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Markovian Defaultable HJM Term Structure Models with Unspanned Stochastic Volatility

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Abstract

This paper presents a class of defaultable term structure models within the HJM framework with stochastic volatility. By modelling the connection between default-free and defaultable term structures, namely the credit spread, a correlation structure between the credit spread, the default-free interest rate and the stochastic volatility is also accommodated. Under certain volatility specifications, the model admits finite dimensional Markovian structures and consequently provides tractable solutions for default-free and defaultable bond prices. Furthermore, a bond pricing formula is obtained in terms of market observable quantities, specifically in terms of discrete tenor defaultable forward rates. The effect of stochastic volatility and of correlations between the stochastic volatility, defaultable short rate and credit spreads on the defaultable bond prices and returns is also investigated.

Key Words: stochastic volatility, Heath-Jarrow-Morton framework, defaultable forward rates, credit spreads.

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1 Introduction

The Heath, Jarrow and Morton (1992) framework (hereafter HJM) is considered as the most general and flexible setting for the study of interest rate dynamics and the pricing of fixed income derivatives. The inputs required for the model are the currently observed forward rate curve and the volatility structure of the forward interest rates or bonds. The study of defaultable term structures within the HJM framework was first examined by Jarrow and Turnbull (1995) and Duffie and Singleton (1999). Schönbucher (1998) proposed a model for the spread of defaultable interest rates over default free interest rates that essentially adds a default risk module to a default free HJM interest rate model. The shortcoming of HJM term structure models is that they are Markovian in the entire yield curve so requiring, in principle, an infinite number of state variables. Since the initial forward rate and the initial credit spread are completely determined by the market, the only remaining flexibility for obtaining finite dimensional realisations (hereafter FDR) within the HJM framework rests in a pertinent specification of the volatility functions.

Several empirical studies support the existence of an additional source of risk in the interest rate volatility that is independent of the risk associated with the term structure, see for instance Collin-Dufresne and Goldstein (2002), Casassus, Collin-Dufresne and Goldstein (2005) and Trolle and Schwartz (2009). These empirical findings suggest the suitability of stochastic volatility term structure models within the framework of the Hull and White (1987) or Heston (1993) models. This is a so-called unspanned stochastic volatility framework, in which an additional state variable is introduced to model the stochastic volatility factor. Volatility specifications within the HJM framework in the presence of stochastic volatility that lead to a finite dimensional Markovian term structure of interest rates were introduced by Chiarella and Kwon (2000). Using a more abstract setting, Björk, Landén and Svensson (2004) provide the necessary and sufficient conditions on the volatility structure for default-free HJM models with stochastic volatility to admit FDR. To our knowledge, HJM models with unspanned stochastic volatility have only been studied in a default-free setting.

The innovation in this paper lies in the development of defaultable HJM term structure models with unspanned stochastic volatility that are finite dimensional and aim at retaining model tractability. More specifically, we introduce unspanned stochastic volatility into the general defaultable HJM framework developed by Schönbucher (1998). The proposed framework models the default time exogenously through a Poisson process and at each default time the value of the defaultable bonds alters by a fractional recovery. Then the dynamics of the defaultable interest rates are derived by assuming a diffusion model for the default-free interest rates and the credit spread. Additionally the HJM drift restriction for the forward credit spread is obtained, similar to Pugachevsky (1999), but with the difference that it accommodates stochastic volatility. The volatilities of both the default-free term structure and the forward credit spread are stochastic as they depend on a hidden Markov volatility process. The Wiener processes that determine the uncertainty in the default-free and the defaultable forward curve may be independent of the Wiener processes driving the uncertainty in the stochastic volatility process, a feature that does not usually arise within traditional defaultable HJM models.

Furthermore, we present the necessary conditions on the volatility structure that allow the proposed defaultable term structure model to admit FDR. Precisely in the spirit of Björk et al. (2004), we assume that the default-free forward rate volatility depends on a hidden Markov volatility process, the current default-free short rate and a quasi exponential time factor. Similarly, the volatility of the credit spread depends on the same hidden Markov volatility process, the current default intensity process and a quasi exponential time factor. Note that these volatility specifications are level dependent. Markovian defaultable term structure models with level dependent volatility[1] were introduced in Chiarella, Nikitopoulos and Schlögl (2007). Berndt, Ritchken and Sun (2010) recently have also considered Markovian defaultable term structure models with level dependent volatility and demonstrate the importance of the correlations between interest rates and credit spreads. The aforementioned models do not consider stochastic volatility and we aim to contribute to this issue with the present paper. Analytical expressions for the default-free and the defaultable bond prices are obtained which are exponentially affine in terms of a finite number of state variables. In order to assign some economic meaning to the state space, the state variables are expressed in terms of fixed tenor defaultable forward rates.

The model additionally preserves the following important features. Firstly, is it consistent by construction with the currently observed default-free term structure and the currently observed credit spread curve. Secondly, the volatility structures of both

[1]Conditions on level dependent volatility specifications, in a default-free HJM setting, have been extensively studied. Some works studying volatility structures for diffusion processes include Ritchken and Sankarasubramanian (1995), Bhar and Chiarella (1997), Björk and Svensson (2001), Björk and Landèn (2002) and Chiarella and Kwon (2001). Extension to volatility structures for jump-diffusion processes have been studied by Björk and Gombani (1999), Chiarella and Nikitopoulos (2003) and Filipovič, Tappe and Teichmann (2010).

default-free interest rates and credit spreads accommodate exponentially decaying functions, level dependence and stochastic volatility and they can be time homogeneous. Thirdly, by modelling the connection between default-free and defaultable term structures, namely the credit spread, we obtain an arbitrage free model[2] and we are able to accommodate a correlation structure between the credit spread and the interest rate as well as the stochastic volatility. The correlation between interest rates and their volatilities has a significant effect on the implied volatilities and some explanatory power for the implied cap skew, as has been demonstrated recently in Trolle and Schwartz (2009) in a default-free setting. On the other hand, the correlation between credit spreads and default-free interest rates has a substantial impact on the prices of options on defaultable bonds and credit default swaps as illustrated by Berndt et al. (2010) in the absence of stochastic volatility. Dai and Singleton (2003) show that affine term structure models with general volatility specifications impose unreasonable restrictions on correlation structures. Although, our model accommodates stochastic and level dependent volatility, the correlation structure remains flexible enough to capture observed features of these correlations.

Thus, the class of models considered here offers great potential for assessing and measuring the extent to which volatility is unspanned in credit risk markets. Such a model may be used to efficiently manage volatility risk. These models would also be able to capture the joint dynamics of the default-free interest rates, credit spreads and fixed income derivatives and due to its tractability, the estimation of these models should be relatively uncomplicated.

We conclude by conducting a Monte-Carlo simulation experiment to gauge the potential of the proposed model and its responsiveness to changes in the underlying correlation structure and stochastic volatility specifications.

The structure of the paper is as follows: In Section 2, we introduce stochastic volatility into the defaultable term structure model developed by Schönbucher (1998) and we generalise it to allow for a correlation structure between the default-free forward rate, the forward credit spread and stochastic volatility. In Section 3, we assume specific volatility structures and derive a Markovian representation of the default-free and the defaultable term structure in terms of a finite number of state variables. Furthermore, we express the state variables as finite dimensional realisations in terms of economic

[2] When the defaultable term structure and the default-free term structure are modelled independently then a no-arbitrage model can be obtained where negative spreads are possible, see Schönbucher (1998), Section 2.4.

quantities observed in the market, specifically in terms of discrete tenor defaultable forward rates. Section 4 presents some simulation results on the distributional properties of the defaultable bond price and bond returns. Section 5 concludes. Technical details are given in the Appendices.

2 The Model Setup

We consider the filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ where $\mathbb{P}$ is the real world probability measure and the filtration $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^N$, $t \geq 0$ satisfies the usual conditions.[3] The sub-filtration $\mathcal{F}_t^W$ is the $\sigma-$algebra generated by the standard $\mathbb{P}-$Wiener process $W(t)$,

$$(\mathcal{F}_t^W)_{t\geq 0} = \{\sigma(W(s) : 0 \leq s \leq t)\}_{t\geq 0}, \tag{2.1}$$

and represents the flow of all background information except from default itself, which generates the sub-filtration $\mathcal{F}_t^N$.

We denote as $P(t, T, \omega)$ the price at time t of the default-free zero coupon bond with maturity $T > t$. We assume a more general modelling setup, where the entire forward rate curve depends on $\omega \in \Omega$ which represents the dependence of the forward rate process on the Wiener paths. This quite general structure will allow us later on to easily introduce the uncertainty associated with stochastic volatility.

Definition 2.1 *1. The instantaneous default-free forward rate of interest prevailing at time t for instantaneous borrowing at T, is defined as*[4]

$$f(t, T, \omega) = -\frac{\partial}{\partial T} \ln P(t, T, \omega), \quad \textit{for all} \quad t \in [0, T]. \tag{2.2}$$

2. The instantaneous default-free short rate is defined as the instantaneously maturing forward rate, so that

$$r(t, \omega) = f(t, t, \omega). \tag{2.3}$$

[3]The usual conditions satisfied by a filtered complete probability space are: (a) $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$ and (b) the filtration is right continuous. See Protter (2004) for technical details.

[4]Note that, the default-free zero coupon bond price is then defined as

$$P(t, T, \omega) = \exp\Big(-\int_t^T f(t, s, \omega) ds\Big).$$

We introduce next the defaultable term structure. We denote as $P^d(t,T,\omega)$ the price at time t of the defaultable zero coupon bond with maturity $T > t$.

Definition 2.2 *1. The instantaneous defaultable forward rate at time t for instantaneous borrowing at T is defined as*

$$f^d(t,T,\omega) = -\frac{\partial}{\partial T}\ln P^d(t,T,\omega), \quad \textit{for all} \quad t \in [0,T]. \tag{2.4}$$

2. The instantaneous defaultable short rate is defined as

$$r^d(t,\omega) = f^d(t,t,\omega). \tag{2.5}$$

3. In addition, the continuously compounded instantaneous forward credit spread is defined as

$$\lambda(t,T,\omega) = f^d(t,T,\omega) - f(t,T,\omega), \tag{2.6}$$

and the instantaneous short-term credit spread is defined as

$$c(t,\omega) = \lambda(t,t,\omega). \tag{2.7}$$

The default process is modelled via a marked point process, see Jeanblanc, Yor and Chesney (2009). Let (E,ξ) be a measurable (mark) space. A random measure μ on the space $\mathbb{R}_+ \times E$ is a family of positive measures $\big(\mu(\omega; dt, dx); \omega \in \Omega\big)$ defined on $\mathbb{R}_+ \times E$ such that, for $[0,t] \times A \in \mathcal{B} \bigotimes \xi$, the map $\omega \longrightarrow \mu(\omega; [0,t], A)$ is $\mathcal{F}^N-$measurable, and $\mu(\omega; \{0\}, E) = 0$. For Borel sets $\mathcal{B}$, note that $E = [0,1]$.

