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Heterogeneous beliefs and adaptive behaviour in a continuous-time asset price model

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HETEROGENEOUS BELIEFS AND ADAPTIVE BEHAVIOUR IN A CONTINUOUS-TIME ASSET PRICE MODEL

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Abstract. This paper extends the analysis of the seminal paper of Brock and Hommes (1998) on heterogeneous beliefs and routes to chaos in a simple asset price model in discrete-time to a model in continuous-time. The resulting model characterized mathematically by a system of stochastic delay differential equations provides a unified approach to deal with adaptive behaviour of heterogeneous agents and stability impact of lagged price information used by chartists to form their expectations. For the underlying deterministic model, we show not only that the result of Brock and Hommes on rational routes to market instability in discrete-time holds in continuous time but also a double edged effect of an increase in lagged price information used by the chartists on market stability. For the stochastic model, we demonstrate that the model is able to display various market phenomena such as bubbles and crashes and replicate stylized facts including volatility clustering, and long range dependence in volatility.

Key words: Heterogeneous beliefs, bounded rationality, adaptiveness, fundamentalists, chartists, stability, stochastic delay differential equations.

JEL Classification: G12, G14, E32

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1. Introduction

It is well recognized that the traditional view of homogeneity and perfect rationality in financial markets faces a number of limitations theoretically and challenges empirically. Over the last two decades, there is a growing research on heterogeneity and bounded rationality in financial markets. With different groups of traders having different expectations about future prices, asset price fluctuations can be caused by an endogenous mechanism. For instance, by considering two types of traders, typically fundamentalists and chartists, Beja and Goldman (1980), Day and Huang (1990), Chiarella (1992) and Lux (1995) among many others have shown that interaction of agents with heterogeneous expectations may lead to market instability. More significantly, Brock and Hommes (1997, 1998) introduce the concept of an adaptively rational equilibrium. A key aspect of their models is that they exhibit expectations feedback. Agents adapt their beliefs over time by choosing from different predictors or expectation functions based upon their past performance (such as realized profits). They show that such bounded rational behaviour of agents can lead to market instability and the resulting nonlinear dynamical system is capable of generating complex behaviour from local staibility to high order cycles and chaos as the intensity of choice to switch predictors increases.

Following the seminal work of Brock and Hommes, various heterogeneous agent models (HAMs) have been developed to incorporate adaptation, evolution, heterogeneity, wealth effect, and even learning with both Walrasian and market maker market clearing scenarios. For example, Chiarella and He (2001), Chiarella et al (2002) and Anufriev and Dindo (2010) consider both asset price and wealth dynamics with heterogeneous beliefs; Farmer and Joshi (2002) and Chiarella and He (2003*b*) extend the Brock and Hommes's framework to a market maker scenario; Chiarella and He (2002, 2003*a*) consider the impact of heterogeneous risk aversion and learning; Chiarella et al (2006) examine the dynamics of moving averages (MA); Westerhoff (2004), Chiarella et al (2005) and Westerhoff and Dieci (2006) show that complex price dynamics may also result within a multi-asset market framework. Those models have successfully explained many market behaviour (such as market booms and crashes, long deviations of the market price from the fundamental price), the stylized facts (such as skewness, kurtosis, volatility clustering and fat tails of returns), and various power laws behaviour (such as the long range dependence in return volatility[1]) observed in financial markets. We refer the reader to Hommes (2006), LeBaron (2006), Chiarella et al. (2009), Lux (2009), and Chen et al. (2011) for surveys of the recent development in this literature.

The framework of Brock and Hommes and its various extension are in a discrete-time setup. The setup facilities economic understanding of the role of heterogeneous expectations and mathematical analysis, it however faces a limitation when dealing with expectations formed from lagged prices over different time horizons. In discrete-time model, different time horizon used to form the expectation or trading strategy leads to different dimensions of the systems which need to be analyzed individually. In particular, when the time horizon of historical information used is long, the resulting models are high dimensional systems. For example, to examine the role of different moving average rules used by chartists on market stability, Chiarella *et al* (2006) propose a discrete-time HAM whose dimension depends on the time horizon of chartists used in moving average. Very often, a theoretical analysis of the impact of lagged prices over different time horizon is difficult when the dimension of the system is high. This paper aims to overcome this challenge in discrete-time HAMs by developing a HAM in a continuous-time setup in which the time horizon of historical price information used by chartists is simply presented by a time delay. The resulting model is characterized mathematically by a system of delay differential equations. It provides an uniform treatment on various time horizons used in the discrete-time model. Development of deterministic delay differential equation models to characterize fluctuation of commodity prices and cyclic economic behavior has a long historical, see, for example, Haldane (1932), Kalecki (1935), Goodwin (1951), Larson (1964), Howroyd and Russell (1984) and Mackey (1989). The development further leads to the studies on the effect of policy lag on macroeconomic stability, see,

[1]For example, Alfarano, Lux and Wagner (2005), Gaunersdorfer and Hommes (2007) and He and Li (2007) have provided some insight into the underlying mechanism on volatility clustering and long range dependence in volatility.

for example, Phillips (1954, 1957), Yoshida and Asada (2007), and on neoclassical growth model in Matsumoto (2011).

Within the proposed model, this paper has three aims. The first is to examine if the result of Brock and Hommes (1998) on rational routes to market instability still holds in a continuous-time setup. The second is to study the effect of an increase of time horizon on market stability. The third is to explore potential of the model to replicate those market behaviour, stylized facts and long range dependent observed in financial markets. The model developed in this paper extends a recent development of HAMs in continuous time in He *et al* (2009) and He and Zheng (2010), which have exhibited a powerful advantage to accommodate different time horizons used by chartists. However, in order to focus the analysis on the roles of time horizons, they do not consider adaptive behaviour of agents. In this paper, we follow Brock and Hommes (1998) to introduce adaptive behaviour of agents who switch their strategies in a boundedly rational way according to some 'performance' or 'fitness' measure such as cumulated profits of strategies over a past time horizon. For the corresponding deterministic model, we first show that the result of Brock and Hommes on rational routes to market instability in discrete-time holds in continuous time. That is, adaptive switching behaviour of agents can lead to market instability as the switching intensity increases. We then show a double edged effect of an increase in the lagged price information used by the chartists on market stability, meaning that an increase in time delay can not only destabilize the market but also stabilize the market, a very different feature of the continuous time HAM from discrete time HAMs. Finally, by including noise agents in the market and imposing a stochastic process on fundamental prices, we demonstrate that the model is able to generate many market phenomena, such as long deviations of the market price from the fundamental price, bubbles, crashes, and the stylized facts, including non-normality in asset return, volatility clustering, and long range dependence of high-frequency returns, observed in financial markets.

