belief $\bar{M}_{i,t}$, that is,

$$\max_{c_{1,t}} \mathbb{E}\left[\int_0^\infty \bar{M}_{1,t} e^{-\beta t} \ln(c_{1,t}) dt \right], \tag{2.20}$$

[15]All the proofs are in the appendix.

then agent 1's share of aggregate consumption is given by $\lambda_{1,t}$ in (2.18). However, strictly speaking, the equivalent beliefs $\bar{M}_{i,t}$ cannot be considered as a proper probability belief, because in general it is not a martingale with respect to the objective measure $\mathcal{P}$, see equation (2.16). Hence a probability measure defined by $\bar{M}_{i,t}$ may not be equivalent to the objective probability measure. To deal with this issue, we decompose $\bar{M}_{i,t}$ into a proper probability belief $\hat{M}_{i,t}$ and a discount factor $A_{i,t}$ so that $\bar{M}_{i,t} = A_{i,t}\hat{M}_{i,t}$ with

$$A_{i,t} = \exp\left(\int_0^t -(\bar{\beta}_i - \beta)ds\right), \qquad \hat{M}_{i,t} = \exp\left(\int_0^t -\frac{1}{2}\bar{\theta}_i^2 ds + \int_0^t \bar{\theta}_i d\omega_s\right). \tag{2.21}$$

Then the *equivalent probability belief* $\hat{M}_{i,t}$ is a martingale with respect to the objective measure $\mathcal{P}$, hence a probability measure defined by $\hat{M}_{i,t}$ is equivalent to $\mathcal{P}$. Secondly, from equation (2.21), the equivalent maximization problem in (2.20) of agent i can be re-written as

$$\max_{c_{i,t}} \hat{\mathbb{E}}_i\left[\int_0^\infty e^{-\bar{\beta}_i t}\ln(c_{i,t})dt\right], \tag{2.22}$$

where $\hat{\mathbb{E}}_i$ is the expectation operator defined by the martingale $\hat{M}_{i,t}$ and $\bar{\beta}_i$ is agent i's subjective discount rate for *future absolute consumption*.

Based on these observations, we can examine the joint effect of the KUJ preference and heterogeneous risk aversions on the equivalent beliefs and subjective discount rates of agents, which helps to understand agents' behaviour compared to the standard homogeneous economy. First, under the KUJ, the heterogeneity in risk aversions affects agents' optimal consumption policy when beliefs are heterogeneous. In fact, suppose $\alpha_1 < 1$ and $\alpha_2 > 1$, then $q_1 < 0$ and $p_2 > 0$. This implies that agent 1's equivalent belief has a *negative* weight on agent 2's subjective belief whereas agent 2's equivalent belief has a *positive* weight on agent 1's subjective belief. The intuition comes from agents' marginal utility under KUJ,

$$\frac{\partial u_{i,t}(c_{i,t}, c_{k,t})}{\partial c_{i,t}} = e^{-\beta t} c_{i,t}^{-\alpha_i} c_{k,t}^{\alpha_i - 1},$$

which suggests that, when the risk aversion of agent i is larger (smaller) than one, his marginal utility of absolute consumption $c_{i,t}$ increases (decrease) with the absolute consumption of the other agent in the economy $c_{k,t}$. This means that agent 1 with $\alpha_1 < 1$ would prefer to consume in those states in which agent 2 thinks are less likely to occur. In contrast, agent 2 with $\alpha_2 > 1$ would prefer to consume in those states where agent 1 thinks are more likely to occur. Therefore, if agent 1 who is less risk averse is relatively more optimistic (pessimistic) than agent 2, then he would behave even *more* optimistically (pessimistically) compared to the situation under his actual subjective belief. In contrast, agent 2 who is relatively more risk averse and pessimistic (optimistic) compare to agent 1 would behave *less* pessimistically (optimistically) compared to his actual subjective belief. The difference between agents' equivalent beliefs about the aggregate endowment growth is measured by

$$\bar{\delta} \equiv \bar{\theta}_1 - \bar{\theta}_2 = \frac{\delta}{2\bar{\alpha} - 1}. \tag{2.23}$$

Therefore, when the average risk aversion $\bar{\alpha} < 1$, the difference between the equivalent probability beliefs is greater (in absolute value) than the difference between agents' subjective beliefs. Therefore the KUJ magnifies the effect of the heterogeneity in beliefs and, from equation (2.19), a greater difference in equivalent probability beliefs leads to high fluctuations in the aggregate consumption share of the agents. When the difference in beliefs $\delta = 0$, agents' optimal consumption no longer depends on their risk aversions since $\bar{M}_{i,t} = 1$ for $i = 1, 2$, see equation (2.16). Therefore, when beliefs are homogeneous, the economy under KUJ and heterogeneous risk aversions is equivalent to the standard logarithmic representative agent economy.

Next, under the KUJ, there is a *negative relation* between *patience* and *risk aversion* of agents. Indeed, agent i's subjective discount rate for future absolute consumption is higher (lower) than the discount rate for future relative consumption, that is $\bar{\beta}_i > (<)\beta$ when $p_i q_i > (<)0$, which is equivalent to $\alpha_i > (<)1$ when $\bar{\alpha} > \frac{1}{2}$. When the risk aversions are the same, we have $\bar{\beta}_1 = \bar{\beta}_2$, hence agents' subjective discount rates for the future absolute consumption are the same. Furthermore, difference between agents' discount rate for the future absolute consumption is given by

$$\bar{\beta}_1 - \bar{\beta}_2 = \frac{1}{2}(p_1 q_1 - p_2 q_2)\delta^2 = -\frac{\Delta\alpha}{2}\bar{\delta}^2 \leq 0,$$

which is strictly negative (because agent 1 is assumed to be less risk averse than agent 2, hence $\Delta\alpha > 0$). This leads to a negative relation between patience and risk aversion of agents since the less risk averse agent (agent 1) always has a smaller discount rate for future absolute consumption (therefore more patient) than the more risk averse agent (agent 2). This result seems counter-intuitive since one would expect a more risk averse agent to be more patient, see van Pragg and Booji (2003). However, under the KUJ, agents' utility is a function of their relative consumption rather than absolute consumption. In our model, the subjective discount rate for future relative consumption is actually common between agents, hence agents' patience for future relative consumption is independent from their risk aversions. Therefore, the dependence of agents' patience parameters on the risk aversions is the result of the KUJ under disagreement. In addition, the patience parameters of agents for future absolute consumption are endogenously determined by their risk aversions and beliefs, instead of an exogenous variable as in the standard consumption-based asset pricing models.

In summary, under the KUJ, agents behave very differently under heterogeneous beliefs. An increase in the disagreement can increase the fluctuations in the aggregate consumption shares of agents and make the less risk averse agent to become more patient for the future (absolute) consumption. These behaviour can have significant impact on the long-run survival, Sharpe ratio, price-dividend ratio, equity premium, and risk-free rate in market equilibrium, as discussed in the rest of the paper.

3. Long-run Survival and Stationarity

When agents have heterogeneous beliefs, their survival can have significant impact on markets (see Kogan et al. (2006)). With a constant disagreement in our model, the heterogeneity in beliefs persists. The economy is said to be *stationary* if both agents survive in the long run. In this section, we provide conditions for the long-run survival of agents and show that the speed of market selection when one agent does not survive in long-run can be very slow. Following Kogan et al. (2006) and Yan (2008), we first introduce the definition of survival.