Definition 2.3 *1. A marked point process N is a random sequence (with stochastic jumps) defined by the pair $\{(\tau_i, q_i), i \in \mathbb{N}\}$ with $\tau_i \in \mathbb{R}_+$ and marks $q_i := q(\tau_i) \in E$.*

2. A random measure μ is associated with the marked point process N by

$$\mu(\cdot; [0,t], A) = N_A(t),$$

such that

$$\mu(\omega; [0,t], E) = \int_0^t \int_E \mu(\omega; ds, dq) := \sum_{i=1}^{\infty} \mathbb{1}_{\{\tau_i(\omega) \le t\}} \mathbb{1}_{\{q_i(\omega) \in E\}}.$$

The measure $\mu(\omega; X \times A)$ denotes the number of arrivals during the time set $X \subset \mathbb{R}_+$ that have marks with values in the mark set $A \subset \xi$. The random measure μ is characterized by the (predictable) compensator random measure ν on the space $\mathbb{R}_+ \times E$, so that for every predictable function $g(\omega; t, q)$, the process defined by

$$M(\omega; t) = \int_0^t \int_E g(\omega; s, q)\mu(\omega; ds, dq) - \int_0^t \int_E g(\omega; s, q)\nu(\omega; ds, dq),$$

is a local martingale. The marked point process N has an intensity h, when the compensator ν has the form

$$\nu(\omega; dt, dq) = h(\omega; t, dq)dt,$$

where $h(\cdot; t, A)$ is a predictable process. The recovery rate $\mathcal{R}(t)$ given default is defined as the extent to which the value of an obligation can be recovered once the obligor has defaulted. The increasing sequence of default times $\{\tau_i\}_{i \in \mathbb{N}}$ is driven by a Cox process. At each default time τ_i, the defaultable bond's face value is reduced by the loss rate $q(\tau_i) \in [0, 1]$, which can be a random variable. At maturity T, the defaultable bond subject to multiple defaults has a final payoff

$$\mathcal{R}(T) := \prod_{\tau_i \leq T} (1 - q(\tau_i)), \tag{2.8}$$

where $\mathcal{R}(T)$ is the product of the face reductions after all defaults until maturity T. The pre-default price $\bar{P}^d(t, T, \omega)$ at time t of a defaultable zero coupon bond with maturity T, the so-called 'pseudo' bond, is given by

$$\bar{P}^d(t, T, \omega) = \exp\Big(- \int_t^T f^d(t, s, \omega) ds \Big). \tag{2.9}$$

This is the price of the defaultable zero-coupon bond given that it has not defaulted before time t. It then follows that the price of the defaultable bond can be written as

$$\begin{aligned} P^d(t, T, \omega) &= \mathcal{R}(t) \exp\Big(- \int_t^T f^d(t, s, \omega) ds \Big) \\ &= \mathcal{R}(t) \bar{P}^d(t, T, \omega). \end{aligned} \tag{2.10}$$

2.1 Embedding Stochastic Volatility within the Defaultable HJM framework

Although the volatility processes in the standard HJM framework can be path dependent, they are not considered to be stochastic in the sense of Hull and White (1987), Heston (1993) and Scott (1997). In a stochastic volatility model for interest rates, the volatility processes should be driven by Wiener processes which are independent of the Wiener processes driving the term structure of interest rates. We adapt this stochastic volatility framework to the defaultable Schönbucher (1998) term structure model, where a model for the spread between defaultable forward rates and default-free forward rates is proposed.

Assumption 2.1 *The dynamics of the stochastic volatility process* $V = \{V(t), t \in [0, T]\}$ *are*

$$dV(t) = \alpha^V(t, V)dt + \sigma^V(t, V)dW^V(t), \tag{2.11}$$

where the drift and diffusion depend only on V*.*

We assume further that for any function $g(t, T, \omega)$ there exists a function z such that $g(t, T, \omega) = \mathrm{z}(t, T, V)$. However, for notional convenience, we write $g(t, T, V)$ in place of $\mathrm{z}(t, T, V)$ from now on.

Assumption 2.2 *The instantaneous default-free forward rate* $f(t, T, V)$ *and the instantaneous forward credit spread* $\lambda(t, T, V)$ *satisfy the stochastic differential equations*

$$df(t, T, V) = \alpha^f(t, T, V)dt + \sigma^f(t, T, V)dW^f(t), \tag{2.12}$$

$$d\lambda(t, T, V) = \alpha^\lambda(t, T, V)dt + \sigma^\lambda(t, T, V)dW^\lambda(t), \tag{2.13}$$

respectively, where $W^f(t)$ *and* $W^\lambda(t)$ *are two correlated Wiener processes.*

The details on the correlation structure are given in Section 2.2. Note that we have consequently assumed that the filtration $\mathcal{F}_t^W$, see (2.1), includes $\mathcal{F}_t^W = \mathcal{F}_t^f \vee \mathcal{F}_t^\lambda \vee \mathcal{F}_t^V$, where

$$(\mathcal{F}_t^f)_{t\geq 0} = \{\sigma(W^f(s) : 0 \leq s \leq t)\}_{t\geq 0},$$
$$(\mathcal{F}_t^\lambda)_{t\geq 0} = \{\sigma(W^\lambda(s) : 0 \leq s \leq t)\}_{t\geq 0},$$
$$(\mathcal{F}_t^V)_{t\geq 0} = \{\sigma(W^V(s) : 0 \leq s \leq t)\}_{t\geq 0}.$$

By using the equivalent stochastic integral equations imposed by Assumption 2.2, the stochastic integral equations for the instantaneous default-free short rate $r(t, V) := f(t, t, V)$ and the instantaneous short-term credit spread $c(t, V) := \lambda(t, t, V)$ are given by

$$r(t, V) = f(0, t) + \int_0^t \alpha^f(u, t, V)du + \int_0^t \sigma^f(u, t, V)dW^f(u), \tag{2.14}$$

$$c(t, V) = \lambda(0, t) + \int_0^t \alpha^\lambda(u, t, V)du + \int_0^t \sigma^\lambda(u, t, V)dW^\lambda(u), \tag{2.15}$$

respectively, where for notional convenience, we have set $f(0,t) = f(0,t,V_0)$ and $\lambda(0,t) = \lambda(0,t,V_0)$ with V_0 being the initial volatility.

By using equation (2.6) and the dynamics specified in Assumption 2.2, the stochastic integral equation for the defaultable forward rate may be expressed as

$$f^d(t,T,V) = f^d(0,T) + \int_0^t \alpha^d(u,T,V)du + \int_0^t \sigma^f(u,T,V)dW^f(u) + \int_0^t \sigma^\lambda(u,T,V)dW^\lambda(u), \tag{2.16}$$

where the initial defaultable forward curve is

$$f^d(0,T) = f(0,T) + \lambda(0,T), \tag{2.17}$$

and the drift coefficient is given by the sum of the individual drift coefficients

$$\alpha^d(t,T,V) = \alpha^f(t,T,V) + \alpha^\lambda(t,T,V). \tag{2.18}$$

In addition, equation (2.16), for $T = t$, provides the dynamics for the instantaneous defaultable short rate $r^d(t,V) := f^d(t,t,V) = f(t,t,V) + \lambda(t,t,V)$ as

$$r^d(t,V) = f^d(0,t) + \int_0^t \alpha^d(u,t,V)du + \int_0^t \sigma^f(u,t,V)dW^f(u) + \int_0^t \sigma^\lambda(u,t,V)dW^\lambda(u). \tag{2.19}$$

2.2 Correlation Structure

Evidence of the effects of the correlation between stochastic volatility and the short rate on the bond price were investigated in Heston (1993). Jarrow and Turnbull (2000) showed that the correlation between the short rate and the credit spread represents an empirically relevant correlation observed between market risk and credit risk. Changes in the default free short rate force investors to reassess the probability of default of defaultable bonds and therefore change the credit spreads.

We define the correlation matrix between the Wiener processes $W^f(t)$, $W^\lambda(t)$ and $W^V(t)$ by

$$\mathbb{E}\big[(dW^V, dW^\lambda, dW^f)^\top (dW^V, dW^\lambda, dW^f)\big] = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix}, \tag{2.20}$$

where the correlation coefficients ρ_{ij} are given by

$$\rho_{12}dt = \mathbb{E}[dW^V(t) \cdot dW^\lambda(t)], \tag{2.21a}$$

$$\rho_{13}dt = \mathbb{E}[dW^V(t) \cdot dW^f(t)], \tag{2.21b}$$

$$\rho_{23}dt = \mathbb{E}[dW^\lambda(t) \cdot dW^f(t)]. \tag{2.21c}$$

To apply the techniques of the HJM approach, it is convenient to replace the correlated Wiener processes $W^f(t)$, $W^\lambda(t)$ and $W^V(t)$ with uncorrelated processes. We define the uncorrelated Wiener process $W(t) = \big(W_1(t), W_2(t), W_3(t)\big)$ under $\mathbb{P}$ such that

$$\begin{bmatrix} dW^V(t) \\ dW^\lambda(t) \\ dW^f(t) \end{bmatrix} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix} \begin{bmatrix} dW_1(t) \\ dW_2(t) \\ dW_3(t) \end{bmatrix}. \tag{2.22}$$

Note that the γ_{ij}'s are chosen such that the correlation structure of the Wiener processes $W^f(t)$, $W^\lambda(t)$ and $W^V(t)$ is preserved with

$$\sum_{k=1}^{3} \gamma_{ik}\gamma_{jk} = \rho_{ij}, \quad \text{for} \quad i < j, \quad j = 1,2,3, \quad \text{and} \quad \sum_{j=1}^{3} \gamma_{ij}^2 = 1, \quad \text{for} \quad i = 1,2,3. \tag{2.23}$$

Then, equations (2.12), (2.13) and (2.11) can be expressed in terms of independent Wiener processes as

$$df(t,T,V) = \alpha^f(t,T,V)dt + \sum_{i=1}^{3} \tilde{\sigma}_i^f(t,T,V)dW_i(t), \tag{2.24a}$$

$$d\lambda(t,T,V) = \alpha^\lambda(t,T,V)dt + \sum_{i=1}^{3} \tilde{\sigma}_i^\lambda(t,T,V)dW_i(t), \tag{2.24b}$$

$$dV(t) = \alpha^V(V,t)dt + \sum_{i=1}^{3} \tilde{\sigma}_i^V(t,V)dW_i(t), \tag{2.24c}$$

where by using transformation (2.22) and for $i = 1,2,3$, the volatility functions are defined as

$$\tilde{\sigma}_i^f(t,T,V) = \gamma_{3i}\sigma^f(t,T,V), \quad \tilde{\sigma}_i^\lambda(t,T,V) = \gamma_{2i}\sigma^\lambda(t,T,V), \quad \tilde{\sigma}_i^V(t,V) = \gamma_{1i}\sigma^V(t,V). \tag{2.25}$$

Then, equation (2.16) is expressed as

$$f^d(t,T,V) = f^d(0,T) + \int_0^t \alpha^d(u,T,V)du + \sum_{i=1}^{3} \int_0^t \tilde{\sigma}_i^d(u,T,V)dW_i(u), \tag{2.26}$$

where, for $i = 1,2,3$, the volatility functions are defined as

$$\tilde{\sigma}_i^d(t,T,V) = \tilde{\sigma}_i^f(t,T,V) + \tilde{\sigma}_i^\lambda(t,T,V). \tag{2.27}$$

The random loss $q(\tau_i)$ is considered as a random draw at default time τ_i. The fractional recovery process $\mathcal{R}(t)$ can be represented as a Doléans-Dade exponential of the stochastic differential equation[5]

$$d\mathcal{R}(t) = -\mathcal{R}(t-) \int_E q\, \mu(dt, dq), \tag{2.28}$$

where recall that $\mu(dt, dq)$ is the random measure associated to the marked point process. We then have the following result for the price dynamics of the default-free bond and the defaultable bond. We show how multiple defaults and recoveries can be incorporated within the HJM framework when there is no jump in the forward rate dynamics.