The paper is organized as follows. We first introduce a stochastic HAM in continuous time with heterogeneous agents who are allowed to switch among two types of strategies, fundamentalists and chartists, based on a performance measure of accumulated profits of the strategies in Section 2. In Section 3, we apply stability and bifurcation theory of delay differential equations, together with numerical analysis of the nonlinear system, to examine the impact of switching and time horizon used by the chartists on market stability. Section 4 provides some numerical simulation results of the stochastic model in exploring the potential of the model to generate various market behavior and the stylized facts. Section 5 concludes.

2. The Model

Consider a financial market with a risky asset (such as stock market index) and let $P(t)$ denote the (cum dividend) price per share of the risky asset at time t. The modelling of the dynamics of the risky asset follows closely to the current HAM framework of Brock and Hommes (1998). However, instead of using a discrete-time setup and Walrasian scenario, we consider a continuous-time setup and a market maker scenario (as in Beja and Goldman (1980) and Chiarella and He (2003*b*)). The market consists of fundamentalists who trade according to fundamental analysis, chartists who trade based on price trend calculated from weighted moving averages of historical prices over a time horizon, and a market maker who clears the market by providing liquidity. The behaviour of the fundamentalists and chartists is modelled as in He *et al* (2009) and He and Zheng (2010). For completeness, we introduce the demand functions of the fundamentalists and chartists briefly and refer the reader to He *et al* (2009) and He and Zheng (2010) for details.

The fundamentalists believe that the market price $P(t)$ is mean-reverting to the fundamental price $F(t)$ that they can estimate based on various types of fundamental information. They buy the stock when the current price $P(t)$ is below the fundamental price $F(t)$ and sell the stock when $P(t)$ is above $F(t)$. For simplicity, the demand of the fundamentalists, $Z_f(t)$ at time t, is assumed to be proportional to the price deviation from the fundamental price, namely,

$$Z_f(t) = \beta_f[F(t) - P(t)], \tag{2.1}$$

where $\beta_f > 0$ is a constant parameter, measuring the mean-reverting speed of the market price to the fundamental price, which may be weighted by the risk aversion coefficient of the fundamentalists. For simplicity, we assume that the fundamental price is given by an exogenous random process to be specified in Section 4.

The chartists are modelled as trend followers. They believe that the future market price follows a price trend $u(t)$. When the current price is above the trend, the trend followers believe the price will rise and they like to hold a long position of the risky asset; otherwise, the trend followers take a short position. In this paper we assume that the demand of the chartists is given by

$$Z_c(t) = \tanh\big(\beta_c[P(t) - u(t)]\big). \tag{2.2}$$

The S-shaped demand function capturing the trend following behavior is well documented in the HAM literature (see, for example, Chiarella et al. (2009)), where the parameter β_c represents the extrapolation rate of the trend followers on the future price trend when the price deviation from the trend is small. Among various price trends used in practice, in this paper, we assume that the price trend $u(t)$ of the trend followers at time t is calculated by an exponentially decaying weighted average of historical prices over a time interval $[t-\tau, t]$,

$$u(t) = \frac{k}{1 - e^{-k\tau}} \int_{t-\tau}^{t} e^{-k(t-s)} P(s) ds, \tag{2.3}$$

where time delay $\tau \in (0, \infty)$ represents a memory length used to calculate the price trend, $k > 0$ measures the decay rate of the weights on the historical prices. Equation (2.3) implies that, when forming the price trend, the trend followers believe the more recent prices contain more information about the future price movement so that the weights associated to the historical prices decay exponentially with a decay rate k. In particular, when $k \to 0$, the weights are equal and the price trend $u(t)$ in equation (2.3) is simply given by the standard MA with equal weights,

$$u(t) = \frac{1}{\tau} \int_{t-\tau}^{t} P(s) ds. \tag{2.4}$$

When $k \to \infty$, all the weights go to the current price so that $u(t) \to P(t)$. For the time delay, when $\tau \to 0$, the trend followers regard the current price as their price trend. When $\tau \to \infty$, the trend followers use all the historical prices (with infinite memory length) to form the price trend

$$u(t) = \frac{1}{k} \int_{-\infty}^{t} e^{-k(t-s)} P(s) ds. \tag{2.5}$$

In general, for $0 < k < \infty$, equation (2.3) can be expressed as a delay differential equation with time delay τ

$$du(t) = \frac{k}{1 - e^{-k\tau}} \big[P(t) - e^{-k\tau} P(t - \tau) - (1 - e^{-k\tau}) u(t)\big] dt. \tag{2.6}$$

In the spirit of Brock and Hommes (1997, 1998) and Chiarella *et al*, (2006), we now introduce the evolution of market population of agents by assuming that agents can switch their strategies based on some fitness measure. Let $N_f(t)$ and $N_c(t)$ be the number of agents who use the fundamental and chartist strategies, respectively, at time t. Assume that market population of agents $N_f(t)+N_c(t) = N$ is a constant. Denote

$$n_f(t) = N_f(t)/N, \qquad n_c(t) = N_c(t)/N. \tag{2.7}$$

Then $n_f(t)$ and $n_c(t)$ satisfying $n_f(t) + n_c(t) = 1$ for all t represent the market fractions of agents who use the fundamental and chartist strategies, respectively. The net profits of the fundamental and chartist strategies over a short time interval $[t - dt, t]$ can be measured by

$$\pi_f(t)dt = Z_f(t)dP(t) - C_f dt, \qquad \pi_c(t)dt = Z_c(t)dP(t) - C_c dt,$$

where $C_f,\ C_c \geq 0$ are constant costs of the strategies. To measure performance of the strategies, we introduce a cumulated profit over a time interval[2] $[t - \tau, t]$ by

$$U_i(t) = \frac{\eta}{1 - e^{-\eta\tau}} \int_{t-\tau}^{t} e^{-\eta(t-s)} \pi_i(s) ds, \qquad i = f, c, \tag{2.8}$$

where $\eta > 0$ represents a decay parameter of the historical profits. That is the performance is defined by a cumulated net profit of the strategy decaying exponentially over a past time interval. Consequently,

$$dU_i(t) = \eta \left[\frac{\pi_i(t) - e^{-\eta\tau} \pi_i(t - \tau)}{1 - e^{-\eta\tau}} - U_i(t) \right] dt, \qquad i = f, c. \tag{2.9}$$

[2]The time delay used to calculate the accumulated net profits can be different from the delay used by the chartists to calculate the price trend in general. For simplicity, we assume both delays are the same in this paper.

