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HETEROGENEOUS BELIEFS AND THE PERFORMANCES OF OPTIMAL PORTFOLIOS

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ABSTRACT. The market selection depends on agent's survival index, which is a function of agent's belief and risk preference. When preferences are identical, the survival index of an agent is a decreasing function of his belief accuracy and therefore agent survives if and only if he has the lowest survival index. Following this result, one maybe tempted to think that an agent is expected to perform at least as good as the market if he survives, and he is expected to outperform the market if his belief is more accurate than all other agents' beliefs. We show that the these statements are false in general. In terms of long-run performance, market outperforms those agents who do not have the minimum survival index in the long-run. When multiple agents survive, we show that no agent can outperform the market in the long-run. In terms of the expected performance, all agents are expected to underperform the market even when they all survive in the long-run. When survival indices differ, the fittest agent with the lowest survival index is expected to outperform the market consistently with any given finite investment horizons if and only if his subjective belief is much more accurate than the other agents' beliefs.

JEL Classification: G12, D84.

Keywords: heterogeneous beliefs; equilibrium asset prices; survival; consensus belief; portfolio performance.

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1. INTRODUCTION

There is a growing evidence that investors are heterogeneous in their expectations and invest in subjectively optimal portfolios that deviate significantly from the market portfolio. The goal of this paper is to compare the performance of the optimal portfolios held by investors with heterogeneous expectations to the performance of the market portfolio. More specifically, we ask the following question; under which situation is it possible for investors to beat the market?

It has long been argued that investors with irrational beliefs do not survive in the long run since they consistently lose money to the rational investors. In other words, investors with irrational beliefs are expected to perform poorly compared to the investors with rational beliefs. This evolutionary argument started with Friedman (1953), who argued that irrational investors need not be considered in determining asset prices since they will be eliminated in the long run. In a partial equilibrium model, DeLong, Shleifer, Summers and Waldmann (1991) show that irrational investors with wrong beliefs may survive in the long run because they may hold portfolios with high growth rates, which outgrow the portfolios of the rational investors. In a general equilibrium model, it turns out that small differences in modeling choices can lead to quite different results. Firstly, Blume and Easley (1992) show that incorrect beliefs may help an investor to survive when saving decisions are exogenous. Kogan, Ross, Wang and Westerfield (2006) consider an equilibrium model with no intermediate consumption, so that investors have to save 100% of their wealth. They show that investors with wrong belief may survive, and even when they do not, they can have significant price impact with negligible shares of aggregate wealth. In contrast, when portfolio choices and saving decisions are endogenized and intermediate consumption is considered, having a wrong belief is always disadvantageous for an investor's survival, see Sandroni (2000), Blume and Easley (2006) and Yan (2008). Secondly, Sandroni (2000) and Blume and Easley (2006) show that, in an economy without growth, the survival is independent of investor's utility function. However, in a growth economy, Yan (2008) shows that an investor survives in the long run if and only if his survival index is the lowest. The market selection process can be extremely slow in an economy with *one* risky asset

(see for example Yan (2008) and Dumas, Kurshev and Uppal (2009)), however, the selection process is much faster in a *multi-asset* economy (see Fedyk, Heyerdahl-Larsen and Walden (2012)). An investor's survival index depends on his belief, patience for future consumption, the growth rate in the economy and the elasticity of intertemporal substitution.

The above discussion shows the literature has made some progress in confirming the market selection hypothesis. Essentially, when preferences are the same, only the investors with the most accurate beliefs will survive and determine the asset prices in the long run. In the market selection literature, they are called the *fittest* investors. Although the fittest investors have the most accurate beliefs, their beliefs do not necessarily coincide with the objective (true) belief. Blume and Easley (2009) and Yan (2008) show that when none of the beliefs held by the investors coincides with the objective belief, there can be multiple survivors with different beliefs but the same survival indices. Jouini and Napp (2011) considers an equilibrium model with multiple survivors and analyze their impact on asset prices. By assuming that individual investors are irrational, but are on average rational, they obtain waves of optimism and pessimism due to wealth fluctuations between investors with different beliefs, leading to a modified long-run risk-return relation. In terms of the portfolio performance, *conventional wisdom* in the market is that investors with relatively more accurate beliefs should consistently outperform the market while the investors with relatively less accurate beliefs consistently underperform the market. A second idea is articulated by the *index fund premise* in Sharpe (2007), "*Few of us are as smart as all of us, it is hard to identify them in advance, and they may charge more than they are worth.*" Ignoring fees charged by the fund managers, the index fund premise suggests that an investor (or fund manager) can beat (outperform) the market if he is *smarter than everyone else* in the market (or fitter than all other investors).

In this paper, we examine these two ideas, namely the conventional wisdom and the index fund premise in a *dynamic equilibrium model*. By assuming that agents have logarithmic preference and the same discount rate for future consumption, we show that an agent's long-run performance is closely linked to his long-run survival. We introduce two criteria to measure

investors' performance relative to the market, namely, *long-run performance* and *expected performance*. In terms of long-run performance, if an investor survives, he will *not* underperform the market in the long run. However, in order to survive, it is not enough just to have a relatively more accurate belief, he must have the *most accurate* belief, which is measured by his survival index. Hence, in terms of long-run performance, the conventional wisdom is *invalid*. Furthermore, if an investor does not survive, then the indexer who holds the market portfolio will outperform the investor in the long run. In terms of expected performance, we show that even when all investors have the same survival index, that is, they all survive in the long-run, they are expected to underperform the market given a finite investment horizon. Moreover, when survival indices differ and there is only *one* survivor in the economy who is fitter than everyone else, he can consistently outperform the market given any investment horizons *if and only if* he is a lot smarter than everyone else in terms of his belief accuracy. Otherwise, the fittest investor can still be expected to underperform the market given short investment horizons. This means that it is not enough to be smarter than everyone else, in order to *beat* the market consistently, one needs to be *a lot* smarter than everyone else. Therefore, the index fund premise is also invalid.

The rest of the paper is organized as follows. In Section 2, we present a dynamic equilibrium model in continuous time where investors have logarithmic preferences and heterogeneous beliefs about the expected dividend growth rates. The market equilibrium is then determined by constructing a consensus belief. Section 3 evaluates the performance of the subjectively optimal portfolios held by investors relative to the market by two criteria; long run performance and the expected performance. Section 4 discusses potential problems and challenges in generalizing the results. Section 5 concludes. All proofs are provided in the Appendix.

