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The Small and Large Time Implied Volatilities in the Minimal Market Model

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The Small and Large Time Implied Volatilities in the Minimal Market Model

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Abstract

This paper derives explicit formulas for both the small and large time limits of the implied volatility in the minimal market model. It is shown that interest rates do impact on the implied volatility in the long run even though they are negligible in the short time limit.

Keywords:
Small and large time implied volatility, benchmark approach, square-root process, the minimal market model.
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1. Introduction

Proposed by Platen [22, 23] to model well diversified stock indices, the minimal market model (**MMM**) is a flexible one-factor model for capturing real-world price dynamics. As a local volatility model underpinned by the square-root process, the MMM is not only complete with respect to hedging but also mathematically tractable, with closed form formulas available for forward rates, zero coupon bonds, digital and European options [24]. Further, the MMM has its own volatility feedback mechanism, so unlike stochastic volatility models it does not need an extra volatility process to generate negative correlation between the local volatility and the index, the so-called leverage effect. In addition, the MMM can be extended to model volatility swaps [6], commodities and exchange rates in multicurrency markets; and random scaling and jumps can also be embedded in the model to reflect realistic randomness of the market activity [24, Chapters 13 and 14].

More importantly, what sets the MMM apart from the other local/stochastic volatility models is its adoption of the benchmark approach [24], instead of the usual risk-neutral

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method for derivatives pricing. The benchmark approach does not assume or rely on the existence of risk-neutral equivalent martingale measures to exclude arbitrage. Rather, it achieves the elimination of, so called, strong arbitrage by utilizing the growth optimal portfolio of the market as a benchmark for securities and portfolios. Indeed, despite the nonexistence of equivalent risk-neutral measures in the MMM, the benchmark approach accommodates direct arbitrage-free pricing under the original probability measure associated with the underlying asset [24, Chapters 10, 13].

In this article we derive both the small and the large time limits of the implied volatility in the MMM. The derivation of the small time limit takes advantage of an extended Roper–Rutkowski formula [26] for small time implied volatilities. As explained in Section 4, applying a forward price transform can easily extend the model-free Roper–Rutkowski formula to regimes with nonzero interest rates and dividend yields. In contrast, the derivation of the large time limit is based on direct comparisons with the lower and upper bounds of the implied volatility, where the bounds are established by appealing to the asymptotics of the noncentral chi-square distributions.

Following the breakthrough by Berestycki et al. [2, 3], small time implied volatility asymptotics have been investigated in [1, 18, 17, 12, 26, 14, 16, 15], to name a few studies in the still expanding literature. Whilst covering a diverse range of models, these studies typically assumed zero interest rates, martingale asset prices, or risk neutral regimes. The exception appears to be the paper of Gao and Lee [15], of which we learnt after the completion of our work. In [15], nonzero interest rates were explicitly allowed and absorbed into forward prices — a well-known tool that we also use in (21) below — and implied volatilities were expanded in terms of option prices in a model-free manner, like that in [26]. Yet, it does not appear that their zeroth order expansion [15, Remark 7.4] implies the Roper–Rutkowski formula or our small time limit. Separately, the article of Gatheral et al. [16] had also come to our attention. Our small time limit agrees with theirs [16, (3.21)], although we arrived at our result by using a different pricing approach and different techniques.

Comparing to the studies of the small time asymptotics, research in large time implied volatilities has been a more recent event. Rogers and Tehranchi [25] and Tehranchi [28] examined martingale models. Forde and his coworkers [13, 14, 9, 10, 11, 8] looked at various stochastic volatility models under the assumption of large-time-large-strike, large-time-large-moneyness, and zero interest rate with fixed strike. Besides the aforementioned small time expansion, Gao and Lee [15] in the same work obtained formulas for large time and extreme strike expansions of the implied volatility in arbitrary order. However, our large time limit complements as much as it is independent of these works. In particular, our explicit formulas for the benchmark approach based limits have demonstrated that interest rates do impact on implied volatilities in the long run, even though they are negligible in the short time limit, see Theorem 1 and 2. So far, this characterization of the influence of interest rates on implied volatility has not appeared elsewhere.

The organization of this article is as follows. In Section 2 we set up the model and state the main theorems. In Section 3 we present a calibrated implied volatility surface and the corresponding small and large time limits. The extension of the Roper–Rutkowski formula is given in Section 4, and the proofs of the main theorems are in Sections 5 and 6. Lastly, in Section 7 we conclude the article with a brief outline of the future research.

2. Model and main results

In the stylized MMM [24, Chapter 13] there exist a savings account and a diversified accumulation index approximating the growth optimal portfolio of the market. The value of the savings account A_t grows according to the function

$$A_t = \mathrm{e}^{rt}, \quad r, t \in [0, \infty),$$

where respectively r and t are the risk-free interest rate and time. The index price S_t is a square-root process satisfying the equation

$$\mathrm{d}S_t = [(r + \sigma^2(S_t, t)]S_t \mathrm{d}t + S_t \sigma(S_t, t)\mathrm{d}W_t,$$

where W_t is a standard Wiener process on a complete filtered probability space $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, \mathbb{P})$. Dividends for the accumulation index are assumed to be continuously reinvested in the index. The deterministic function $\sigma : (0, \infty) \times [0, \infty) \to (0, \infty)$ is called the local volatility; it is defined by

$$\sigma(S, t) = \sqrt{\alpha \mathrm{e}^{(r+\eta)t}/S}, \quad (S, t) \in (0, \infty) \times [0, \infty).$$

The strictly positive constants α and η are, respectively, the initial value and the net growth rate of the growth optimal portfolio of the market. The local volatility σ provides volatility feedback to the index price and produces the often observed leverage effects: relatively high (low) asset price leads to relatively low (high) volatility. Figure 1 below displays the leverage effect in the SP500 index.

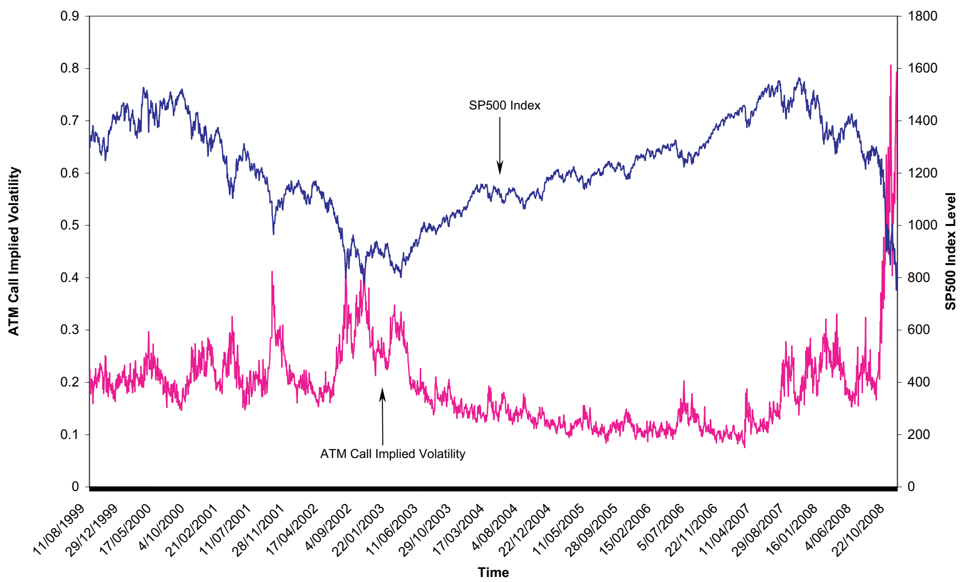

Figure 1: Leverage effect: negative correlation between the SP500 Index and its at the money call implied volatility, 11/08/1999–22/10/2008. Data source: Datastream.