Following Hofbauer and Sigmund (1998) (Chapter 7), the evolution dynamics of the market populations are governed by

$$dn_i(t) = \beta n_i(t)[dU_i(t) - d\bar{U}(t)], \qquad i = f, c, \tag{2.10}$$

where

$$d\bar{U}(t) = n_f(t)dU_f(t) + n_c(t)dU_c(t)$$

is the average performance of the two strategies and $\beta > 0$ is a constant, measuring the intensity of choice. In particular, if $\beta = 0$, there is no switching between strategies, while for $\beta \to \infty$ all agents immediately switch to the better strategy. For $0 < \beta < \infty$, agents change their strategy to a better perform strategy and the speed of changing is measured by the switching intensity parameter.

The above switching mechanism in continuous-time setup is consistent with the one used in discrete-time HAMs. In fact, it can be verified that the dynamics of the market fraction $n_f(t)$ satisfies

$$dn_f(t) = \beta n_f(t)(1 - n_f(t))[dU_f(t) - dU_c(t)], \tag{2.11}$$

leading to

$$n_f(t) = \frac{e^{\beta U_f(t)}}{e^{\beta U_f(t)} + e^{\beta U_c(t)}}, \tag{2.12}$$

which is the discrete choice model used in Brock and Hommes (1998). In addition, when $\tau \to 0$, $U_i(t) \approx \pi_i(t)$ for $i = f, c$, defining the performance by the current profit. When $\tau \to \infty$, $U_i(t + dt) \approx U_i(t) + \delta\pi_i(t)$ with $\delta = \eta dt$ and $i = f, c$, defining the performance as cumulated historical profits that decay geometrically at a rate of δ.

Finally, the price $P(t)$ at time t is adjusted by the market maker according to the aggregate market excess demand, that is,

$$dP(t) = \mu\big[n_f(t)Z_f(t) + n_c(t)Z_c(t)\big]dt + \sigma_M dW_M(t),$$

where $\mu > 0$ represents the speed of the price adjustment by the market maker, $W_M(t)$ is a standard Wiener process capturing the random excess demand process either driven by unexpected market news or noise agents, and $\sigma_M > 0$ is a constant.

To sum up, by letting $U(t) = U_f(t) - U_c(t)$, $\pi(t) = \pi_f(t) - \pi_c(t)$ and $C = C_f - C_c$, the market price of the risky asset is determined according to the following stochastic delay differential system

$$\begin{cases} dP(t) = \mu\Big[n_f(t)Z_f(t) + \big(1 - n_f(t)\big)Z_c(t)\Big]dt + \sigma_M dW_M(t), \\ du(t) = \dfrac{k}{1 - e^{-k\tau}}\Big[P(t) - e^{-k\tau}P(t-\tau) - (1 - e^{-k\tau})u(t)\Big]dt, \\ dU(t) = \dfrac{\eta}{1 - e^{-\eta\tau}}\Big[\pi(t) - e^{-\eta\tau}\pi(t-\tau) - (1 - e^{-\eta\tau})U(t)\Big]dt, \end{cases} \tag{2.13}$$

where

$$n_f(t) = \frac{1}{1 + e^{-\beta U(t)}}, \qquad Z_f(t) = \beta_f\big(F(t) - P(t)\big), \qquad Z_c(t) = \tanh\Big[\beta_c\big(P(t) - u(t)\big)\Big],$$
$$\pi(t) = \mu\Big[n_f(t)Z_f(t) + \big(1 - n_f(t)\big)Z_c(t)\Big]\big[Z_f(t) - Z_c(t)\big] - C.$$

In summary, we have extended the adaptive heterogeneous belief model of asset price in discrete time to an adaptive heterogeneous belief model in continuous time in this section. The resulting model is characterized by a three dimensional systems of nonlinear stochastic delay differential equations, which can be difficult to analyze directly. To understand the interaction of the deterministic dynamics and noise process, we first study the dynamics of the corresponding deterministic model in in Section 3. The stochastic model (2.13) is then analyzed in Section 4. Section 5 concludes.

3. Dynamics of the deterministic delay model

By assuming that the fundamental price is a constant $F(t) \equiv \bar{F}$ and there is no market noise $\sigma_M = 0$, the system (2.13) becomes a deterministic delay differential system

$$\begin{cases} \dfrac{dP}{dt} = \mu\Big[n_f(t)\beta_f(\bar{F} - P(t)) + \big(1 - n_f(t)\big)\tanh\Big(\beta_c\big(P(t) - u(t)\big)\Big)\Big], \\ \dfrac{du}{dt} = \dfrac{k}{1 - e^{-k\tau}}\Big[P(t) - e^{-k\tau}P(t-\tau) - (1 - e^{-k\tau})u(t)\Big], \\ \dfrac{dU}{dt} = \dfrac{\eta}{1 - e^{-\eta\tau}}\Big[\pi(t) - e^{-\eta\tau}\pi(t-\tau) - (1 - e^{-\eta\tau})U(t)\Big]. \end{cases} \tag{3.1}$$

It is easy to see that $(P, u, U) = (\bar{F}, \bar{F}, -C)$ is a unique steady state of (3.1), which consists of the constant fundamental price and the strategies cost disparity. We therefore call $(P, u, U) = (\bar{F}, \bar{F}, -C)$ the *fundamental-steady-state*. In this section, we study the dynamics of the deterministic model (3.1), including the stability and bifurcation of the fundamental steady state.

At the fundamental steady state, the market fractions of fundamentalists and chartists become $n_f^* := 1/(1 + e^{\beta C})$ and $n_c^* := 1/(1 + e^{-\beta C})$ respectively. Obviously, when $C = 0$, $n_f^* = n_c^* = 0.5$, meaning that the market fractions at the steady state is independent of the switching intensity parameter β. However, if it costs agents more to use fundamentalists' strategy, that is $C > 0$, then there are more chartists than fundamentalists at the steady state, that is $n_c^* \geq n_f^*$. Furthermore, when $C > 0$, an increase in β leads to a decrease in the market fraction n_f^* of the fundamentalists at the steady state.

