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Abstract

In this paper we model a financial market composed of agents with heterogeneous beliefs who change their strategy over time. We propose two different solution methods which lead to two different types of endogenous dynamics. The first makes use of the maximum entropy approach to obtain an exponential type probability function for strategies, analogous to the well known Brock and Hommes (1997) model, but with the endogenous specification for the intensity of choice parameter, which varies over time as a consequence of the relative performances of each strategy. The second type of dynamics is obtained by setting up a master equation and solving it using recently developed asymptotic solution techniques, which yield a system of differential equations describing the evolution of the share of each strategy in the market. The performances of the two solutions are then compared and contrasted with the empirical evidence.

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1 Introduction

The development of market models with heterogeneous agents having less than perfect rationality has gained increasing momentum since the late eighties for two main reasons. The first is the development of analytical methods (nonlinear dynamics, chaos theory and complex systems) and computational tools which have allowed researchers to build and solve heterogeneous agents models with higher degrees of freedom. The possibilities of analytical investigation have been widened also by the introduction of statistical mechanics tools in financial analysis (Mantegna and Stanley, 1999). The second reason concerns the inadequacy of the full rationality paradigm both from a theoretical and from an empirical point of view. In particular, as regards the first aspect, the internal consistency of the framework has been questioned by the no-trade theorems since Rubenstein (1975). From the point of view of empirical analysis, recent models using heterogeneous and bounded rationality agents (see for example Lux, 1995, 1998) are able to reproduce real market behaviour remarkably better than traditional ones[1], in particular as regards the fat tail distribution of returns. A class of models which yields notable results in this respect classifies the agents into chartist and fundamentalist[2].

Within this stream of research, the most popular framework is probably the one introduced by the influential paper of Brock and Hommes (1997). Their evolutionary switching probability model has been applied in a number of financial market models and, recently, also in macroeconomics to study the behaviour of agents (see de Grauwe, 2010; Pfajfar and Santoro, 2010). Brock and Hommes (1997) propose a model with evolutionary switching between the two rules of formation of expectations. The probability of an agent choosing one or the other strategy is given by a multinomial logit model (Mansky and McFadden, 1981). Given a proper partition function Z_t, the share n_j of agents choosing the price predictor j will be equal to

$$n_{j,t} = \frac{\exp\left(\beta U_{j,t}\right)}{Z_t}. \tag{1}$$

The quantity U is a weighted average of past net profits and $\beta \in [0; +\infty)$ is a parameter that measures the *intensity of choice*, which can be defined as the sensitivity of investors to the relative performance of each strategy. If $\beta \to +\infty$, the model corresponds to the neoclassical deterministic model with all agents choosing the optimal predictor, while for $\beta \to 0$ there is no switching between strategies and agents are spread uniformly across the strategies.

Among the large literature that sprung from the Brock and Hommes (1997) contribution, the work by Chiarella et al. (2006) (hereafter CHH) uses this probabilistic formalization to study the potentially destabilising effects of the adoption of a moving average pricing rule by chartists. This paper reconsiders the

[1]See Hommes (2006) for an exhaustive survey.

[2]See Zeeman (1974); Day and Huang (1990); Chiarella and He (2003); Chiarella et al. (2009) among many others. As demonstrated by Aoki and Yoshikawa (2006, ch. 9) this classification can approximate the totality of the different possible strategies in a market.

model of CHH to study the impact of the distributions of heterogeneous strategies (with switching) on the dynamics of asset prices. We modify this original approach by providing two alternative dynamics for the proportions of the two agents (and consequently for the price evolution) using two different methodologies.

First, we propose a different specification of the probability (1) with an endogenous specification of the intensity of choice parameter, obtained by means of a maximum entropy (MaxEnt) inference model. In particular, the intensity of choice is computed as a function of the profits associated with the different strategies. Moreover, while the intensity of choice parameter can assume only positive values, the variable introduced here varies over the full real line $(-\infty; +\infty)$, with the two extremes corresponding to a situation where all agents are choosing, respectively, one or the other strategy. This different formulation leads to the identification of a proper dynamic path for the shares of the two agents. This method, originally developed in statistical physics, has found a few applications in economics and finance (Foster, 2004; Liossatos, 2004; Landini et al., 2008). In a treatment similar to the one presented in this paper, Nadal et al. (1998) obtain a logit function for the possible distribution of different strategies.

The second type of dynamics is obtained by a quite different approach. The dynamics of the probability of each strategy is modelled by means of a master equation. The use of the master equation is definitely not new in this context (see for example Alfarano et al., 2008), but the particular solution algorithm we use, introduced by Di Guilmi (2008) for macroeconomic applications, has never been applied in a pricing model. The main advantage of this method is that it yields an asymptotic closed form solution, composed of an ordinary differential equation, describing the evolution of the proportion of the two types of agents, to which an endogenous stochastic term is added. In this case the intensity of switching is an argument of the transition rates and it is assumed to be exogenous.

The performances of the two different solutions are then compared to evaluate the two different dynamics of price and their sensitivity to the variations in the intensity of choice and to the proportion of the different strategies in the market. Their outcomes are also contrasted with some well known stylised facts of asset prices and returns in order to assess their relative performance in matching empirical evidence.

The contribution of the present work is twofold. First, we introduce into a pricing model two alternative approaches which are definitely original in this context. The MaxEnt inference derives the probability of switching by endogenising all the relevant quantities, while in the master equation approach, in a more traditional fashion, this probability is exogenously specified. The former type of dynamics introduces an extension to an established and very popular approach, and thus can open interesting perspectives for the development of research on this topic. The second aspect of novelty concerns the fact that we compare two methods on the basis of their ability to replicate well known stylised facts of financial markets. As a consequence, we are able to consistently evaluate the actual improvement of the analysis provided by the endogenisation of the intensity of choice parameter.

The paper is structured as follows: the next section introduces the basic assumptions of the framework; sections 3 and 4 detail, respectively, the MaxEnt approach and the master equation solution method, specifying the dynamical systems obtained by each method and studying their properties; section 5 integrates the results and compares the two solutions with the empirical evidence; finally section 6 offers some concluding considerations.

2 Basic assumptions

This section introduces the hypotheses for modelling our market which are common to the two methods. The further specifications needed for the implementation and solution of each method are presented in sections 3 and 4.

The basic structure of the framework is the same as presented in CHH with some adjustments.

- the number of agents N is constant;
- the agents adopt one of two possible strategies: a proportion $n^f = \frac{N^f}{N}$ of traders estimates the price according to its supposed fundamental value, while n^c is the proportion of chartists who engage in some kind of trend chasing trading rule;
- the agents can change their strategies;
- the excess demands for each type of agent are formulated as in CHH:
  - for fundamentalists:
$$d^f(t) = \alpha(P^* - P(t)), \tag{2}$$
    where $P(t)$ is the price at time t and P^* is the fundamental price;
  - for trend chasers:
$$d^c(t) = \tanh(a\psi^L(t)), \tag{3}$$
    where $\psi^L(t) = P_t - ma^L(t)$, with $ma^L(t)$ standing for the moving average of prices over a period L. This functional form for representing chartists excess demand has been proposed by Chiarella (1992) and, as CHH point out, it picks up two important features of filtered moving average rules. First, with a low value of a, the technical analysts react only when the change in the price signal is confirmed, filtering frequent changes in a short time period. Second, since the function is limited to the interval $(-1, 1)$, it captures the limited long/short positions, risk averting behaviour and traders budget constraints;
  - accordingly, $D(t) = N^f(t)d^f(t) + N^c(t)d^c(t)$ is the total excess demand;

- The profit functions associated with each strategy at time t are, respectively:

$$\pi^f(t) = d^f(t-dt)[P(t) - P(t-dt)] \tag{4}$$

$$\pi^c(t) = d^c(t-dt)[P(t) - P(t-dt)] \tag{5}$$

  For the sake of simplicity, the costs of each strategy are assumed to be negligible. Positive costs have the effect of modifying the steady state values of n^f and n^c, but do not otherwise change the analysis.