2. THE MODEL

We develop a dynamic general equilibrium model with an infinite time horizon and N investors ($i = 1, 2, \cdots, N$). The uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$

on which a K-dimensional Wiener process $\mathbf{Z}(t) = (Z_1(t), \cdots, Z_K(t))^T$ is defined, and $\mathcal{F}_t$ is the information generated by the Wiener process $\mathbf{Z}(t)$. We assume $\mathbf{Z}(t)$ has a constant correlation matrix given by $\boldsymbol{\rho} = (\rho_{jk})_{K\times K}$. The cumulative dividend from time zero to T is given by $\int_0^T D_k(t)dt$ where $D_k(t)$ follows a geometric Brownian motion,

$$dD_k(t) = D_k(t)[\mu_{D,k}dt + \sigma_{D,k}dZ_k(t)] \tag{2.1}$$

with constant $\mu_{D,k}$ and $\sigma_{D,k}$. The *aggregate dividend process* is defined by $D(t) \equiv \sum_{k=1}^K D_k(t)$. Let $\bar{\mu}(t) \equiv (\sum_{k=1}^K \mu_{D,k}D_k)/D(t)$ and $\bar{\boldsymbol{\sigma}}(t) \equiv (\bar{\sigma}_1(t), \bar{\sigma}_2(t), \cdots, \bar{\sigma}_K(t))^T$, where $\bar{\sigma}_k(t) \equiv (\sigma_{D,k}D_k)/D(t)$. Then the aggregate dividend process satisfies the *stochastic differential equation* (SDE),

$$dD(t) = D(t)[\bar{\mu}(t)dt + \bar{\boldsymbol{\sigma}}(t)^T d\mathbf{Z}(t)]. \tag{2.2}$$

We assume $\mathcal{P}$ is the hypothetically *true* probability measure, which is called the *objective* probability measure. Moreover, $\mu_{D,k}$ is the objective belief about the expected dividend growth rate of asset k. Investors in the model do not know about $\mathcal{P}$ and $\mu_{D,k}$ and their optimal portfolio/consumption strategies are based on their subjective probability measure $\mathcal{P}_i$ and their perceived expected dividend growth rates $\mu_{D,k}^{(i)}$.

2.1. **Belief System.** We consider a *simple* belief system, in which investors have *constant* disagreements about the expected dividend growth rates. More precisely, investor i lives in a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P}_i)$ in which $\mathbf{Z}_i(t) \equiv \mathbf{Z}(t) - \boldsymbol{\theta}_i t$ is a Wiener process, hence *believes* that $D_k(t)$ follows

$$dD_k(t) = D_k(t)[\mu_{D,k}^{(i)}dt + \sigma_{D,k}dZ_k^{(i)}(t)], \tag{2.3}$$

where $\mu_{D,k}^{(i)} \equiv \mu_{D,k} + \theta_k^{(i)}\sigma_{D,k}$ is the *perceived* expected dividend growth rate of asset k by investor i. Furthermore, $\theta_k^{(i)}$ measures the *deviation* of investor i's perceived expected dividend growth rate from the objective one. When $\theta_k^{(i)} > (<)0$, investor i is *optimistic (pessimistic)* about asset k's dividend growth relative to the objective belief. Under this simple belief system, investor i's subjective probability measure can be defined by a positive density process $M_i(t)$,

which is a $\mathcal{P}$-martingale (see Jouini and Napp (2007)). Let $\hat{\boldsymbol{\theta}}_i \equiv \boldsymbol{\rho}^{-1}\boldsymbol{\theta}_i$ and

$$M_i(t) \equiv \frac{d\mathcal{P}_i}{d\mathcal{P}} = \exp\left\{ -\frac{1}{2}\boldsymbol{\theta}_i^T\boldsymbol{\rho}^{-1}\boldsymbol{\theta}_i \ t + \hat{\boldsymbol{\theta}}_i^T\mathbf{Z}(t)\right\}, \tag{2.4}$$

Then $M_i(t)$ satisfies

$$\frac{dM_i(t)}{M_i(t)} = \hat{\boldsymbol{\theta}}_i^T d\mathbf{Z}(t). \tag{2.5}$$

2.2. **Securities Market.** There is a risk-free bond with price $B(t)$ with $B(0) = 1$ and

$$dB(t) = r(t)B(t)\ dt, \tag{2.6}$$

where $r(t)$ is the interest rate determined in equilibrium. There are K risky assets indexed by $k = 1, 2, \cdots, K$, each asset k is a claim to the kth dividend process. The ex-dividend price for asset k is denoted by $S_k(t)$ and the instantaneous return of asset k is denoted by $dR_k(t) \equiv (dS_k(t) + D_k(t))/S_k(t)$. Define the vector of instantaneous returns of risky assets as $d\mathbf{R}(t) \equiv (dR_1(t), \cdots, dR_K(t))^T$, which satisfies

$$d\mathbf{R}(t) = \boldsymbol{\mu}(t)dt + \boldsymbol{\sigma}(t)d\mathbf{Z}(t), \tag{2.7}$$

where $\boldsymbol{\mu}(t) \equiv (\mu_1(t), \cdots, \mu_K(t))^T$ is the *vector of expected asset returns* under the objective probability measure and $\boldsymbol{\sigma}(t) \equiv \{\sigma_{j,k}(t)\}_{K\times K}$ is the *volatility matrix*, which is assumed to be invertible. Both the expected return vector and the volatility matrix are determined in equilibrium. Following Merton (1980), high-frequency data allows investors to estimate volatilities to any desired precision. However, the growth rate of a process is much harder to estimate. Accurate estimation of the growth rate requires a long time-series of data. This means that investors in the economy can disagree about the growth rate of the dividend processes even though they have the same information (observe the same processes) given by the filtration $\mathcal{F}_t$. More precisely, given investors' belief system, the consistency relation implies that investor i believes that the instantaneous returns of risky assets follow

$$d\mathbf{R}(t) = \boldsymbol{\mu}_i(t)dt + \boldsymbol{\sigma}(t)d\mathbf{Z}_i(t), \tag{2.8}$$

where $\boldsymbol{\mu}_i(t) \equiv \boldsymbol{\mu}(t) + \boldsymbol{\sigma}(t)\boldsymbol{\theta}_i$ is the vector of expected returns under the subjective probability measure of investor i.