It is known (see Gopalsamy (1992)) that[3] the stability is characterized by the eigenvalues of the characteristic equation of the system at the steady state. Denote $\gamma_f = \mu n_f^* \beta_f$, $\gamma_c = \mu(1 - n_f^*)\beta_c$. Then the characteristic equation of the system (3.1) at the fundamental steady state $(P, u, U) = (\bar{F}, \bar{F}, -C)$ is given by

$$\Delta(\lambda) = (\lambda + \eta)\widetilde{\Delta}(\lambda) = 0, \tag{3.2}$$

where

$$\widetilde{\Delta}(\lambda) = \lambda^2 + (k + \gamma_f - \gamma_c)\lambda + k\gamma_f - k\gamma_c + \frac{k\gamma_c}{1 - e^{-k\tau}} - \frac{k\gamma_c e^{-(\lambda+k)\tau}}{1 - e^{-k\tau}}. \tag{3.3}$$

Note that equation (3.3) has the same form of the characteristic equation of the model studied in He *et al* (2009) except that γ_f and γ_c are defined differently. Hence we can apply Theorems 3.2, 3.3 and 3.4 in He *et al* (2009) to system (3.1). For completeness, we summarize the result as follows and refer the details to He *et al* (2009).

First, the stability of the fundamental steady state do not change for time delay $\tau > \tilde{\tau}$, where

$$\widetilde{\tau} = \frac{1}{k} \ln \left[1 + \frac{2k\gamma_c}{(k + \gamma_f - \gamma_c)^2 + 2 \mid k + \gamma_f - \gamma_c \mid \sqrt{k\gamma_c}} \right].$$

[3]For a general theory of delay differential equations, we refer the reader to Hale (1997).

That is, there is an upper bound (which can be either bounded or unbounded) on the time delay for stability change. Secondly, the change in stability and bifurcation values in time delay are associated with the zero solutions of functions $S_n^{\pm}$ defined by

$$S_n^{\pm}(\tau) = \tau - \frac{\theta_{\pm}(\tau) + 2n\pi}{\omega_{\pm}(\tau)},$$

where

$$\omega_{\pm} = \left(\frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}\right)^{\frac{1}{2}}, \quad \theta_{\pm}(\tau) = \begin{cases} \arccos(a_{4\pm}), & \text{for } a_{3\pm} \geq 0; \\ 2\pi + \arcsin(a_{3\pm}), & \text{for } a_{3\pm} < 0, \ a_{4\pm} \geq 0; \\ 2\pi - \arccos(a_{4\pm}), & \text{for } a_{3\pm} < 0, \ a_{4\pm} < 0 \end{cases}$$

and

$$a_1 = k^2 + \gamma_f^2 + \gamma_c^2 - 2\gamma_f\gamma_c - \frac{2k\gamma_c}{1 - e^{-k\tau}}, \qquad a_2 = k^2\gamma_f^2 + \frac{2k^2\gamma_f\gamma_c e^{-k\tau}}{1 - e^{-k\tau}},$$

$$a_{3\pm} = \frac{-\omega_{\pm}(\tau)(1 - e^{-k\tau})(k + \gamma_f - \gamma_c)}{k\gamma_c e^{-k\tau}}, \qquad a_{4\pm} = 1 - \frac{(1 - e^{-k\tau})(\omega_{\pm}^2(\tau) - k\gamma_f)}{k\gamma_c e^{-k\tau}}.$$

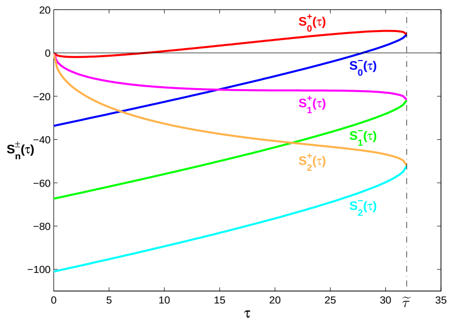
Denote $\{\tau_i \mid \tau_i < \tau_{i+1}, i = 0, 1, 2, \dots i_0\}$ to be the set of all zeros of $S_n^{\pm}(\tau)$ satisfying $\frac{dS_n^{\pm}(\tau_i)}{d\tau} \neq 0$. Then the local stability and bifurcation of the steady state with respect to the time delay of system (3.1) is summarized in the following proposition.

Proposition 3.1. *Assume $k \neq \gamma_c - \gamma_f$. If $2k\gamma_c < k^2 + (\gamma_f - \gamma_c)^2$, then the fundamental steady state $(\bar{F}, \bar{F}, -C)$ of system* (3.1) *is asymptotically stable for $\tau \geq -\frac{1}{k}\ln\left[1 - \frac{2k\gamma_c}{k^2 + (\gamma_f - \gamma_c)^2}\right]$. Otherwise,*

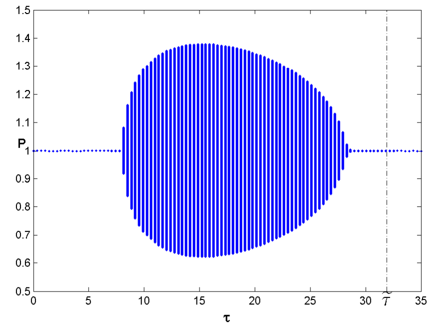
(i) *if function $S_0^{\pm}(\tau)$ has no positive zero on $I = (0, \tilde{\tau})$, then the fundamental steady state $(\bar{F}, \bar{F}, -C)$ is asymptotically stable for all $\tau > 0$;*

(ii) *if function $S_n^{+}(\tau)$ or $S_n^{-}(\tau)$ has positive zeros on I for some $n \in N_0$, then the fundamental steady state $(\bar{F}, \bar{F}, -C)$ is asymptotically stable for $0 < \tau < \tau_0$ and becomes unstable for τ staying in a right neighborhood of τ_0. In addition, system* (3.1) *undergoes a Hopf bifurcation when $\tau = \tau_i, i = 0, 1, 2, \dots i_0$.*