Lemma 2.1 *Given the dynamics (2.24a) for the default-free forward rate $f(t, T, V)$, the default-free bond price satisfies the SDE*

$$\frac{dP(t, T, V)}{P(t-, T, V)} = \big[r(t, V) + b(t, T, V)\big] dt - \sum_{i=1}^{3} \int_t^T \tilde{\sigma}_i^f(t, s, V) ds dW_i(t), \tag{2.29}$$

where

$$b(t, T, V) = -\int_t^T \alpha^f(t, s, V) ds + \frac{1}{2} \sum_{i=1}^{3} \big(\int_t^T \tilde{\sigma}_i^f(t, s, V) ds \big)^2. \tag{2.30}$$

Given the dynamics (2.26) for the defaultable forward rate $f^d(t, T, V)$, the defaultable bond price satisfies the SDE

$$\frac{dP^d(t, T, V)}{P^d(t-, T, V)} = \big[r^d(t, V) + b^d(t, T, V)\big] dt - \sum_{i=1}^{3} \int_t^T \tilde{\sigma}_i^d(t, s, V) ds dW_i(t) - \int_E q\mu(dt, dq), \tag{2.31}$$

where

$$b^d(t, T, V) = -\int_t^T \alpha^d(t, s, V) ds + \frac{1}{2} \sum_{i=1}^{3} \big(\int_t^T \tilde{\sigma}_i^d(t, s, V) ds \big)^2. \tag{2.32}$$

Proof: These results are derived along the lines of Schönbucher (1998), which involves an application of Ito's formula to (2.2) and (2.10) respectively, by using the dynamics (2.24a) of the default-free forward rate and the dynamics (2.26) of the defaultable forward rate. ■

[5] For the details of proof to this key result, the reader is referred to Jacod and Shiryaev (2003), Theorem 4.61, pg.59.

2.3 Risk-Neutral Dynamics

The absence of arbitrage opportunities implies that there exists an equivalent probability measure $\tilde{\mathbb{P}}$, namely the risk-neutral measure (which is not unique due to market incompleteness). For every finite maturity T, there exist a 3-dimensional predictable process $\Phi(t) = \{\phi_1(t), \phi_2(t), \phi_3(t), t \in [0,T]\}$ and a strictly positive measurable function $\psi(t,q)$ satisfying the integrability conditions

$$\int_0^t ||\phi_i(s)||^2 ds < \infty, \quad \text{for} \quad i = 1,2,3, \qquad \int_0^t \int_E |\psi(s,q)| h(s,dq) ds < \infty, \tag{2.33}$$

such that

$$d\tilde{W}_i(t) = dW_i(t) - \phi_i(t)dt, \quad \text{for} \quad i = 1,2,3, \tag{2.34}$$

is a $\tilde{\mathbb{P}}$-Wiener process and the default indicator process $N(t)$ has a $\tilde{\mathbb{P}}$-intensity

$$\tilde{h}(t,dq) = \psi(t,q)h(t,dq). \tag{2.35}$$

Lemma 2.2 *Using Girsanov's theorem such that the integrability conditions* (2.33) *and equation* (2.34) *are satisfied, then a risk-neutral measure* $\tilde{\mathbb{P}}$ *exists, if and only if,*

$$\alpha^f(t,T,V) = -\sum_{i=1}^{3} \tilde{\sigma}_i^f(t,T,V)\Big(\phi_i(t) - \int_t^T \tilde{\sigma}_i^f(t,s,V)ds\Big), \tag{2.36}$$

where $\phi_i(t)$ *denotes the market price of interest rate risk associated with the noise process* $W_i(t)$*. Then the risk-neutral dynamics of the default-free forward rate are*

$$df(t,T,V) = \sum_{i=1}^{3} \tilde{\sigma}_i^f(t,T,V) \int_t^T \tilde{\sigma}_i^f(t,s,V)dsdt + \sum_{i=1}^{3} \tilde{\sigma}_i^f(t,T,V)d\tilde{W}_i(t), \tag{2.37}$$

and the risk-neutral dynamics of the default-free bond price are

$$\frac{dP(t,T,V)}{P(t,T,V)} = r(t,V)dt - \sum_{i=1}^{3} \int_t^T \tilde{\sigma}_i^f(t,s,V)dsd\tilde{W}_i(t), \tag{2.38}$$

Proof: Follows along the lines of Heath et al. (1992). ■

Lemma 2.3 *Using Girsanov's theorem such that the integrability condition* (2.33) *and equations* (2.34),(2.35) *are satisfied, then a risk-neutral measure* $\tilde{\mathbb{P}}$ *exists, if and only if,*

$$[r^d(t,V) + b^d(t,T,V)] - \sum_{i=1}^{3} \phi_i(t) \int_t^T \tilde{\sigma}_i^d(t,s,V)ds - \int_E q\,\psi(t,q)h(t,dq) = r(t,V), \tag{2.39}$$

where $\phi_i(t)$ is the market price of interest rate risk associated with the noise process $W_i(t)$ and $\psi(t,q)$ is the market price of default risk.

Proof: Follows using similar arguments to those of Björk, Kabanov and Runggaldier (1997). ■

Taking the derivative of (2.39) with respect to T and performing some standard manipulations then yields

$$\alpha^d(t,T,V) = -\sum_{i=1}^{3} \tilde{\sigma}_i^d(t,T,V)\Big(\phi_i(t) - \int_t^T \tilde{\sigma}_i^d(t,s,V)ds\Big), \tag{2.40}$$

which is the corresponding HJM forward rate drift restriction condition for the defaultable bond price. As noted in Schönbucher (2003), the precise knowledge of the nature of the default process N and its compensator M is not necessary in setting up an arbitrage free model for the term structure of defaultable bonds. Although, the proposed model for the defaultable forward rate implies a condition on the drift of the credit spread[6].

Corollary 2.4 *The credit spread drift restriction implied by the proposed model is*

$$\begin{aligned}\alpha^\lambda(t,T,V) = &-\sum_{i=1}^{3} \phi_i(t)\tilde{\sigma}_i^\lambda(t,T,V) + \sum_{i=1}^{3} \tilde{\sigma}_i^\lambda(t,T,V)\int_t^T \tilde{\sigma}_i^\lambda(t,s,V)ds \\ &+ \sum_{i=1}^{3}\Big(\tilde{\sigma}_i^\lambda(t,T,V)\int_t^T \tilde{\sigma}_i^f(t,s,V)ds + \tilde{\sigma}_i^f(t,T,V)\int_t^T \tilde{\sigma}_i^\lambda(t,s,V)ds\Big).\end{aligned} \tag{2.41}$$

Proof: Substitute (2.18) and (2.27) into equation (2.40) and use condition (2.36). See also Appendix A. ■

Equation (2.41) expresses the drift of the credit spread in terms of the volatilities of the default free forward rate and the credit spread. This condition guarantees that the spread cannot become negative, see Bielecki and Rutkowski (2002). Substituting $b^d(t,T,V)$ as given in equation (2.32) into equation (2.39), as well as using (2.40), it follows that the short term spread is the product of the market price of jump risk, the default intensity and the expected loss quota, that is

$$r^d(t,V) - r(t,V) = \int_E q\,\psi(t,q)h(t,dq). \tag{2.42}$$

[6]See for similar results Schönbucher (1998), and Pugachevsky (1999), equation (35).

Taking into account the fact that the intensity of the default process under the risk-neutral measure is given by (2.35) then

$$r^d(t,V) - r(t,V) = \int_E q\,\tilde{h}(t,dq). \tag{2.43}$$

From definitions (2.6) and (2.7), we have $c(t,V) =: \lambda(t,t,V) = r^d(t,V) - r(t,V)$ and from (2.43) the short term credit spread, $c(t,V)$ under fractional recovery can be expressed as

$$c(t,V) \equiv \int_E q\,\tilde{h}(t,dq). \tag{2.44}$$

Formulating the intensity rate as a stochastic process allows rich dynamics for the credit spread process and is flexible enough to capture the empirically observed stochastic credit spreads. As cited in Jarrow and Turnbull (2000), there is considerable empirical evidence suggesting that the credit spread is a function of at least default intensity and the recovery process.

Definition 2.4 *We define the subsidiary state variables*

$$S_1(t,V) = \sum_{i=1}^{3} \int_0^t \tilde{\sigma}_i^f(u,t,V) \int_u^t \tilde{\sigma}_i^f(u,v,V)dvdu, \tag{2.45a}$$

$$S_2(t,V) = \sum_{i=1}^{3} \int_0^t \tilde{\sigma}_i^\lambda(u,t,V) \int_u^t \tilde{\sigma}_i^\lambda(u,v,V)dvdu, \tag{2.45b}$$

$$S_3(t,V) = \sum_{i=1}^{3} \int_0^t \tilde{\sigma}_i^f(u,t,V) \int_u^t \tilde{\sigma}_i^\lambda(u,v,V)dvdu, \tag{2.45c}$$

$$S_4(t,V) = \sum_{i=1}^{3} \int_0^t \tilde{\sigma}_i^\lambda(u,t,V) \int_u^t \tilde{\sigma}_i^f(u,v,V)dvdu, \tag{2.45d}$$

$$\psi_1(t,V) = \sum_{i=1}^{3} \int_0^t \tilde{\sigma}_i^f(u,t,V)d\tilde{W}_i(u), \tag{2.45e}$$

$$\psi_2(t,V) = \sum_{i=1}^{3} \int_0^t \tilde{\sigma}_i^\lambda(u,t,V)d\tilde{W}_i(u). \tag{2.45f}$$

Lemma 2.5 *Consider the path-dependent state space variables $S_i(t,V)$ and $\psi_i(t,V)$ presented in Definition 2.4. Under $\tilde{\mathbb{P}}$, the defaultable short rate satisfies the stochastic integral equation*

$$r^d(t,V) = f^d(0,t) + \sum_{i=1}^{4} S_i(t,V) + \sum_{i=1}^{2} \psi_i(t,V), \tag{2.46}$$

the default-free short rate satisfies the stochastic integral equation

$$r(t,V) = f(0,t) + S_1(t,V) + \psi_1(t,V), \tag{2.47}$$

and the short term spread satisfies the stochastic integral equation

$$c(t,V) = \lambda(0,t) + \sum_{i=2}^{4} S_i(t,V) + \psi_2(t,V). \tag{2.48}$$

Additionally, the state variable $V = \{V(t), t \in 0, T\}$ *satisfies the SDE*

$$dV(t) = \big[\alpha^V(V,t) + \sum_{i=1}^{3} \phi_i(t)\tilde{\sigma}_i^V(t,T,V)\big]dt + \sum_{i=1}^{3} \tilde{\sigma}_i^V(t,T,V)d\tilde{W}_i(t). \tag{2.49}$$

Proof: See Appendix B. ■

Corollary 2.6 *The defaultable bond price can be expressed as*

$$P^d(t,T,V) = \tilde{\mathbb{E}}\Big[\mathcal{R}(t)\exp\Big(-\int_t^T r^d(s,V)ds\Big)\Big|\mathcal{F}_t^W\Big]. \tag{2.50}$$

Proof: See Appendix C. ■

The market price of risk $\phi_i(t)$ of the risk factor $W_i(t)$ appears in the drift of the volatility process. Thus, our model shares the common feature of the class of Heston (1993) stochastic volatility models. These models do not imply a complete market as they cannot be fully hedged by a portfolio of bonds.

Note that, the expectation in equation (2.50) could be calculated numerically by simulating the SDE resulting from (2.46) for $r^d(t,V)$ and stochastic volatility process in equation (2.49). By additionally simulating the recovery process $\mathcal{R}(t)$, see (2.28), the defaultable bond price can be obtained.