- the return associated with each strategy is evaluated by means of the fitness functions:

$$U^f(t) = \pi^f(t) + \eta U^f(t-dt), \qquad U^c(t) = \pi^c(t) + \eta U^c(t-dt) \tag{6}$$

  where η is a parameter incorporating the memory of the cumulated fitness function. Accordingly we define

$$\begin{aligned} U(t) = \quad & U^f(t) - U^c(t) = \\ & \left[d^f(t-dt) - d^c(t-dt)\right][P(t) - P(t-dt)] + \eta U(t-dt). \end{aligned} \tag{7}$$

- the interaction of the agents within the market is modelled by introducing a third type of agent, the market maker (Beja and Goldman, 1980), who determines the evolution of price according to the excess demands. This agent can be regarded as the institutional setting within which the market operates. The dynamics of price is described by the following equation

$$\begin{aligned} & P(t+dt) = \\ & P(t)[1+\sigma_\varepsilon \varepsilon] + \tfrac{\rho}{2}\left[(1+\nu(t))\alpha(P^* - P(t)) + (1-\nu(t))h(P(t) - ma^L(t))\right] \end{aligned} \tag{8}$$

where ρ is the velocity of the adjustment, $\nu = n^f - n^c$ and σ_ϵ is the constant standard deviation for the noise term $\epsilon \sim \mathcal{N}(0,1)$.

3 The MaxEnt dynamics

In this section we present the first of the two solution methods introduced in section 1: the maximum entropy inference method. In subsection 3.1 we briefly explain the main features of the approach and then we apply it to our model. Then subsection 3.2 deals with the dynamical system obtained by using this inference method and discusses its stability properties.

3.1 MaxEnt inference

The MaxEnt inference method is widely adopted in the natural sciences to infer probability distributions when little or no information about the population is available. In such a situation, applying Laplace's principle of insufficient reason, the best possible choice is to assign to all the possible configurations of the system the same probability, that is to assume a uniform distribution. The number of possible configurations or states of the system depends upon two factors: first, the number of its constituents and, second, the level of their heterogeneity. This latter can be quantified by the number of the possible states which an agent can occupy (the so-called micro-states).

Accordingly, Shannon defines the entropy of a system as the average logarithm of the probability of occupation numbers for configurations of a state space or, in other words, the number of ways in which a macro-configuration can be realised. Given a space of M possible micro states and indicating with N^j the occupation number of a micro-state j, the Shannon entropy can be expressed as

$$H(N^j) = \sum_{j}^{M} -N^j log(N^j). \tag{9}$$

It has been demonstrated (van Campenhout and Cover, 1981) that the distribution function that maximizes entropy is preferred since distributions that display a low level of entropy yield lower levels of fitting when applied to data.

For the present application, we need to modify the fitness functions in order to avoid negative values. Thus we express them as logistic functions of the form

$$U^f(t) = \frac{1}{1 + exp(-\rho\pi^f(t))} + \eta U^f(t - dt), \tag{10}$$

$$U^c(t) = \frac{1}{1 + exp(-\rho\pi^c(t))} + \eta U^c(t - dt), \tag{11}$$

with ρ as a parameter. This functional form ensures a positive dependence of U^j on profits whilst at the same time, given the proper initial conditions, they take only positive values, which are necessary for the consistency of the developments presented below.

In order to integrate the model with other relevant arguments for the estimation of the probability, it is possible to impose some constraints on the maximization of the function (9). For this particular model we need to introduce a constraint which ensures the consistency of the probability so that $n^f + n^c = 1$ or, equivalently, $N^f + N^c = N$. We also need the probability to be dependent on the performance of each strategy measured by the fitness. In particular we assume a direct proportionality between the relative performance of a strategy and the number of investors adopting it. This proportionality can be quantified according to

$$\frac{N^f}{U^f} = \frac{N^c}{U^c} \tag{12}$$

which can be also expressed as

$$N^f U^c - N^c U^f = 0.$$

Knowing that $N^f = N - N^c$, we can write

$$U^f(N^f - N) - U^c(N^c - N) = 0.$$

After simple manipulations and using the first equality of equation (7) we obtain

$$N^f U^f - N^c U^c = NU. \tag{13}$$

We are now able to propose a MaxEnt model for the estimation of the probabilities of choosing one of the two available strategies: The MaxEnt problem for this particular model can be formulated in the following way:

$$\max_{N^f, N^c} H(N^f, N^c) = -N^f log(N^f) - N^c log(N^c) \tag{14}$$

s.t.

$$\begin{cases} N^f(t) + N^c(t) = N \\ N^f(t)U^f(t) - N^c(t)U^c(t) = N\,U(t). \end{cases} \tag{15}$$

The associated Lagrangean is:

$$\begin{aligned} \ell = \; & -N^f(t)log(N^f(t)) - N^c(t)log(N^c(t)) + \delta_1(t)N^f(t) + \delta_1(t)N^c(t) - \delta_1(t)N + \\ & +\delta_2(t)N^f(t)U^f(t) - \delta_2(t)N^c(t)U^c(t) - \delta_2(t)N\,U(t) \end{aligned}$$

with first order conditions[3]:

$$\begin{cases} \frac{\partial \ell}{\partial N^f(t)} = -log(N^f(t)) - 1 + \delta_1(t) + \delta_2(t)U^f(t) \\ \frac{\partial \ell}{\partial N^c(t)} = -log(N^c(t)) - 1 + \delta_1(t) + \delta_2(t)U^c(t) \\ \frac{\partial \ell}{\partial \delta_1(t)} = N - N^f(t) - N^c(t) \\ \frac{\partial \ell}{\partial \delta_2(t)} = N\,U(t) - N^f(t)U^f(t) + N^c(t)U^c(t). \end{cases} \tag{16}$$

Equating each term in (16) to zero, and substituting $\delta_1(t) = 1 - \psi(t)$ and $\delta_2(t) = \gamma(t)$

$$\begin{cases} N^f(t) = e^{-\psi(t)+\gamma(t)U^f(t)} \\ N^c(t) = e^{-\psi(t)+\gamma(t)U^c(t)} \\ N^f(t) + N^c(t) = N \\ N^f(t)U^f(t) - N^c(t)U^c(t) = N\,U(t) \end{cases} \tag{17}$$

Substituting the first two equations into the third and rearranging, we obtain

$$e^{-\psi(t)} = \frac{N}{e^{\gamma(t)U^f(t)} + e^{\gamma(t)U^c(t)}}$$

which, when substituted into the last equation of (17), generates

$$e^{\gamma(t)U^f(t)}U^f(t) - e^{\gamma(t)U^c(t)}U^c(t) = N\,U(t)\frac{e^{\gamma(t)U^f(t)} + e^{\gamma(t)U^c(t)}}{N}.$$

Rearranging we obtain

$$e^{-\gamma(t)U^f(t)}U^c(t) - e^{\gamma(t)U^c(t)}U^f(t) = 0 \tag{18}$$

The variable $\gamma(t)$ measures the elasticity of the occupation numbers to the performances of each strategy or, in other words, the sensitivity of agents to market

[3] As demonstrated in Di Guilmi et al. (2010) the first order conditions are also sufficient.

conditions. This variable is therefore an *intensity of switching variable* and plays the same role as the intensity of switching parameter β in the adaptation by CHH of the Brock and Hommes (1997) model. Solving equation (18) we get a formulation of $\gamma(t)$ dependent on the fitness functions, namely

$$\gamma(t) = -log\left[\frac{U^f(t)}{U^c(t)}\right]\left[U^c(t) + U^f(t)\right]^{-1}. \tag{19}$$

Then, using the first two equations in (17) we can compute the theoretical probability of an agent choosing one strategy, conditioned on the present value of N^f and N^c. Thus we have

$$p^f(t) = \frac{N^f(t)}{N} = n^f(t) = \frac{e^{-U^f(t)\gamma(t)}}{e^{-U^f(t)\gamma(t)} + e^{U^c(t)\gamma(t)}} \tag{20}$$

$$p^c(t) = \frac{N^c(t)}{N} = n^c(t) = \frac{e^{U^c(t)\gamma(t)}}{e^{-U^f(t)\gamma(t)} + e^{U^c(t)\gamma(t)}} \tag{21}$$

Since from equation (19), $\gamma \in (-\infty, \infty)$, while in CHH and in Brock and Hommes (1997) by assumption $\beta \in [0, \infty)$, the formulation of the probabilities needs to be different from that of these models to be consistent. Precisely, we have that when $\gamma \to \infty$, the chartist strategy is performing better and therefore agents switch to chartism. In the opposite case, when $\gamma \to -\infty$, the fitness function for fundamentalist is relatively large and thus we expect a greater number of agents to adopt that strategy.