2.1. European option prices under the MMM

Without loss of generality we will consider European option prices at time $t = 0$. In [24, (10.4.1), (13.3.16)], it is shown that under the MMM, the European call option price C, denominated in units of the domestic currency, is given by

$$C(K,T) = S\mathbb{E}\left[\left.\frac{(S_T - K)_+}{S_T}\right| S_0 = S\right], \qquad 0 < S, K < \infty, \quad 0 \le T < \infty, \tag{1}$$

where $X_+ = \max(X, 0)$, S is the current index price, K the strike, and T the time to expiry. Note that no equivalent risk neutral measure exists in the MMM and $\mathbb{E}$ is taken directly under the measure $\mathbb{P}$, see [24, Chapters 10, 13]. More explicitly, the MMM call price can be written as

$$C(K,T) = S\tilde{\chi}^2(y;4,x) - K\mathrm{e}^{-rT}\tilde{\chi}^2(y;0,x), \tag{2}$$

where

$$\begin{cases} x = \dfrac{S}{\varphi(T)}, \quad y = \dfrac{K\mathrm{e}^{-rT}}{\varphi(T)}, \quad \varphi(T) = \dfrac{\alpha}{4\eta}\left(\mathrm{e}^{\eta T} - 1\right), \quad \alpha, \eta > 0, \\ \tilde{\chi}^2(y;\delta,x) = 1 - \chi^2(y;\delta,x), \qquad \delta \ge 0, \\ \chi^2(y;\delta,x) = \displaystyle\int_0^y p(z;\delta,x)\,\mathrm{d}z, \quad \delta > 0, \quad \chi^2(y;0,x) = \mathrm{e}^{-x/2} + \int_0^y p(z;0,x)\,\mathrm{d}z, \\ p(y;\delta,x) = \dfrac{1}{2}\left(\dfrac{y}{x}\right)^{(\delta-2)/4} \exp\left(-\dfrac{x+y}{2}\right) I_{(\delta-2)/2}\left(\sqrt{xy}\right), \end{cases} \tag{3}$$

and $I_\nu(\cdot)$ is the modified Bessel function of the first kind with index ν; see [24, (13.3.17)–(13.3.19)]. For nonnegative y, δ, and x, the function $\chi^2(y;\delta,x)$ denotes the cumulative distribution function, evaluated at y, of a noncentral chi-square random variable with δ degrees of freedom and noncentrality parameter x. See e.g. [20, Chapter 29] for details of the distribution with $\delta > 0$; see [27] for the distribution with zero degrees of freedom.

In [24, (13.3.5), (13.3.20), (13.3.21)], it is also shown that the European put price P and zero coupon bond price Z are respectively given by

$$\begin{aligned} P(K,T) &= K\mathrm{e}^{-rT}\left(\chi^2(y;0,x) - \mathrm{e}^{-x/2}\right) - S\chi^2(y;4,x), &(4)\\ Z(T) &\equiv Z(0,T) = \mathrm{e}^{-rT}\left(1 - \mathrm{e}^{-x/2}\right), &(5) \end{aligned}$$

and the following put-call parity relation holds:

$$C(K,T) + KZ(T) = P(K,T) + S. \tag{6}$$

2.2. The Black–Scholes price and the implied volatility in the MMM

Assuming a constant dividend yield $\kappa \in \mathbb{R}$ and a nonincreasing risk-free zero coupon bond price function $T \mapsto \mathsf{Z}(T) \equiv \mathsf{Z}(0,T)$, a general Black–Scholes call price at time $t = 0$ can be represented by the formula

$$\mathsf{C}_{\mathrm{BS}}(K,T;v) = S\mathrm{e}^{-\kappa T}N(d_1) - K\mathsf{Z}(T)N(d_2), \tag{7}$$

where v is the volatility parameter,

$$\begin{cases} N(d) = \displaystyle\int_{-\infty}^{d} n(\vartheta)\,\mathrm{d}\vartheta, \quad n(\vartheta) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\vartheta^2/2}, \\ d_1(K,T;v) = \dfrac{\ln(S/K) - \ln\mathsf{Z}(T) - \kappa T + v^2T/2}{v\sqrt{T}}, \\ d_2(K,T;v) = d_1(K,T;v) - v\sqrt{T}. \end{cases} \tag{8}$$

See e.g. [7, (17.9)]. In Section 4 we will use this call price to extend the Roper–Rutkowski formula [26]. In the MMM, this general formula can be simplified. Firstly, the MMM is concerned with an accumulation index, so $\kappa = 0$ in the model. Secondly, at time $t = 0$ the MMM zero coupon bond price Z and the yield-to-maturity of the bond $\hat{r}$ are related by the identities

$$\hat{r}(T) \equiv \hat{r}(0,T) = -\frac{1}{T}\ln\big(Z(T)\big) = r - \frac{1}{T}\ln\left(1 - \mathrm{e}^{-x/2}\right), \tag{9}$$

where x is defined in (3) and the last equality results from (5); see [24, (12.2.57) and (13.3.4)]. (Brigo and Mercurio [4, p. 6] call $\hat{r}$ the continuously compounded spot interest rate. Here we follow the terminology of Musiela and Rutkowski [21, p. 266] and call it the yield-to-maturity.) Therefore, in the MMM the Black–Scholes price (7) can be simplified to

$$C_{\mathrm{BS}}(K,T;v) = SN(d_1) - K\mathrm{e}^{-\hat{r}T}N(d_2),$$

where $N(\cdot)$ is the same as before and

$$\begin{cases} d_1(K,T;v) = \dfrac{\ln(S/K) + (\hat{r} + v^2/2)T}{v\sqrt{T}}, \\ d_2(K,T;v) = \dfrac{\ln(S/K) + (\hat{r} - v^2/2)T}{v\sqrt{T}}. \end{cases}$$

We are now ready to define implied volatility.

Definition 1 (Implied volatility in the MMM). Under the MMM, the implied volatility is defined as the unique nonnegative function $(K,T) \mapsto \phi(K,T)$ satisfying the equation

$$C(K,T) = C_{\mathrm{BS}}\big(K,T;\phi(K,T)\big) \tag{10}$$

for all $K, T \in (0,\infty)$.

For $0 < T < \infty$, the existence and uniqueness of the implied volatility ϕ is guaranteed by the implicit function theorem. To see this, let $J = C(K,T) - C_{\mathrm{BS}}(K,T;v)$. Then the Jacobian determinant $|J_v| = \partial_v C_{\mathrm{BS}}(K,T;v) = Sn\big(d_1(K,T;v)\big)\sqrt{T}$ is strictly positive for all $0 < T < \infty$. However, as $T \to 0$ or $T \to \infty$, the Jacobian determinant becomes zero. So it is not apparent that the implied volatility possesses a limit in small time, by which we mean $\lim_{T\to 0}\phi$, or a limit in large time, by which we mean $\lim_{T\to\infty}\phi$.