3 Markovian Term Structure Model with Stochastic Volatility

Stochastic volatility specifications within the HJM framework that allow finite dimensional Markovian representations (FDR) have been studied by Chiarella and Kwon (2000) and Björk et al. (2004). By employing Lie algebra theory, Björk et al. (2004) examined the necessary and sufficient conditions on stochastic volatility for diffusion default-free HJM models to admit FDR. They demonstrated that a sufficient condition for the existence of FDR is that the volatility function should be the product of a quasi exponential function of the time to maturity and an arbitrary function of the forward rate and the volatility process.[7] We adopt these volatility specifications and consider an application to the defaultable term structure model proposed by Schönbucher (1998). Thus we consider a class of functional forms for the volatility functions $\sigma^f(t,T,V)$ and $\sigma^\lambda(t,T,V)$, as proposed by Björk et al. (2004), that will allow the non-Markovian representation of $r^d(t,V)$, given by (2.46), to be reduced to a finite dimensional Markovian system of SDEs.

Assumption 3.1 *The volatility functions of the default-free forward interest rate, the forward credit spread and the volatility are of the form*

$$\sigma^f(t,T,V) = \bar{\sigma}_f \sqrt{V(t)}\sqrt{r(t,V)}e^{-\kappa_f(T-t)}, \tag{3.1a}$$

$$\sigma^\lambda(t,T,V) = \bar{\sigma}_\lambda \sqrt{V(t)}\sqrt{c(t,V)}e^{-\kappa_\lambda(T-t)}, \tag{3.1b}$$

$$\sigma^V(t,V) = \bar{\sigma}_V \sqrt{V(t)}, \tag{3.1c}$$

respectively, where $\bar{\sigma}_f \geq 0$, $\bar{\sigma}_\lambda \geq 0$, $\bar{\sigma}_V \geq 0$, κ_f *and* κ_λ *are constants.*

The volatility specifications (3.1a) and (3.1b) can be considered as an extension of the Ritchken and Sankarasubramanian (1995) volatility structures to stochastic volatility with an application to the defaultable term structure.

Definition 3.1 *Under the volatility specifications of Assumption 3.1, we define the additional subsidiary state variables*

$$\eta_1(t,V) = \sum_{i=1}^{3} \gamma_{3i}^2 \int_0^t \bar{\sigma}_f^2\, r(u,V)V(u)e^{-2\kappa_f(t-u)}du, \tag{3.2a}$$

$$\eta_2(t,V) = \sum_{i=1}^{3} \gamma_{2i}^2 \int_0^t \bar{\sigma}_\lambda^2\, c(u,V)V(u)e^{-2\kappa_\lambda(t-u)}du, \tag{3.2b}$$

$$\eta_3(t,V) = \sum_{i=1}^{3} \gamma_{2i}\gamma_{3i} \int_0^t \bar{\sigma}_f\bar{\sigma}_\lambda\sqrt{r(u,V)c(u,V)}V(u)e^{-(\kappa_f+\kappa_\lambda)(t-u)}du. \tag{3.2c}$$

Lemma 3.1 *For* $i = 1,2,3$, *the subsidiary state variables* $\eta_i(t,V)$ *of Definition 3.1*

[7] See Proposition 5.2 of Björk et al. (2004).

satisfy the SDEs

$$d\eta_1(t,V) = \Big(\sum_{i=1}^{3}\gamma_{3i}^2\bar{\sigma}_f^2\, r(t,V)V(t) - 2\kappa_f\eta_1(t,V)\Big)dt, \tag{3.3a}$$

$$d\eta_2(t,V) = \Big(\sum_{i=1}^{3}\gamma_{2i}^2\bar{\sigma}_\lambda^2\, c(t,V)V(t) - 2\kappa_\lambda\eta_2(t,V)\Big)dt, \tag{3.3b}$$

$$d\eta_3(t,V) = \Big(\sum_{i=1}^{3}\gamma_{2i}\gamma_{3i}\bar{\sigma}_f\bar{\sigma}_\lambda\sqrt{r(t,V)c(t,V)}V(t) - (\kappa_f+\kappa_\lambda)\eta_3(t,V)\Big)dt, \tag{3.3c}$$

and the state variable $S_3(t,V)$, see (2.45c), *satisfies the SDE*

$$dS_3(t,V) = \Big[\eta_3(t,V) - \kappa_f S_3(t,V)\Big]dt. \tag{3.4}$$

Proof: Follows from differentiation of the state variables defined in Definition 3.1. ■

The following proposition shows that the defaultable short rate, the default-free short rate and the short term spread are completely determined by seven state space variables, namely $r(t,V)$, $c(t,V)$, $\eta_i(t,V)$, $i=1,2,3$, $S_3(t,V)$, and $V(t)$. Note that, an alternative representation that employs ten state space variables is also feasible, see Appendix D, though we propose the formulation that requires fewer state space variables and includes model factors such as $r(t,V)$ and $c(t,V)$ in the set of the state space.

Proposition 3.2 *Under the volatility specification of Assumption 3.1, the defaultable short rate satisfies the SDE*

$$\begin{aligned} dr^d(t,V) = \big[&\theta_d(t) + \eta_1(t,V) + \eta_2(t,V) + 2\eta_3(t,V) - (\kappa_f-\kappa_\lambda)S_3(t,V) + (\kappa_f-\kappa_\lambda)c(t,V) \\ &- \kappa_f r(t,V)\big]dt + \Big(\sum_{i=1}^{3}\gamma_{3i}\bar{\sigma}_f\sqrt{r(t,V)V(t)} + \sum_{i=1}^{3}\gamma_{2i}\bar{\sigma}_\lambda\sqrt{c(t,V)V(t)}\Big)d\tilde{W}_i(t), \end{aligned} \tag{3.5}$$

where the deterministic coefficient $\theta_d(t)$ in the drift is given by

$$\theta_d(t) = f_2^d(0,t) + \kappa_f f(0,t) + \kappa_\lambda\lambda(0,t).$$

The state variables $\eta_i(t,V)$, $i=1,2,3$, and $S_3(t,V)$ satisfy the system of SDEs presented in Lemma 3.1. The default-free short rate, the short rate spread and the volatility *function $V(t)$ satisfy the SDEs*

$$dr(t,V) = \big[\theta_f(t,V) + \eta_1(t,V) - \kappa_f r(t,V)\big]dt + \sum_{i=1}^{3} \gamma_{3i}\bar{\sigma}_f \sqrt{r(t,V)V(t)}d\tilde{W}_i(t), \quad (3.6)$$

$$dc(t,V) = \big[\theta_\lambda(t,V) + \eta_2(t,V) + 2\eta_3(t,V) - (\kappa_f - \kappa_\lambda)S_3(t,V) - \kappa_\lambda c(t,V)\big]dt$$
$$+ \sum_{i=1}^{3} \gamma_{2i}\bar{\sigma}_\lambda \sqrt{c(t,V)V(t)}d\tilde{W}_i(t), \quad (3.7)$$

$$dV(t) = \big[\alpha^V(V,t) + \sum_{i=1}^{3} \gamma_{1i}\phi_i(t)\bar{\sigma}^V \sqrt{V(t)}\big]dt + \sum_{i=1}^{3} \gamma_{1i}\bar{\sigma}^V \sqrt{V(t)}d\tilde{W}_i(t). \quad (3.8)$$

where the functions in the drifts are given by

$$\theta_f(t,V) = f_2(0,t) + \kappa_f f(0,t),$$
$$\theta_\lambda(t,V) = \lambda_2(0,t) + \kappa_\lambda \lambda(0,t).$$

Proof: The proof to this proposition is found in Appendix D. ■

We show next that the default-free and the defaultable bond prices across all maturities can be expressed in terms of the default-free short rate, the short rate spread and a set of Markovian state variables.

Proposition 3.3 *The price at time t of a default-free bond with maturity T is exponential affine and is given by*

$$P(t,T,V) = \frac{P(0,T)}{P(0,t)} \exp\Big(\beta_f(t,T)f(0,t) - \frac{1}{2}\beta_f^2(t,T)\eta_1(t,V) - \beta_f(t,T)r(t,V)\Big). \quad (3.9)$$

The price at time t of a defaultable bond with maturity T is also exponential affine and is given by

$$P^d(t,T,V) = \frac{\bar{P}^d(0,T,V_0)}{\bar{P}^d(0,t,V_0)} \exp\Big[\zeta(t,T) - \frac{1}{2}\beta_f^2(t,T)\eta_1(t,V) - \frac{1}{2}\beta_\lambda^2(t,T)\eta_2(t,V) \quad (3.10)$$
$$-\mathfrak{a}(t,T)\eta_3(t,V) - \big[\beta_f(t,T) + \beta_\lambda(t,T)\big]S_3(t,V) - \beta_f(t,T)r(t,V) - \beta_\lambda(t,T)c(t,V)\Big],$$

where

$$\zeta(t,T) = \ln \mathcal{R}(t) + \beta_f(t,T)f(0,t) + \beta_\lambda(t,T)\lambda(0,t),$$

and

$$\mathfrak{a}(t,T) = \frac{1}{\kappa_f}\beta_f(t,T) + \frac{1}{\kappa_\lambda}\beta_\lambda(t,T) + \Big(\frac{1}{\kappa_f} + \frac{1}{\kappa_\lambda}\Big)\Big(\frac{1}{\kappa_f + \kappa_\lambda}\Big)\Big(1 - e^{-(\kappa_f+\kappa_\lambda)(T-t)}\Big),$$

with

$$\beta_f(t,T) = \int_t^T e^{-\kappa_f(v-t)}dv \quad \text{and} \quad \beta_\lambda(t,T) = \int_t^T e^{-\kappa_\lambda(v-t)}dv. \quad (3.11)$$

Proof: See Appendix E. ■

Essentially, we have shown that the defaultable bond price takes an exponential affine form in the sense of Duffie and Kan (1996). Indeed by considering level dependent volatilities, we allow the volatility functions, σ^f and σ^λ to depend on the state space, a feature that is not included in the Duffie and Kan (1996) models. We should stress that the bond pricing formula in Proposition 3.3 will depend on a particular realisation of the path for the volatility process V, as the notation indicates. Thus we would need to simulate the entire system to obtain values of $r(t,V)$, $c(t,V)$, $S_3(t,V)$, $\eta_1(t,V)$, $\eta_2(t,V)$ and $\eta_3(t,V)$ which would then be substituted into the formula to obtain the bond price for that particular realisation of V. In fact, the distributions of bond prices and bond returns displayed in Section 4 below have been calculated in this way.

3.1 Finite Dimensional Realisations in Terms of Defaultable Forward Rates

Some of the Markovian state variables obtained in the proposed Markovian defaultable HJM model do not posses an economic meaning, namely $\eta_i(t,V)$ for $i = 1,2,3$ and $S_3(t,V)$. Under a default-free term structure setting, Björk and Svensson (2001) and Chiarella and Kwon (2003) have shown that it is possible to express these types of state variables as a linear combination of fixed tenor forward rates, thus obtaining finite dimensional affine realisations in terms of forward rates. In this section, by adopting their idea, we are able to express the six state variables of the proposed defaultable term structure model, namely $r(t,V)$, $c(t,V)$, $S_3(t,V)$ and $\eta_i(t,V)$ for $i = 1,2,3$, in terms of defaultable forward rates of six fixed tenors.[8] Consequently, the proposed defaultable term structure is expressed as an exponentially affine term structure in terms of fixed tenor defaultable forward rates. This representation establishes a connection between the defaultable bond price (3.10) and market observable quantities.

Definition 3.2 *We define the deterministic functions*

$$a_1(t,T) = e^{-\kappa_f(T-t)}, \quad a_2(t,T) = e^{-\kappa_\lambda(T-t)},$$
$$a_3(t,T) = \beta_f(t,T)e^{-\kappa_f(T-t)}, \quad a_4(t,T) = \beta_\lambda(t,T)e^{-\kappa_\lambda(T-t)},$$
$$a_5(t,T) = [e^{-\kappa_f(T-t)}\beta_\lambda(t,T) + e^{-\kappa_\lambda(T-t)}\beta_f(t,T)], \quad a_6(t,T) = a_1(t,T) + a_2(t,T).$$

[8]Note that the state variables $r(t,V)$, $c(t,V)$ have an economic meaning. The model could allow to express only the remaining state variables $\eta_i(t,V)$ for $i = 1,2,3$ and $S_3(t,V)$ in terms of fixed tenor defaultable forward rates, if this is required, see for instance Chiarella and Nikitopoulos (2003).