3.2 The dynamical system

The pricing mechanism is set up according to CHH. The dynamics of $\nu(t) = n^f(t) - n^c(t)$ is obtained by substituting for the frequencies n^f and n^c from (20) and (21). Hence we can write $\nu(t) = \frac{e^{-\gamma(t)U^f(t)} - e^{\gamma(t)U^c(t)}}{e^{-\gamma(t)U^f(t)} + e^{\gamma(t)U^c(t)}}$ which can be easily transformed into

$$\nu(t) = \frac{e^{-\gamma(t)(U^f(t)-U^c(t))} - 1}{e^{-\gamma(t)(U^f(t)-U^c(t))} + 1} = \tanh\left[-\frac{\gamma(t)}{2}\left(U^f(t) - U^c(t)\right)\right]$$

The difference in the fitness functions is calculated using (10) and (11), while the price evolution is given by the (8). Thus the dynamical system is

$$\begin{cases} \nu(t) = \tanh\left[-\frac{\gamma(t)}{2}\left(U^f(t) - U^c(t)\right)\right] \\ \gamma(t) = -log\left[\frac{U^f(t)}{U^c(t)}\right]\left[U^c(t) + U^f(t)\right]^{-1} \\ P(t) = P(t-dt)[1+\sigma_\epsilon] + \frac{\varrho}{2}\left[(1+\nu(t-dt))\alpha(P^* - P(t-dt)) + \right. \\ \left. +(1-\nu(t-dt))h(P(t-dt) - ma^L(t-dt))\right] \end{cases} \tag{22}$$

While the price equation (the last) is the same as CHH, the first two equations impacting on its dynamics are different. First, we compare the stability conditions. In CHH the general condition for the steady state ($P = P*, \nu = 0, U = 0$) to be locally asymptotically stable is

$$2\bar{a} < \bar{\alpha} < 2 \ \ \forall L, \tag{23}$$

with $\bar{a} = a\rho n^{c*}$ and $\bar{\alpha} = \alpha\rho n^{f*}$. In our case the study of the stability is complicated by the fact that γ is a variable. In particular when $\alpha > a$ the system becomes more sensitive to the stochastic noise and may turn out to be unstable when $\sigma \geq 0.05$. One of the roots in CHH is

$$\frac{\partial U(t)}{\partial U(t-dt)} = \frac{\eta\gamma(t)}{2}.$$

For the stability of their system it is enough to set $\eta, \gamma \in (0, 1]$. Here, even though $\eta \in (0, 1]$, instability may arise given that $\gamma \in (-\infty, \infty)$. Figure 1 displays the phase plots for different lags in the moving average. The variations of the price are symmetrical with respect of its starting value but not with respect to γ. The values of the intensity of switching variable are typically lower than the ones used by CHH in their simulations. The 8-shape plot is shifted to positive values of γ, in particular for low values of L. For $L \geq 50$ the asymmetry remains but the concentration is bigger around negative values and more dispersed for positive values for $0.02 < \gamma < 0.05$. Recalling that a positive γ is the consequence of the higher fitness of the trend chasing strategy, the system then appears to be dominated by the chartists. High deviation from price are determined by a relatively higher proportion of trend chasers, and this effect is bigger the larger is L. This may explain why, for $\alpha > a$, that is a higher sensitivity of fundamentalists, the system can become unstable[4].

A further insight is provided by figure 2 that reports the phase diagram for different values of α with the lag fixed at 5. As α increases, the initial 8-shaped figure becomes U-shaped. An asymmetry is evident in this case. In particular, the largest deviation from the fundamental price happens for the largest values of γ. This means that for a large γ, and thus a relatively better performance of the chartist strategy, the deviation of price are bigger the larger is the sensitivity of fundamentalists to it.

Figure 3 reports the bifurcation diagrams for a with $\alpha = 1$ and different values of L. In this case the system is always asymptotically stable for $a < 2$ and low L, while for large L stability requires that $a < \alpha$. For a bigger α and large length for the moving average, the price displays complex dynamics. For $L = 2$, the first panel in figure 3 shows orbits with six different attractors and a region of possibly chaotic behaviour for $2.7 < a < 2.8$.

We can conclude that the introduction of the intensity of switching as a variable makes the system more volatile and, in particular, more sensitive to the reactions of fundamentalist traders. For a relatively high sensitivity of fundamentalists to the difference between fundamental and current price, the system explodes even without stochastic noise. Thus, in this treatment, it seems that the moving average rule is a less relevant source of instability than the over-reaction of fundamental traders.

[4]Setting $\rho = 2$ and considering that $n^{f*} = n^{c*} = 0.5$ we have that $\bar{\alpha} = \alpha$ and $\bar{a} = a$.

4 The master equation dynamics

This section presents the price dynamics obtained from the asymptotic solution of the master equation. In subsection 4.1, the different assumptions and the model based on the master equation are introduced. Then, in subsection 4.2, we present the solution and the dynamical system of equations that describes the evolution of the price.

4.1 Master equation formulation

We denote by κ the transition probability of switching from chartist to fundamentalist and by ι the probability of the inverse transition. Following Lux (1995), the probabilities may be quantified according to

$$\kappa(t) = ve^{bU(t)}, \tag{24}$$

$$\iota(t) = ve^{-bU(t)}, \tag{25}$$

where v and b are parameters and $U(t)$ is defined as in (7). As with the previous formulation in section 2 and 3 the probability of an agent changing pricing rule is dependent upon the performances of the two strategies. We note that the parameter b in the above expression measures the sensitivity of agents to the difference in the performance of their strategy and, therefore, represents an exogenous intensity of choice. Consequently, the comparison between the two models presented in this paper can also shed light into the additional insights and flexibility that the endogenisation of the intensity of choice can provide to the modelling of evolutionary switching in a heterogeneous agents framework.

We indicate with ϖ the unconditional probability of choosing the strategy based on fundamental price, with λ the probability of observing a change of strategy from chartist to fundamentalist and with μ the probability of recording the opposite transition. Then the transition rates can be expressed as

$$\lambda = (1 - \varpi)\kappa, \tag{26}$$

$$\mu = \varpi\iota. \tag{27}$$

The only additional hypothesis that we need to specify with regard to the process is that it is a jump Markov process and, accordingly, its macro dynamics can be analytically identified by means of the master equation. The master equation can be defined as a first-order differential difference equation that describes the dynamics of the probability of a system to occupy each one of a pre-defined set of macro-states. For our purposes it is convenient to specify it as a balance flow equation between probability inflows and outflows into and out of a generic macro-state[5]. Namely, taking as state variable the number of fundamentalists, the variation of probability in a unit of time can be quantified by

$$\frac{dp(N^f,t)}{(t-dt)} = \lambda\, p(N^f - 1, t) + \mu\, p(N^f + 1, t) - (\lambda + \mu)\, p(N^f, t), \tag{28}$$

[5]For derivation and different formulations of the master equation useful references are Kelly (1979), Aoki (2002, chap. 3) and Di Guilmi et al. (2010).

writing $p(N^f, t)$ to denote the probability of recording a number N^f of fundamentalists at time t. The structure of the equation is straightforward: in a unit of time the probability of observing a number N^f of agents choosing the fundamentalist strategy is given by the probability of observing, in the previous unit of time, a number of fundamentalists equal to either $N^f + 1$ or $N^f - 1$, weighted by the probabilities of a jump, respectively, out of or into state f, less the probability to have already a number of fundamentalists equal to N^f and observe a transition. In order to identify the dynamics of switching and price, we need to solve equation (28).