Remark 1. For any finite $T > 0$, the existence and uniqueness of the implied volatility can also be deduced by using the general arbitrage bounds for call price and the monotonicity of $\mathsf{C}_{\mathrm{BS}}(K,T;v)$ in v; see Section 4 below. We omit arbitrage bounds in the definition of the implied volatility because they are automatically satisfied by the MMM call price C; see Step (i) of the proof in Section 5.

2.3. Main results

For small time asymtotics we have the following theorem:

Theorem 1. *Under the MMM, the implied volatility has the small time limit*

$$\lim_{T\to 0}\phi(K,T) = \frac{\sqrt{\alpha}\ln(S/K)}{2(\sqrt{S}-\sqrt{K})}, \qquad K \in (0,\infty). \tag{11}$$

This theorem makes clear that the risk-free rate does not affect the implied volatility in the small time limit. It confirms the intuition that the time value of money diminishes in infinitesimal time spans and thus has negligible bearing on the option price. The theorem is proved in Section 5.

For large time asymtotics we have the following theorem:

Theorem 2. *Under the MMM, the implied volatility has the large time limit*

$$\lim_{T\to\infty}\phi(K,T) = \sqrt{2(3-2\sqrt{2})(r+\eta)}, \qquad K \in (0,\infty). \tag{12}$$

As a result of this large time limit, the MMM implied volatility in the long run is determined by the risk-free rate r and the net growth rate η of the growth optimal portfolio of the market. This is not surprising given that the (long term) increases in the index and option prices are dictated by these two rates. This theorem is proved in Section 6.

2.4. Notation

If $F(\cdot)$ is a probability distribution, then $\tilde{F}(\cdot) = 1 - F(\cdot)$ is the complementary distribution function of F. Except in the introduction, subscript letters generally denote partial derivatives, e.g. $x_T = \partial x/\partial T$. Limits and asymptotics have the following meanings:

$$f(T) \xrightarrow{T\to l} h \quad \Longleftrightarrow \quad \lim_{T\to l} f(T) = h.$$

$$f(T) \sim g(T) \quad (T \to l) \quad \Longleftrightarrow \quad \lim_{T\to l}[f(T)/g(T)] = 1, \quad l \in [-\infty,\infty].$$

$$f(T) = \mathrm{O}(g(T)) \quad (T\to\infty) \qquad \text{if} \qquad |f(T)/g(T)| \text{ is bounded in the limit.}$$

Given strike K and expiry T, the MMM call price is $C(K,T)$, and the corresponding Black–Scholes price with implied volatility ϕ is $C_{\mathrm{BS}}(K,T,\phi(K,T))$. $C_{\mathrm{BS}}(K,T;v)$ stands for a generic Black–Scholes price with volatility $v \in [0,\infty]$.

3. Implied volatility calibration

To estimate the model parameters we calibrated the MMM on the SP500 total return index (SPX) using data obtained from Datastream for the period 04/01/1988–27/01/2009. A similar calibration procedure has been performed in [19].

On 27/01/2009, the SP500 index value had a value of 1362.18 and used as a proxy for the (annualized) risk-free rate r, the effective 3-month U.S. T-bill rate on the same day was 0.0011154 per annum. The calibration returned the estimates $\alpha = 43.307$ and $\eta = 0.089896$. These calibrated parameters were then fed into the MMM formula to produce call prices on the SPX. Figure 2 shows an implied volatility surface generated from the MMM call prices on 27/01/2009. Also plotted in the graph are the theoretical small and large time implied volatility limits.

From Figures 2 and 3 it can be seen that as the maturity shortens, the implied volatility decreases to the theoretical limit and the skew becomes more pronounced. In comparison, Figures 2–4 illustrate that as the maturity lengthens, the implied volatility converges to the theoretical large time limit, and the skew flattens at a decreasing speed. Our observation of the large time asymptotics is consistent with the findings of Rogers and Tehranchi [25] and Forde and Jacquier [13], even though these authors work in risk-neutral regimes.

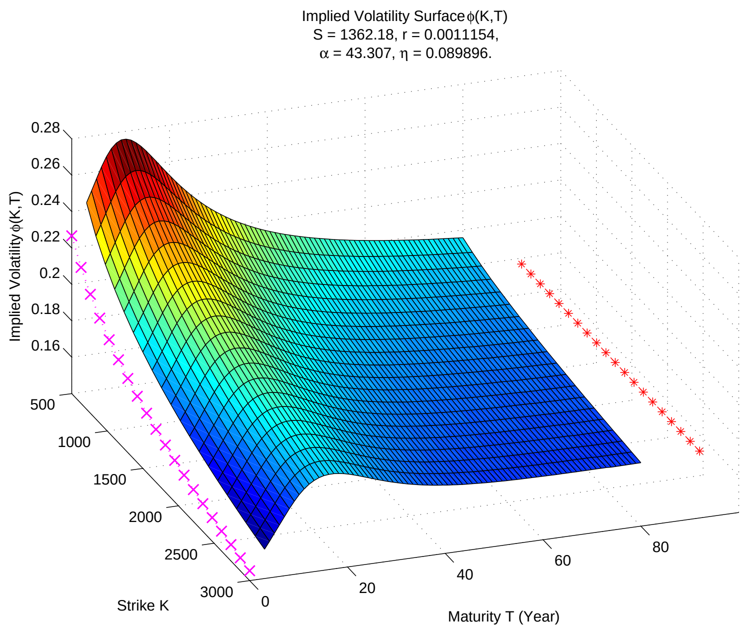

Figure 2: SP500 index implied volatility under the MMM and the theoretical small and large time limits on 27/01/2009.

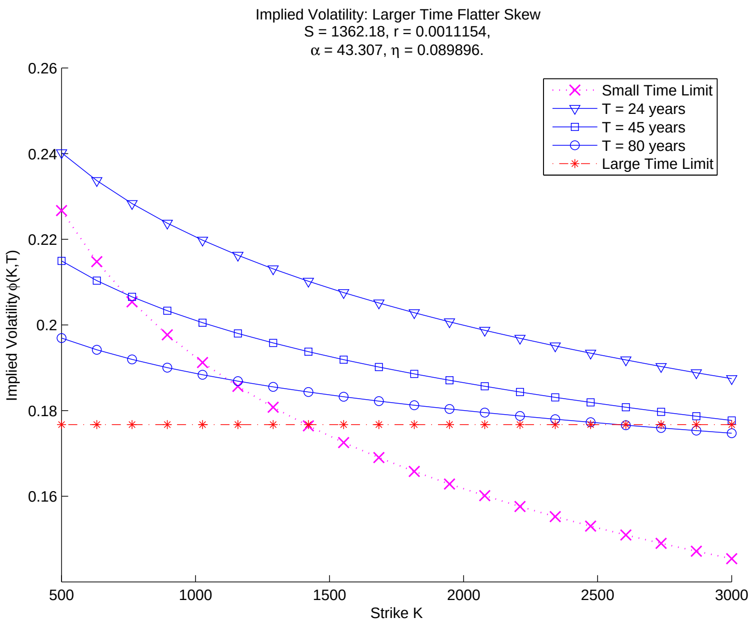

Figure 3: The small and large time behavior of the SP500 index implied volatility skew under the MMM on 27/01/2009.