Proposition 3.4 *The defaultable forward rate of any maturity can be expressed in terms of six fixed tenor forward rates as*

$$f^d(t,T,V) = D(t,T) + \sum_{m=1}^{6}\sum_{j=1}^{6} a_j(t,T)\hat{a}_{jm} f^d(t,T_m,V), \tag{3.12}$$

where

$$D(t,T) = \widetilde{f}^d(0,t;0,T) + \sum_{m=1}^{6}\sum_{j=1}^{6} a_j(t,T)\hat{a}_{jm}\widetilde{f}^d(0,t;0,T_m),$$

and[9]

$$\widetilde{f}^d(0,t;0,T) = f^d(0,T) - e^{-\kappa_f(T-t)} f(0,t) - e^{-\kappa_\lambda(T-t)}\lambda(0,t). \tag{3.13}$$

In addition, $\hat{a}_{jm}$ *denotes the* jm^{th} *element of the matrix* $A(t)^{-1}$*, which is the inverse of the* 6×6 *square matrix* $A(t)$ *defined as*

$$A(t) = [a_{jm}] \tag{3.14}$$

with $a_{jm} = a_m(t,T_j)$ *as given in Definition 3.2. Assume that* $A(t)$ *is invertable for all* $t \in \{t'; t' = min_i[T_i]\}$*. The state variable* $V = \{V(t), t \in \{t'; t' = min_i[T_i]\}\}$ *satisfies the SDE (3.8).*

Proof: See Appendix F. ■

Corollary 3.5 *The defaultable bond price can be expressed in an exponential affine form in terms of fixed tenor defaultable forward rates as*

$$P^d(t,T,V) = \frac{\bar{P}^d(0,T,V_0)}{\bar{P}^d(0,t,V_0)} \exp\left[-\int_t^T D(t,s)ds - \sum_{m=1}^{6}\sum_{j=1}^{6}\hat{a}_{jm} f^d(t,T_m,V)\int_t^T a_j(t,s)ds\right]. \tag{3.15}$$

Proof: Substitution of (3.12) into the definition (2.10) yields the result. ■

The defaultable bond price expression (3.15) offers an important advantage especially for calibration applications. Market information related to a distinct set of defaultable forward rates can be embedded into the formula for the defaultable term structure in a very convenient manner due to the tractability of the proposed model. Note also that under this model, the stochastic volatility quantities are the only quantities that are not traded in the market.

[9] The term $\widetilde{f}^d(0,t;0,T)$ includes information of the initial term structure of the default-free forward rate and the forward credit spread, thus of the defaultable forward rate.

4 Numerical Experiments

In this section, we examine the effect of variations of the parameters of the stochastic volatility, correlation and default intensity on the distribution of the defaultable bond price and the defaultable bond returns.

4.1 Model Inputs

In our numerical investigations we use a typical choice of the system (2.22) so that $\gamma_{12} = \gamma_{13} = \gamma_{23} = 0$. Then using (2.23), (2.22) yields the transformation

$$\begin{bmatrix} dW^V(t) \\ dW^\lambda(t) \\ dW^f(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \rho_{12} & \sqrt{1-\rho_{12}^2} & 0 \\ \rho_{13} & \frac{\rho_{23}-\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}} & \sqrt{\frac{1-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2+2\rho_{12}\rho_{13}\rho_{23}}{1-\rho_{12}^2}} \end{bmatrix} \begin{bmatrix} dW_1(t) \\ dW_2(t) \\ dW_3(t) \end{bmatrix}. \tag{4.1}$$

By specifying the drift $\alpha^V(V,t) = \kappa_V\big(\bar{V} - V(t)\big)$, and the market price of risk by $\phi_1(t) = \bar{\phi}\sqrt{V(t)}$ with the scaling factor $\bar{\phi} = 1$,[10] then the stochastic volatility process (3.8) is reduced to

$$dV(t) = \big[\kappa_V\bar{V} - (\kappa_V - \bar{\sigma}^V)V(t)\big]dt + \bar{\sigma}^V\sqrt{V(t)}d\tilde{W}_1(t). \tag{4.2}$$

We use the set of parameters given in Table 1[11] and the initial term structures of forward rates and forward credit spreads[12] given by

$$f(0,T,V_0) = 0.05 - 0.04\sqrt{V_0}e^{-1.8T} \quad \text{and} \quad \lambda(0,T,V_0) = 0.03 - 0.01\sqrt{V_0}e^{-1.6T}$$

respectively, with the initial volatility chosen to be $V_0 = 0.08$. This implies that the initial short rate and the initial short term credit spread will be $r(0,V_0) = 0.0387$ and $c(0,V_0) = 0.0272$ respectively. The proposed initial term structures gives forward rates between 3.8% and 5% over a period of 20 years and a forward credit spread between 2.7% and 3%, which are typical values (at least before GFC). We make a simplifying assumption that the initial credit spread remains the same irrespective of

[10] Note that this is one possible parameterization that eventually rescales other parameters in the volatility drift term.

[11] For the risk-free forward rate specifications, we have consulted the estimated parameter values presented in Trolle and Schwartz (2009), where a default-free interest rate model with stochastic volatility has been estimated, by fitting the model to caps and swaption data.

[12] Traditionally initial forward curves are specified within the Nelson-Siegel family and our specifications fit in to this family. The resulting forward curves and forward credit spreads are typical examples of observed forward curves and forward credit spreads.

$\bar{\sigma}^f$	$\bar{\sigma}^\lambda$	$\bar{\sigma}^V$	$\bar{V}$	κ_f	κ_λ	κ_V	ρ_{12}	ρ_{13}	ρ_{23}
0.65	0.45	0.30	1.00	0.25	0.3	0.85	0.30	0.45	-0.40

Table 1: The parameter values used in the simulation experiment.

downgrades on default and subsequent restructuring. For recovery,[13] we simulate the process $d\mathcal{R}(t) = -\mathcal{R}(t-)q(t)dM(t)$ where the compensated process $dM(t) = dN(t) - \tilde{h}(t)dt$ is a martingale and $N(t)$ is a Cox process governing the default dynamics. This is a special case of the general marked point processes defined in (2.28).

For the simulation experiment, we use an Euler-Maruyama approximation and discretise into 250 subintervals. By considering $t = 1$ and $T = 2$, we generate $200,000$ simulated paths, to obtain the simulated distributions of the defaultable bond price $P^d(1, 2)$, as expressed in Proposition 3.3.

4.2 Simulation Results

The defaultable bond is assumed to have an average default intensity $\tilde{h}(t) = 0.30$. This risk-neutral default intensity is backed out of bond prices using the formula $\tilde{h} = \frac{s}{1-\mathcal{R}}$, with a yield spread of $s = 1321bps$ as the average yield spread of defaultable bonds over treasuries. This choice falls between $B-$rated bonds risk-neutral default intensity approximated to be 0.0902 and $CAA-$rated bonds whose intensity was estimated to be 0.2130 in Hull, Predescu and White (2005). In addition, the loss given default e is assumed to be distributed according to $LGD \sim \mathcal{N}(0.6839, 0.07)$.[14] Figure 1 illustrates the effect of the correlation ρ_{12} between the stochastic volatility process $V(t)$ and the credit spread process $c(t, V)$ on the distribution of defaultable bond prices and returns. Increasing the correlation ρ_{12} from -0.6 to 0.6 tends to increase the skewness of the defaultable bond prices and the (negative) skewness of the bond returns. A similar observation but with a bigger impact is made in Figure 2, where the effect of the correlation ρ_{13} between the stochastic volatility process $V(t)$ and the short rate process $r(t, V)$ is considered. This could be attributed to the fact that the short rate

[13]Although our model allows for multiple defaults and recovery, we assume that the firm's default intensity and recovery rate remain the same even after default and restructuring. A more realistic specification would allow for downgrades in the credit quality thereby increasing the default intensity and reducing the recovery rate in the eventuality of future events. This would require a more general migration model.

[14]This is documented in Moody's (2003) report which gives average recoveries for different rating classes over the time period 1982 – 2003.

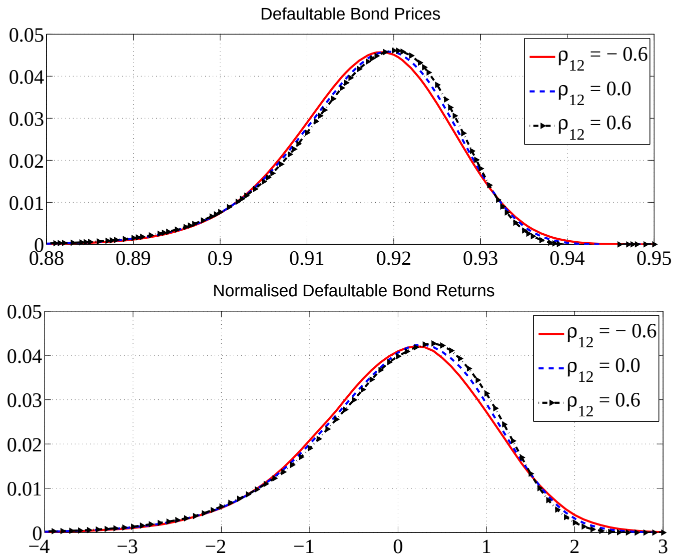

Figure 1: The effect of the correlation coefficient ρ_{12} to the distribution of defaultable bond prices and normalised defaultable bond returns.

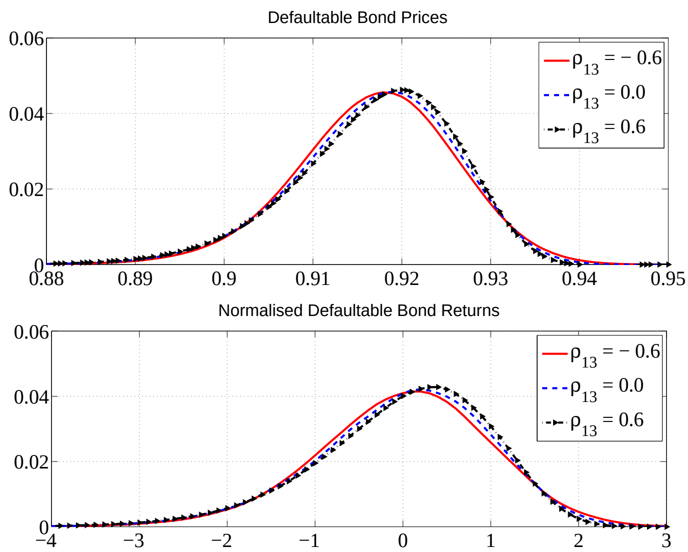

Figure 2: The effect of the correlation coefficient ρ_{13} to the distribution of defaultable bond price and normalised defaultable bond returns.

has a higher average volatility than the credit spread process. Figure 3 illustrates

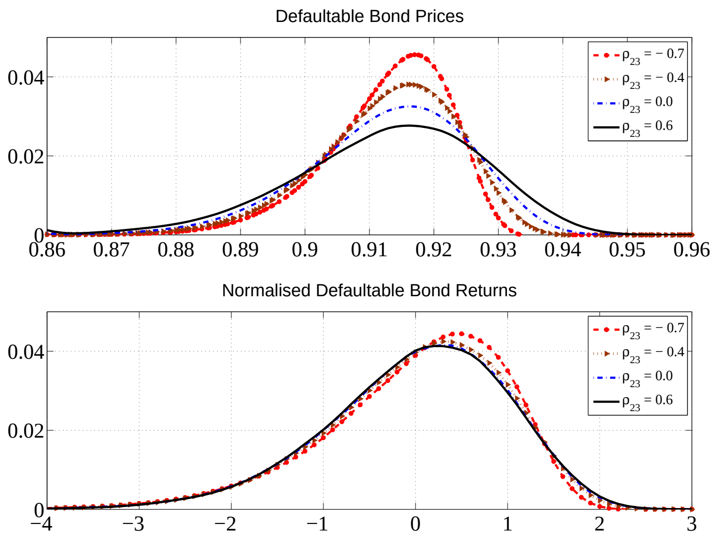

Figure 3: Effect of the correlation coefficient ρ_{23} to the distribution of defaultable bond price and normalised defaultable bond returns.

the effect of the correlation ρ_{23} between the short-term credit spread $c(t, V)$ and the short rate process $r(t, V)$ on the distribution of defaultable bond prices and defaultable bond returns. Increasing the correlation ρ_{23} tends to decrease both the kurtosis and the (negative) skewness of the distributions. The correlation ρ_{23} conveys information about the covariation between default-free discount rates and the market's perception of default risk. In Longstaff and Schwartz (1995) and Duffee (1998), it was shown that this relationship is negative for investment-grade, noncallable corporate bonds and strongly negative for lower rated and callable bonds.