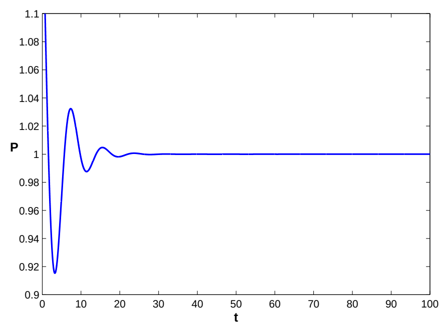
Intuitively, Proposition 3.1 implies that the steady state is stable for either small or large time delay when the market is dominated by the fundamentalists measured by γ_f and the decay parameter k. Otherwise, the steady state becomes unstable through Hopf bifurcations when time delay increases. This result is in line with the results obtained in discrete-time HAMs. However, Proposition 3.1 also indicates



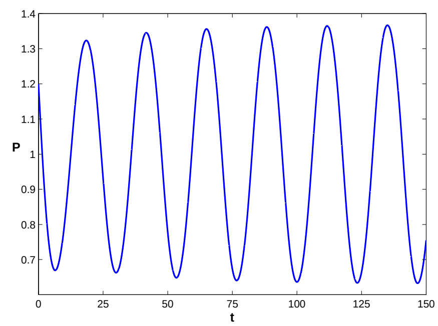
(a) Function $S_n^{\pm}$



(b) Price bifurcation



(c) $\tau = 3$

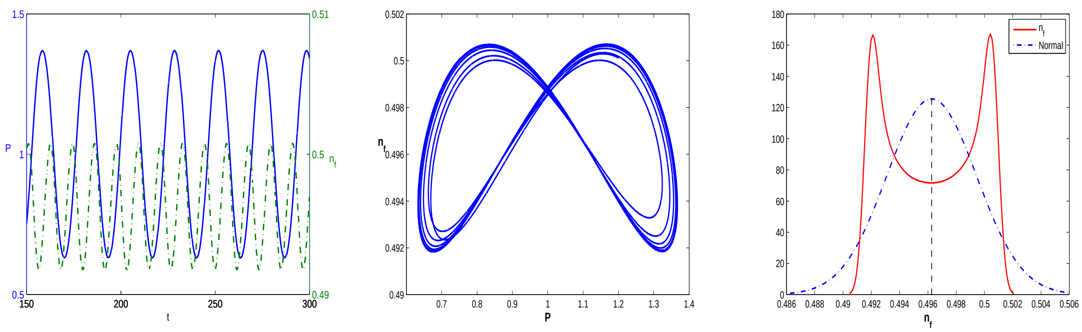


(d) $\tau = 16$

FIGURE 3.1. (a) The plot of $S_n^{\pm}$ as functions of τ; (b) the corresponding bifurcation diagram of the market price with respect to τ; and the market price for (c) $\tau = 3$ and (d) $\tau = 16$. Here $k = 0.05$, $\mu = 1$, $\beta_f = 1.4$, $\beta_c = 1.4$, $\eta = 0.5$, $\beta = 0.5$, $C = 0.02$ and $\bar{F} = 1$.

a very interesting phenomena of the continuous-time model that is not easy to obtain in discrete-time model, which is the stability switching. That is, the system becomes unstable as time delay increases initially, but the stability can be recovered when the time delay increases further. Fig. 3.1 illustrates such interesting stability switching phenomena. Fig. 3.1 (a) indicates two Hopf bifurcation values in τ, say $\tau_0 < \tau_1$, determined by two zero roots of $S_0^{\pm}$. The first one occurs when $S_0^+(\tau)$ crosses 0 at $\tau = \tau_0 \approx 8.2369$ and the second one occurs when $S_0^-(\tau)$ crosses 0 at $\tau = \tau_1 \approx 27.8874$. Fig. 3.1 (b) plots the corresponding bifurcation diagram of the market price with respect to τ showing that the fundamental steady state is stable for $\tau \in [0, \tau_0) \cup (\tau_1, \infty)$ and Hopf bifurcations occur at $\tau = \tau_0$ and $\tau = \tau_1$. Figs 3.1 (c) and (d) illustrate that the fundamental steady state is asymptotically stable for

$\tau = 3\,(< \tau_0)$ and unstable for $\tau = 16\,(\in (\tau_0, \tau_1))$. Numerical simulations for $\tau > \tau_1$ (not reported here) verify the stability of the steady state. The difference of the stability between small τ ($\tau < \tau_0$) and large τ ($\tau > \tau_1$) is that the speed of the convergence is high for small delay and low for large delay. We can see that it is the continuous-time model that facilitates such analysis on the stability effect of lagged price information and stability switching, an advantage of continuous-time model over discrete-time model.



(a) Plot of P (b) Phase plot of $P - n_f$ (c) The density of n_f

FIGURE 3.2. (a) The plots of market price P (solid line) and the market fraction $n_f(t)$ of fundamentalists (dotted line); (b) the phase plot of $(P(t), n(t))$; (c) the density distribution of the market fraction n_f of fundamentalists. Here $k = 0.05$, $\mu = 1$, $\beta_f = 1.4$, $\beta_c = 1.4$, $\eta = 0.5$, $\beta = 0.5$, $C = 0.02$, $\bar{F} = 1$ and $\tau = 16$.

When the steady state becomes unstable, it bifurcates stable periodic solutions through a Hopf bifurcation. The periodic fluctuations of the market prices are associated with periodic fluctuations of market fractions. To illustrate this feature, Fig. 3.2 provides plots of time series of price $P(t)$ and the market fraction of the fundamentalists $n_f(t)$, a phase plot of price, and the distribution of market fraction n_f of the fundamentalists for time delay $\tau = 16$. Based on the bifurcation diagram in Fig. 3.1 (b), the steady state is unstable for $\tau = 16$. Figs 3.2 (a) show that both price and market fraction fluctuate periodically. The phase plot in Fig. 3.2 (b) shows that price and fraction converge to a *figure-eight shaped* attractor, a phenomenon which is also observed in the discrete-time model in Chiarella *et al* (2006). More interestingly, the period of the fluctuation of the market price is twice as much as that of the market fraction. The corresponding distribution of the market fraction $n_f(t)$ of the fundamentalists illustrated in Fig. 3.2 (c) show clearly the switching of agents' trading strategies over the time.