2.3. **Investors' Preferences.** We assume all investors have the same utility function, which is *time additive* and *logarithmic* with a common *subjective discount rate* β. More precisely, investor i's objective at time zero is to maximize his expected lifetime utility. We distinguish between two measures of investor i's *expected utility* of consumption, one is measured by investor i's subjective probability measure $\mathcal{P}_i$,

$$U_i(0,T) \equiv \mathbb{E}^{(i)}\left[\int_0^T e^{-\beta t}\ln(c_i(t))dt\right], \tag{2.9}$$

where $\mathbb{E}^{(i)}$ is the expectation operator[1] defined by $M_i(t)$, and the other is measured under the objective probability measure $\mathcal{P}$,

$$\hat{U}_i(0,T) \equiv \mathbb{E}\left[\int_0^T e^{-\beta t}\ln(c_i(t))dt\right]. \tag{2.10}$$

Intuitively, the expected utility, $U_i(0,T)$, measured under $\mathcal{P}_i$ is investor i's *perceived expected utility* based on his subjective belief about the expected dividend growth rates $\{\mu_{D,k}^{(i)}\}$ and the vector of expected returns $\boldsymbol{\mu}_i(t)$. In contrast, $\hat{U}_i(0,T)$ is investor i's expected utility based on the objective belief about the expected dividend growth rates $\{\mu_{D,k}\}$ and the vector of expected returns $\boldsymbol{\mu}(t)$. In general, investor i would not be able to compute $\hat{U}_i(0,T)$ since he does not know about the objective belief. However, suppose one is able to run simulations of the economy as independent experiments, $\hat{U}_i(0,T)$ can be interpreted as the *expected realized utility* of investor i (see Brunnermeier and Parker (2005)).

Investor i's objective is to maximize $U_i(0,\infty)$ subjected to the *dynamic budget constraint*,

$$dW_i(t) = \Big(W_i(t)[r(t) + \boldsymbol{\pi}_i(t)^T(\boldsymbol{\mu}(t) - r(t)\mathbf{1})] - c_i(t)\Big)dt + W_i(t)[\boldsymbol{\pi}_i(t)^T\boldsymbol{\sigma}(t)d\mathbf{Z}(t)], \tag{2.11}$$

[1]The probability under $\mathcal{P}_i$ of an event A that occurs at time T conditional on the information at time t can be calculated by $\mathbb{E}_t^{(i)}[1_A] = \mathbb{E}_t[\frac{M_i(T)}{M_i(t)}1_A]$.

where $\boldsymbol{\pi}_i(t) \equiv (\pi_1^{(i)}(t), \cdots, \pi_K(t))^T$ is the vector of proportions of investor i's wealth invested in each risky asset, and $\boldsymbol{\alpha}(t) \equiv \boldsymbol{\mu}(t) - r(t)\mathbf{1}$ is the vector of *expected excess returns*. Since the market is *complete*, there exists a *state price density* (SPD) process $\xi(t)$ with $\xi(0) = 1$ satisfying

$$d\xi(t) = -\xi(t)[r(t)dt + \boldsymbol{\kappa}^T(t)d\mathbf{Z}(t)], \tag{2.12}$$

where

$$\boldsymbol{\kappa}(t) = (\kappa_1(t), \cdots, \kappa_K(t))^T \equiv (\boldsymbol{\sigma}(t)\boldsymbol{\rho})^{-1}\boldsymbol{\alpha}(t)$$

is the vector of *market prices of risk*. Using the SPD process, investor i's budget constraint becomes a static one given by

$$\mathbb{E}\bigg[\int_0^\infty \xi(t)c_i(t)dt\bigg] \leq W_i(0) = \sum_{k=1}^K x_k^{(i)} S_k(0), \tag{2.13}$$

where $x_k^{(i)}$ denotes the endowment of investor i in number of shares of asset k (see Cox and Huang (1989) and Karatzas, Lehoczky and Shreve (1987), Cvitanic and Zapatero (2004)). Moreover, we assume each risky asset is in one unit supply and the risk-free bond is in zero net supply, and investors have the same initial wealth[2], that is,

$$W_i(0) = \bar{W} \qquad \text{for } i = 1, 2, \cdots, N. \tag{2.14}$$

Lemma 2.1. *(Optimal Portfolio/Consumption Strategy)*

(i). *Investor i's optimal rate of consumption is given by*

$$c_i(t) = \beta\bar{W}\frac{M_i(t)e^{-\beta t}}{\xi(t)}. \tag{2.15}$$

(ii). *Investor i's share of aggregate consumption is given by*

$$\lambda_i(t) \equiv \frac{c_i(t)}{\sum_{h=1}^N c_h(t)} = \bigg(\sum_{h=1}^N \frac{M_h(t)}{M_i(t)}\bigg)^{-1}. \tag{2.16}$$

[2]Investor i's initial wealth is jointly determined by his endowments $\{x_k^{(i)}\}$ and equilibrium asset prices $\{S_k(0)\}$. One sufficient condition for (2.14) to hold is $x_k^{(i)} = \frac{1}{N}$ for all k, however it is not a necessary condition.

(iii). *Investor i's optimal portfolio is given by*

$$\boldsymbol{\pi}_i(t) = \mathbf{V}(t)^{-1}\boldsymbol{\alpha}_i(t), \tag{2.17}$$

where $\mathbf{V}(t) \equiv \boldsymbol{\sigma}(t)\boldsymbol{\rho}\boldsymbol{\sigma}(t)$ is the covariance matrix of the instantaneous asset returns and $\boldsymbol{\alpha}_i(t) \equiv \boldsymbol{\alpha}(t) + \boldsymbol{\sigma}(t)\boldsymbol{\theta}_i$ is the perceived vector of expected excess returns under investor i's subjective probability measure.

Lemma 2.1 shows that since investors have identical preferences, their optimal consumption process only differ by their subjective probability measures. Moreover, investor i's share of aggregate consumption depends on the ratio between his subjective probability measure to that of the other investors in the economy. In terms of optimal portfolio strategies, since each investor is *myopic* due to the assumption of logarithmic preference, investor i simply chooses the efficient *mean-variance* portfolio under his subjective belief about the expected excess returns.

2.4. **Consensus Belief and Market Equilibrium.** In this section, we follow the approach in Jouini and Napp (2007) to construct a consensus belief and to determine market equilibrium. First, we present a multi-asset version of the definition of equilibrium in Basak (2005).

Definition 2.2. *An equilibrium is a price system $(\{S_k(t)\}, B(t))$ and consumption-portfolio strategies $(c_i(t), \boldsymbol{\pi}_i(t))$ such that*

(i) *investors choose their optimal consumption-portfolio under their subjective probability measures;*

(ii) *the perceived asset prices are consistent across investors, that is,*

$$S_k(t) = \mathbb{E}_t^{(i)}\left[\int_0^\infty e^{-\beta(s-t)}\frac{c_i(t)}{c_i(s)}D_k(s)\ ds\right]$$

for all $k = 1, 2, \cdots, K$ and $i = 1, 2, \cdots, N$, and

(iii) *goods and security markets clear, meaning that*

$$\sum_{i=1}^{N} c_i(t) = D(t), \qquad \sum_{i=1}^{N} W_i(t) = \sum_{k=1}^{K} S_k(t),$$
$$\sum_{i=1}^{N} \pi_k^{(i)}(t) W_i(t) = S_k(t) \text{ for } k = 1, 2, \cdots, K.$$