4.2 Analytical solution

Since an analytical solution for master equations can be obtained only under very specific and restrictive conditions, we solve it using the approximation method introduced by Aoki (2002) and further developed and detailed in Di Guilmi (2008) and Chiarella and Di Guilmi (2011), using a formulation identical to that in (28). We refer the reader to these works for full details of the derivation of the solution. The asymptotic solution of the master equation (28) allows us to quantify and to express in explicit form the stochastic dynamics of the market, identifying its trend and cycle components. As a result, we can quantify the long-run path dynamics (that eventually leads to a steady state equilibrium, if it exists), and the fluctuations around this trend. In order to obtain this information, we assume that the fraction of fundamentalists in a given moment is determined by its expected mean (m), the drift, and, according to Aoki (2002), by an additive fluctuation component s of order $N^{1/2}$ around this value. Thus we can write

$$N^f = Nm + \sqrt{N}s. \tag{29}$$

The subsequent basic steps of the method are: first, the homogenization of the transition fluxes by using lead and lag operators; second, the Taylor's expansion of the modified master equation and third, equating the terms with same order of power of N. In this way we can obtain two different equations, one describing the dynamics of the drift and the other quantifying the evolution of the probability of fluctuations around the drift.

The asymptotically approximate solution of the master equation is given by the system of coupled differential equations

$$\frac{dm}{d\tau} = \lambda m - (\lambda + \mu)m^2, \tag{30}$$

$$\frac{\partial Q}{\partial \tau} = [2(\lambda + \mu)m - \lambda] \frac{\partial}{\partial s}(sQ(s)) + \frac{\left[\lambda m(1-m) + \mu m^2\right]}{2} \left(\frac{\partial}{\partial s}\right)^2 Q(s). \tag{31}$$

where $Q(s, \tau)$ is the transition density function of the spread s denoted with respect to τ, which denotes the time rescaled by the factor N, so that $\tau = tN$.

Equation (30) is a deterministic ordinary differential equation which displays logistic dynamics for the trend. Equation (31) is a second order stochastic partial differential equation, known as the *Fokker-Planck equation* that drives the spread component (i.e. the fluctuations around the trend) of the probability flow. As one can see, the dynamics is convergent to a steady state. Setting the left hand side of the macroscopic equation (30) to zero, we can calculate the steady state value m^* as

$$m^* = \frac{\lambda}{\lambda + \mu}. \tag{32}$$

The solution of the equation for the spread component allows us to calculate the distribution function θ for the spread s, determining, in this way, the stationary probability distribution of fluctuations. We find that

$$\theta(s) = C \exp\left(-\frac{s^2}{2\sigma^2}\right) \quad \text{with} \quad \sigma^2 = \frac{\lambda\mu}{(\lambda + \mu)^2}, \tag{33}$$

which is a Gaussian density. We point out that fluctuations depend only on transition rates.

4.3 The dynamical system

The previous results can be used to build a new system in order to study the price dynamics. As one can see from equation (8), the evolution of price depends on the difference in the fitness function, determined by equation (7), and on the proportion of agents following the different strategies, which is quantified by equations (30) plus a stochastic noise distributed according to equation (33). Thus the system can be written as

$$\begin{cases} n^f(t) = n^f(t - dt) - [\lambda(t) + \mu(t)]\,[n^f(t)]^2 + \lambda(t) n^f(t) + \sigma\, dW \\ U(t) = U^f(t) - U^c(t) = \left[d^f(t - dt) - d^c(t - dt)\right][P(t) - P(t - dt)] + \eta U(t - dt) \\ P(t) = P(t - dt)[1 + \sigma_\epsilon] + \rho\left[n^f(t)\alpha(P^* - P(t - dt)) + \right. \\ \left. + (1 - n^f(t)) h(P(t - dt) - ma^L(t - dt))\right] \end{cases} \tag{34}$$

where dW is a stationary Wiener increment and $\sigma\, dW$ is the stochastic fluctuation component in the proportion of fundamentalist investors, coming from the distribution (33).

Using the result in equation (32), the first of the equations in (34) has a stable stationary point $n^{f*} = \frac{\lambda}{\lambda+\mu}$. Therefore, a stable stationary point for the system is $(n^{f*} = \frac{\lambda}{\lambda+\mu}, U^* = 0, P = P^*)$. From the simulations of the dynamical system, it appears that the asymptotic stability does not hold if $\alpha > a$ when we add the stochastic noise.

The phase plot displayed in figure 4 is a reverse U-shape graph and the effect of an increase in the length of the moving average seems to have a minor effect on stability compared to the MaxEnt case.

In this case the system displays a wider range of stability ($\alpha < 2.4$) and a low sensitivity to the transition parameters v and b. However the pattern appears to be the same as the previous solution with asymptotic stability for $a < \alpha$ and low L; for $a \geq \alpha$ the dynamics is complex and maybe chaotic. The dynamics show bifurcations for low L and a behaviour that appears to be chaotic for large L (figures 5 and 6). Also in this case, for $L = 2$, six attractors are identifiable, even though with overlapping pairs of lines in the diagram. The bifurcation analysis and the study of the Lyapunov exponents can be helpful in more precisely defining the region of complex and chaotic dynamics associated with the different values of the parameters, but the study of these issues goes beyond the purpose of the present paper and is part of our future research agenda.

5 Simulations

Simulations have been performed in order to provide visual insights of the two dynamics and contrast their outcomes. The list of the parameters and their reference values are reported in table 1. Figure 7 contrasts the distributions of prices generated by the two dynamics, showing quite different patterns. While for the master equation solution the price distribution is relatively concentrated around the fundamental value, for the MaxEnt case the variations are of larger amplitude and the distribution appears to be noticeably more dispersed. In both cases the price seems to be affected by the length of the moving average interval.

As far as the MaxEnt solution is concerned, the dynamics of γ reveals a consistent pattern: it is positively correlated with the number of chartists, in particular figure 8 indicates that the intensity of switching variable anticipates the switching of agents. When it is below zero there will be a bigger transition from the chartist to the fundamentalist strategy than the opposite case. The contrary holds when $\gamma > 0$ in the previous period[6]. For both solutions, changes in the memory of the fitness function η do not appear to alter the dynamics of price.

As regards the performances of the two approaches in replicating empirical evidence, both are able to generate a unit root process for the price, as confirmed by the augmented Dickey-Fuller test. The series of returns generated by the MaxEnt method clearly displays volatility clustering, while for the master equation this result is not so clear. For both procedures the square and absolute returns display high autocorrelation while raw returns do not.

As long as the distribution of returns is concerned, it is fat-tailed under both dynamics. The MaxEnt solution produces a distribution of returns that can be well approximated by a power law in the upper tail, both for positive (figure 9) and negative (figure 10) returns. The estimated slope coefficients for the power law probability function are 2.6231 for positive returns and 2.056 for the negative ones, and therefore comparable with the empirical literature, which reports estimates between 2 and 3 (Lux, 2008). Plots and estimates are obtained by using the algorithm proposed in Clementi et al. (2006) who uses a

[6]The values of the plot inf figure 8 come from different sets of simulations in order to explore the variability of the intensity of switching over the full range of variation for n^c.

formulation for the probability function of the type

$$P(x) = (x/x_0)^{-\alpha}, \tag{35}$$

where x_0 is the threshold and α the slope coefficient. This method is particularly accurate since the estimate of the threshold of the tail x_0 is data driven. The tail is chosen in order to obtain a region of acceptance of the Kolmogorov-Smirnov test for the distribution (35).

As far as the master equation solution is concerned, the power law cumulative distribution function (35) satisfactorily fits only a small part of the upper tail, with coefficients equal, respectively, to 2.786 for positive returns and 3.009 for negative returns (figures 11 and 12). The fit for this type of dynamics can be improved by using a generalised Pareto distribution, as shown in figure 13, whose density is given by

$$p(x) = \left(\frac{1}{\sigma}\right)\left(1 + \alpha\frac{(x - x_0)}{\sigma}\right)^{(-1-1/\alpha)} \tag{36}$$

Equation (36) is equivalent to equation (35) for $x_0 = \sigma$.

6 Concluding remarks

In this paper we apply two stochastic methods, recently introduced into economics from statistical physics, to an existing model in order to study the dynamics of price in a market composed of agents with heterogeneous beliefs. The two methods we employ in this paper are not completely new in the field, but the developments and the application we propose here are original. The two procedures are also tested by comparing their outcomes with the empirical evidence.