Definition 3.1. *Agent i vanishes in the long-run when $\mathcal{P}(\lim_{t\to\infty} \ \lambda_{i,t} = 0) = 1$. Agent i survives in the long run if he does not vanish.*

A survival index can be defined as a function of agent's belief, risk aversion and subjective discount rate. According to Yan (2008), an agent survives if and only if he has the lowest survival index. Under the KUJ, equation (2.22) implies that agents behave as a logarithmic utility maximizer of their absolute level of consumption with subjective discount rate $\bar{\beta}_1$ and belief $\hat{M}_{i,t}$. Therefore, using Proposition 2 in Yan (2008) and setting agents' relative risk aversion coefficients to one, we obtain the following survival index of agent i

$$I_i := \frac{1}{2}\bar{\theta}_i^2 + \bar{\beta}_i = \frac{1}{2}(p_i\theta_1^2 + q_i\theta_2^2) + \beta, \qquad i = 1, 2. \tag{3.1}$$

Then the difference between agents' survival indices is given by

$$I_1 - I_2 = \frac{\theta_1^2 - \theta_2^2}{\alpha_1 + \alpha_2 - 1}, \tag{3.2}$$

which leads to the following conditions for the long-run survival.

Proposition 3.2. *(Long-run Survival) Assume $\bar{\alpha} \neq \frac{1}{2}$. Then both agents survive if and only if $|\theta_1| = |\theta_2|$. When $|\theta_1| \neq |\theta_2|$, there is only one agent survives, in particular,*

- *when $\alpha_1 + \alpha_2 > 1$, agent 2 survives if and only if $|\theta_2| < |\theta_1|$ and agent 1 survives if and only if $|\theta_1| < |\theta_2|$;*
- *when $\alpha_1 + \alpha_2 < 1$, agent 2 survives if and only if $|\theta_1| < |\theta_2|$ and agent 1 survives if and only if $|\theta_2| < |\theta_1|$.*

Proposition 3.2 provides some insight into the long-run survival of heterogeneous agents under the KUJ preferences. First, it shows that the market is stationary if and only if $|\theta_1| = |\theta_2|$, that is agents have the same absolute disagreement with the objective belief. In order for market stationarity to occur with persistence of heterogeneity in beliefs, we need one agent to be over-optimistic and the other agent to be over-pessimistic about the growth rate of aggregate endowment so that $\theta_1 = -\theta_2$; that is the amount of over-pessimism exactly offsets the amount of over-optimism. Note that under the KUJ, both agents survive as long as $|\theta_1| = |\theta_2|$ and market stationarity does not depend on the agents' risk aversions.[16] This is in contrast to Yan's

[16] In equation (3.2), the difference in the survival indices is zero if and only if the numerator is zero.

result that being less risk averse helps an agent to survive if the expected aggregate endowment growth under the objective probability measure is strictly positive.

Secondly, agent with more accurate belief survives when both agents are sufficiently risk averse (so that $\bar{\alpha} > \frac{1}{2}$), but market selection does not work when agents are less risk averse (so that $\bar{\alpha} < \frac{1}{2}$), under which the agent with a more accurate belief actually does not survive in the long run, while the agent with a less accurate belief survives. This is a surprising results and here is the intuition behind this result. When $\bar{\alpha} < \frac{1}{2}$, from Lemma 2.1, we have $p_1 < 0$ and $q_2 < 0$, which means that the optimal consumption of an agent is not based on his own subjective belief, but rather on the subjective belief of the other agent in the economy. Therefore, although an agent may have a more accurate belief of the expected aggregate endowment growth, his optimal consumption policy is actually based on the less accurate belief of the other agent in the economy. This is in contrast to the results in Blume and Easley (2006) and Yan (2008) that in a complete market with endogenous savings and portfolio choices, incorrect beliefs is always a disadvantage for survival.

When one agent vanishes in the long run, the question is how fast is the market selection process. The answer to this question is important to understand the impact of non-surviving agent on equilibrium asset prices. If the selection process is "fast", then the heterogeneity in beliefs would have little or no effect on equilibrium prices. However, if the selection process is "slow", although heterogeneity in beliefs cannot persist in the long run, but its impact on equilibrium prices cannot be simply ignored, see Yan (2008). Consider a situation that agent 1 vanishes. We measure the speed of the selection process by the expected first time that agent 1's share of aggregate consumption $\lambda_{1,t}$ reaches a level below his initial level $l < \lambda_{1,0}$. Mathematically, the vanishing time is defined by

$$\tau_l = \mathbb{E}[\inf\{t : \ \lambda_{1,t} = l\}]. \tag{3.3}$$

The following result characterizes the speed the market selection.

Proposition 3.3. *(Speed of Market Selection) Given that agent* 1 *vanishes in the long run, then the expected first time that agent* 1*'s share of aggregate consumption* $\lambda_{1,t}$ *will hit the level* $l(< \lambda_{1,0})$ *is given by*

$$\tau_l = \frac{2\bar{\alpha} - 1}{\frac{1}{2}(\theta_2^2 - \theta_1^2)} \ln \left[\frac{(1-l)}{l} \frac{\lambda_{1,0}}{\lambda_{2,0}} \right]. \tag{3.4}$$

Proposition 3.3 shows that the speed of the selection process depends not only on the difference in beliefs, but also on the average risk aversion $\bar{\alpha}$. Equation (3.4) shows that even when the difference in beliefs δ is small, the speed of the selection process can still be sufficient fast if the average risk aversion is close to a half. Denote $l^* = \frac{\lambda_{1,0}}{2}$ and let τ_{l^*} be the expected time that agent 1's share of aggregate consumption reduces to half of its initial level. In Fig. 3.1, agent 2's subjective belief is assumed to coincide with the objective belief, that is $\theta_2 = 0$. Fig. 3.1 shows that as the average risk aversion $\bar{\alpha}$ becomes closer to $\frac{1}{2}$, agent 1's *half life* τ_{l^*} is much shorter given the same difference in subjective beliefs δ. For example, when agent 1 underestimates

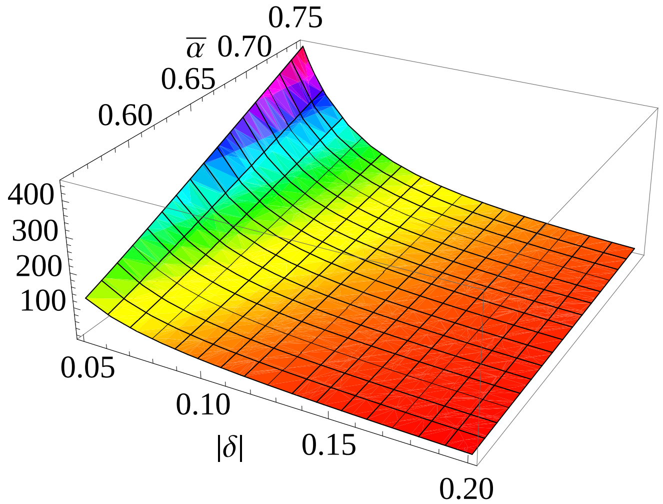


FIGURE 3.1. Agent 1's half life as a function of the disagreement measured by δ. We set $\theta_1 = 0$, the difference in risk aversion $\Delta\alpha = 0.1$ and the subjective discount rate for future relative consumption to $\beta = 0.01 - \frac{1}{2}p_1 q_1 \delta^2$.

or overestimates the MPR or Sharp ratio of the stock by $\delta = 0.1$, his half life is approximately 116 years when $\bar{\alpha} = 0.75$, but only 42 years when $\bar{\alpha} = 0.55$. Therefore, under KUJ, when the average risk aversion is small, in order for heterogeneity in beliefs to have significant impact on market equilibrium, it is important to have a stationary market, that is $|\theta_1| = |\theta_2|$.