We follow Jouini and Napp (2007) to construct a *consensus investor* with logarithmic preference and a *consensus belief* $M(t)$ who has an initial wealth of $\bar{W}$. Jouini and Napp (2007) show that, with

$$M(t) = \sum_{i=1}^{N} \omega_i M_i(t), \text{ where } \omega_i = \frac{1/\eta_i}{\sum_{h=1}^{N} 1/\eta_h}, \tag{2.18}$$

an economy with a common belief $M(t)$ generates the same set of equilibrium prices as an economy with heterogeneous beliefs $\{M_i(t)\}$, where η_i is investor i's Lagrange multiplier. From equation (2.18), the consensus belief satisfies $M(0) = 1$ and

$$dM(t) = M(t)[\hat{\boldsymbol{\theta}}(t)^T d\mathbf{Z}(t)], \tag{2.19}$$

where $\hat{\boldsymbol{\theta}}(t) \equiv \sum_{i=1}^{N} \lambda_i(t)\hat{\boldsymbol{\theta}}_i$, and $\boldsymbol{\theta}(t) = \boldsymbol{\rho}\hat{\boldsymbol{\theta}}(t)$ measures the deviation of the consensus belief from the objective belief about expected dividend growth rates. When $\theta_k(t) > (<)0$, a consensus consumer is optimistic (pessimistic) about asset k's dividend growth relative to the objective belief. Note that although each investor's optimism/pessimism is assumed to be constant, the optimism/pessimism of the consensus consumer depends on investors' share of aggregate consumption, thus is time-varying. Intuitively, when the market is dominated by the optimists (pessimists), a consensus investor who invests in the market portfolio is also optimistic (pessimistic).

In the economy with a common belief $M(t)$, each investor optimally invests in the *market portfolio* and consumes a fraction of the aggregate dividend. Thus we can determine the SPD

process $\xi(t)$ from the first order condition of the consensus consumer,

$$D(t) = \frac{M(t)e^{-\beta t}}{\xi(t)} \qquad \Rightarrow \qquad \xi(t) = \frac{M(t)e^{-\beta t}}{D(t)} \tag{2.20}$$

and determine the equilibrium asset price of each asset k,

$$S_k(t) = \mathbb{E}_t\bigg[\int_0^\infty \frac{\xi(s)}{\xi(t)} D_k(s)\ ds\bigg] = \mathbb{E}_t\bigg[\int_0^\infty\ e^{-\beta(s-t)}\frac{M(s)}{M(t)}\frac{D(t)}{D(s)}D_k(s)\ ds\bigg]. \tag{2.21}$$

Consequently, we can characterize the market equilibrium.

Proposition 2.3. *(Market Equilibrium)*

(i) *The market prices of risk is given by*

$$\boldsymbol{\kappa}(t) = \bar{\boldsymbol{\sigma}}(t) - \boldsymbol{\rho}^{-1}\boldsymbol{\theta}(t). \tag{2.22}$$

(ii) *The risk-free rate is given by*

$$r(t) = \beta + \bar{\mu}(t) - \bar{\boldsymbol{\sigma}}(t)^T\boldsymbol{\rho}\bar{\boldsymbol{\sigma}}(t) + \boldsymbol{\theta}(t)^T\bar{\boldsymbol{\sigma}}(t). \tag{2.23}$$

(iii) *The equilibrium price of asset k is given by*

$$S_k(t) = \sum_{i=1}^{N}\lambda_i(t)S_k^{(i)}(t), \qquad k = 1, 2, \cdots, K, \tag{2.24}$$

where

$$S_k^{(i)}(t) = \mathbb{E}_t^{(i)}\bigg[\int_0^\infty e^{-\beta(s-t)}\frac{D(t)}{D(s)}D_k(s)ds\bigg]$$

is the equilibrium of asset k in an economy populated by investor i alone.

Proposition 2.3 gives explicit solutions for the market prices of risk, the risk-free rate and the equilibrium asset prices in terms of the consensus belief, that is $\boldsymbol{\theta}(t)$. Equation (2.22) shows that the vector of market prices of risk is negatively related to level of optimism in the consensus belief about the expected dividend growth rates weighted by the inverse of the correlation matrix of the dividend processes, that is $\boldsymbol{\rho}^{-1}\boldsymbol{\theta}(t)$. The risk-free rate, as shown in equation (2.23) is

the standard one under homogeneous beliefs plus the optimism in consensus belief about the aggregate dividend growth, which is measured by $\boldsymbol{\theta}(t)^T\bar{\boldsymbol{\sigma}}(t)$. Furthermore, equation (2.24) shows that the equilibrium price for each risky asset k is a consumption share weighted average of asset prices that would prevail in economies in each of which there is only one investor, which is consistent with Detemple and Murthy (1994), Yan (2010), Xiong and Yan (2010) and Jouini and Napp (2011).

3. PERFORMANCE EVALUATION

In this section, we compare the performances between the market portfolio and the optimal portfolios held by the investors. First, we define the *cumulative continuous return* of investor i's optimal portfolio from time zero to time T as

$$g_i(T) \equiv \ln\left[\frac{W_i(T)}{W_i(0)}\right]$$

and the cumulative continuous return of the *market portfolio* as

$$g(T) \equiv \ln\left[\frac{W(T)}{W(0)}\right],$$

where $W(t) \equiv \sum_{i=1}^{N} W_i(t) = \sum_{k=1}^{K} S_k(t)$ is the aggregate market wealth at time t. Note that since investors have logarithmic preferences, each investor consumes the same constant fraction of their wealth at time t, that is

$$W_i(t) = \mathbb{E}_t\left[\int_t^\infty \frac{\xi(u)}{\xi(t)}\frac{c_i(u)}{c_i(t)}du\right]c_i(t) = \frac{c_i(t)}{\beta}.$$

Therefore, investor i's performance relative to the market is given by

$$g_i(T) - g(T) = \ln\left[\frac{\lambda_i(T)}{\lambda_i(0)}\right], \tag{3.1}$$

where $\lambda_i(t)$ is investor i's share of aggregate consumption, which is also *his share of aggregate wealth* at time t. Essentially, investor i outperforms (underperforms) the market when there is an increase (decrease) in his wealth share, that is $\lambda_i(T) > (<)\lambda_i(0)$. In the following,

we analyze both the *long-run performance* and *expected performance* of investors' optimal portfolios relative to the market based on the deviations of their subjective probability measures from the objective probability measure.

3.1. **Long-run Performance.** We first consider a criterion to compare long-run performance.