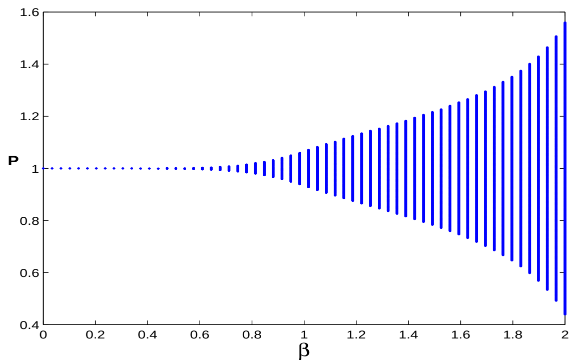
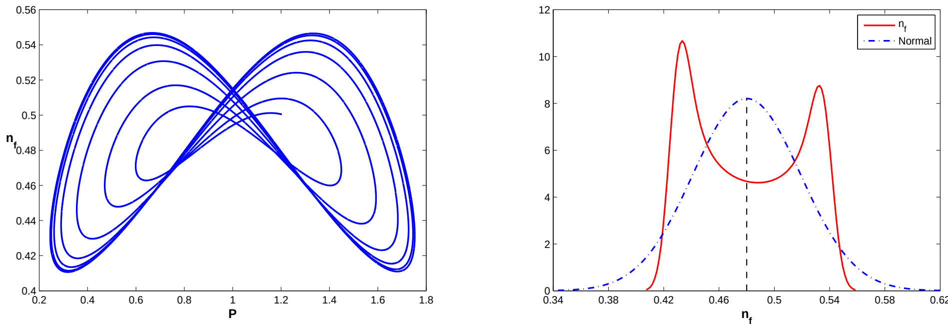


FIGURE 3.3. The bifurcation diagram with respect to β. Here $k = 0.05$, $\mu = 1$, $\beta_f = 1.4$, $\beta_c = 1.4$, $\eta = 0.5$, $C = 0.02$, $\bar{F} = 1$ and $\tau = 8$.

In the discrete-time Brock and Hommes framework, the rational routes to complicated price dynamics are characterized by the complicated bifurcations as the switching intensity β increases. For the continuous time model developed in this paper, this result also holds. In Fig. 3.3, we plot the price bifurcation diagram with respect to the switching intensity parameter β. It shows that the fundamental steady state is stable when the switching intensity β is low. It then becomes unstable as the switching intensity increases, bifurcating to periodic price with increasing fluctuations.

To better understand the impact of the switching intensity when the steady state becomes unstable, we provide a numerical simulation result for $\beta = 2$ in Fig. 3.4. Fig. 3.4 (a) show a similar *figure-eight shaped* attractor in a phase plot of price and market fraction. Comparing to Fig. 3.2 (c), the size of the attractor increases, implying high fluctuations in the market price and fractions. The distribution plot of the market fraction of fundamentalists in Fig. 3.4 (b) show the fluctuations of the market fraction (of the fundamentalists) over interval $(41\%, 55\%)$ for high intensity $\beta = 2$ comparing to the fluctuations within the interval $(49\%, 50.2\%)$ for



(a) Phase plot of $P - n_f$

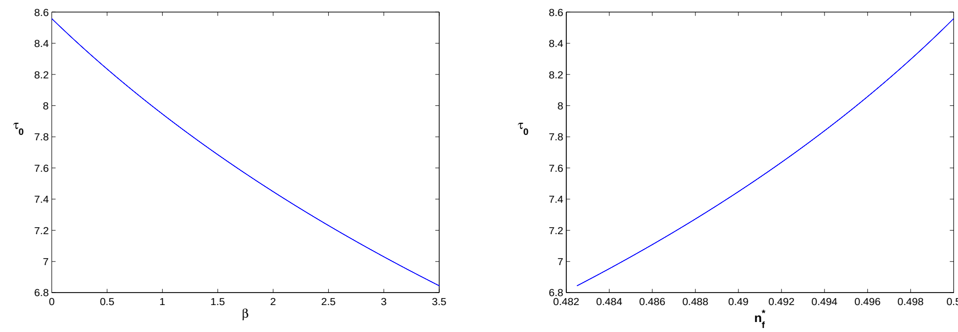
(b) The density of n_f

FIGURE 3.4. The phase plot and the corresponding density distribution of the market fraction n_f of fundamentalists. Here $k = 0.05$, $\mu = 1$, $\beta_f = 1.4$, $\beta_c = 1.4$, $\eta = 0.5$, $\beta = 2$, $C = 0.02$, $\bar{F} = 1$ and $\tau = 16$.

low intensity $\beta = 0.5$ in Fig. 3.2 (d); that is, an increase of four times in β leads to an increase of about ten times the range of fluctuations.

It is interesting to understand the joint impact of the time delay, the switching, and the steady state market fractions on market stability. We have shown that an initial increase in time delay destabilize the fundamental price. Intuitively, on the one hand, a high market fraction of the fundamentalists at the steady state stabilizes the price and hence a positive relation between the market fraction of the fundamentalists and time delay with respect to the stability of the steady state is expected. On the other hand, an increase in switching intensity destabilizes the fundamental price and hence we expect a negative relation between the switching intensity and time delay with respect to the stability of the steady state. The above intuitions have been verified in Fig. 3.5 which plots the first bifurcation value τ_0's relationship with respect to the intensity β in Fig. 3.5 (a) and the market fraction of fundamentalists n_f^* in Fig. 3.5 (b).

We complete the analysis of this section with an observation. The *twin peaks shaped* density distributions in Fig. 3.2 (d) and 3.4 (b) imply that, when the fundamental price is unstable, the market fractions tend to stay away from the steady state market fraction level most of the time and a mean of n_f below 0.5 clearly indicates the dominance of the chartist strategies.



(a) The first bifurcation value τ_0 as a function of β

(b) The first bifurcation value τ_0 as a function of n_f^*

FIGURE 3.5. The first bifurcation value τ_0's relationship with β and n_f^*. Here $k = 0.05$, $\mu = 1$, $\beta_f = 1.4$, $\beta_c = 1.4$, $C = 0.02$ and $\forall \eta \in [0, 1]$.

In summary, the analysis in this section shows that the continuous-time HAM provides a better understanding of market dynamics. Apart from providing some consistent results to the discrete-time HAMs on rational routes to market instability, we are able to study the impact of lagged prices used by the chartists on market stability. In particular, we show a double edged effect of an increases in time lag, which is characterized by stability switchings.