4. Consensus Belief, Market Equilibrium, and Cyclical Behaviour

To characterize the market equilibrium, we construct a *consensus consumer* who is endowed with the aggregate endowment, and can generate the same set of asset prices as in the original economy under KUJ and heterogeneous beliefs. In a standard pure exchange economy with heterogeneous agents who maximize the expected utility from their absolute consumption under their heterogeneous beliefs, Jouini and Napp (2007) provide a method for constructing such a consensus consumer. We now follow Jouini and Napp (2007) to construct a consensus consumer for the economy in Section 2.

Proposition 4.1. *(Consensus Consumer) The consensus consumer has logarithmic utility with a subjective discount rate β so that the instantaneous utility from aggregate endowment is measure by*

$$u_{M,t}(D_t) \equiv e^{-\beta t} \ln(D_t)$$

and a consensus belief $M_t = (\bar{M}_{1,t} + \bar{M}_{2,t})/2$ that satisfies

$$\frac{dM_t}{M_t} = -(\bar{\beta}_t - \beta)dt + \bar{\theta}_t d\omega_t,$$

where

$$\bar{\beta}_t = \lambda_{1,t}\bar{\beta}_1 + \lambda_{2,t}\bar{\beta}_2, \qquad \bar{\theta}_t = \lambda_{1,t}\bar{\theta}_1 + \lambda_{2,t}\bar{\theta}_2.$$

Proposition 4.1 reduces significantly the analysis of market equilibrium in the economy of Section 2 to a standard consumption problem with logarithmic utility and homogeneous beliefs. It has the following interesting implications. First, the SPD process derived from the first order condition of the consensus consumer is given by

$$\frac{\xi_s}{\xi_t} = e^{-\beta(s-t)} \frac{M_s}{M_t} \frac{D_t}{D_s}. \tag{4.1}$$

The consensus belief M_t is in general not a martingale, hence not a proper probability belief. It is a martingale only when $\alpha_1 = \alpha_2 = 1$ and $\bar{\beta}_t = \beta$. In general, as in Jouini and Napp (2007), we can decompose M_t into a proper probability belief $\hat{M}_t$ and a discount factor A_t so that $M_t = A_t \hat{M}_t$, where

$$A_t = \exp\left(\int_0^t -(\bar{\beta}_s - \beta) ds\right), \qquad \hat{M}_t = \exp\left(\int_0^t -\frac{1}{2}\bar{\theta}_s^2 ds + \int_0^t \bar{\theta}_s d\omega_s\right).$$

The disagreement between the consensus probability belief and the objective belief is measured by $\bar{\theta}_t$, which is a consumption-share weighted average of agents' disagreement with the objective belief under their equivalent probability beliefs. Secondly, the subjective discount rate of the consensus consumer $\bar{\beta}_t$ is a consumption-share weighted average of agents' discount rate for future absolute consumption. Because of the stochastic nature of the aggregate consumption shares of agents, this endogenously generated discount rate implies a stochastically discount factor A_t. Withe the help of Proposition 4.1, we can obtain the equilibrium market characteristics, including MPR (or Sharpe ratio), equilibrium risk-free rate, and price-dividend (PD) ratio in market equilibrium.

Proposition 4.2. *(Market Equilibrium)*

(i) *The equilibrium MPR process is given by*

$$\kappa_t = \sigma_D - \bar{\theta}_t, \tag{4.2}$$

(ii) *The equilibrium risk-free interest rate is given by*

$$r_t = \bar{\beta}_t + \mu_D - \sigma_D(\sigma_D - \bar{\theta}_t), \tag{4.3}$$

(iii) *The price-dividend (PD) ratio in equilibrium is given by*

$$\phi_t \equiv \frac{S_t}{D_t} = \lambda_{1,t}\phi_1 + \lambda_{2,t}\phi_2, \qquad \phi_i = \frac{1}{\bar{\beta}_i} \text{ for } i = 1, 2. \tag{4.4}$$

Proposition 4.2 shows that the equilibrium MPR and risk-free rate are determined by the consensus belief about the expected growth rate of aggregate endowment. Intuitively, if the consensus belief is pessimistic (optimistic), then the Sharpe ratio κ_t would be higher (lower) and the risk-free rate r_t would be lower (higher) compare to a standard economy under homogeneous beliefs. Furthermore, equation (4.4) shows that the equilibrium stock price is consumption share weighted average of the prices that would prevail in economies in which there is only agent. Note that the PD ratios ϕ_i $(i = 1, 2)$in single-agent economies depends on agents' patience for future absolute consumption $\bar{\beta}_i$, which is jointly determined by agents' beliefs, risk aversions and the common discount rate for future relative consumption. Since there is a negative correlation between risk aversion and patience implied by the model, the PD ratio is always higher in the single-agent economy of the less risk averse agent. When risk aversion are the same, the equilibrium PD ratio becomes constant. Note that the same result holds in a standard economy in which logarithmic agents with heterogeneous beliefs maximize expected utility of absolute consumption. Li (2007) derives similar results in a pure-exchange economy in which agents maximizes expected utility of absolute consumption under different subjective beliefs and subjective discount rates. However, in his model, the subjective discount rates are given exogenously, as a result, the market may not be stationary even when agents have the same absolute disagreement with the objective belief, which is the case in our model.

The cyclical behaviour of market characteristics includes a wide variety of dynamic asset pricing phenomena identified in financial markets. Proposition 4.2, together with the consumption share dynamics in Lemma 2.1, implies the following results on the cyclical behaviour of market characteristics.

Corollary 4.3. *(Cyclical Behaviour)*

(a) *the MPR or Sharpe ratio κ_t exhibits a countercyclical behaviour;*
(b) *the risk-free rate r_t is countercyclical when $\sigma_D < (\Delta\alpha)\bar{\delta}/2$ and procyclical when $\sigma_D > (\Delta\alpha)\bar{\delta}/2$;*
(c) *the P/D ratio ϕ_t is procyclical, which means that the dividend yield $1/\phi_t$ is countercyclical when the less risk averse agent is relatively more optimistic than the more risk averse agents.*

The results in Corollary 4.3 reconcile the cyclical behaviour of market characteristics obtained in Campbell and Cochrane (1999) and Chan and Kogan (2002).