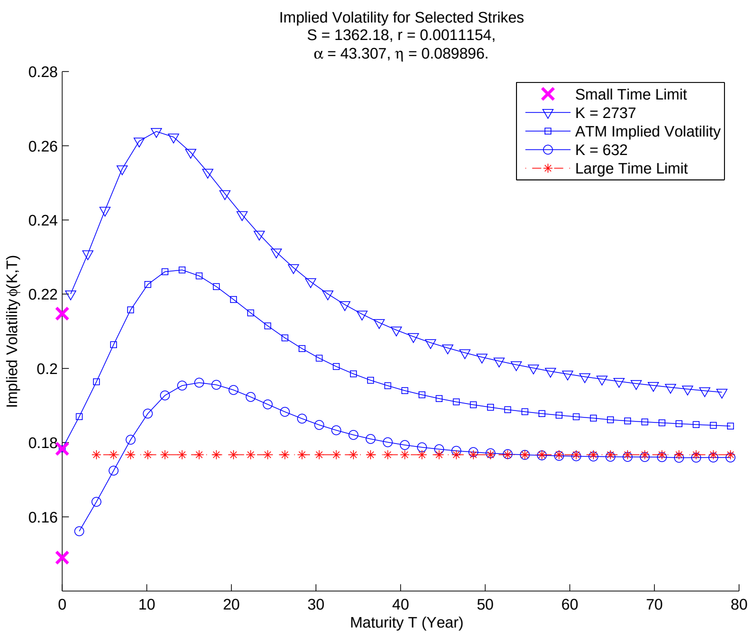

Figure 4: The small and large time limits of the SP500 index implied volatility under the MMM on 27/01/2009 for different strikes.

4. An extended Roper–Rutkowski formula

Under the assumption of zero risk-free interest rate and some minimal conditions on the call option prices, Roper and Rutkowski [26, Theorem 5.1] derived a model-free zeroth order asymptotic formula for the implied volatility in small time.

In this section we extend their formula to markets with nonzero dividend yields and interest rates. Since bond prices can be parametrized by risk-free interest rates, we will, instead of specifying a risk-free rate, introduce a risk-free zero coupon bond into the Roper–Rutkowski setup. We will derive the extended formula by applying a well-known forward price transform. After the variable change, it will become clear that the Roper–Rutkowski proof can be repeated here almost line by line. For this reason we will only sketch our proof of the result.

4.1. The general market model and the extended Roper–Rutkowski formula

For ease of referencing we shall call our setup a general market model (**GMM**). Consider a market that has a continuum of zero coupon bond prices and call option prices for an asset. Without loss of generality we study the market at time $t = 0$. Let the constant dividend yield be $\kappa \in \mathbb{R}$ and current asset price $S > 0$. For the bond price function $T \mapsto \mathsf{Z}(T)$ we have the following assumptions.

Assumption 1. The bond price $\mathsf{Z} : [0, \infty) \to (0, 1]$ satisfies the following conditions.

($\mathsf{Z}1$) No arbitrage bounds:

$$0 < \mathsf{Z}(T) \leq 1, \quad \forall\, T \in [0, \infty). \tag{13}$$

($\mathsf{Z}2$) Convergence to payoff:

$$\lim_{T \to 0} \mathsf{Z}(T) = \mathsf{Z}(0) = 1. \tag{14}$$

($\mathsf{Z}3$) Time value of money:

$$T \mapsto \mathsf{Z}(T) \quad \text{is nonincreasing.} \tag{15}$$

For the call prices $(K, T) \mapsto \mathsf{C}(K, T)$ the following conditions are also assumed.

Assumption 2. The call price $\mathsf{C} : (0, \infty) \times [0, \infty) \to [0, \infty)$ fulfils the following conditions:

($\mathsf{C}1$) No arbitrage bounds:

$$(S\mathrm{e}^{-\kappa T} - K\mathsf{Z}(T))_+ \leq \mathsf{C}(K, T) \leq S\mathrm{e}^{-\kappa T}, \quad \forall\, S, K > 0,\ T \geq 0. \tag{16}$$

($\mathsf{C}2$) Convergence to payoff:

$$\lim_{T \to 0} \mathsf{C}(K, T) = \mathsf{C}(K, 0) = (S - K)_+. \tag{17}$$

($\mathsf{C}3$) Time value of the option:

$$T \mapsto \mathsf{C}(K, T) \quad \text{is nondecreasing.} \tag{18}$$

If we set $\kappa = 0$ and $\mathsf{Z}(T) \equiv 1$ in the setup above, then we recover the zero dividend yield and zero interest rate setup of Roper and Rutkowski [26, Section 2]. With some abuse of notation we can now define implied volatility for the GMM.

Definition 2 (Implied volatility in the GMM). Under the GMM, the implied volatility is defined as the unique nonnegative function $(K, T) \mapsto \phi(K, T)$ satisfying the equation

$$\mathsf{C}(K, T) = \mathsf{C}_{\mathrm{BS}}\big(K, T; \phi(K, T)\big) \qquad \forall\, K, T \in (0, \infty),$$

where C_{BS} is defined in (7).

As mentioned earlier in Remark 1, the existence and uniqueness of the implied volatility is guaranteed by the arbitrage bounds for the call price in (16) and the monotonicity of $\mathsf{C}_{\mathrm{BS}}(K, T; v)$ in v. Here is our result.

Theorem 3 (Extended Roper–Rutkowski formula)**.** *Under the GMM, if there exists a constant $T_1 > 0$ such that $\mathsf{C}(K, T) > (S\mathrm{e}^{-\kappa T} - K\mathsf{Z}(T))_+$ for every fixed $K > 0$ and $T \in (0, T_1)$, then as $T \to 0$,*

$$\lim_{T\to 0} \phi(K, T) = \begin{cases} \displaystyle\lim_{T\to 0} \sqrt{2\pi}\frac{\mathsf{C}(K, T)}{K\sqrt{T}}, & K = S, \\ \displaystyle\lim_{T\to 0} \frac{|\ln(S/K)|}{\{-2T\ln[\mathsf{C}(K, T) - (S\mathrm{e}^{-\kappa T} - K\mathsf{Z}(T))_+]\}^{1/2}}, & K \neq S, \end{cases} \tag{19}$$

where the equality of the limits is understood in the sense that the left-hand side limit exists (is infinite) if the right-hand side limit exists (is infinite).

Remark 2. Similar to the case discussed in [26, Section 5.2], if there exists a T_0 such that $\mathsf{C}(K, T) = (S\mathrm{e}^{-\kappa T} - K\mathsf{Z}(T))_+$ for every fixed $K > 0$ and $T \in (0, T_0)$, then obviously $\phi(K, T) = 0$ for every $T \in (0, T_0)$.

Remark 3. If $\kappa = 0$ and $\mathsf{Z}(T) \equiv 1$ for all $T \in [0, \infty)$, then (19) is reduced to the Roper–Rutkowski formula [26, Corollary 5.1].

4.2. Proof of the extended Roper–Rutkowski formula

To prove their formula Roper and Rutkowski rely on a representation formula for the Black–Scholes call price, which states that

$$\mathsf{B}(K, T; v) = (S - K)_+ + S\int_0^{v\sqrt{T}} N'\left(\frac{\ln(S/K)}{\tau} + \frac{\tau}{2}\right) \mathrm{d}\tau, \tag{20}$$

where $N'(\cdot) \equiv n(\cdot)$ is the standard normal density in (8); see [26, Lemma 3.1]. A variant of this formula had earlier appeared in Carr and Jarrow [5].