Definition 3.1. *Consider two feasible portfolio/consumption strategies, A and B. Strategy A outperforms strategy B in the long-run if and only if* $\lim_{T\to\infty}\ [g_B(T) - g_A(T)] < 0$ *almost surely under the objective probability measure.*

Definition 3.1 indicates that one portfolio-consumption strategy A outperforms another strategy B in the long run if and only if the cumulative continuous return of A is higher than that of B in the long run. In the market selection literature, it has been confirmed that an investor's long run survival depends on his *survival index*, defined by (see Blume and Easley (2006) and Yan (2008))

$$s_i(\boldsymbol{\theta}_i) = \boldsymbol{\theta}_i^T \boldsymbol{\rho}^{-1} \boldsymbol{\theta}_i \geq 0. \tag{3.2}$$

It is known that (i) only the *fittest* investors with the *minimum* survival index survive in the long run; (ii) investors who are not the fittest do not survive; and (iii) when there are multiple investors with the same survival index, there can be *multiple survivors*. The next proposition shows the relationship between long-run survival and long-run performance.

Proposition 3.2. *(Long-run performance)*

(i) *If investor i does not survive, then investor i under-performs the market in the long-run,*

$$\lim_{T\to\infty}\ [g_i(T) - g(T)] < 0.$$

(ii) *If investor i is the only survivor, then investor i outperforms the market in the long run,*

$$\lim_{T\to\infty}\ [g(T) - g_i(T)] < 0.$$

(iii) *If investor i survives, but there are multiple survivors, then investor i neither outperforms nor under-performs the market; that is, the limit $\lim_{T\to\infty}\ [g(T) - g_i(T)]$ does not exist.*

The economic intuition for Proposition 3.2 is simple. An investor m who holds the market portfolio consumes a constant fraction of the aggregate dividend, therefore his wealth share remains fixed for all time. If an investor i is the only fittest investor, then he will eventually drive out all the other investors who are not the fittest except for investor m. Therefore, there will a positive growth almost surely in investor i's wealth share in the long run. When there are multiple survivors, for example, both investor h and investor i survive in the long run, then although all the other investors' shares of aggregate consumption go to zero, the distribution of those consumption shares between investors h and i is not stationary in the long run, hence the limit does not exist.

3.2. **Expected Performance.** We now consider an alternative criterion to compare the *expected performance* of the optimal portfolio/consumption strategies given a finite investment horizon.

Definition 3.3. *Consider two feasible portfolio-consumption strategies, A and B. Strategy A is expected to outperforms strategy B from time 0 to time T if and only if $\mathbb{E}[g_B(T)] < \mathbb{E}[g_A(T)]$ under the objective probability measure.*

According to Definition 3.3, strategy A is expected to outperform strategy B when the expected average continuous return under A is higher than that under B. The above criterion can be written in term of the expected growth in wealth shares,

$$\mathbb{E}\left[\ln\left(\frac{\lambda_B(T)}{\lambda_B(0)}\right)\right] < \mathbb{E}\left[\ln\left(\frac{\lambda_A(T)}{\lambda_A(0)}\right)\right]. \tag{3.3}$$

Furthermore, if strategy B is to hold the market portfolio and consumes a constant fraction of the aggregate dividend, then the wealth share λ_B under strategy B is a constant, and equation

(3.3) becomes

$$\mathbb{E}\left[\ln\left(\frac{\lambda_A(T)}{\lambda_A(0)}\right)\right] > 0. \tag{3.4}$$

Hence, strategy A is expected to outperform the market within the period $[0, T]$ if and only if the expected growth in wealth share under the objective probability measure is strictly positive.

In the following, we compare the expected performances of the *speculators* with an *indexer*. Assume there are three investors in the economy (investors A, B and m). Investors A and B have subjective beliefs denoted by $M_A(t)$ and $M_B(t)$ respectively, and investor m holds the consensus belief $M(t)$. Investor m is essentially an *indexer* who holds the market portfolio and consumes a constant fraction of the aggregate dividend. Investors A and B are *speculators* because they speculate on the market based on their subjective beliefs about the expected dividend growth rates, denoted by $\boldsymbol{\theta}_A$ and $\boldsymbol{\theta}_B$ respectively. We assume investors have the same initial wealth, hence their initial shares of aggregate consumption are given by $\bar{\lambda} = \frac{1}{3}$. The consensus belief in this case is given by

$$M(t) = \frac{1}{2}(M_A(t) + M_B(t)), \tag{3.5}$$

and the share of aggregate dividend by investors A and B are given respectively by

$$\lambda_A(t) = \frac{2}{3}\left[1 + \frac{M_B(t)}{M_A(t)}\right]^{-1} \text{ and } \lambda_B(t) = \frac{2}{3}\left[1 + \frac{M_A(t)}{M_B(t)}\right]^{-1}. \tag{3.6}$$

When the speculators have the same survival indices, that is $s_A = s_B$, we can compare their expected performance with the indexer.

Proposition 3.4. *When the speculators have different subjective beliefs, $\boldsymbol{\theta}_A \neq \boldsymbol{\theta}_B$, but have the same survival indices, that is $s_A = s_B$, the indexer (investor m) is expected to outperform both speculators, that is $\mathbb{E}[g_i(T)] < \mathbb{E}[g(T)]$ for $i = A, B$.*

Proposition 3.4 is arguably the most surprising result of the paper, in the sense that although both speculators survive in the long-run, however they are expected to underperform the indexer. The intuition is as follows. Since the speculators have subjective beliefs which deviate from the

objective belief by the same *distance*, their expected wealth share at time T is the same as their initial wealth share, that is $\mathbb{E}[\lambda_i(T)] = \bar{\lambda}$ for $i = A, B$. However, since $\ln(\lambda_i(T))$ is strictly concave, a second order Taylor expansion

$$\mathbb{E}[\ln(\lambda_i(T))] \approx \mathbb{E}[\lambda_i(T)] - \frac{1}{2}Var\left[\frac{\lambda_i(T)}{\mathbb{E}[\lambda_i(T)]}\right] = \bar{\lambda} - \frac{1}{2}Var\left[\frac{\lambda_i(T)}{\bar{\lambda}}\right] \tag{3.7}$$

shows that the fluctuations in speculators' wealth share have a negative impact on their expected performance. The indexer investor m holds the market portfolio, thus his wealth share is always equal to $\bar{\lambda}$. Also note that in this case, because of

$$\mathbb{E}\bigg[\int_0^T e^{-\beta t}\ln(c_i(t))dt\bigg] < \mathbb{E}\bigg[\int_0^T e^{-\beta t}\ln(\bar{\lambda}D(t))dt\bigg],$$

the expected utility from time zero to T under the objective probability measure is higher for the indexer than for the speculators, that is $\hat{U}_i(0,T) < \hat{U}_m(0,T)$ for $i = A, B$. Furthermore, we can calculate the probability of outperformance.