5. Stock and Bond Returns

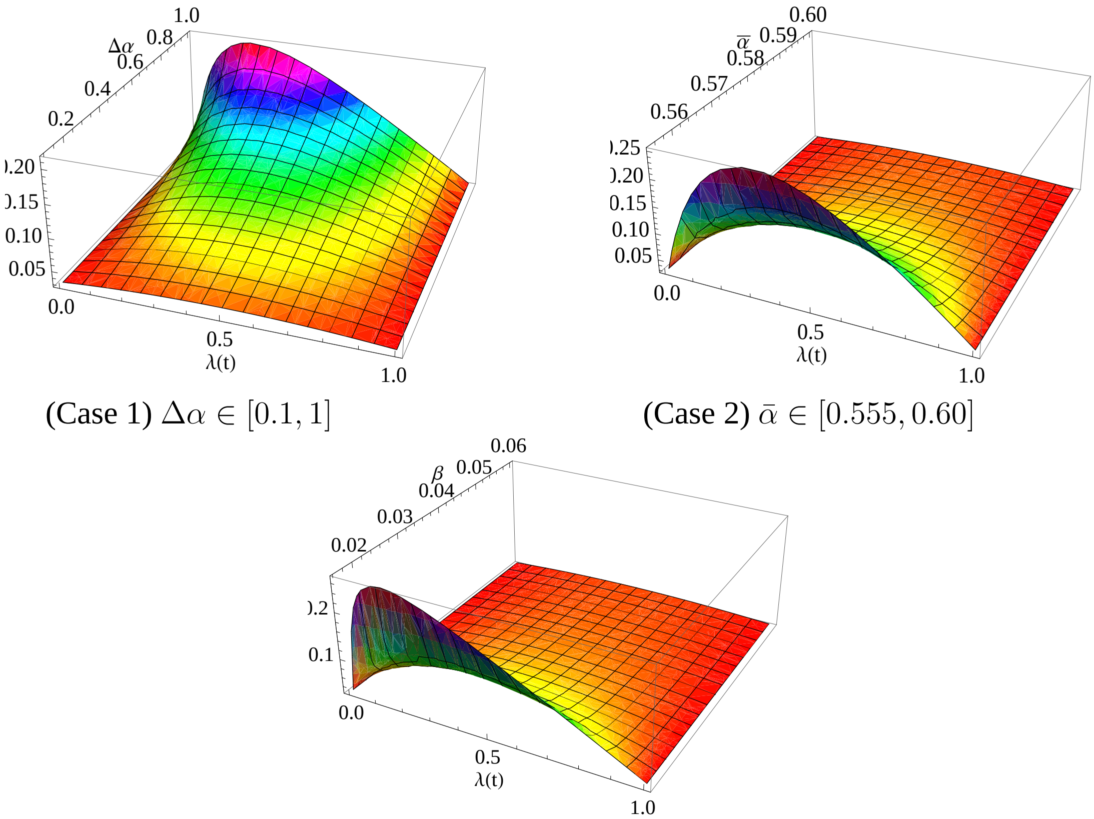
In this section, we examine the impact of the KUJ and disagreement of agents on stock volatility, equity premium and risk-free rate when the market is stationary (that is $|\theta_1| = |\theta_2|$). It is found that even a *small* disagreement can have a significant impact.

5.1. **Stock Return Volatility.** Applying Itô's lemma to the equilibrium stock price in equation (4.4), we can obtain the contribution of disagreement to the excess volatility of stock return.

Corollary 5.1. *(Stock Volatility) The volatility of instantaneous stock return is given by*

$$\sigma_t = \sigma_D + \sigma_{\phi,t}, \qquad \textit{where} \qquad \sigma_{\phi,t} = \sigma_{\lambda,t}\left[\frac{\phi_1 - \phi_2}{\phi_t}\right], \tag{5.1}$$

and $\sigma_{\lambda,t} \equiv \lambda_{1,t}\lambda_{2,t}\bar{\delta}$ measures the fluctuations in the shares of aggregate consumption.



(Case 1) $\Delta\alpha \in [0.1, 1]$ (Case 2) $\bar{\alpha} \in [0.555, 0.60]$

(Case 3) $\beta \in [0.0164, 0.06]$

FIGURE 5.1. Stock return volatility σ_t as a function of agent 1's share of aggregate consumption $\lambda_{1,t}$. The parameters values are $\mu_D = 0.02$, $\sigma_D = 0.03$, $\mu_{D,1} = 0.021$ and $\mu_{D,2} = 0.019$. In Case 1, we assume $\bar{\alpha} = 0.60$, $\beta = 0.06$. In Case 2, we assume $\Delta\alpha = 0.1$, $\beta = 0.06$. In Case 3, we assume $\Delta\alpha = 0.1$, $\bar{\alpha} = 0.60$.

Corollary 5.1 shows that the stock volatility can be decomposed into two components; *the volatility of the aggregate dividend* and the *volatility of relative changes in PD ratios*. While the former is assumed to be a constant (σ_D), the latter is stochastic and positive when the optimistical agent is less risk averse compared to the pessimistic agent (see Li (2007)), leading to excess volatility. We now provide an explanation for the excess volatility. When the less risk averse agent 1 is optimistic compared to agent 2, we have $\sigma_{\lambda,t} > 0$; meaning that agent 1's share of aggregate consumption is positively (and perfectly) correlated with aggregate consumption growth. Since the more risk averse agent is always less patient compared to less risk averse agent, we must have $\bar{\beta}_1 < \bar{\beta}_2$. This then implies $\phi_1 > \phi_2$. Therefore $\sigma_t > \sigma_D$, implying excess volatility. When agents have the same stock evaluations ($\phi_1 = \phi_2$), we have $\Delta\alpha = 0$ and hence $\sigma_t = \sigma_D$, that is stock volatility coincides with that of the aggregate dividend.

To quantify the excess volatility, we examine the impact of *difference in risk aversions* ($\Delta\alpha$), *average risk aversion* ($\bar{\alpha}$) and the *discount rate for relative consumption* (β) on stock volatility (σ_t) numerically and report the results in Fig. 5.1. With a small disagreement in the expected growth rate in dividend $\mu_{D,1} - \mu_{D,2} = 0.2\%$, Fig. 5.1 *Case 1* shows that the difference in risk aversions is positively related to stock volatility. Intuitively, greater difference in risk aversions corresponds to larger difference in patience, which increases stock volatility.

Fig. 5.1 *Case 2* shows that given $\bar{\alpha} > \frac{1}{2}$, average risk aversion is negatively related to stock volatility. Intuitively, an increase in agents' average risk aversion reduces the difference in the equivalent beliefs about expected divined growth, which leads to less speculation between agents and smaller stock volatility. Note that an increase (decrease) in average risk aversion $\bar{\alpha}$ has the same effect as a decrease (increase) in agents' disagreement δ since only the difference in the equivalent beliefs $\bar{\delta}$ matters. Therefore, we do not necessarily need a large disagreement between agents to generate excess volatility in stock returns, it can be simply the case when average risk aversion is low.

In Fig. 5.1 *Case 3*, given that the subjective discount rate for relative consumption $\beta > -\min[\mu_{M,1}, \mu_{M,2}]$ with $\mu_{M,i} = (1/2)p_i q_i \delta^2$ for $i = 1, 2$ (such that equilibrium exists), stock volatility is negatively related to the level of β. Intuitively, since agent 1 is assumed to be less risk averse (and therefore more patient) than agent 2, as the level of β decreases, both agents' stock valuation increases due to the increase in patience, however more so for agent 1 because his subjective discount rate for absolute consumption $\bar{\beta}_1$ can be very close to zero such that his valuation (ϕ_1) is much larger than agent 2's (ϕ_2). For example, when $\beta = 0.0164$, we obtain $\bar{\beta}_1 = 0.00015$ and $\bar{\beta}_2 = 0.0057$, which implies that $\phi_1 - \phi_2 = 6491.4$. In comparison, when $\beta = 0.06$ so that $\bar{\beta}_1 = 0.044$ and $\bar{\beta}_2 = 0.049$, which implies the difference in stock valuations is only $\phi_1 - \phi_2 = 2.575$. Hence, stock volatility is very sensitive to agent's patience for future relative consumption. The large difference in stock valuations created by choosing β closes to the threshold also generates a *negative skewness* in stock volatility, which means that volatility is higher (lower) when the less risk aversion and more optimistic agent (agent 1) has a smaller (larger) share of aggregate consumption.