Figure 4 illustrates the effect of the volatility of volatility $\bar{\sigma}^V$ on the distribution of defaultable bond price and returns, respectively. When $\bar{\sigma}^V = 0$, the volatility process is deterministic and an increasing $\bar{\sigma}^V$ implies that the market has a higher chance of extreme movements. Increasing volatility of volatility tends to skew the defaultable bond price and return distribution to the right and to increase the kurtosis of both bond price and bond returns. Note that, it has been empirically observed that negatively skewed returns (with heavy downside tails) are characteristic of portfolios of defaultable bonds, see D'Souza, Amir-Atefi and Racheva-Jotova (2004). We also investigated the effects of the speed of mean reversion κ_V of the volatility process on the defaultable bond price and return distributions. From Figure 5 we observe that increasing the speed of mean

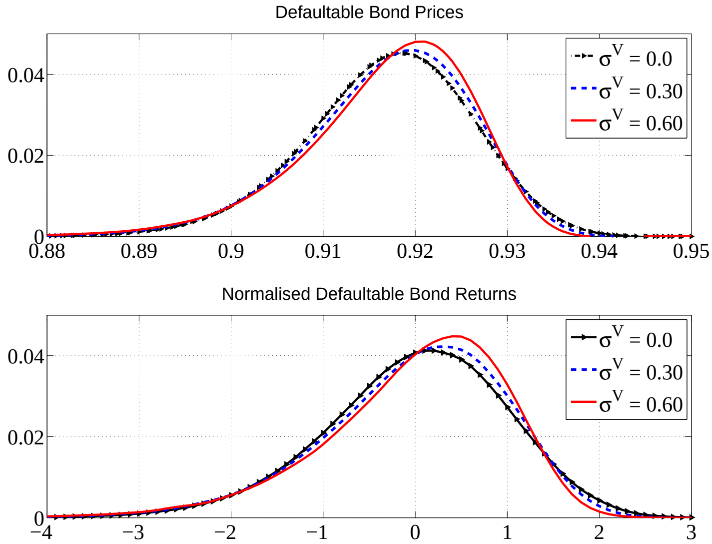

Figure 4: Distribution of defaultable bond price and normalised bond returns under varying $\bar{\sigma}^V$.

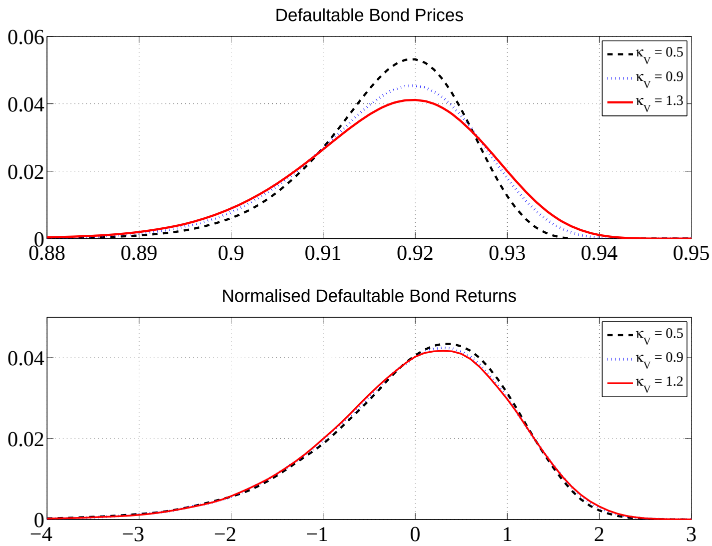

Figure 5: Distribution of defaultable bond price and normalised bond returns under varying κ_V.

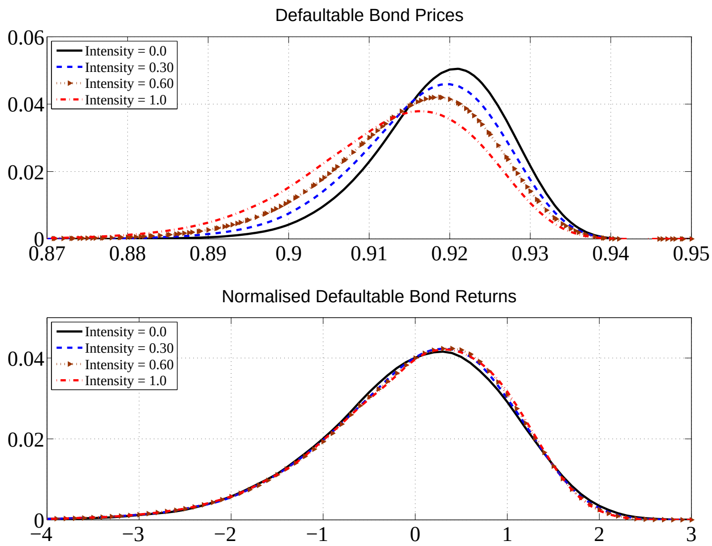

Figure 6: Distribution of defaultable bond prices under varying default intensity $\tilde{h}(t)$.

reversion of the volatility process reduces the kurtosis of both the defaultable bond price and bond returns. Figure 6 presents the effect of the default intensity to the defaultable bond price and bond returns.

5 Conclusion

This paper develops a class of defaultable HJM interest rate models with unspanned stochastic volatility and considers volatility specifications that reduce the proposed model to Markovian form. By modelling the credit spreads, a connection between the default-free and the defaultable forward term structure has been established and a correlation structure between credit spreads, interest rates and stochastic volatility has been incorporated. In addition, an explicit exponential affine formula for defaultable bond prices in the presence of unspanned stochastic volatility has been derived. The paper also attempts to provide a link between the state variables and the market observed quantities, in particular forward rates. This is of significant value when implementing the model and further research into calibration and evaluation of these models is still required.

Some numerical investigations have been performed to illustrate the effect of the volatility of volatility and correlation between the Wiener processes driving the defaultable forward rates, credit spreads and the stochastic volatility to the distributions of defaultable bond prices and defaultable bond returns.

The proposed model is broad enough to accommodate multiple defaults and recovery, yet it cannot capture bond downgrades or upgrades over time, as it does not allow the updating of the default intensity and recovery rates between default events. This will require the inclusion of a migration model, see for instance the affine Markov chain framework proposed in Hurd and Kuznetsov (2007) which opens a line for future research. Another direction for further research is the estimation of the proposed defaultable term structure model via filtering methods, along the lines of Trolle and Schwartz (2009), where a default-free term structure model was estimated. The proposed framework can also be applied to price credit derivatives, see for instance Chiarella, Fanelli and Musti (2011) for credit spread options pricing and this will be the topic of subsequent work.

Acknowledgements

The authors would like to thank Professor Wolfgang Runggaldier and Professor Tomas Björk for fruitful discussions and useful suggestions on an earlier version of this work. The usual caveat applies.

Appendices

A Credit spread drift restriction.

By using equation (2.27), the drift restriction condition (2.40) can be expanded to

$$\begin{aligned}\alpha^d(t,T,V) = &-\sum_{i=1}^{3}\phi_i(t)\tilde{\sigma}_i^f(t,T,V) - \sum_{i=1}^{3}\phi_i(t)\tilde{\sigma}_i^\lambda(t,T,V) \\ &+\sum_{i=1}^{3}\tilde{\sigma}_i^f(t,T,V)\int_t^T \tilde{\sigma}_i^f(t,s,V)ds + \sum_{i=1}^{3}\tilde{\sigma}_i^\lambda(t,T,V)\int_t^T \tilde{\sigma}_i^\lambda(t,s,V)ds \\ &+\sum_{i=1}^{3}\tilde{\sigma}_i^\lambda(t,T,V)\int_t^T \tilde{\sigma}_i^f(t,s,V)ds + \sum_{i=1}^{3}\tilde{\sigma}_i^f(t,T,V)\int_t^T \tilde{\sigma}_i^\lambda(t,s,V)ds.\end{aligned} \tag{A.1}$$

Substitute (2.18) and (2.27) into equation (2.40) and use condition (2.36) to obtain (2.41). ♦

B Proof of Lemma 2.5

By using equation (2.26), (2.34) and the drift restriction condition (2.41), the defaultable forward rate risk-neutral dynamics are

$$
\begin{aligned}
f^d(t,T,V) = f^d(0,T) &+ \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^f(u,T,V)\int_u^T \tilde{\sigma}_i^f(u,s,V)dsdu \\
&+ \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^\lambda(u,T,V)\int_u^T \tilde{\sigma}_i^\lambda(u,s,V)dsdu + \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^\lambda(u,T,V)\int_u^T \tilde{\sigma}_i^f(u,s,V)dsdu \\
&+ \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^f(u,T,V)\int_u^T \tilde{\sigma}_i^\lambda(u,s,V)dsdu \\
&+ \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^f(u,T,V)d\tilde{W}_i(u) + \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^\lambda(u,T,V)d\tilde{W}_i(u). \qquad \text{(B.1)}
\end{aligned}
$$

It follows from equation (B.1) that the instantaneous defaultable short rate dynamics $r^d(t,V)$ under the risk neutral measure are given by the stochastic integral equation

$$
\begin{aligned}
r^d(t,V) = f^d(0,t) &+ \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^f(u,t,V)\int_u^t \tilde{\sigma}_i^f(u,s,V)dsdu + \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^\lambda(u,t,V)\int_u^t \tilde{\sigma}_i^\lambda(u,s,V)dsdu \\
&+ \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^\lambda(u,t,V)\int_u^t \tilde{\sigma}_i^f(u,s,V)dsdu + \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^f(u,t,V)\int_u^t \tilde{\sigma}_i^\lambda(u,s,V)dsdu \\
&+ \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^f(u,t,V)d\tilde{W}_i(u) + \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^\lambda(u,t,V)d\tilde{W}_i(u). \qquad \text{(B.2)}
\end{aligned}
$$

By using the state variables of Definition 2.4 this can be written in differential form as

$$
dr^d(t,V) = \left[f_2^d(0,t) + \sum_{j=1}^{4}\frac{\partial}{\partial t}S_j(t,V)\right]dt + \sum_{j=1}^{2}d\psi_j(t,V). \qquad \text{(B.3)}
$$

Similarly, from condition (2.41) the forward credit spread dynamics $\lambda(t,T,V)$ in equation (2.24b) can be written as

$$
\begin{aligned}
\lambda(t,T,V) = \lambda(0,T) &+ \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^\lambda(u,T,V)\int_u^T \tilde{\sigma}_i^\lambda(u,s,V)dsdu + \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^f(u,T,V)\int_u^T \tilde{\sigma}_i^\lambda(u,s,V)dsdu \\
&+ \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^\lambda(u,T,V)\int_u^T \tilde{\sigma}_i^f(u,s,V)dsdu + \sum_{i=1}^{3}\tilde{\sigma}_i^\lambda(t,t,V)d\tilde{W}_i(t). \qquad \text{(B.4)}
\end{aligned}
$$

By using equation (B.4) and definition (2.7), then equation (2.48) is derived.