Proposition 3.5. *(Probability of Outperformance) The probability that investor A outperforms the market from time zero to time T is given by*

$$\mathbb{P}\bigg(g(T) < g_A(T)\bigg) = \Phi\bigg(\frac{s_B - s_A}{2}\sqrt{\frac{T}{(\boldsymbol{\theta}^{(A)} - \boldsymbol{\theta}^{(B)})^T\boldsymbol{\rho}^{-1}(\boldsymbol{\theta}^{(A)} - \boldsymbol{\theta}^{(B)})}}\bigg),$$

where $\Phi(x)$ is the standard normal cumulative distribution function.

Proposition 3.5 implies that the probability that investor A will outperform the market depends on the difference in survival indices. When survival indices are equal ($s_A = s_B$), the probability is exactly 50%. In the case of $s_B > (<) s_A$, investor A has a more (less) than 50% chance of out-performing the market. Intuitively, the investor with a more accurate belief has a better chance of outperforming the market than the investor with a less accurate belief, and the probability increases over time.

The question which still remains is: if a speculator is the *only* fittest investor in the market, is he expected to outperform the market? One may be tempted to think yes, however we provide a counter example and the answer is in general *no*.

3.3. **Numerical Simulation.** We numerically compare the expected performances between the indexer and the speculators according to Definition 3.3. We assume that there is *one* risky asset S which is a claim to the aggregate dividend process, following a one-dimensional geometric Brownian motion

$$\frac{dD(t)}{D(t)} = \mu_D dt + \sigma_D dZ_D(t).$$

The dividend growth rate under the objective belief is $\mu_D = 1.789\%$ and the volatility is set to $\sigma_D = 3.218\%$ to match the moments of the U.S. consumption data 1891-1998 (see Campbell (2003) Table 2). We assume the speculators have constant disagreement about the value of μ_D, specified by $\theta_D^{(i)} = (\mu_D^{(i)} - \mu_D)/\sigma_D$ for $i = A, B$. Furthermore, investor A has a more accurate belief than investor B, that is $\theta^{(A)} = -\theta^{(B)}(1-\delta)$ where $0 \leq \delta \leq 1$, δ measures the belief accuracy of investor A relative to investor B. When $\delta = 1$, investor A is *rational* in the sense that his subjective belief coincides with the objective belief, when $\delta = 0$, investor A has the same survival index as investor B. We assume investor B's deviation from the objective belief is given by $\theta^{(B)} = 0.01/\sigma_D = 0.3048$. The subjective discount factor is given by $\beta = 0.02$ and each investor is endowed with an equal share of the stock. Note that the risk-free rate is given by $r(t) = \beta + \mu_D - \sigma_D^2 + \theta(t)\sigma_D$, the equilibrium stock price is proportional to the dividend, more precisely, $S(t) = D(t)/\beta$ and the market price of risk is given by $\kappa(t) = \sigma_D - \theta(t)$. Therefore, in this case, we can compute the optimal portfolio strategies in equilibrium explicitly. The optimal portfolio of the speculators are given by

$$\pi_A(t) = 1 + \frac{\theta_A - \theta(t)}{\sigma_D} \text{ and } \pi_B(t) = 1 + \frac{\theta_B - \theta(t)}{\sigma_D},$$

where $\theta_A - \theta(t) < 0$ and $\theta_B - \theta(t) > 0$. Therefore, in equilibrium, investor B always borrow at the risk-free rate to invest in the stock and investor A always lend at the risk-free rate, whether

he has a long or short position in the stock depends on the deviation of this subjective belief from the consensus belief. Moreover, the optimal portfolio for the indexer is obviously $\pi_m(t) = 1$, and the optimal portfolio under the objective probability measure is given by $\pi_o(t) = 1 - \frac{\theta(t)}{\sigma_D}$. The optimal mean-variance portfolio $\pi_o(t)$ under the objective probability measure depends on the consensus belief which is jointly determined by investors' subjective beliefs in equilibrium. Intuitively, it is optimal to invest more (less) into the stock when the consensus belief is pessimistic (optimistic) about future stock returns. Fama and French (2007) makes the same point in a static CAPM framework.

We use Monte-Carlo simulations to compute the average difference in *expected performance* $\frac{1}{T}\mathbb{E}[g_A(T) - g(T)]$ and also the difference in *expected utility* $\hat{U}_A(0,T) - \hat{U}_m(0,T)$ between investor A and the indexer for different *investment horizons* T and *difference in belief accuracies* δ. Note that if $\frac{1}{T}\mathbb{E}[g_A(T) - g(T)]$ is positive (negative), then investor A is expected to outperform (underperform) the market from time zero to time T. If $\hat{U}_A(0,T) - \hat{U}_m(0,T)$ is positive (negative), then investor A is expected to have a greater (lower) level of utility than the indexer. We use a time increment of $\Delta t = \frac{1}{12}$ and $50,000$ paths. Results are reported in Tab. 3.1. We can see that, when investors have the same belief accuracy ($\delta = 0$), consistent with Proposition 3.4, investor A is expected to underperform the market for all investment horizons and also has a lower expected utility compare to the indexer. In this case, given any finite investment horizons, each investor always has a 50% chance of outperforming the market.

When $\delta = 0.50$, investor A has a more accurate belief than investor B and will eventually drive out investor B in the long run. Also agent A has more than 50% chance of outperforming the market at any time. However, results show that investor A is still expected to under-perform the market for investment horizons up to 50 years. Also, his expected performance is only just higher than the market's when the investment horizon reaches 100 years[3]. Therefore, when the difference in beliefs accuracies (measured by δ) is not large enough, it may take a *very long* time before the fittest investor is expected to outperform the market. Furthermore, even though

[3]The actual value of $\mathbb{E}[g_A(T) - g(T)]$ when $T = 100$ is 8.70637×10^{-6}.

investor A is expected to outperform the market over a long time horizon, his expected utility is still less than the indexer's. This is because the instantaneous utility is being discounted by the subjective discount factor, therefore the expected out-performance more distant in the future may not be able to offset the expected under-performance at short time horizons.

When $\delta = 1$, investor A is rational since his subjective probability measure coincides with the objective probability measure. In this case, he is expected to consistently outperform the market and also obtain a higher expected utility compared to the indexer for any finite investment time horizons. Intuitively, the wealth share of investor A must on average grow fast enough to dominate the negative effect from the fluctuation in wealth share.