Now we explain the negative skewness in volatility. Intuitively, as equation (5.1) shows, volatility of relative changes in the PD ratio $\sigma_{\phi,t}$ is inversely related to the current level of the PD ratio ϕ_t. When there is a large difference in agents' stock valuations, ϕ_t can be very small (large) and $\sigma_{\phi,t}$ very large (small) when agent 1's share of aggregate consumption is closer to zero (one). This negative skewness in the relationship between stock volatility and agent 1's share of aggregate consumption is the key ingredient in generating large equity premium, which we will discuss in the next subsection.

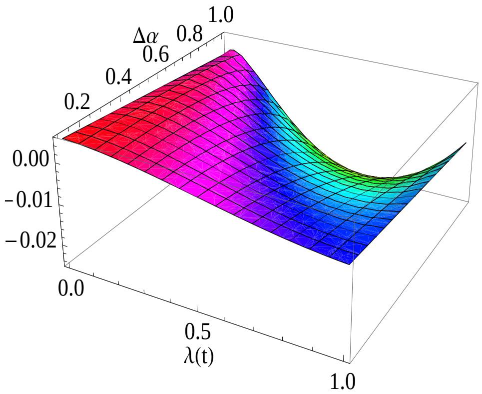
5.2. **Equity Premium.** The equity premium of instantaneous stock return is defined as

$$\mathbb{E}_t[dR_t - r_t dt] = (\mu_t - r_t)dt,$$

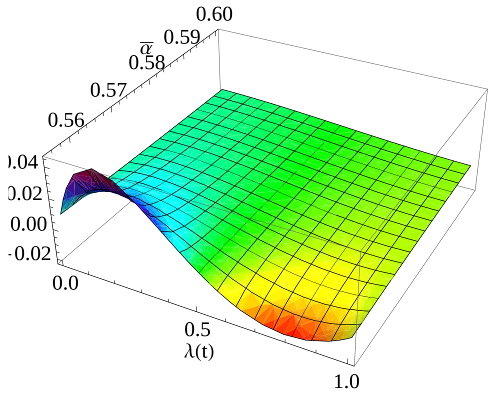
hence the annualized equity premium is equal to the product of the MPR or the Sharp ratio and the stock volatility, that is

$$\mu_t - r_t = \kappa_t \sigma_t. \tag{5.2}$$

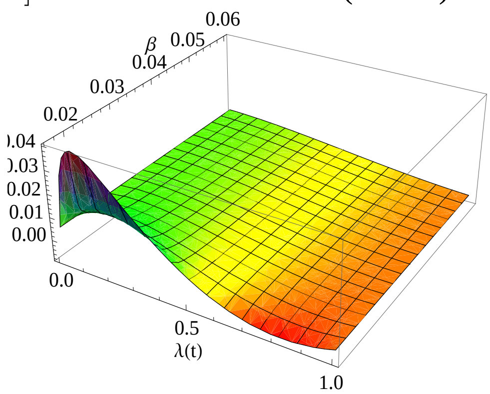
The estimated equity premium in the US market is 5% to 6% p.a based on the last one hundred years of data. When beliefs are homogenous, that is $\theta_i = 0$ for $i = 1, 2$, the market price of risk and the stock volatility are given by $\kappa_t = \sigma_D$ and $\sigma_t = \sigma_D$, thus the equity premium is *constant* and given by $\mu_t - r_t = \sigma_D^2$. If we assume volatility of dividend growth σ_D is identical to the volatility of aggregate consumption growth, which is around 3% p.a according to US consumption data, then the equity premium is equal to 0.09% p.a, which is *too low* compared to the historically observed level. When beliefs are heterogeneous, both stock volatility σ_t and the market price of risk κ_t depend on the fluctuations in agents' share of consumption.



(Case 1) $\Delta\alpha \in [0.1, 1]$



(Case 2) $\bar{\alpha} \in [0.555, 0.60]$



(Case 3) $\beta \in [0.0164, 0.06]$

FIGURE 5.2. Equity premium $\mu_t - r_t$ as a function of agent 1's share of aggregate consumption $\lambda_{1,t}$. The parameters values are $\mu_D = 0.02$, $\sigma_D = 0.03$, $\mu_{D,1} = 0.021$ and $\mu_{D,2} = 0.019$. In Case 1, we assume $\bar{\alpha} = 0.60$, $\beta = 0.06$. In Case 2, we assume $\Delta\alpha = 0.1$, $\beta = 0.06$. In Case 3, we assume $\Delta\alpha = 0.1$, $\bar{\alpha} = 0.60$.

To quantify the impact of disagreement on equity premium, we consider three cases in Fig. 5.2 based on a numerical analysis. Fig. 5.2 *Case 1* shows that *difference in risk aversions* does not contribute to higher equity premium, though it can produce excess volatility. When $\Delta\alpha = 1$, the equity premium is negative for most values of $\lambda_{1,t}$. The reason is that when the pessimist is much more risk averse than the optimist, in equilibrium, both agents' equivalent probability belief are more optimistic than their subjective beliefs, that is $\theta_i < \bar{\theta}_i$, for $i = 1, 2$. Thus the consensus belief is more likely to be over-optimistic compare to the objective belief, which leads to a negative market price of risk. In *Case 2*, when the average risk aversion is small, equity premium is positive when $\lambda_{1,t} < 0.5$ and negative when $\lambda_{1,t} > 0.5$. Intuitively, a smaller average risk aversion leads to greater stock volatility and a positive (negative) market price of risk when the consensus belief is over-pessimistic (over-optimistic). Of course, this intuition can also apply to standard differences-in-opinion models without KUJ. However the difference here is that under KUJ, there can be a large variation in the market price of risk even when the difference in subjective beliefs is relatively small, only 0.20% p.a. in this case. For example, when $\bar{\alpha} = 0.555$, under KUJ, the range for market price of risk is given by $(-0.3033, 0.3027)$ is *significantly* larger than the range without KUJ ($\alpha_1 = \alpha_2 = 1$), which is $(-0.0067, 0.06)$. *Case 3* shows that when the discount rate for relative consumption β is close to the threshold, there is a large positive equity premium when $\lambda_{1,t} < 0.5$ and a small negative equity premium when $\lambda_{1,t} > 0.5$. The asymmetry is due to the negative skewness in stock volatility (see Fig. 5.1 Case 3), a *high* stock volatility is accompanied by a *positive* market price of risk while a *low* stock volatility is accompanied by a *negative* market price of risk.