However, (20) does not hold when the risk-free interest rate is not zero, or equivalently when there is a nontrivial zero coupon bond. Indeed, if in the GMM, $\kappa = 0$ and $\mathsf{Z}(T) = \mathrm{e}^{-rT}$ for $T \geq 0$ and some $r > 0$, then $\mathsf{C}_{\mathrm{BS}}(K, T; v) \neq \mathsf{B}(K, T; v)$ for $K, T > 0$. Yet, using the forward price we can derive a representation formula similar to (20) for the Black–Scholes price C_{BS} in the GMM. For $S, K > 0$ and $T \geq 0$, the forward price ξ in the MMM is

$$\xi = \ln(S/K) - \ln\big(\mathsf{Z}(T)\big) - \kappa T. \tag{21}$$

In the (ξ, T) coordinates, let $\mathcal{C}$ be the transformed call price

$$\mathcal{C}(\xi, T) = \frac{\mathsf{C}\big(K(\xi, T), T\big)}{K(\xi, T)\mathsf{Z}(T)}. \tag{22}$$

Then in (ξ, T) the conditions (16)–(18) can be written as

$$\begin{cases} (\mathrm{e}^{\xi} - 1)_+ \leq \mathcal{C}(\xi, T) \leq \mathrm{e}^{\xi}, \quad \xi \in \mathbb{R}, \quad T \in [0, \infty), \\ \lim\limits_{T \to 0} \mathcal{C}(\xi, T) = (\mathrm{e}^{\xi} - 1)_+, \quad \xi \in \mathbb{R}, \\ T \mapsto \mathcal{C}(\xi, T) \quad \text{is nondecreasing.} \end{cases} \tag{23}$$

Note that the corresponding Black–Scholes price in (ξ, T) is

$$\mathcal{C}_{\mathrm{BS}}(\xi, T; v) = \frac{\mathsf{C}_{\mathrm{BS}}\big(K(\xi, T), T; v\big)}{K(\xi, T)\mathsf{Z}(T)} = \mathrm{e}^{\xi} N\big(d_1(\xi, T; v)\big) - N\big(d_2(\xi, T; v)\big),$$

where v is the volatility parameter, $d_1(\xi, T; v) = (\xi + v^2 T/2)/(v\sqrt{T})$, and $d_2(\xi, T; v) = d_1(\xi, T; v) - v\sqrt{T}$.

Let ψ be the implied volatility in the (ξ, T) coordinates, i.e.,

$$\psi(\xi, T) = \phi(K(\xi, T), T). \tag{24}$$

Then by the definition of the implied volatility, in either the (K, T) or (ξ, T) coordinates, we have, for all $\xi \in \mathbb{R}$ and $T \in (0, \infty)$,

$$\mathcal{C}(\xi, T) = \frac{\mathsf{C}\big(K(\xi, T), T\big)}{K(\xi, T)\mathsf{Z}(T)} = \frac{\mathsf{C}_{\mathrm{BS}}\big(K(\xi, T), T; \phi(K(\xi, T), T)\big)}{K(\xi, T)\mathsf{Z}(T)} = \mathcal{C}_{\mathrm{BS}}(\xi, T; \psi(\xi, T)). \tag{25}$$

Moreover, by following [26, Lemma 3.1] we deduce the representation formula

$$\mathcal{C}_{\mathrm{BS}}(\xi, T; v) = (\mathrm{e}^{\xi} - 1)_+ + \mathrm{e}^{\xi} \int_0^{v\sqrt{T}} N'\left(\frac{\xi}{\tau} + \frac{\tau}{2}\right) \mathrm{d}\tau, \tag{26}$$

where again $N'(\cdot) = n(\cdot)$ is the standard normal density; c.f. (20).

Now we are ready to present the proof of Theorem 3. Since we will largely make use of the results in [26], our proof will be brief.

Proof of Theorem 3. Let

$$F(\xi, \theta) = \int_0^{\theta} N'\left(\frac{\xi}{\tau} + \frac{\tau}{2}\right) \mathrm{d}\tau, \qquad \xi \in \mathbb{R}, \quad \theta \geq 0. \tag{27}$$

Then by [26, Lemmas 5.1 and 5.2], we get, as $\theta \to 0$,

$$F(\xi,\theta) \sim \begin{cases} \dfrac{\theta}{\sqrt{2\pi}}, & \xi = 0, \\ \dfrac{\theta^3}{\sqrt{2\pi}\xi^2} \exp\left(-\dfrac{\theta^2\xi + \xi^2}{2\theta^2}\right), & \xi \neq 0. \end{cases} \tag{28}$$

By (26), we have

$$\mathcal{C}_{\mathrm{BS}}(\xi, T; \psi(\xi,T)) = (\mathrm{e}^{\xi} - 1)_+ + \mathrm{e}^{\xi} \int_0^{\psi(\xi,T)\sqrt{T}} N'\left(\frac{\xi}{\tau} + \frac{\tau}{2}\right) \mathrm{d}\tau. \tag{29}$$

By Assumption 1, and Assumption 2 in the form expressed by (23), and by following [26, Proposition 4.1], we get

$$\psi(\xi,T)\sqrt{T} \xrightarrow{T\to 0} 0, \qquad \xi \in \mathbb{R}. \tag{30}$$

Now from (25) and (29) we get

$$\frac{\mathcal{C}(\xi,T) - (\mathrm{e}^{\xi} - 1)_+}{\mathrm{e}^{\xi}} = F\left(\xi, \psi(\xi,T)\sqrt{T}\right). \tag{31}$$

Then a combination of (31), (30), (28), and an application of the same procedure in [26, Theorem 5.1, Corollary 5.1] would show that as $T \to 0$,

$$\psi(\xi,T) \sim \begin{cases} \sqrt{2\pi}\dfrac{\mathcal{C}(\xi,T)}{\sqrt{T}}, & \xi = 0, \\ \dfrac{|\xi|}{\left\{-2T\ln[\mathcal{C}(\xi,T) - (\mathrm{e}^{\xi}-1)_+]\right\}^{1/2}}, & \xi \neq 0. \end{cases}$$

Respectively, $\xi = 0$ and $\xi \neq 0$ correspond to the at the money ($K = S$) and the not at the money ($K \neq S$) cases. In fact, by (24) and (25), a back transformation of the above asymptotic formula to the (K,T) coordinates gives, as $T \to 0$,

$$\phi(K,T) \sim \begin{cases} \sqrt{2\pi}\dfrac{\mathsf{C}(K,T)}{K\mathsf{Z}(T)\sqrt{T}}, & K = S, \\ \dfrac{|\ln(S/K) - \ln(\mathsf{Z}(T)) - \kappa T|}{\left\{-2T\ln\left[\dfrac{\mathsf{C}(K,T)}{K\mathsf{Z}(T)} - \left(\dfrac{S}{K\mathsf{Z}(T)\mathrm{e}^{\kappa T}} - 1\right)_+\right]\right\}^{1/2}}, & K \neq S. \end{cases}$$

Taking the limits then gives the desired expressions in (19). □

5. Proof of the small time limit: Theorem 1

The proof of the small time limit is an application of Theorem 3. It takes the following steps:

(i) verification of Assumptions 1 and 2;
(ii) computation of the at the money limit;
(iii) computation of the out of the money limit;
(iv) computation of the in the money limit.