Expected performance relative to the market $\frac{1}{T}\mathbb{E}[g_A(T) - g(T)]$

	$T=1$	$T=5$	$T=10$	$T=20$	$T=50$	$T=100$
$\delta = 0$	-4.45%	-3.93%	-3.53%	-3.04%	-2.34%	-1.84%
$\delta = 0.50$	-0.82%	-0.67%	-0.54%	-0.38%	-0.14%	0.00%
$\delta = 1$	1.15%	1.10%	1.04%	0.95%	0.74%	0.53%

Expected utility relative to the indexer $\hat{U}_A(0,T) - \hat{U}_m(0,T)$

	$T=1$	$T=5$	$T=10$	$T=20$	$T=50$	$T=100$
$\delta = 0$	-0.021	-0.480	-1.666	-5.236	-18.528	-36.625
$\delta = 0.50$	-0.005	-0.885	-0.284	-0.783	-2.002	-2.534
$\delta = 1$	0.004	0.124	0.462	1.543	5.748	10.964

TABLE 3.1. The expected performance and expected utility of investor A relative to the indexer (the market) with different investment horizons T (years) and differences in belief accuracies δ.

4. DISCUSSION AND GENERALIZATIONS

This section discusses some generalizations of the model in Section 2 to incorporate market frictions such as transaction costs, more general type of utility functions and learning. The first generalization is to incorporate transaction costs. One can imagine that this would favor the indexer who holds the market portfolio and never re-balances his portfolio (assuming the supply of stocks remain the same). However transaction costs are disadvantageous for the speculators because their portfolios have a high turn-over rate. Therefore, incorporating transaction costs

into the current framework would strengthen our results and it becomes more difficult for the speculators to out-perform the market.

The second generalization is to consider a more general power utility function $u(c) = c^{1-\gamma}/(1-\gamma)$, where γ is the constant relative risk aversion coefficient. When assuming this type of utility function for all the investors in the economy, Jouini and Napp (2007) show that the consensus belief is in general not a martingale (unless $\gamma = 1$ which correspond to the case of logarithmic utility), hence not a probability belief. In this case, the consensus belief can be decomposed into a probability belief and a discount factor, which means that the representative investor in general has a different discount rate for future consumption from all the investors in the economy. As a result, an investor may choose to hold the consensus belief, but his optimal portfolio may not be the market portfolio. In particular, if investors have different relative risk aversion coefficients, the risk aversion of the representative investor is a wealth weighted average of the risk aversions of the investors, hence time-varying (see Weinbaum (2009, 2010)). Therefore, to setup a general preference structure such that it is optimal for at least some of the investors to hold the market portfolio in equilibrium is a challenging task, which we leave for future research.

Another generalization is learning. In this paper, we assume that the speculators have constant deviations from the objective belief. However, investors may update their beliefs when new information becomes available. Under learning, the long-run survival of an investor depends on his prior belief and rate of learning (the speed to which his belief converges to the objective belief). From our analysis, one may expect that the investor with a faster rate of learning is expected to perform better than the investor with a slower rate of learning and may be able to "beat" the market. However, the conclusion at the end of Section 3.3 is unlikely to be over-turned. Learning faster than all other investors in the economy is not enough to beat the market. In order to beat the market consistently, one needs to learn *a lot* faster. Incorporating learning with a more complex information structure and under a more general class of utility functions is again a very difficult task, which we leave to future research.

5. CONCLUSION

This paper considers a general equilibrium model in continuous time and studies the impact of disagreement on the performance of the optimal portfolio held by investors relative to the market portfolio. Investors have logarithmic utility, and optimally holds the market portfolio and consume a constant fraction of the aggregate dividend if and only if his belief coincides with the consensus belief. In terms of long-run performance, the market outperforms all investors who do not survive in the long run. If there is only one speculator who is the fittest and survives, then he outperforms the market in the long run. If there are multiple survivors with different beliefs, then the long-run limit does not exist, in the case of two investor, each has a 50% chance of outperforming the market. In terms of the expected performance, when both speculators have the same survival index (hence both survive), they are expected to underperform the market and obtain an expected utility less than that of the indexer. When one speculator has a more accurate belief or fitter than the other, whether the fittest investor is expected to outperform the market depends on the time horizon and the more importantly the difference in belief accuracies. When the difference is small, as demonstrated by a numerical example, the fittest investor is still expected to underperform the market over short time horizons and has less expected utility under the objective probability measure compare to the indexer. Only when the difference in belief accuracies is large enough is the fittest investor expected to consistently outperform the market for any time horizons.

APPENDIX A. PROOFS

A.1. **Proof of Lemma 2.1.** *(Optimal Portfolio/Consumption Strategy)*

(i) The first order condition of investor i's optimization problem is given by

$$e^{-\beta t}\frac{M_i(t)}{c_i(t)} = \eta_i \xi(t),$$

which leads to investor i's optimal rate of consumption in (2.15), where η_i is investor i's Lagrange multiplier. By substituting (2.15) into the budget constraint in (2.13), we obtain $1/\eta_i = \beta W_i(0) = \beta \bar{W}$.

(ii) Investor i's share of aggregate consumption can be obtained by

$$\lambda_i(t) = \frac{\eta_i^{-1} M_i(t)}{\sum_{h=1}^N \eta_h^{-1} M_h(t)} = \frac{M_i(t)}{\sum_{h=1}^N M_h(t)},$$

which leads to equation (2.16).

(iii) Since investor i's wealth at time t is the present value of all his future consumption, that is

$$W_i(t) = \mathbb{E}_t\left[\int_t^\infty \frac{\xi(u)}{\xi(t)} c_i(u) du\right] = \frac{c_i(t)}{\beta}.$$

Therefore, given that $c_i(t)$ is given by (2.15),

$$Y(t) \equiv e^{\beta t}\frac{W_i(t)\xi(t)}{M_i(t)}$$

is a *constant*, hence $dY(t) = 0$. We first have

$$\frac{d(W_i(t)\xi(t))}{W_i(t)\xi(t)} = \beta dt + (\boldsymbol{\pi}_i(t)^T \boldsymbol{\sigma}(t) - \boldsymbol{\kappa}(t)^T) d\mathbf{Z}(t),$$

from which we then obtain

$$\frac{dY(t)}{Y(t)} = \left[\boldsymbol{\theta}_i^T \boldsymbol{\rho}^{-1} \boldsymbol{\theta}_i - (\boldsymbol{\pi}_i(t)^T \boldsymbol{\sigma}(t) - \boldsymbol{\kappa}(t)^T)\boldsymbol{\theta}_i\right] dt + (\boldsymbol{\pi}_i(t)^T \boldsymbol{\sigma}(t) - (\boldsymbol{\kappa}(t) + \hat{\boldsymbol{\theta}}_i)^T) d\mathbf{Z}(t).$$

Substituting $\boldsymbol{\pi}_i(t) = (\boldsymbol{\sigma}(t)\boldsymbol{\rho}\boldsymbol{\sigma}(t))^{-1}(\boldsymbol{\alpha}(t) + \boldsymbol{\sigma}(t)\boldsymbol{\theta}_i)$ into the above equation leads to $dY(t) = 0$, which completes the proof. □