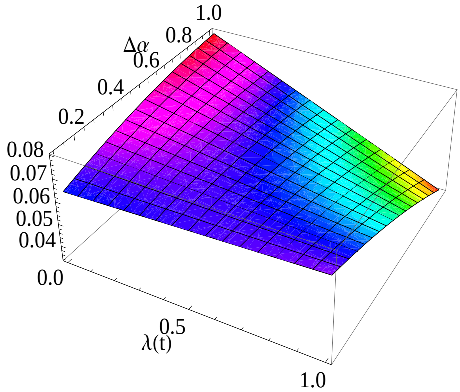
5.3. **Risk-free Rate.** The level of risk-free rate in the economy is given by equation (4.3), which is an aggregate consumption share weighted average of r_1 and r_2, where $r_i \equiv \bar{\beta}_i + \bar{\mu}_{D,i} - \sigma_D^2$ for $i = 1, 2$ and

$$r_1 - r_2 = (\bar{\beta}_1 - \bar{\beta}_2) + (\bar{\mu}_{D,1} - \bar{\mu}_{D,2}),$$

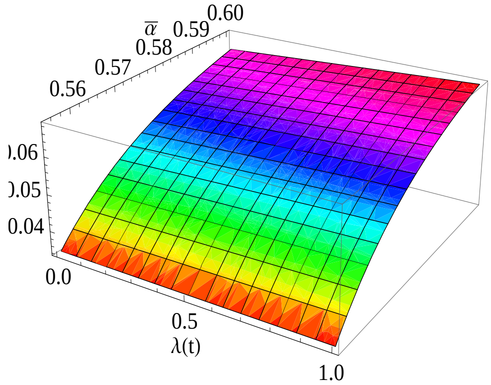
which shows that the range for the risk-free rate depends on the difference between agents' patience for future consumption and the difference between agents' equivalent beliefs about the expected dividend growth rate. Note that the risk-free rate can become a constant when $\bar{\beta}_1 - \bar{\beta}_2 = \bar{\mu}_{D,2} - \bar{\mu}_{D,1}$, that is when the optimistic agent is less risk averse than the pessimistic agent. The above condition can be re-written in terms of agents' risk aversions and the disagreement between their subjective beliefs, $\frac{1}{2}\Delta\alpha\bar{\delta} = \sigma_D$. The level of risk-free rate is also constant when beliefs are homogeneous, the benchmark risk-free rate under homogeneous beliefs is given by

$$r_t = \beta + \mu_D - \sigma_D^2. \tag{5.3}$$

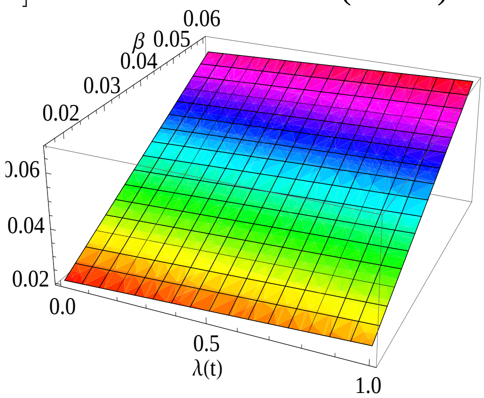
In general, risk-free rate depends on the fluctuations in the aggregate consumption shares of agents. Fig. 5.3 illustrates the impact of difference in risk aversions, average risk aversion and the subjective discount rate for relative consumption on the level of risk-free rate. *Case 1* shows a large difference in risk aversions leads to greater variations in the risk-free rate. Intuitively, under the KUJ, a large difference in risk aversions leads to greater difference in agents' patience for future consumption. *Case 2* shows that a smaller risk aversion reduces the variation in the risk-free rate, because a smaller average risk aversion leads to a larger in difference in agents' equivalent beliefs, which offsets the difference in patience for future consumption in this case. *Case 3* shows that the level of risk-free rate is negatively related to the discount rate for relative consumption. This is expected since agents' subjective discount



(Case 1) $\Delta\alpha \in [0.1, 1]$



(Case 2) $\bar{\alpha} \in [0.555, 0.60]$



(Case 3) $\beta \in [0.0164, 0.06]$

FIGURE 5.3. Risk-free rate r_t as a function of agent 1's share of aggregate consumption $\lambda_{1,t}$. The parameters values are $\mu_D = 0.02$, $\sigma_D = 0.03$, $\mu_{D,1} = 0.021$ and $\mu_{D,2} = 0.019$. In Case 1, we assume $\bar{\alpha} = 0.60$, $\beta = 0.06$. In Case 2, we assume $\Delta\alpha = 0.1$, $\beta = 0.06$. In Case 3, we assume $\Delta\alpha = 0.1$, $\bar{\alpha} = 0.60$.

rate for future absolute consumption increases in β. When $\beta = 0.0164$, the risk-free rate varies between 2.03% and 2.48%, which is close to the average short-term interest rate in the U.S (see Campbell (2003) Tab. 1). Note that when beliefs are *homogeneous*, the risk-free in (5.3) is equal to 3.55% for $\beta = 0.0164$. Therefore, in this case, a small disagreement of 0.2% together with difference in risk aversions can reduce the risk-free rate by more than 1%.

5.4. **Monte Carlo Simulation.** To quantify the overall impact of disagreement on equity premium, volatility, and risk-free rate, we conduct a Monte Carlo simulation in this section. Assume under the objective probability measure, the expected growth rate and volatility of the aggregate dividend are given by $\mu_D = 0.02$ and $\sigma_D = 0.03$. Furthermore, agents' subjective beliefs are given by $(\mu_{D,1}, \mu_{D,2}) = (0.021, 0.019)$ with a small disagreement of 0.2%. For a small time increment $\Delta t = T/n$, define the stock return and average stock return respectively as

$$\Delta R_{t+\Delta t} \equiv \frac{\Delta S_{t+\Delta t} + D_t \Delta t}{S_t} - 1 = \frac{\phi_{t+\Delta t} D_{t+\Delta t}}{\phi_t D_t} + \frac{1}{\phi_t}\Delta t - 1, \;\; \overline{\Delta R} \equiv \frac{1}{n}\left[\sum_{t=0}^{T-\Delta t} \Delta R_{t+\Delta t}\right].$$

We use Monte Carlo simulations to compute for each path, the *average equity premium*

$$\bar{\varepsilon}_T \equiv \frac{1}{n}\sum_{t=0}^{T-\Delta t}[\Delta R_{t+\Delta t} - r_t],$$

the *volatility of stock returns* $\bar{\sigma}_T$,

$$\bar{\sigma}_T^2 \equiv \frac{1}{n-1}\left[\sum_{t=0}^{T-\Delta t}(\Delta R_{t+\Delta t} - \overline{\Delta R})^2\right],$$

and the *average risk-free rate*

$$\bar{r}_T \equiv \frac{1}{n}\left[\sum_{t=0}^{T-\Delta t} r_t\right].$$

Tab. 5.1 reports the unconditional mean and standard deviation (in parenthesis) for each quantity for different combinations of difference in risk aversion ($\Delta\alpha$), average risk aversion ($\bar{\alpha}$), and subjective discount rate for relative consumption (β). We assume that the planning horizon is $T = 100$ years. Results show that *Case 3* produces the most desirable results. When agents have a small difference of 0.1 in risk aversions, a small average risk aversion and a discount rate for relative consumption close to the threshold, average equity premium has mean of 3.30% with a 1.68% standard deviation, average risk-free rate has a mean of 2.07% with 0.06% standard deviation, and volatility in stock returns has a means of 16.83% with a standard deviation of 5.36%. This demonstrates that, under the KUJ preferences, a *small* amount of disagreement among agents can produce on average significantly higher equity premium, lower risk-free rate and excess stock volatility compare to the market under homogeneous beliefs.

	$(\Delta\alpha, \bar{\alpha}, \beta)$	$\bar{\varepsilon}_T$	$\bar{\sigma}_T$	$\bar{r}_T$
Case 1	$(1, 0.6, 0.06)$	0.15%	14.22%	7.12%
		(0.82%)	(3.90%)	(0.88%)
Case 2	$(0.1, 0.555, 0.06)$	1.58%	13.75%	3.42%
		(1.55%)	(4.14%)	(0.005%)
Case 3	$(0.1, 0.6, 0.0164)$	3.30%	16.83%	2.07%
		(1.68%)	(5.36%)	(0.06%)

TABLE 5.1. Unconditional mean and standard deviations (in parenthesis) of the average equity premium $\bar{\varepsilon}_T$, volatility in stock returns $\bar{\sigma}_T$ and the average risk-free rate $\bar{r}_T$ for a planning horizon of $T = 100$ years.