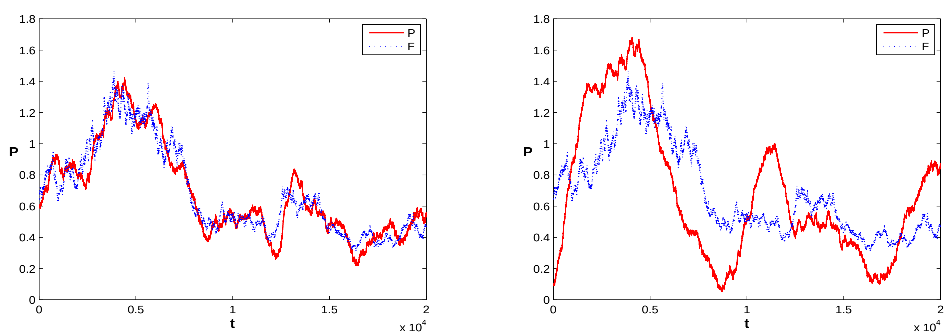
4. Price Behavior of the Stochastic Model

In this section, through numerical simulations, we focus on the interaction between the market dynamics of the deterministic model and noise processes in order to explore the potential capability of the model to generate various market behaviors, such as long deviation of the market price from the fundamental price, and the stylized facts, including volatility clustering, and long range dependence in asset returns observed in financial markets.

To complete the stochastic model (2.13), we introduce the stochastic fundamental price process

$$dF(t) = \frac{1}{2}\sigma_F^2 F(t)dt + \sigma_F F(t) dW_F(t), \;\; F(0) = \bar{F}, \tag{4.1}$$

where $\sigma_F > 0$ represents the volatility of the fundamental return and $W_F(t)$ is the standard Wiener process, which is independent of the standard Wiener process for the market noise $W_M(t)$ introduced in (2.13). The reasons for the selection of (4.1) as the fundamental price process are two folds. The first is that the fundamental price follows a non-growing random walk process, which is in line with the market price process with no growth discussed in Section 3. The second is that the fundamental return defined by $d\big(\ln(F(t))\big)$ is a pure white noise process following the normal distribution with mean of 0 and standard deviation of $\sigma_F\sqrt{dt}$. This ensures that any non-normality and volatility clustering of market returns that the model could generate are not carried from fundamental returns.



(a) $\tau = 3$ (b) $\tau = 16$

FIGURE 4.1. Time series of the fundamental price (the blue dot line) and the market price (the red solid line) for two delays (a) $\tau = 3$; (b) $\tau = 16$. Here $k = 0.05$, $\mu = 1$, $\beta_f = 1.4$, $\beta_c = 1.4$, $\eta = 0.5$, $\beta = 0.5$, $C = 0.02$, $\bar{F} = 1$, $\sigma_F = 0.12$ and $\sigma_M = 0.05$.

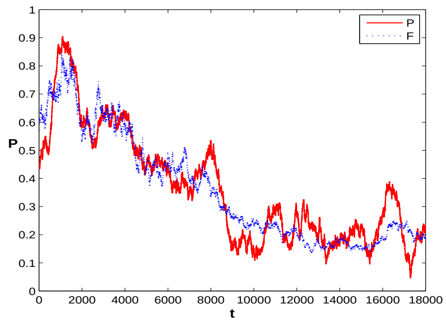
Firstly, we explore the joint impact of the time horizon, τ, of the chartist trading strategy and the two noise processes on the market price dynamics. For the corresponding deterministic model (3.1), Figs 3.1 (c) and (d) show that the fundamental steady state is stable for $\tau = 3$ and unstable for $\tau = 16$ leading to periodic fluctuations of the market price. For the stochastic model, with the same random draws of the fundamental price and the market noise processes, we plot the market price (the red solid line) and the fundamental price (the blue dot line) in Fig. 4.1 for two different values of τ. For $\tau = 3$, Fig. 4.1 (a) demonstrates that the market price follows the fundamental price closely. For $\tau = 16$, Fig. 4.1 (b) indicates that the market price fluctuates around the fundamental price in cyclic way. This demonstrate that the stochastic price behaviour is underlined by the dynamics of the corresponding deterministic model.

Secondly, we explore the potential of the stochastic model in generating the stylized facts for daily data observed in financial markets. We choose $\tau = 3$ at first so that the fundamental steady state is stable[4], as illustrated in Fig. 3.1 (c). For the stochastic model with both noise processes, Fig. 4.2 represents the results of a typical simulation where the time step is one day. Fig. 4.2 (a) shows that the market price (the red solid line) follows the fundamental price (the blue dot line) in general, but companied with large deviations from time to time. The returns of the market prices in Fig. 4.2 (b) show significant volatility clustering. Comparing to the corresponding normal distribution, the return density distribution in Fig. 4.2 (c) displays high kurtosis. The returns show almost insignificant autocorrelations (ACs) in Fig. 4.2 (d), but the ACs for the absolute returns and the squared returns in Figs. 4.2 (e)-(f) are significant with strong decaying patterns as time lag increases, leading to long range dependence. These results demonstrate that the stochastic model established in this paper has a great potential to generate most of the stylized facts observed in financial markets.

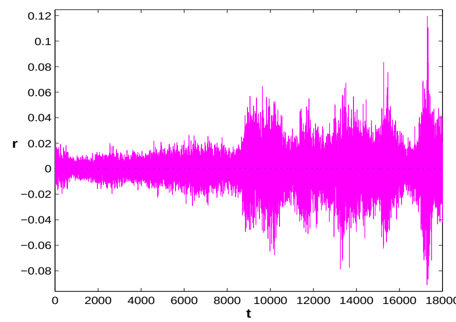
We may argue that the above features of the stochastic model is a joint outcome of the interaction of the nonlinear HAM and the two stochastic process similar to He and Zheng (2010). The numerical simulations (not reported here) when there is only one stochastic process involved show that the potential of the model in generating the stylized facts is not due to either one of the two stochastic processes, rather than to both processes. The underlying mechanism in generating the stylized facts, long range dependence, and the interplay between the nonlinear deterministic dynamics and noises are very similar to the one explored in He and Li (2007) for a discrete-time HAM.