Note that the dividend yield $\kappa = 0$ in the MMM; see the discussion following (8).

Proof of Theorem 1.

Step (i): Verification of Assumptions 1 and 2. It is easy to verify that the bond price Z satisfies Assumption 1. We omit the details.

To verify Assumption 2 we will check (16) first. By (5), in the MMM (16) becomes

$$\big(S - K\mathrm{e}^{-rT}(1-\mathrm{e}^{-x/2})\big)_+ \le C(K,T) \le S, \quad \forall\, S, K > 0,\ T \ge 0.$$

Since $\chi^2(y;4,x)$ and $\chi^2(y;0,x)$ are distributions, (2) implies that $C(K,T) \le S$ for all $K > 0$ and $T \ge 0$. This proves the upper bound for C. To derive the lower bound we will check two cases. When $S \le K\mathrm{e}^{-rT}(1-\mathrm{e}^{-x/2})$, we need $C \ge 0$. This is obviously true considering that in (1) the payoff function is nonnegative and S_T is a nonnegative process. When $S > K\mathrm{e}^{-rT}(1-\mathrm{e}^{-x/2})$, the lower bound in (16) can be derived by noting that $x/y \le [\chi^2(y;0,x) - \mathrm{e}^{-x/2}]/\chi^2(y;4,x)$, which holds for all $S, K > 0$ and $T \ge 0$. So C satisfies (16).

Next, C also satisfies condition (17) by the continuity and the Markov property of the diffusion S_T.

Moreover, simple differentiation gives

$$C_T(K,T) = -2Sx_T p(y;4,x)/x + rK\mathrm{e}^{-rT}\tilde{\chi}^2(y;0,x). \tag{32}$$

This implies that $C_T(K,T) \ge 0$ for all $K, T \in (0,\infty)$ because $x_T/x = -\eta\mathrm{e}^{\eta T}/(\mathrm{e}^{\eta T}-1)$, and the density $p(y;\delta,x)$ and the distribution $\tilde{\chi}^2$ are nonnegative. So (18) is also satisfied by C.

In sum, C satisfies all the conditions (16)–(18) in Assumption 2.

Step (ii): At the money small time limit. When $K = S$, the small time limit in (11) becomes

$$\Big[\lim_{T\to 0}\phi(K,T)\Big]_{K=S} = \lim_{K\to S}\frac{\sqrt{\alpha}\ln(S/K)}{2(\sqrt{S}-\sqrt{K})} = \sqrt{\frac{\alpha}{S}}, \quad S \in (0,\infty). \tag{33}$$

For $K = S$, the extended Roper–Rutkowski formula (19) gives

$$\phi(S,T) \sim \sqrt{2\pi}\frac{C(S,T)}{S\sqrt{T}} \qquad (T \to 0).$$

When $K = S$, $C(S,T) \xrightarrow{T\to 0} 0$. So L'Hopital's rule implies that

$$\lim_{T\to 0}\sqrt{2\pi}\frac{C(S,T)}{S\sqrt{T}} = \lim_{T\to 0}\sqrt{2\pi}\frac{C_T(S,T)}{S/(2\sqrt{T})} = \lim_{T\to 0}\frac{2\sqrt{2\pi}}{S}\sqrt{T}C_T(S,T),$$

provided the last limit exists. Recalling that C_T is given by (32) and taking note that $\sqrt{T}\tilde{\chi}^2(y;0,x) \xrightarrow{T\to 0} 0$, $[x_T\sqrt{T}p(y;4,x)/x]_{K=S} \xrightarrow{T\to 0} -\sqrt{\alpha/S}/(4\sqrt{2\pi})$, we get the at the money small time limit

$$\lim_{T\to 0}\phi(S,T) = \lim_{T\to 0}\frac{2\sqrt{2\pi}}{S}\sqrt{T}C_T(S,T) = -4\sqrt{2\pi}\lim_{T\to 0}\left[\sqrt{T}\frac{x_T}{x}p(y;4,x)\right]_{K=S} = \sqrt{\frac{\alpha}{S}}.$$

Step (iii): Out of the money small time limit. In this case (19) gives

$$\lim_{T\to 0}\phi(K,T) = \lim_{T\to 0}\frac{|\ln(S/K)|}{\{-2T\ln[C(K,T)-(S-KZ(T))_+]\}^{1/2}}.$$

Since $S < K$, $(S-KZ(T))_+ = 0$ for all sufficiently small T. Consequently

$$\lim_{T\to 0}\{-2T\ln[C(K,T)-(S-KZ(T))_+]\} = \lim_{T\to 0}\{-2T\ln[C(K,T)]\}. \tag{34}$$

Since TC_T also tends to zero as $T \to 0$, applying L'Hopital's rule twice gives

$$\lim_{T\to 0}\{-2T\ln C\} = -2\lim_{T\to 0}\frac{\ln C}{T^{-1}} = 2\lim_{T\to 0}\frac{T^2C_{TT}}{C_T},$$

provided the last limit exists. It can be shown by straightforward calculation that

$$T^2\frac{C_{TT}}{C_T} = T^2\frac{1}{R_1}(R_2+R_3+R_4+R_5),$$

where

$$\begin{aligned}
R_1 &= 1 - \frac{rK\mathrm{e}^{-rT}x\tilde{\chi}^2(y;0,x)}{2Sx_Tp(y;4,x)},\\
R_2 &= -\frac{\eta}{\mathrm{e}^{\eta T}-1},\\
R_3 &= \frac{1}{2}\left[\frac{p(y;2,x)}{p(y;4,x)}-1\right]y_T + \frac{1}{2}\left[\frac{p(y;6,x)}{p(y;4,x)}-1\right]x_T,\\
R_4 &= \frac{r^2K\mathrm{e}^{-rT}x\tilde{\chi}^2(y,0,x)}{2Sx_Tp(y;4,x)},\\
R_5 &= \frac{rK\mathrm{e}^{-rT}(\mathrm{e}^{\eta T}-1)}{2S\eta}\left[\frac{-p(y;0,x)}{p(y;4,x)}y_T + \frac{-p(y;2,x)}{p(y;4,x)}x_T\right].
\end{aligned}$$

Then the properties of the chi-square distributions imply that $R_1 \xrightarrow{T\to 0} 1$; T^2R_2, R_4, $T^2R_5 \xrightarrow{T\to 0} 0$; and $T^2R_3 \xrightarrow{T\to 0} 2(\sqrt{S}-\sqrt{K})^2/\alpha$. From these asymptotics the out of the money small time limit follows.