6. CONCLUSION

Habit preference models play very important role in asset pricing literature. With external habit preferences and countercyclical variation in risk aversion, Campbell and Cochrane (1999) explain many of those market characteristics and puzzles and Chan and Kogan (2002) further show that such risk aversion variation can be endogenously generated through wealth redistribution of multiple agents with different risk aversion coefficients. However, Xiouros and Zapatero (2010) develop a discrete-time model of heterogeneous agents similar to that of Chan and Kogan (2002) and show that the heterogeneous risk aversion alone can only have a marginal, almost negligible, effect. They show that the variation required by the stochastic discount factor is unlikely to be produced by such a model with reasonable parameter values. They also point to heterogeneity in agents' expectation or beliefs as a promising alternative.

In this paper, we reconcile the findings in Campbell and Cochrane (1999) and Chan and Kogan (2002) by considering a combination of differences-in-opinion and *Keep up with the Joneses* (KUJ) models. Following the differences-in-opinion models, we assume that agents agree to disagree. But different from the current literature, the KUJ in this paper is characterized by a boundedly rational feature of agent to maximize the expected utility of his relative consumption to the other agent. When two agents disagree about the expected growth rate of the aggregate endowment process, we solve the optimal consumption and derive the market price of risk, risk-free rate, and price-dividend ration in closed form. We provide conditions for agents' long-run survival. We show that the model is able to characterize the cyclical behaviour of Sharpe ratio, volatility, risk premium, and P/D ratio. Our analysis show that a small disagreement can have significant impact on the market price of risk, risk-free rate, price-dividend ratio in market equilibrium. In particular, when the less risk averse agent is relatively optimistic, allowing a small amount of disagreement between agents can explain many market characterizes including excess volatility, a high equity premium and a low risk-free rate identified in financial markets. We show that heterogeneous beliefs and the KUJ preferences introduced in this paper is a promising alternative to explain why agents require high compensation for taking ex-post risk and why the market price of risk appears to very so dramatically over time.

Appendix A. Proofs

A.1. **Proof of Lemma 2.1.** (Optimal consumption policy)

By agent 1's first-order condition for optimal consumption,

$$\frac{c_{1,t}}{c_{2,t}} = I_{1,t}(\eta_1 \xi_t c_{2,t}/M_{1,t}),$$

where η_1 is the Lagrange multiplier to satisfy agent 1's budget constraint and $I_{1,t}(c_{1,t}/c_{2,t}) = u_{1,t}^{'-1}(c_{1,t}/c_{2,t})$ is the inverse function of agent 1's marginal utility of relative consumption. Similarly,

$$\frac{c_{2,t}}{c_{1,t}} = I_{2,t}(\eta_2 \xi_t c_{1,t}/M_{2,t}).$$

These lead to

$$c_{1,t} = \eta_1^{-\frac{1}{\alpha_1}} M_{1,t}^{\frac{1}{\alpha_1}} e^{-\frac{\beta t}{\alpha_1}} \xi_t^{\frac{-1}{\alpha_1}} c_{2,t}^{\frac{\alpha_1-1}{\alpha_1}},$$
$$c_{2,t} = \eta_2^{-\frac{1}{\alpha_2}} M_{2,t}^{\frac{1}{\alpha_2}} e^{-\frac{\beta t}{\alpha_2}} \xi_t^{\frac{-1}{\alpha_2}} c_{1,t}^{\frac{\alpha_2-1}{\alpha_2}}. \tag{A.1}$$

Solving this system of equations for $c_{1,t}$ and $c_{2,t}$ assuming $\alpha_1 + \alpha_2 \neq 1$ leads to

$$c_{1,t} = \bar{\eta}_1^{-1} e^{-\beta t} \bar{M}_{1,t} \xi_t^{-1},$$
$$c_{2,t} = \bar{\eta}_2^{-1} e^{-\beta t} \bar{M}_{2,t} \xi_t^{-1}, \tag{A.2}$$

where $\bar{\eta}_1 = \eta_1^{p_1}\eta_2^{q_1}$ and $\bar{\eta}_2 = \eta_1^{p_2}\eta_2^{q_2}$. Furthermore, the set of weights $\{p_i, q_i\}$ and $\bar{M}_{i,t}$ for $i = 1, 2$ are as defined in Lemma 2.1. Therefore, agent 1's share of aggregate consumption is given by

$$\lambda_{1,t} = \frac{c_{1,t}}{c_{1,t} + c_{2,t}} = \frac{\bar{\eta}_1^{-1}\bar{M}_{1,t}}{\bar{\eta}_1^{-1}\bar{M}_{1,t} + \bar{\eta}_2^{-1}\bar{M}_{2,t}}.$$

To solve for the modified Lagrange multipliers $\bar{\eta}_1$ and $\bar{\eta}_2$, we substitute $c_{1,t}$ into agent 1's budget constraint in (2.14), from which we obtain

$$\bar{\eta}^{-1}\mathbb{E}\left[\int_0^\infty e^{-\beta t}\bar{M}_{1,t}dt\right] = x_1 S_0.$$

Since $e^{-\beta t}\mathbb{E}[\bar{M}_{1,t}] = e^{-\bar{\beta}_1 t}$ where $\bar{\beta}_1$ is defined in Lemma 2.1 and x_1 is assumed to be $1/2$. We obtain $\bar{\eta}_1^{-1} = \bar{\beta}_1 S_0/2$. Similary, $\bar{\eta}_2^{-1} = \bar{\beta}_2 S_0/2$. Hence, we have the expression of $\lambda_{1,t}$ in equation (2.18). The stochastic differential equation (SDE) for $\lambda_{1,t}$ can be obtained by applying Itô's lemma to equation (2.18). □

A.2. **Proof of Proposition 3.2.** (Long-run Survival) Agent 1's share of aggregate consumption can be written as

$$\lambda_{1,t} = \left(1 + \frac{\bar{\beta}_2\bar{M}_{2,t}}{\bar{\beta}_1\bar{M}_{1,t}}\right)^{-1}.$$

Therefore, agent 1's long-run survival depends on the ratio of agents' equivalent beliefs $\bar{M}_{2,t}/\bar{M}_{1,t}$, which is given by

$$\frac{\bar{M}_{2,t}}{\bar{M}_{1,t}} = \exp\left\{\frac{-\frac{1}{2}(\theta_2^2 - \theta_1^2) - \delta\omega_t}{\alpha_1 + \alpha_2 - 1}\right\}. \tag{A.3}$$

If (A.3) converges to zero as $t \to \infty$, then $\lambda_{1,t} \overset{a.s}{\to} 1$ and agent 1 survives and agent 2 vanishes. If (A.3) diverges to infinity as $t \to \infty$, then $\lambda_{1,t} \overset{a.s}{\to} 0$ and agent 1 vanishes while agent 2 survives. Using the strong Law of Large Numbers for Brownian Motion (see Karatzas and Shreve (1991, Sec.2.9.A)), for any value of σ,

$$\lim_{t\to 0} \ \exp\{at + \sigma\omega_t\} = \begin{cases} 0, & a < 0 \\ \infty, & a > 0, \end{cases}$$

where the convergence takes place almost surely. In our case,

$$a = -\frac{1}{2}\frac{\theta_2^2 - \theta_1^2}{\alpha_1 + \alpha_2 - 1}.$$

We consider two cases, (i) $\alpha_1 + \alpha_2 > 1$ and (ii) $\alpha_1 + \alpha_2 < 1$. Under case (i), $a > 0$ if and only if $|\theta_2| < |\theta_1|$, under which agent 1 vanishes and agent 2 survives. Similarly, $a < 0$ if and only if $|\theta_1| < |\theta_2|$, under which agent 1 survives and agent 2 vanishes. Under case (ii), a has the opposite sign compare to case 1. Lastly, $a = 0$ if and only if $|\theta_1| = |\theta_2|$ under which there does not exist an stationary distribution for $\bar{M}_{2,t}/\bar{M}_{1,t}$, hence both agent 1 and agent 2 survive in the long-run. □