A.2. **Proof of Proposition 2.3.** *(Market Equilibrium)*

From the first order condition of the consensus consumer in equation (2.20), we obtain

$$\frac{d\xi(t)}{\xi(t)} = \Big[-\beta - \bar{\mu}(t) + \bar{\boldsymbol{\sigma}}(t)^T \boldsymbol{\rho} \bar{\boldsymbol{\sigma}}(t) - \boldsymbol{\theta}(t)^T \bar{\boldsymbol{\sigma}}(t) \Big] dt + (\hat{\boldsymbol{\theta}}(t) - \bar{\boldsymbol{\sigma}}(t))^T d\mathbf{Z}(t).$$

Note that $d\xi(t)$ must also satisfy equation (2.12). Equating the diffusion and drift coefficients leads to the vector of market prices of risk in (2.22) and the equilibrium risk-free rate in (2.23), respectively. The equilibrium price of asset k is computed by equation (2.21). The consensus belief can be written as

$$\frac{M(s)}{M(t)} = \sum_{i=1}^{N} \lambda_i(t) \frac{M_i(s)}{M_i(t)},$$

which leads to the explicit form of equilibrium price in equation (2.24). □

A.3. **Proof of Proposition 3.2.** *(Long-run Performance)*

Since the survival index of investor i is given by $s_i = \boldsymbol{\theta}_i^T \boldsymbol{\rho}^{-1} \boldsymbol{\theta}_i$, we obtain

$$\exp\{g_i(T) - g(T)\} = \left(\sum_{h=1}^{N} \omega_h \exp\left\{ -\frac{1}{2}(s_h - s_i)T + (\hat{\boldsymbol{\theta}}_h - \hat{\boldsymbol{\theta}}_i)^T \mathbf{Z}(T) \right\} \right)^{-1} \tag{A.1}$$

for $i = 1, 2, \cdots, N$. Using the strong Law of Large Numbers for Brownian motion (see Karatzas and Shreve (1991) Sec. 2.9.A), for any value of σ,

$$\lim_{T\to\infty} \exp\{aT + \sigma Z(T)\} = \begin{cases} 0, & a < 0 \\ \infty, & a > 0, \end{cases}$$

where $Z(T)$ is an one-dimensional Wiener process and convergence takes place almost surely. In equation (A.1), $(\hat{\boldsymbol{\theta}}_h - \hat{\boldsymbol{\theta}}_i)^T \mathbf{Z}(T)$ can be re-written as $\sqrt{(\boldsymbol{\theta}^{(A)} - \boldsymbol{\theta}^{(B)})^T \boldsymbol{\rho}^{-1} (\boldsymbol{\theta}^{(A)} - \boldsymbol{\theta}^{(B)})} Z(T)$, where

$$Z(T) \equiv \frac{(\hat{\boldsymbol{\theta}}_h - \hat{\boldsymbol{\theta}}_i)^T \mathbf{Z}(T)}{\sqrt{(\boldsymbol{\theta}^{(A)} - \boldsymbol{\theta}^{(B)})^T \boldsymbol{\rho}^{-1} (\boldsymbol{\theta}^{(A)} - \boldsymbol{\theta}^{(B)})}}$$

is a one-dimensional Wiener process. Therefore, if investor i's survival index is not the minimum, $s_i \neq \min(s_1, s_2, \cdots, s_N)$, then $\exp\{g_i(T) - g(T)\} \overset{a.s}{\to} 0$ and $g_i(T) - g(T) \overset{a.s}{\to} -\infty$ as $T \to \infty$. Moreover, for the ***fittest*** investor i with the minimum survival index, that is $s_i \leq s_h$

for all $h \neq i$, we have

$$\exp\{g(T) - g_i(T)\} = \sum_{h=1}^{N} \omega_h \exp\left\{ -\frac{1}{2}(s_h - s_i)T + (\hat{\boldsymbol{\theta}}_h - \hat{\boldsymbol{\theta}}_i)^T \mathbf{Z}(T) \right\}$$
$$= \omega_i + \sum_{h=1, h\neq i}^{N} \lambda_h(t) \exp\left\{ -\frac{1}{2}(s_h - s_i)T + (\hat{\boldsymbol{\theta}}_h - \hat{\boldsymbol{\theta}}_i)^T \mathbf{Z}(T) \right\}. \tag{A.2}$$

This leads to the limit

$$\lim_{T\to\infty} \exp\{g(T) - g_i(T)\} = \begin{cases} \omega_i < 1, & s_i < s_h \text{ for all } h \neq i. \\ \text{does not exist,} & \text{otherwise.} \end{cases}$$

□

A.4. **Proof of Proposition 3.4.** *(Expected Performance)*

Since $s_A = s_B$, from equation (3.6), we have investors' wealth shares explicitly,

$$\lambda_A(T) = \frac{2}{3}\left[1 + \exp\left\{(\hat{\boldsymbol{\theta}}_B - \hat{\boldsymbol{\theta}}_A)^T \mathbf{Z}(T)\right\}\right]^{-1},$$
$$\lambda_B(T) = \frac{2}{3}\left[1 + \exp\left\{(\hat{\boldsymbol{\theta}}_A - \hat{\boldsymbol{\theta}}_B)^T \mathbf{Z}(T)\right\}\right]^{-1}.$$

Therefore, the wealth shares of the speculators are equal in distribution, that is $\lambda_A(T) \overset{d}{=} \lambda_B(T)$ and $\frac{1}{2}[\lambda_A(T) + \lambda_B(T)] = \bar{\lambda}$. Therefore, by symmetry we obtain $\mathbb{E}[\lambda_A(T)] = \mathbb{E}[\lambda_B(T)] = \bar{\lambda}$ (a similar proof can be found in Jouini and Napp (2011)). Applying Jensen's inequality, we obtain the desired result from $\boldsymbol{\theta}_A \neq \boldsymbol{\theta}_B$. □

A.5. **Proof of Proposition 3.5.** *(Probability of Outperformance)*

Given that investor A's wealth share in equation (3.6), we have $\lambda_A(T) > \lambda_A(0)$ if and only if $M_B(T)/M_A(T) < 0$, which is equivalent to $(\hat{\boldsymbol{\theta}}_B - \hat{\boldsymbol{\theta}}_A)^T \mathbf{Z}(T) < \frac{1}{2}(s_B - s_A)T$. The LHS of the last inequality follows a normal distribution with mean zero and variance of $(\boldsymbol{\theta}^{(A)} - \boldsymbol{\theta}^{(B)})^T \boldsymbol{\rho}^{-1} (\boldsymbol{\theta}^{(A)} - \boldsymbol{\theta}^{(B)})T$. Therefor the probability $\mathbb{P}(g_A(T) - g(T) > 0)$ can be computed in closed form. □

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