Step (iv): In the money small time limit. When $S > K$, (19) gives

$$\lim_{T\to 0} \phi(K,T) = \lim_{T\to 0} \frac{|\ln(S/K)|}{\{-2T\ln[C(K,T) - (S-KZ(T))_+]\}^{1/2}}.$$

Since $S > K$, $(S - KZ(T))_+ = S - KZ(T)$ for all sufficiently small T. As a result,

$$\begin{aligned}\lim_{T\to 0} \phi(K,T) &= \lim_{T\to 0} \frac{|\ln(S/K)|}{\sqrt{-2T\ln[C(K,T) - S + KZ(T)]}} \\ &= \lim_{T\to 0} \frac{|\ln(S/K)|}{\sqrt{-2T\ln[P(K,T)]}},\end{aligned} \tag{35}$$

where the second equality above results from the put-call parity (6). Since $P(K,T) \xrightarrow{T\to 0} (K-S)_+ = 0$ for $S > K$ and $TP_T \xrightarrow{T\to 0} 0$, we apply L'Hopital's rule twice to get

$$\lim_{T\to 0}\{-2T\ln P\} = -2\lim_{T\to 0} \frac{\ln P}{T^{-1}} = 2\lim_{T\to 0} \frac{T^2 P_{TT}}{P_T}, \tag{36}$$

provided the last limit exists. By the put-call parity (6), we have

$$P_T = C_T + KZ_T \quad \text{and} \quad P_{TT} = C_{TT} + KZ_{TT}.$$

By using these two identities and the chi-square distributions, it can be shown that

$$\begin{aligned}\frac{P_T}{-2Sx_T p(y;4,x)/x} &\xrightarrow{T\to 0} 1, \\ \frac{T^2 P_{TT}}{-2Sx_T p(y;4,x)/x} &\xrightarrow{T\to 0} 2\left(\sqrt{S} - \sqrt{K}\right)^2/\alpha.\end{aligned}$$

By these two limits, (35) and (36),

$$\lim_{T\to 0} \phi(K,T) = \frac{\sqrt{\alpha}\ln(S/K)}{2(\sqrt{S} - \sqrt{K})}, \qquad S, K \in (0,\infty), \quad S > K.$$

This completes both the proof for the in the money case and the proof of the theorem. □

Remark 4. After completing our work we learnt of the results of Gao and Lee [15]. They [15, Remark 7.4] stated that the small time asymptotics of the time-scaled implied volatility V, in their notation, were controlled by $k^2/(2L) \sim V^2$; and they argued that this would imply the Roper–Rutkowski formula [26, Theorem 5.1]. In our notation, their formula becomes $\phi \sim |\ln(K/S)| / \sqrt{-2T\ln C}$, which clearly does not imply the Roper–Rutkowski formula [26, Theorem 5.1] or our extended version in (19). See also Remark 6 regarding their large time result.

Remark 5. Separately, we also became aware of the paper by Gatheral et al. [16]. We noted that for the out of the money case the limit in (34) could have been computed by using their Theorem A.2. Indeed, for all in, at, and out of the money cases, our small time limit agrees with theirs [16, (3.2)]. However, their pricing approach is different from ours; and it is not entirely clear if on a finite interval in $\mathbb{R}_+$ away from the origin the Yoshida heat kernel expansion can be generalized to diffusions with degenerate coefficients like the CEV or CIR/square-root process, which is what seems to have been suggested in [16, Remark 3.5, Appendix A].

6. Proof of the large time limit: Theorem 2

We shall prove the large time limit in two steps:

(I) Proof of the convergence and the upper bound $\limsup_{T\to\infty}\phi(K,T) \le \sqrt{2(r+\eta)}$.

(II) Proof of $\lim_{T\to\infty}\phi(K,T) = v_* \equiv S\sqrt{2(3-2\sqrt{2})(r+\eta)}$.

By definition and by the properties of the Black–Scholes formula, the implied volatility is bounded below by zero. Consequently, the upper bound in Step (I) implies that a large time limit exists in the interval $[0, \sqrt{2(r+\eta)}]$. In turn, the existence of the limit allows Step (II) to take place, where we shall show that as $T\to\infty$, the implied volatility ϕ is bounded below by any $v \in (0, v_*)$ and above by any $v \in (v_*, \sqrt{2(r+\eta)})$.

6.1. Step (I): The convergence in large time and the upper bound $\sqrt{2(r+\eta)}$

We shall prove the following proposition.

Proposition 4. *Assume the MMM. Then*

$$\limsup_{T\to\infty}\phi(K,T) \le \sqrt{2(r+\eta)}$$

and the implied volatility ϕ converges to a limit in $[0, \sqrt{2(r+\eta)}]$ as $T\to\infty$.

Proof. Define $v^\epsilon = \sqrt{2(r+\eta)}/(1-\epsilon)$ for $0<\epsilon\ll 1$. Then by definition

$$\begin{aligned}
&C_{\mathrm{BS}}(K,T,v^\epsilon) - C(K,T)\\
&= SN\big(d_1(K,T;v^\epsilon)\big) - K\mathrm{e}^{-\hat{r}T}N\big(d_2(K,T;v^\epsilon)\big) - S[1-\chi^2(y;4,x)] + K\mathrm{e}^{-rT}\tilde{\chi}^2(y;0,x)\\
&= K\mathrm{e}^{-rT}\tilde{\chi}^2(y;0,x)\\
&\times\left[\frac{S\chi^2(y;4,x)}{K\mathrm{e}^{-rT}\tilde{\chi}^2(y;0,x)} - \frac{S\tilde{N}(d_1(K,T;v^\epsilon))}{K\mathrm{e}^{-rT}\tilde{\chi}^2(y;0,x)} - \frac{K\mathrm{e}^{-\hat{r}T}N(d_2(K,T;v^\epsilon))}{K\mathrm{e}^{-rT}\tilde{\chi}^2(y;0,x)} + 1\right].
\end{aligned}$$

The properties of the chi-square and normal distributions imply that the fractions inside the square brackets all tend to zero as T tends to infinity. As a result, the square bracketed term is strictly positive for all sufficiently large T; and it tends to 1 as T tends to infinity. Now recall that $C(K,T) = C_{\mathrm{BS}}(K,T,\phi(K,T))$ for all $K,T \in (0,\infty)$. This gives

$$C_{\mathrm{BS}}(K,T;v^{\epsilon}) - C_{\mathrm{BS}}(K,T;\phi(K,T)) = C_{\mathrm{BS}}(K,T;v^{\epsilon}) - C(K,T) > 0$$

for all T large enough. Now note that $C_{\mathrm{BS}}(K,T;v)$ is strictly increasing in v, other things being equal. So for any $0 < \epsilon \ll 1$ and for T sufficiently large, $v^{\epsilon} \geq \phi(K,T)$ for each $K \in (0,\infty)$. Since the implied volatility ϕ is bounded below by zero, for each fixed $K \in (0,\infty)$, $\phi(K,T)$ can be considered as a bounded infinite sequence in T as $T \to \infty$. This implies that $\phi(K,T)$ has a convergent subsequence in T as $T \to \infty$. Further, as ϵ can be made arbitrarily small, we must have

$$\limsup_{T\to\infty} \phi(K,T) \leq \lim_{\epsilon\to 0} v^{\epsilon} = \sqrt{2(r+\eta)}.$$

Consequently, ϕ has a large time limit in $[0, \sqrt{2(r+\eta)}]$. □

6.2. Step (II): Proof of the large time limit

Having determined its large time convergence, we now show how to obtain the desired limit v_* by bounding the implied volatility in the interval $(0, \sqrt{2(r+\eta)})$.