A.3. **Proof of Proposition 3.3.** (Speed of Market Selection) An equivalent problem to computing (3.3) is to compute

$$\tau_l = \mathbb{E}[\inf\{t : \ at + \omega_t = l^*\}], \tag{A.4}$$

where

$$a = \frac{1}{2}\frac{\theta_2^2 - \theta_1^2}{\delta} \text{ and } l^* = -\frac{\alpha_1 + \alpha_2 - 1}{\delta} \ln\left[\frac{\lambda_{1,0}}{\lambda_{2,0}}\left(\frac{1-l}{l}\right)\right].$$

This is a well studied problem, see Karatzas and Shreve (1991) Chapter 3.5, the explicit density function of τ_l is given by

$$\mathcal{P}[l \in dt] = \frac{|l^*|}{\sqrt{2\pi t^3}} \exp\left\{ -\frac{(l^* - at)^2}{2t}\right\} dt, \ \ t > 0.$$

To find the expected first hitting time, we compute the integral

$$\int_0^\infty \frac{|l^*|}{\sqrt{2\pi t}} \exp\left\{ -\frac{(l^* - at)^2}{2t}\right\} dt. \tag{A.5}$$

The explicit solution of (A.5) depends on the sign of a and l^*. There are two cases in Proposition 3.2 under which agent 1 vanishes in the long-run, (i) $\alpha_1 + \alpha_2 > 1$ and (ii) $\alpha_1 + \alpha_2 < 1$. If (i), then the sufficient and necessary condition for agent 1 to vanish is $\theta_2^2 < \theta_1^2$. If (ii), then the sufficient and necessary condition for agent 2 to vanish in the long-run is $\theta_1^2 < \theta_2^2$. Therefore, regardless of the sign of δ, a and l^* always have the same sign, under which the integral in (A.5) has the closed-form solution given by

$$\tau_l = \frac{l^*}{a}$$

Substituting in the values for a and l^* completes the proof. □

A.4. **Proof of Proposition 4.1.** (Consensus Consumer) See Jouini and Napp (2007) Example 2.1. □

A.5. **Proof of Proposition 4.2.** (Market Equilibrium) From the first order condition of the consensus consumer, the SPD process $\xi_t = e^{-\beta t} M_t / D_t$, therefore

$$\frac{d\xi_t}{\xi_t} = \left[-\bar{\beta}_t - \mu_D + \sigma_D(\sigma_D - \bar{\theta}_t)\right] dt + (\bar{\theta}_t - \sigma_D) d\omega,$$

which implies that $\kappa_t = \sigma_D - \bar{\theta}_t$ and $r_t = \bar{\beta}_t + \mu_D - \sigma_D(\sigma_D - \bar{\theta}_t)$. To obtain the equilibrium stock price in (4.4), we evaluate the following expected value of an integral

$$S_t = \mathbb{E}_t\left[\int_t^\infty \frac{\xi_s}{\xi_t} D_s ds\right].$$

Given the expression of ξ_s/ξ_t in (4.1), we can rewrite the integral as

$$S_t = \mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)} \frac{M_s}{M_t} ds\right] D_t.$$

Moreover, since

$$\frac{M_s}{M_t} = \lambda_{1,t}\frac{\bar{M}_{1,s}}{\bar{M}_{1,t}} + \lambda_{2,t}\frac{\bar{M}_{2,s}}{\bar{M}_{2,t}} = \lambda_{1,t} e^{-(\bar{\beta}_1 - \beta)(s-t)} \frac{\hat{M}_{1,s}}{\hat{M}_{1,t}} + \lambda_{2,t} e^{-(\bar{\beta}_2 - \beta)(s-t)} \frac{\hat{M}_{2,s}}{\hat{M}_{2,t}},$$

the equilibrium stock price is given by

$$\frac{S_t}{D_t} = \lambda_{1,t}\int_t^{\infty} e^{-\bar{\beta}_1(s-t)}ds + \lambda_{2,t}\int_t^{\infty} e^{-\bar{\beta}_2(s-t)}ds = \frac{\lambda_{1,t}}{\bar{\beta}_1} + \frac{\lambda_{2,t}}{\bar{\beta}_2},$$

which completes the proof. □

A.6. **Proof of Corollary 4.3.** (Cyclical Behaviour)

Following Lemma 2.1, the covariance between agent 1's consumption share and the aggregate endowment process is given by

$$\sigma_{\lambda,t} \equiv \lambda_{1,t}\lambda_{2,t}\bar{\delta},$$

hence the covariance is strictly positive (negative) when $\bar{\delta} > (<)0$, that is agent 1 is *equivalently* more optimistic (pessimistic) than agent 2.

First, note that from equation (4.2), the equilibrium MPR or Sharpe ratio satisfies

$$d\kappa_t = -\bar{\delta}d\lambda_{1,t}$$

Hence $d\kappa_t dD_t < 0$, that is Sharpe ratio is countercyclical. Moreover, the risk-free rate from equation (4.3) satisfies

$$dr_t = [(\bar{\beta}_1 - \bar{\beta}_2) + \sigma_D(\bar{\theta}_1 - \bar{\theta}_2)]d\lambda_{1,t} = \bar{\delta}(\sigma_D - \frac{1}{2}\Delta\alpha\bar{\delta})d\lambda_{1,t}.$$

Therefore, since $\Delta\alpha > 0$, the risk-free interest rate is procyclical if and only if $\sigma_D > \frac{1}{2}\Delta\alpha\bar{\delta}$. Furthermore, the P/D ratio in equation (4.4) satisfies

$$d\phi_t = (\phi_1 - \phi_2)d\lambda_{1,t} = \frac{\Delta\alpha\bar{\delta}^2}{2\bar{\beta}_1\bar{\beta}_2}d\lambda_{1,t}.$$

Therefore, the P/D ratio is procyclical and the dividend yield $1/\phi_t$ is countercyclical if and only if $(\Delta\alpha)\bar{\delta} > 0$, that is the less risk averse agent is equivalently more optimistic than the more risk averse agent.

A.7. **Proof of Corollary 5.1.** (Stock Volatility) Given that the equilibrium stock price is given by (4.4), the equilibrium stock return is given by

$$dR_t = \frac{d\phi_t}{\phi_t} + \frac{dD_t}{D_t} + \frac{d\phi_t}{\phi_t}\frac{dD_t}{D_t} + \frac{1}{\phi_t}dt$$

The instantaneous relative change in the price-dividend ratio is given by

$$\frac{d\phi_t}{\phi_t} = (\phi_1 - \phi_2)\frac{d\lambda_{1,t}}{\phi_t}.$$

Hence the volatility of $d\phi_t/\phi_t$ is given by

$$\sigma_{\lambda,t}\frac{(\phi_1 - \phi_2)}{\phi_t}.$$

Given that the volatility of the stock dividend is σ_D, the signed volatility of stock returns σ_t coincides with equation (5.1). □

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