The MaxEnt method is used to compute the intensity of switching of the orginal model by CHH, by integrating all the relevant quantities in the inference problem. The intensity of switching variable obtained by solving the maximization is a function of the fitness of the two strategies and it is able to anticipate and to drive the switching pattern of agents. The second dynamics is obtained by using the methods proposed by Di Guilmi (2008) for the asymptotic solution of the master equation, obtaining a system of coupled equations which drive the drift and the fluctuations of the number of agents choosing a given strategy. In this case the functional transition probabilities for each individual are imposed by assumption, therefore the intensity of switching is an exogenous parameter, as in the models coming from the Brock and Hommes (1997) tradition.

The comparison between the two dynamics reveals that the endogenisation of the smoothing parameter slightly increases the instability of the system. The range of parameters for which the system is stable is smaller for the MaxEnt solution and the price distribution displays considerably fatter tails. A possible explanation for the higher instability is that an endogenous formulation of the intensity of switching determines feedback effects which can originate a self-sustaining deviation from the fundamentals, as testified by the distribution of prices and the study of stability. This is supported by the fact that the attempts we have made using alternative formulations of the constraints, and therefore different specifications of the inference problem, produced neither reliable nor significantly different results.

Nevertheless, in the stability region, the MaxEnt solution performs to some extent better in the replication of the empirical evidence. For both procedures the series of prices are unit root processes; the raw autocorrelations of returns is not significant while the ones for the absolute and square returns are significant. The MaxEnt dynamics gives somewhat more convincing behaviour in replicating the evidence of volatility clustering and fat tails of returns.

The main aim of this paper is to introduce the two different dynamics and to provide a first assessment. To summarise, we are not able to find conclusive evidence of a relevant improvement of the performance of the model when using an endogenous intensity of switching. However, this approach can potentially have some impact for the development of this class of heterogeneous agents models, especially considering the popularity of the original Brock and Hommes (1997) framework. Indeed, the intensity of switching variable produces a satisfactory replication of the empirical evidence and can provide additional information on the model behaviour. Future research should focus on the possible ways to overcome the limitations pointed out in this paper, without introducing exogenous quantities into the solution. For example, in a more elaborate model, it would be possible to consider a different and more suitable system of constraints for the MaxEnt problem, leading to more stable solutions and, consequently, to more reliable outcomes.

α	sensitivity of fundamentalists to the difference in price	$\in [.5, 1+a]$;
a	sensitivity of chartists to the difference in price	1;
L	lags in the moving average	$\in [1, 100]$;
ρ	price adjustment by the market maker	$\in [1, 2]$;
v	adjustment of probability of switching	0.05;
b	sensitivity of agents to the difference in profits	0.01.

Table 1: List of the parameters and default values used in the simulations.

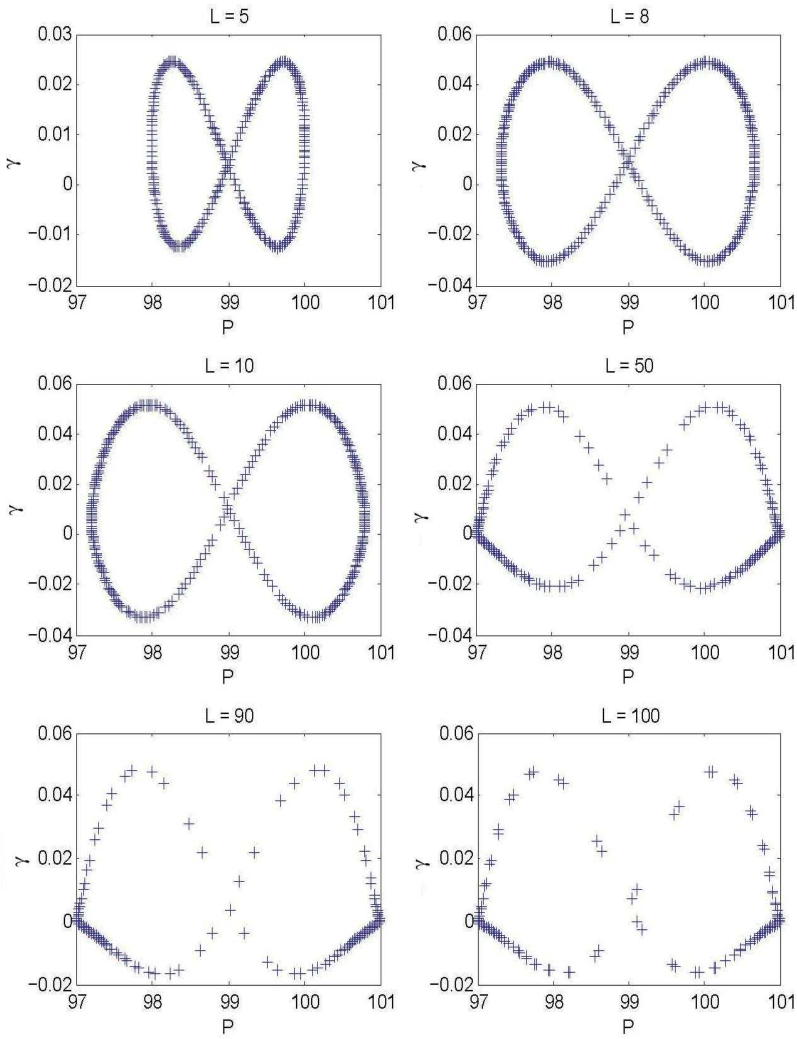

Figure 1: Phase plot of γ and price. Maximum entropy solution

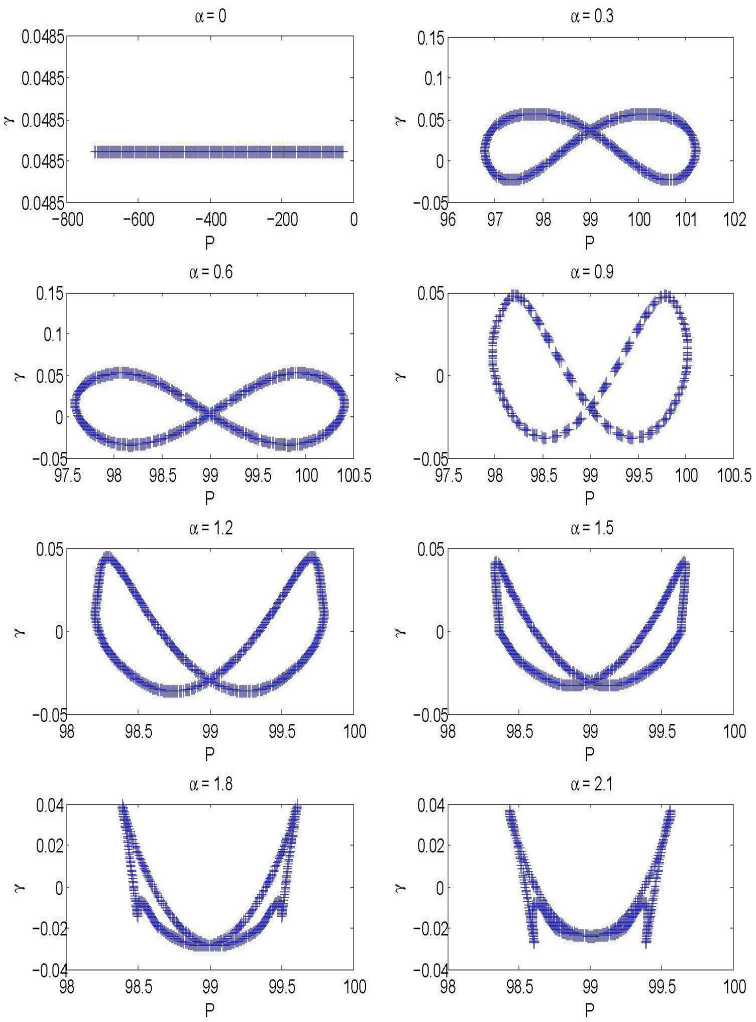

Figure 2: Phase plot for different α with $L = 5$. Maximum entropy solution.