Furthermore, the default free instantaneous short rate in equation (2.14) follows the stochastic integral equation

$$r(t,V) = f(0,t) + \sum_{i=1}^{3} \int_0^t \tilde{\sigma}_i^f(u,t,V) \int_u^t \tilde{\sigma}_i^f(u,s,V) ds du + \sum_{i=1}^{3} \int_0^t \tilde{\sigma}_i^f(u,t,V) d\tilde{W}_i(u). \tag{B.5}$$

Using the quantities defined in (2.45a) - (2.45e), equation (B.5) can be expressed as (2.47). Furthermore, the stochastic volatility process $V(t)$, recall (2.24c), will follow the stochastic differential equation (2.49) under the risk-neutral measure $\tilde{\mathbb{P}}$. ♦

C Proof of Corollary 2.6

By using condition (2.39), equation (2.31) yields

$$\frac{dP^d(t,T)}{P^d(t-,T)} = r(t,V)dt + \sum_{i=1}^{3} \tilde{\sigma}_{B,i}^d(t,T,V)\big(dW_i(t) - \phi_i(t)dt\big) - \big(\int_E q\mu(dt,dq) - \int_E \psi(t,q)h(t,dq)dt\big), \tag{C.1}$$

or from (2.34) and (2.35)

$$\frac{dP^d(t,T)}{P^d(t-,T)} = r(t,V)dt + \sum_{i=1}^{3} \tilde{\sigma}_{B,i}^d(t,T,V)d\tilde{W}_i(t) - d\tilde{M}(\omega;t). \tag{C.2}$$

These are the dynamics of the defaultable bond price under the risk-neutral measure in which

$$\begin{aligned}\tilde{M}(\omega;t) &= \int_0^t \int_E q\,\mu(\omega;ds,dq) - \int_0^t \int_E q\,\psi(s,q)h(s,dq)ds \\ &= \int_0^t \int_E q\,\mu(\omega;ds,dq) - \int_0^t \int_E q\,\tilde{h}(s,dq)ds,\end{aligned}$$

is a martingale. We define the relative defaultable bond price by

$$Z^d(t,T,V) = \frac{P^d(t,T)}{B(t,V)},$$

where $B(t,V) = \exp\Big(\int_0^t r(s,V)ds\Big)$ is the accumulated money market account. Applying Itô's quotient rule, the stochastic differential equation for $Z^d(t,T,V)$ is

$$\frac{dZ^d(t,T,V)}{Z^d(t-,T,V)} = \sum_{i=1}^{3} \tilde{\sigma}_{B,i}^d(t,T,V)d\tilde{W}_i(t) - \int_E q\,\tilde{h}(t,dq)dt. \tag{C.3}$$

If we let $\tilde{\mathbb{E}}$ denote mathematical expectation with respect to the risk neutral probability measure, it then follows that

$$\tilde{\mathbb{E}}\big[dZ^d(t,T,V)\big|\mathcal{F}_t\big] = 0.$$

This implies that

$$\tilde{\mathbb{E}}\big[Z^d(T,T,V)\big|\mathcal{F}_t\big] = Z^d(t,T,V),$$

and given that $P^d(T,T) = \bar{P}^d(T,T,V)\mathcal{R}(T)$, the defaultable bond price satisfies

$$P^d(t,T) = \tilde{\mathbb{E}}\Big[\exp\Big(-\int_t^T r(s,V)ds\Big)\mathcal{R}(T)\Big|\mathcal{F}_t\Big], \tag{C.4}$$

which on following the approach in Lando (1998) reduces to

$$P^d(t,T) = \tilde{\mathbb{E}}\Big[\exp\Big(-\int_t^T \big(r(s,V) + \int_E q\,\tilde{h}(s,dq)\big)ds\Big)\mathcal{R}(t)\Big|\mathcal{F}_t^W\Big], \tag{C.5}$$

where $\mathcal{F}_t^W$ is defined in (2.1). Note that the quantity $\exp\big(-\int_t^T r(s,V)ds\big)$ is the stochastic discount factor under measure $\tilde{\mathbb{P}}$ used to discount back to time t the \$1 payoff to be received at time T. Using the relationship (2.43), the bond price formula under the expectations operator in equation (C.5) can then be written as (2.50). ♦

D Proof of Proposition 3.2

Using the dynamics (2.46) of the defaultable short rate and the volatility specifications of Assumption 3.1 then $r^d(t,V)$ satisfies the stochastic differential equation

$$\begin{aligned} dr^d(t,V) = {} & \Big[\frac{\partial}{\partial t}f(0,t) + \sum_{j=1}^{4}\frac{\partial}{\partial t}S_j(t,V) - \kappa_f\psi_1(t,V) - \kappa_\lambda\psi_2(t,V)\Big]dt \\ & + \sum_{i=1}^{3}\Big(\gamma_{3i}\bar{\sigma}_f\sqrt{r(t,V)V(t)} + \gamma_{2i}\bar{\sigma}_\lambda\sqrt{\lambda(t,V)V(t)}\Big)d\tilde{W}_i(t). \end{aligned} \tag{D.1}$$

Furthermore, by using the additional state variables $\eta_i, i = 1,2,3$, as given in Definition 3.1, equation (D.1) yields

$$\begin{aligned} dr^d(t,V) = {} & \big[f_2(0,t) + \eta_1(t,V) - \kappa_f S_1(t,V) + \eta_2(t,V) - \kappa_\lambda S_2(t,V) \\ & + 2\eta_3(t,V) - \kappa_f S_3(t,V) - \kappa_\lambda S_4(t,V) - \kappa_f\psi_1(t,V) - \kappa_\lambda\psi_2(t,V)\big]dt \\ & + \sum_{i=1}^{3}\Big(\gamma_{3i}\bar{\sigma}_f\sqrt{r(t,V)V(t)} + \gamma_{2i}\bar{\sigma}_\lambda\sqrt{\lambda(t,V)V(t)}\Big)d\tilde{W}_i(t). \end{aligned} \tag{D.2}$$

This formulation employs ten state variables, namely $\eta_i, i = 1,2,3$, $S_j, j = 1,2,3,4$, $\psi_\ell, \ell = 1,2$, and V. However, the dynamics of the defaultable short rate can be expressed as a system with only seven state variables by using (2.47) and (2.48). More specifically, from the short-term credit spread dynamics (2.48), the variable $\psi_2(t,V)$ can be expressed as

$$\psi_2(t,V) = c(t,V) - \lambda(0,t) - \sum_{j=2}^{4} S_j(t,V). \tag{D.3}$$

By rewriting the default-free short rate dynamics (2.47), we have that

$$\psi_1(t,V) = r(t,V) - f(0,t) - S_1(t,V). \tag{D.4}$$

Substituting further equations (D.3) and (D.4) into (D.2), we obtain

$$\begin{aligned} dr^d(t,V) = &\big[f_2(0,t) + \kappa_f f(0,t) + \kappa_\lambda \lambda(0,t) + \eta_1(t,V) + \eta_2(t,V) \\ &+ 2\eta_3(t,V) - (\kappa_f - \kappa_\lambda)S_3(t,V) - \kappa_f r(t,V) - \kappa_\lambda c(t,V)\big]dt \\ &+ \Big(\sum_{i=1}^{3} \gamma_{3i}\bar{\sigma}_f \sqrt{r(t,V)V(t)} + \sum_{i=1}^{3} \gamma_{2i}\bar{\sigma}_\lambda \sqrt{\lambda(t,V)V(t)}\Big) d\tilde{W}_i(t). \end{aligned} \tag{D.5}$$

This can be rearranged to yield the result in Proposition 3.2. Similarly by using the state variables $\eta_i, i = 1,2,3$, as given in Definition 3.1, the results for the default-free short rate and the short-term credit spread as obtained. ♦

E Proof of Proposition 3.3

By substituting the drift condition (2.40) into the dynamics (2.26), the stochastic integral equation for the defaultable forward rate under the risk-neutral measure is expressed as

$$f^d(t,T,V) = f^d(0,T) + \sum_{i=1}^{3} \Big[\int_0^t \tilde{\sigma}_i^{d*}(u,T,V)du + \int_0^t \tilde{\sigma}_i^d(u,T,V)d\tilde{W}_i(u)\Big], \tag{E.1}$$

where

$$\tilde{\sigma}_i^{d*}(t,T,V) = \tilde{\sigma}_i^d(t,T,V) \int_t^T \tilde{\sigma}_i^d(t,s,V)ds. \tag{E.2}$$

Then, from equation (2.9), the 'pseudo' bond is expressed as

$$\begin{aligned} \bar{P}^d(t,T,V) = \exp\Big[-\int_t^T f^d(0,s)ds - \sum_{i=1}^{3} \Big(&\int_t^T \int_0^t \tilde{\sigma}_i^{d*}(u,s,V)duds \\ &+ \int_t^T \int_0^t \tilde{\sigma}_i^d(u,s,V)d\tilde{W}_i(u)ds \Big)\Big]. \end{aligned} \tag{E.3}$$

We define a new variable I such that

$$I = \int_t^T \int_0^t \tilde{\sigma}_i^{d*}(u, s, V) du ds + \int_t^T \int_0^t \tilde{\sigma}_i^{d}(u, s, V) d\tilde{W}_i(u) ds. \tag{E.4}$$

By applying Fubini's theorem, this can be rewritten as

$$I = \underbrace{\int_0^t \int_t^T \tilde{\sigma}_i^{d*}(u, s, V) ds du}_{I_1} + \underbrace{\int_0^t \int_t^T \tilde{\sigma}_i^{d}(u, s, V) ds d\tilde{W}_i(u)}_{I_2}, \tag{E.5}$$

so that, $I = I_1 + I_2$. We note that,

$$\int_t^T \tilde{\sigma}_i^{d*}(u, s, V) ds = \int_t^T \tilde{\sigma}_i^{d}(u, s, V) \int_u^t \tilde{\sigma}_i^{d}(u, v, V) dv ds + \int_t^T \tilde{\sigma}_i^{d}(u, s, V) \int_t^s \tilde{\sigma}_i^{d}(u, v, V) dv ds,$$

and by expanding further, see (2.27), we have that

$$\begin{aligned}
\int_t^T \tilde{\sigma}_i^{d*}(u, s, V) ds = & \int_t^T \tilde{\sigma}_i^{f}(u, s, V) \int_u^t \tilde{\sigma}_i^{f}(u, v, V) dv ds + \int_t^T \tilde{\sigma}_i^{f}(u, s, V) \int_t^s \tilde{\sigma}_i^{f}(u, v, V) dv ds \\
& + \int_t^T \tilde{\sigma}_i^{\lambda}(u, s, V) \int_u^t \tilde{\sigma}_i^{\lambda}(u, v, V) dv ds + \int_t^T \tilde{\sigma}_i^{\lambda}(u, s, V) \int_t^s \tilde{\sigma}_i^{\lambda}(u, v, V) dv ds \\
& + \int_t^T \tilde{\sigma}_i^{f}(u, s, V) \int_u^t \tilde{\sigma}_i^{\lambda}(u, v, V) dv ds + \int_t^T \tilde{\sigma}_i^{f}(u, s, V) \int_t^s \tilde{\sigma}_i^{\lambda}(u, v, V) dv ds \\
& + \int_t^T \tilde{\sigma}_i^{\lambda}(u, s, V) \int_u^t \tilde{\sigma}_i^{f}(u, v, V) dv ds + \int_t^T \tilde{\sigma}_i^{\lambda}(u, s, V) \int_t^s \tilde{\sigma}_i^{f}(u, v, V) dv ds.
\end{aligned} \tag{E.6}$$