Proof of Theorem 2. Throughout this proof the parameter v is assumed to be in $(0, \sqrt{2(r+\eta)})$. Given such a v, the Black–Scholes and the MMM call prices can be written as

$$\begin{cases} C_{\mathrm{BS}}(K,T;v) = S - S\mathcal{R}_{\mathrm{BS}}(K,T;v), \\ \quad C(K,T) = S - S\mathcal{R}(K,T), \end{cases} \tag{37}$$

where $\mathcal{R}_{\mathrm{BS}}$ and $\mathcal{R}$ are given by

$$\begin{cases} \mathcal{R}_{\mathrm{BS}}(K,T;v) = \tilde{N}(d_1(K,T;v)) + \dfrac{K}{S}\mathrm{e}^{-\hat{r}T} - \dfrac{K}{S}\mathrm{e}^{-\hat{r}T}\tilde{N}(d_2(K,T;v)), \\ \quad \mathcal{R}(K,T) = \dfrac{K}{S}\mathrm{e}^{-rT} + \chi^2(y;4,x) - \dfrac{K}{S}\mathrm{e}^{-rT}\chi^2(y;0,x). \end{cases}$$

Next, define $\underline{\mathcal{R}}$ and $\overline{\mathcal{R}}$ to be

$$\underline{\mathcal{R}}(K,T) = \mathrm{e}^{(r+\eta)T-(x+y)/2}\frac{y^2}{4}\left[\frac{1}{2} + \frac{xy}{24} + \frac{(xy)^2}{768}\right] - \frac{K}{S}\mathrm{e}^{\eta T - x/2}\left[\frac{xy}{4} + (xy)^2\right],$$

$$\overline{\mathcal{R}}(K,T) = \mathrm{e}^{(r+\eta)T-x/2}\frac{y^2}{4}\left[\frac{1}{2} + \frac{xy}{24} + (xy)^2\right] - \frac{K}{S}\mathrm{e}^{\eta T-(x+y)/2}\left[\frac{xy}{4} + \frac{(xy)^2}{64}\right].$$

By using the properties of the chi-square and normal distributions, it can be shown that for sufficiently large T,

$$\frac{K}{S}e^{-[\hat{r}-(r+\eta)]T} + \underline{\mathcal{R}}(K,T) \leq \mathrm{e}^{(r+\eta)T}\mathcal{R}(K,T) \leq \frac{K}{S}e^{-[\hat{r}-(r+\eta)]T} + \overline{\mathcal{R}}(K,T). \tag{38}$$

Moreover, as $T \to \infty$,

$$\underline{\mathcal{R}}(K,T) = -\frac{2\eta^2K^2}{\alpha^2}\mathrm{e}^{-(r+\eta)T} - \frac{760SK^3\eta^4}{3\alpha^4}\mathrm{e}^{-2(r+\eta)T-\eta T} + \mathrm{O}(\mathrm{e}^{-3(r+\eta)T-2\eta T}), \tag{39}$$

$$\overline{\mathcal{R}}(K,T) = -\frac{2\eta^2K^2}{\alpha^2}\mathrm{e}^{-(r+\eta)T} - \frac{4SK^3\eta^4}{3\alpha^4}\mathrm{e}^{-2(r+\eta)T-\eta T} + \mathrm{O}(\mathrm{e}^{-3(r+\eta)T-2\eta T}), \tag{40}$$

and

$$\begin{aligned} &\mathrm{e}^{(r+\eta)T}\mathcal{R}_{\mathrm{BS}}(K,T;v) \\ &= \frac{K}{S}\mathrm{e}^{-[\hat{r}-(r+\eta)]T} - \frac{2K\eta}{\alpha}\frac{v\sqrt{T}}{(d_2+v\sqrt{T})d_2\sqrt{2\pi}}\mathrm{e}^{-d_2^2/2} + \frac{2K\eta}{\alpha}n(d_2)\mathrm{O}(d_2^{-3}), \end{aligned} \tag{41}$$

where $d_2 = d_2(K,T;v)$. Similarly, it can be shown that for all large enough T,

$$\begin{cases} \frac{1}{2}d_2^2(K,T;v) - (r+\eta)T > c_1T, & \text{if } v \in (0, v_*), \\ \frac{1}{2}d_2^2(K,T;v) - (r+\eta)T < -c_2T, & \text{if } v \in (v_*, \sqrt{2(r+\eta)}), \end{cases} \tag{42}$$

where c_1 and c_2 are some strictly positive constants dependent only on K, S, v, r, η. Combining (38)–(42) then gives the following inequalities:

(a) if $v \in (0, v_*)$, then for sufficiently large T,

$$\mathrm{e}^{(r+\eta)T}\mathcal{R}_{\mathrm{BS}}(K,T;v) \geq \frac{K}{S}\mathrm{e}^{-[\hat{r}-(r+\eta)]T} + \overline{\mathcal{R}}(K,T) \geq \mathrm{e}^{(r+\eta)T}\mathcal{R}(K,T);$$

(b) if $v \in (v_*, \sqrt{2(r+\eta)})$, then for sufficiently large T,

$$\mathrm{e}^{(r+\eta)T}\mathcal{R}_{\mathrm{BS}}(K,T;v) \leq \frac{K}{S}\mathrm{e}^{-[\hat{r}-(r+\eta)]T} + \underline{\mathcal{R}}(K,T) \leq \mathrm{e}^{(r+\eta)T}\mathcal{R}(K,T).$$

By these inequalities, (37), and the equality that $C(K,T) = C_{\mathrm{BS}}(K,T;\phi(K,T))$, we get

$$\begin{cases} C_{\mathrm{BS}}(K,T;v) \leq C(K,T) = C_{\mathrm{BS}}(K,T;\phi(K,T)), & \text{if } v \in (0, v_*); \\ C_{\mathrm{BS}}(K,T;v) \geq C(K,T) = C_{\mathrm{BS}}(K,T;\phi(K,T)), & \text{if } v \in (v_*, \sqrt{2(r+\eta)}). \end{cases}$$

As the function $v \mapsto C_{\mathrm{BS}}(K,T;v)$ is monotonically increasing in v, we have, for each K,

$$\begin{cases} \phi(K,T) \geq v, & \text{if } v \in (0, v_*), \\ \phi(K,T) \leq v, & \text{if } v \in (v_*, \sqrt{2(r+\eta)}), \end{cases}$$

for all sufficiently large T. This implies that

$$\liminf_{T\to\infty} \phi(K,T) \geq v_* \quad \text{and} \quad \limsup_{T\to\infty} \phi(K,T) \leq v_*.$$

As a result,

$$\lim_{T\to\infty} \phi(K,T) = v_* = \sqrt{2(3-2\sqrt{2})(r+\eta)}.$$

And the proof is complete. □

Remark 6. The results of Gao and Lee [15, Corollary 7.8] do not apply here because the assumptions of their corollary do not hold in the MMM. Specifically, for their Case $(+)$ they require, in their notation, $k/L \xrightarrow{T\to\infty} \text{const} \in [0,\infty)$; but under the MMM $k/L \xrightarrow{T\to\infty} -1$. Similarly, for their Case $(-)$ they need $k/L \xrightarrow{T\to\infty} \text{const} \in (0,\infty)$; but under the MMM $k/L \xrightarrow{T\to\infty} \infty$.

7. Conclusion

We have derived both the small and the large time limits for the implied volatility in the MMM. Although only the zeroth order asymptotics are proved, it seems likely that higher order expansions in time can be achieved along similar lines. This, as well as the extreme strike asymptoics, will be pursued in another work.

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