We conclude this section with a remark on the predictability of asset returns over different trading frequency. As one of the stylized facts, the insignificant ACs of daily returns implies that daily returns are not predictable. However, it is also well documented, see for example Pesaran and Timmermann (1994, 1995), that weekly

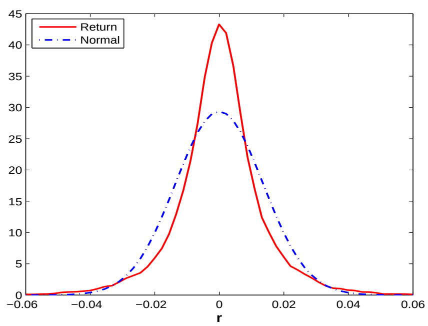
[4]We can also choose τ in its unstable interval to obtain the stylized facts.



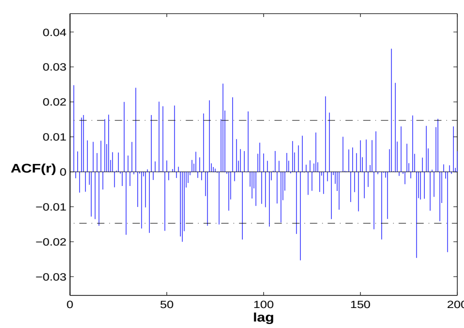
(a) The market price and the fundamental price



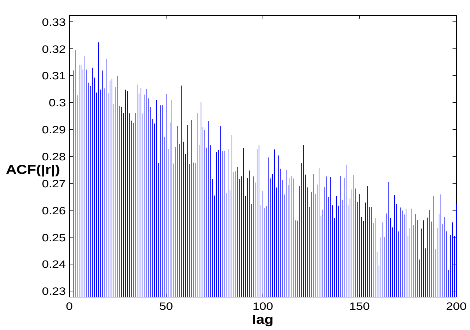
(b) The market return (r)



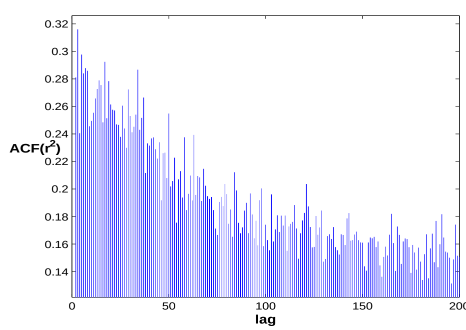
(c) The density of the market return



(d) The ACs of the market return

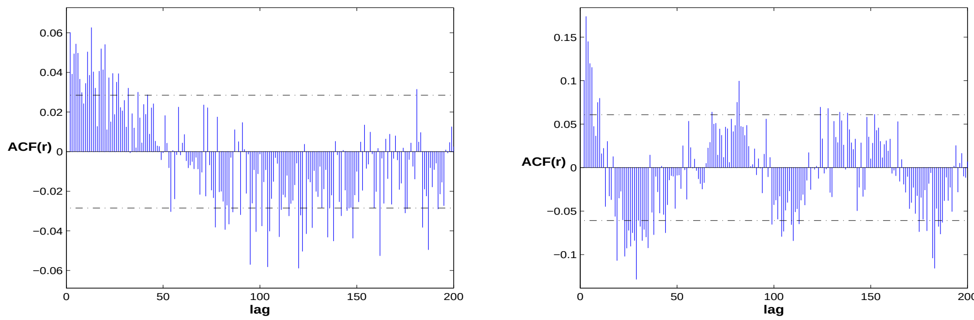


(e) The ACs of the absolute return



(f) The ACs of the squared return

FIGURE 4.2. The time series of (a) the market price (red solid line) and the fundamental price (blue dot line) and (b) the market return; (c) the density distribution of the market returns; the ACs of (d) the market returns; (e) the absolute returns, and (f) the squared returns. Here $k = 0.05$, $\mu = 1$, $\beta_f = 1.4$, $\beta_c = 1.4$, $\eta = 0.5$, $\beta = 0.5$, $C = 0.02$, $\bar{F} = 1$, $\tau = 3$, $\sigma_F = 0.12$ and $\sigma_M = 0.05$.



(a) The ACs of the weekly market return (b) The ACs of the monthly market return

Figure 4.3. The ACs of the weekly and monthly market returns. Here $k = 0.05$, $\mu = 1$, $\beta_f = 1.4$, $\beta_c = 1.4$, $\eta = 0.5$, $\beta = 0.5$, $C = 0.02$, $\bar{F} = 1$, $\tau = 3$, $\sigma_F = 0.12$ and $\sigma_M = 0.05$.

and monthly returns are predictable. Fig. 4.3 illustrates the ACs of weekly (a) and monthly (b) returns. The significant ACs indicates that weekly and monthly returns are predictable. This demonstrates that the different return predictability for different trading frequency can also be replicated in the model established in this paper.

5. Conclusion

This paper contributes to the development of financial market modelling and asset price dynamics with bounded rationality and heterogeneous agents. Most of the heterogeneous agent models developed in the literature are in discrete time setup. Among various issues in this literature, the impact of adaptive behaviour on market stability has been well studied, while the impact of lagged prices (used by chartists to form their expectations) on market stability has not been well understood due to the problem of high dimensional systems. This paper develops a continuous time framework to study the joint impact of lagged prices and adaptive behaviour of heterogeneous agents. By using the replicated dynamics in population evolution literature, we extend the discrete choice model used in discrete-time HAMs to a continuous time model. The underlying delay differential equations provide a uniform approach to study the impact of the lagged prices through a time delay parameter.

The continuous time model developed in this paper studies a financial market consisting of adaptive and heterogeneous agents using fundamentalist and chartist strategies. Agents revise their strategies in a boundedly rational way according to a performance measure of the accumulated profits. The analysis of the model provides not only some consistent results to the discrete-time HAMs, such as stabilizing effect of fundamentalists, destabilizing effect of chartists, and the rational route to market instability, but also a double edged effect of an increases in lagged prices on market stability. An increase in the using of lagged prices can not only destabilize, but also stabilize the market price. By introducing noise agents in the market and imposing a stochastic process on fundamental prices, we demonstrate that the model is able to generate long deviations of the market price from the fundamental price, bubbles, crashes, and most of the stylized facts, including non-normality in return, volatility clustering, and long range dependence of high-frequency returns, observed in financial markets.

The continuous time framework developed in this paper has shown some advantages comparing to discrete-time framework, in particular when dealing with the impact of lagged prices. The framework can be used to study the joint impact of many heterogeneous strategies based on different lagged prices on market stability. We leave this study to the future research.

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