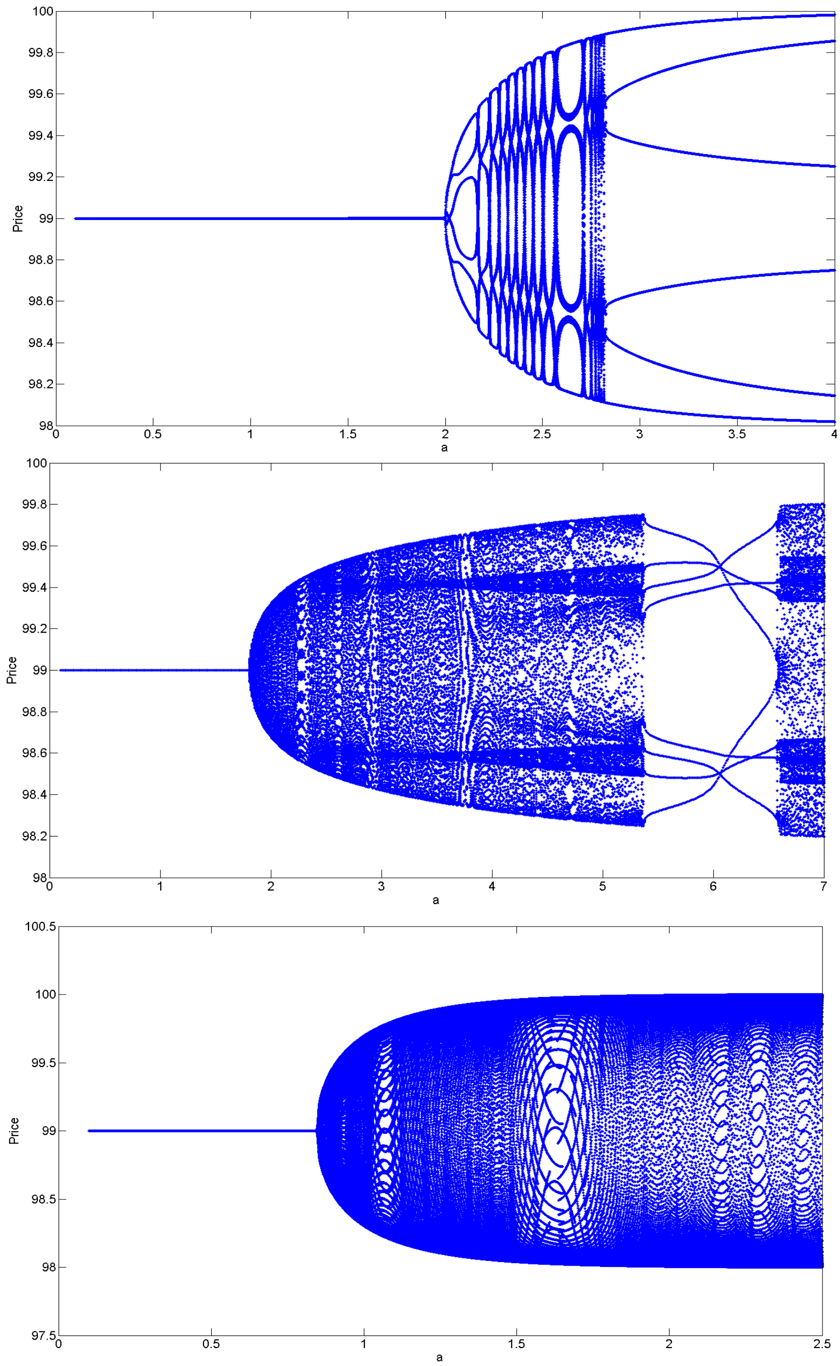

Figure 3: Bifurcation diagrams for $\alpha = 1$ and $L = 2$ (upper panel), $L = 4$ (central panel) and $L = 60$ (lower panel). Maximum entropy solution

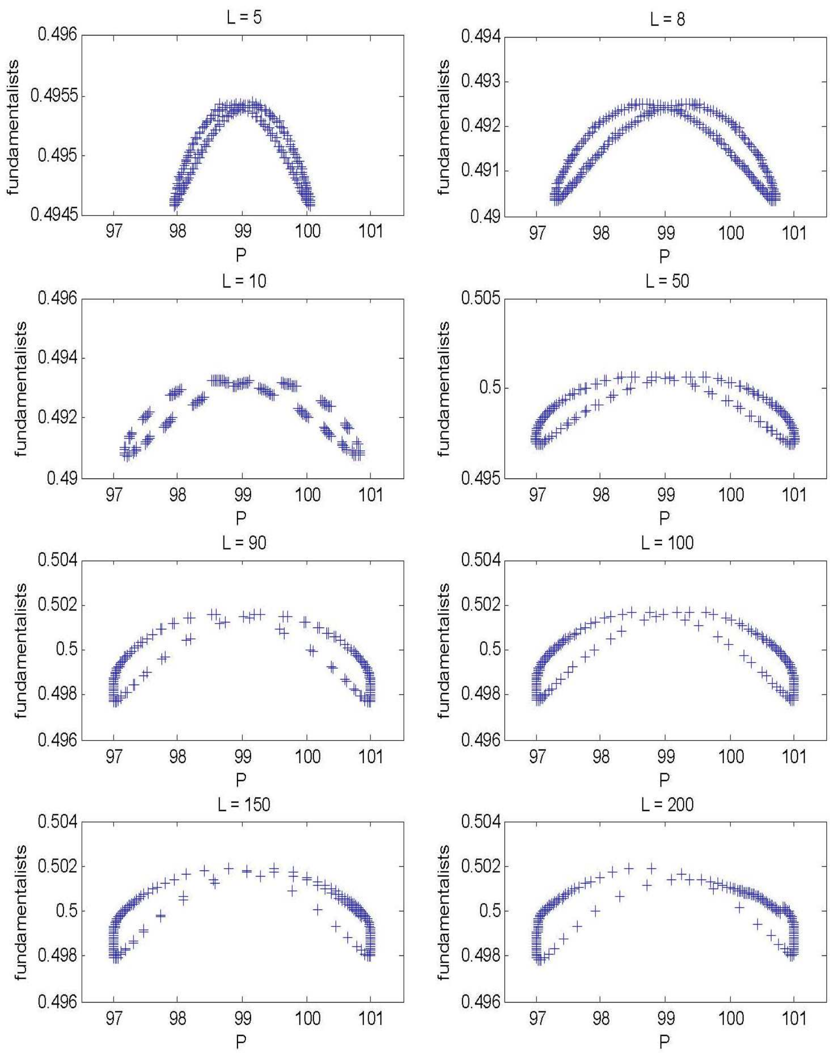

Figure 4: Phase plot of proportion of fundamentalists and price. Master equation solution.

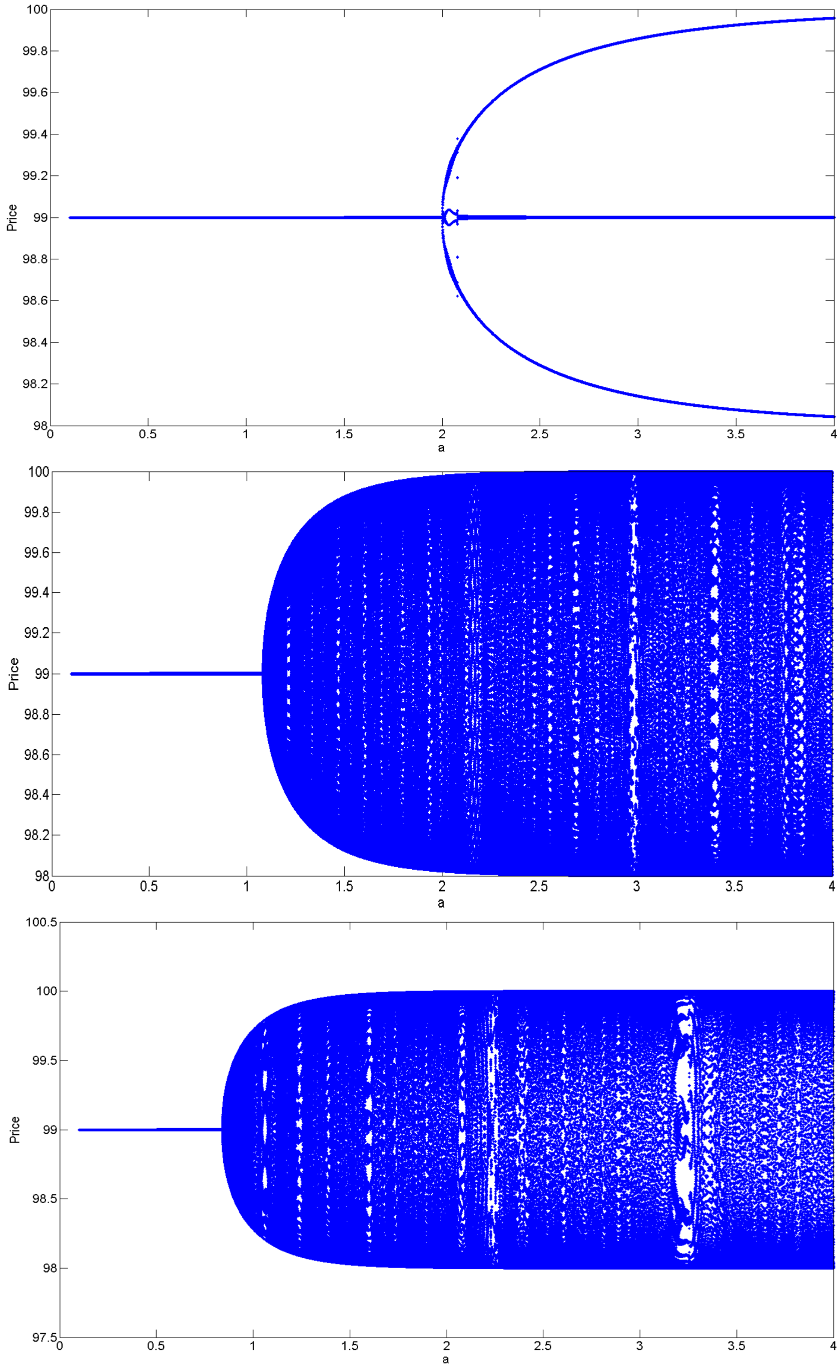

Figure 5: Bifurcation diagrams for $\alpha = 1$ and $L = 2$ (upper panel), $L = 4$ (central panel) and $L = 10$ (lower panel). Master equation solution

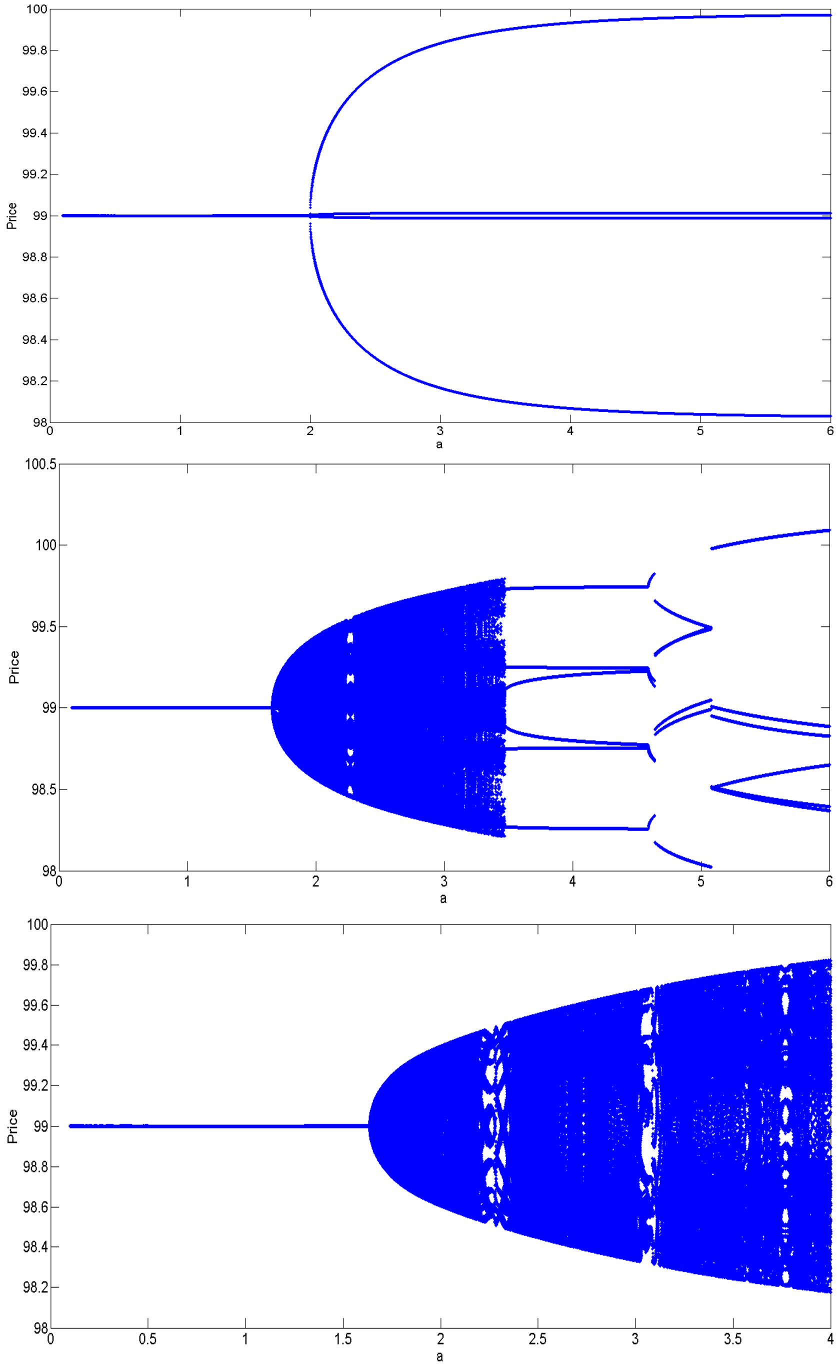

Figure 6: Bifurcation diagrams for $\alpha = 2$ and $L = 2$ (upper panel), $L = 4$ (central panel) and $L = 10$ (lower panel). Master equation solution

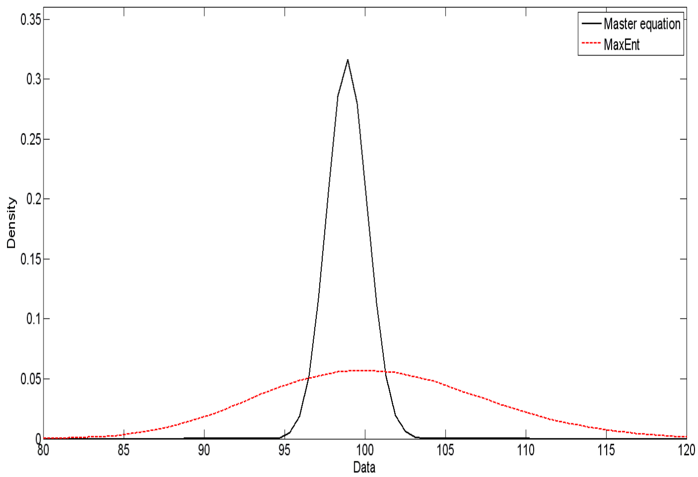

Figure 7: Distribution of the prices generated by the two dynamics.

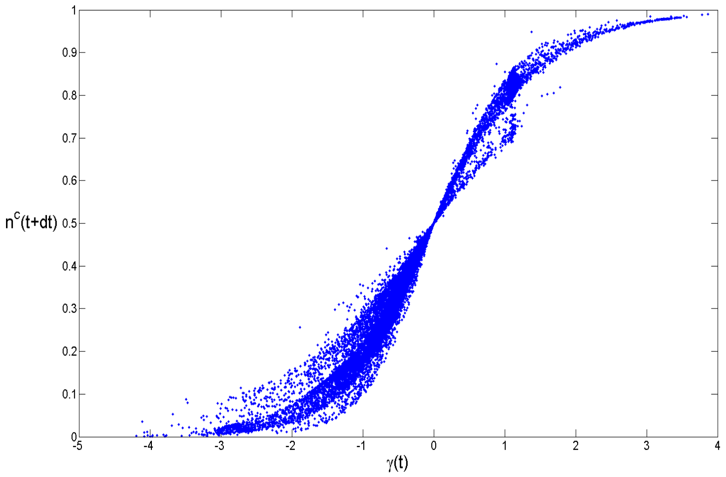

Figure 8: Scatter plot of γ and the proportion of chartists generated by the maximum entropy solution.

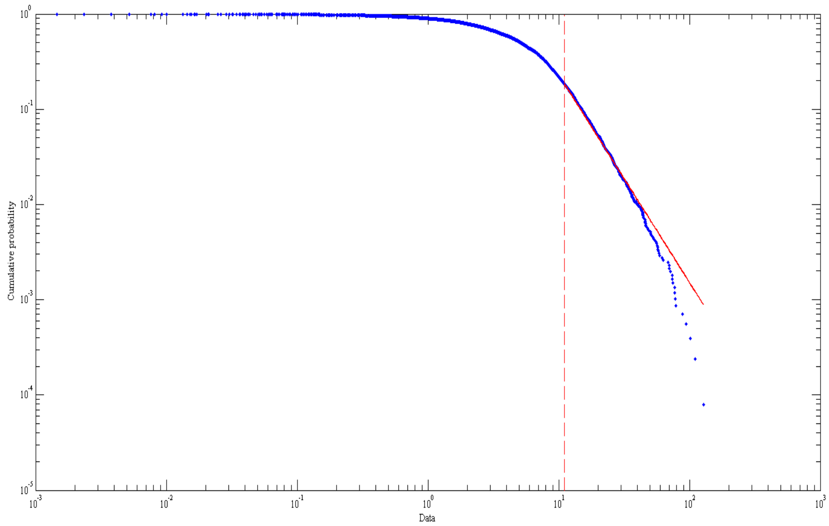

Figure 9: Cumulative probability of positive returns with power law fit for upper tail. Maximum entropy solution.

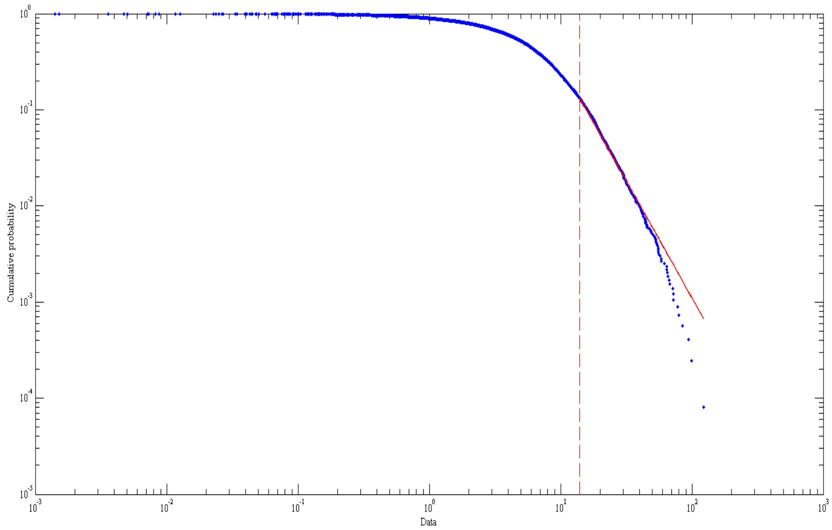

Figure 10: Cumulative probability of negative returns with power law fit for upper tail. Maximum entropy solution.

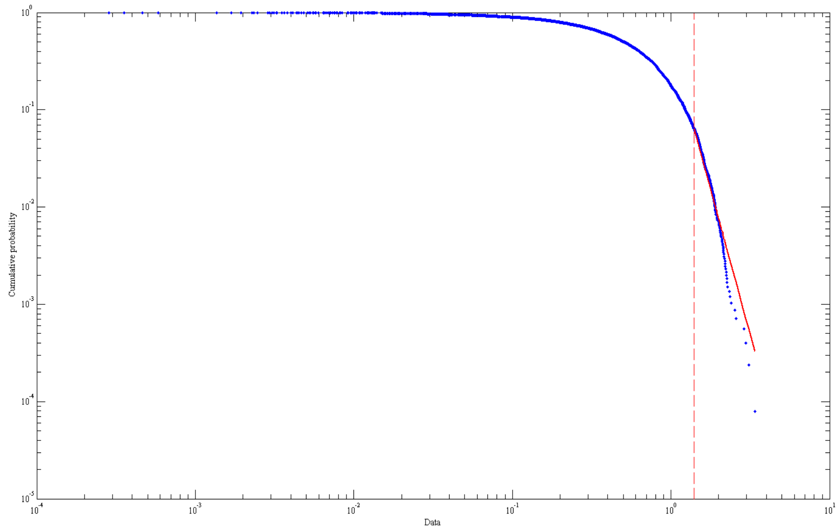

Figure 11: Cumulative probability of positive returns with power law fit for upper tail. Master equation solution.

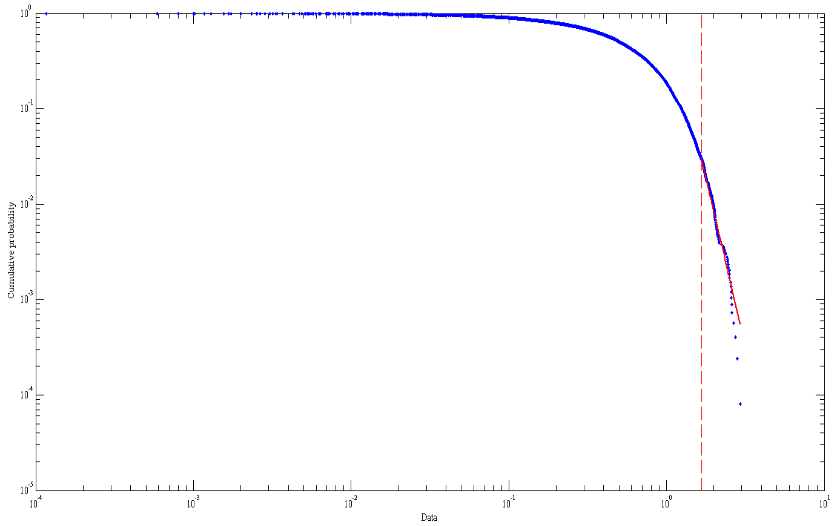

Figure 12: Cumulative probability of negative returns with power law fit for upper tail. Master equation solution.

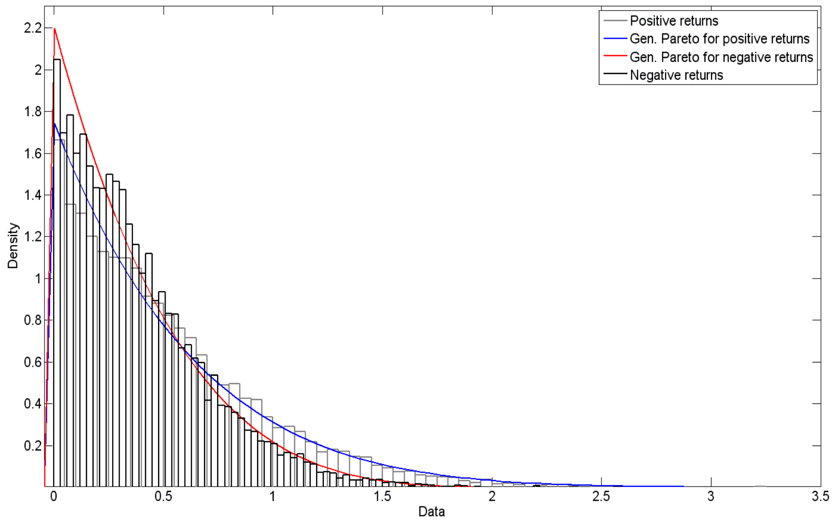

Figure 13: Probability density of returns with generalised Pareto fit. Master equation solution.

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