From (2.25) and by using the volatility specifications of Assumption 3.1 and the deterministic functions $\beta_f(t, T)$ and $\beta_\lambda(t, T)$ defined in equation (3.11), we obtain

$$\begin{aligned}
\int_t^T \tilde{\sigma}_i^{d}(u, s, V) ds &= \int_t^T \tilde{\sigma}_i^{f}(u, s, V) ds + \int_t^T \tilde{\sigma}_i^{\lambda}(u, s, V) ds \\
&= \beta_f(t, T) \tilde{\sigma}_i^{f}(u, t, V) + \beta_\lambda(t, T) \tilde{\sigma}_i^{\lambda}(u, t, V).
\end{aligned} \tag{E.7}$$

Additionally, note that[15]

$$
\begin{aligned}
&\int_t^T \tilde{\sigma}_i^f(u,s,V)\int_t^s \tilde{\sigma}_i^f(u,v,V)dvds = \frac{1}{2}[\beta_f(t,T)\tilde{\sigma}_i^f(u,t,V)]^2,\\
&\int_t^T \tilde{\sigma}_i^\lambda(u,s,V)\int_t^s \tilde{\sigma}_i^\lambda(u,v,V)dvds = \frac{1}{2}[\beta_\lambda(t,T)\tilde{\sigma}_i^\lambda(u,t,V)]^2, \qquad \text{(E.8)}\\
&\int_t^T \tilde{\sigma}_i^f(u,s,V)\int_u^t \tilde{\sigma}_i^\lambda(u,v,V)dvds = \Big(\frac{\beta_f(t,T)}{\kappa_\lambda} + \frac{1-e^{-(\kappa_f+\kappa_\lambda)(T-t)}}{\kappa_\lambda(\kappa_f+\kappa_\lambda)}\Big)\tilde{\sigma}_i^\lambda(u,t,V)\tilde{\sigma}_i^f(u,t,V),\\
&\int_t^T \tilde{\sigma}_i^\lambda(u,s,V)\int_u^t \tilde{\sigma}_i^f(u,v,V)dvds = \Big(\frac{\beta_\lambda(t,T)}{\kappa_f} + \frac{1-e^{-(\kappa_\lambda+\kappa_f)(T-t)}}{\kappa_f(\kappa_f+\kappa_\lambda)}\Big)\tilde{\sigma}_i^\lambda(u,t,V)\tilde{\sigma}_i^f(u,t,V).
\end{aligned}
$$

Thus from (E.7) and (E.8), the equation (E.6) can be altered to

$$
\begin{aligned}
\int_t^T \tilde{\sigma}_i^{d*}(u,s,V)ds &= \beta_f(t,T)\tilde{\sigma}_i^f(u,t,V)\int_u^t \tilde{\sigma}_i^f(u,v,V)dv + \frac{1}{2}[\beta_f(t,T)\tilde{\sigma}_i^f(u,t,V)]^2\\
&+ \beta_\lambda(t,T)\tilde{\sigma}_i^\lambda(u,t,V)\int_u^t \tilde{\sigma}_i^\lambda(u,v,V)dv + \frac{1}{2}[\beta_\lambda(t,T)\tilde{\sigma}_i^\lambda(u,t,V)]^2 \qquad \text{(E.9)}\\
&+ \beta_f(t,T)\tilde{\sigma}_i^f(u,t,V)\int_u^t \tilde{\sigma}_i^\lambda(u,v,V)dv + \beta_\lambda(t,T)\tilde{\sigma}_i^\lambda(u,t,V)\int_u^t \tilde{\sigma}_i^f(u,v,V)dv\\
&+ \Big[\frac{\beta_f(t,T)}{\kappa_\lambda} + \frac{\beta_\lambda(t,T)}{\kappa_f} + \Big(\frac{1}{\kappa_f}+\frac{1}{\kappa_\lambda}\Big)\Big(\frac{1-e^{-(\kappa_f+\kappa_\lambda)(T-t)}}{\kappa_f+\kappa_\lambda}\Big)\Big]\tilde{\sigma}_i^f(u,t,V)\tilde{\sigma}_i^\lambda(u,t,V).
\end{aligned}
$$

By substituting (E.7) and (E.9) into (E.5) follows that

$$
\begin{aligned}
I &= \beta_f(t,T)\int_0^t \tilde{\sigma}_i^f(u,t,V)\int_u^t \tilde{\sigma}_i^f(u,v,V)dvdu + \beta_f(t,T)\int_0^t \tilde{\sigma}_i^f(u,t,V)d\tilde{W}_i(u)\\
&+ \beta_\lambda(t,T)\int_0^t \tilde{\sigma}_i^\lambda(u,t,V)\int_u^t \tilde{\sigma}_i^\lambda(u,v,V)dvdu + \beta_\lambda(t,T)\int_0^t \tilde{\sigma}_i^f(u,t,V)\int_u^t \tilde{\sigma}_i^\lambda(u,v,V)dvdu\\
&+ \beta_\lambda(t,T)\int_0^t \tilde{\sigma}_i^\lambda(u,t,V)\int_u^t \tilde{\sigma}_i^f(u,v,V)dvdu + \beta_\lambda(t,T)\int_0^t \tilde{\sigma}_i^\lambda(u,t,V)d\tilde{W}_i(u) \qquad \text{(E.10)}\\
&- \beta_\lambda(t,T)\int_0^t \tilde{\sigma}_i^\lambda(u,t,V)\int_u^t \tilde{\sigma}_i^f(u,v,V)dvdu + \beta_f(t,T)\int_0^t \tilde{\sigma}_i^\lambda(u,t,V)\int_u^t \tilde{\sigma}_i^f(u,v,V)dvdu\\
&+ \Big[\frac{1}{\kappa_\lambda}\beta_f(t,T) + \frac{1}{\kappa_f}\beta_\lambda(t,T) + \Big(\frac{1}{\kappa_f}+\frac{1}{\kappa_\lambda}\Big)\Big(\frac{1}{\kappa_f+\kappa_\lambda}\Big)\Big(1-e^{-(\kappa_f+\kappa_\lambda)(T-t)}\Big)\Big]\\
&\int_0^t \tilde{\sigma}_i^f(u,t,V)\tilde{\sigma}_i^\lambda(u,t,V)du + \frac{1}{2}\Big[\beta_f^2(t,T)\int_0^t \tilde{\sigma}_i^{2f}(u,t,V)du + \beta_\lambda^2(t,T)\int_0^t \tilde{\sigma}_i^{2\lambda}(u,t,V)du\Big],
\end{aligned}
$$

[15]Note that

$$
\int_t^T e^{-\kappa_f(s-t)}\int_t^s e^{-\kappa_\lambda(v-t)}dvds = \frac{1}{\kappa_\lambda}\beta_f(t,T) + \frac{1}{\kappa_\lambda(\kappa_f+\kappa_\lambda)}\Big(1-e^{-(\kappa_f+\kappa_\lambda)(T-t)}\Big),
$$

and

$$
\int_t^T e^{-\kappa_\lambda(s-t)}\int_t^s e^{-\kappa_f(v-t)}dvds = \frac{1}{\kappa_f}\beta_\lambda(t,T) + \frac{1}{\kappa_f(\kappa_f+\kappa_\lambda)}\Big(1-e^{-(\kappa_\lambda+\kappa_f)(T-t)}\Big).
$$

which by employing (2.47) and (2.48) can be written as

$$I = \beta_f(t,T)[r(t,V) - f(0,t)] + \beta_\lambda(t,T)[c(t,V) - \lambda(0,t)]$$
$$+ [\beta_f(t,T) - \beta_\lambda(t,T)] \int_0^t \tilde{\sigma}_i^\lambda(u,t,V) \int_u^t \tilde{\sigma}_i^f(u,v,V)dvdu + \mathfrak{a}(t,T)\int_0^t \tilde{\sigma}_i^f(u,t,V)\tilde{\sigma}_i^\lambda(u,t,V)du$$
$$+ \frac{1}{2}\Big[\beta_f^2(t,T)\int_0^t \tilde{\sigma}_i^{2f}(u,t,V)du + \beta_\lambda^2(t,T)\int_0^t \tilde{\sigma}_i^{2\lambda}(u,t,V)du\Big], \quad \text{(E.11)}$$

where

$$\mathfrak{a}(t,T) = \frac{\beta_f(t,T)}{\kappa_\lambda} + \frac{\beta_\lambda(t,T)}{\kappa_f} + \Big(\frac{1}{\kappa_f} + \frac{1}{\kappa_\lambda}\Big)\frac{1 - e^{-(\kappa_f+\kappa_\lambda)(T-t)}}{\kappa_f + \kappa_\lambda}.$$

We can then write equation (E.3) as

$$\bar{P}^d(t,T,V) = \frac{\bar{P}^d(0,T)}{\bar{P}^d(0,t)}\exp\Big[- \beta_f(t,T)[r(t,V) - f(0,t)] - \beta_\lambda(t,T)[\lambda(t,V) - \lambda(0,t)]$$
$$- [\beta_f(t,T) - \beta_\lambda(t,T)]\sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^\lambda(u,t,V)\int_u^t \tilde{\sigma}_i^f(u,v,V)dvdu$$
$$- \mathfrak{a}(t,T)\sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^f(u,t,V)\tilde{\sigma}_i^\lambda(u,t,V)du \quad \text{(E.12)}$$
$$- \frac{1}{2}\Big[\beta_f^2(t,T)\sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^{2f}(u,t,V)du + \beta_\lambda^2(t,T)\sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^{2\lambda}(u,t,V)du\Big]\Big].$$

We recall from Appendix D of Proposition 3.2 the definition of the state variables $\eta_1(t,V)$, $\eta_2(t,V)$, $\eta_3(t,V)$ and $S_3(t,V)$. Then, the equation for the pseudo bond above reduces to

$$\bar{P}^d(t,T,V) = \frac{\bar{P}^d(0,T)}{\bar{P}^d(0,t)}\exp\Big(- \frac{1}{2}\beta_f^2(t,T)\eta_1(t,V) - \frac{1}{2}\beta_\lambda^2(t,T)\eta_2(t,V) - \mathfrak{a}(t,T)\eta_3(t,V)$$
$$\text{(E.13)}$$
$$- \big[\beta_f(t,T) - \beta_\lambda(t,T)\big]S_3(t,V) - \beta_f(t,T)[r(t,V) - f(0,t)] - \beta_\lambda(t,T)[\lambda(t,V) - \lambda(0,t)]\Big).$$

By using the definition of the defaultable bond in terms of the pseudo bond given in equation (2.10), the equation for the defaultable bond yields equation (3.10).

Similarly, we recall that the default-free forward rate (2.37) under the risk-neutral measure can be written as

$$f(t,T,V) = f(0,T) + \sum_{i=1}^{3}\Big[\int_0^t \tilde{\sigma}_i^{f*}(u,T,V)du + \int_0^t \tilde{\sigma}_i^f(u,T,V)d\tilde{W}_i(u)\Big], \quad \text{(E.14)}$$

where

$$\tilde{\sigma}_i^{f*}(t,T,V) = \tilde{\sigma}_i^f(t,T,V)\int_t^T \tilde{\sigma}_i^f(t,s,V)ds. \quad \text{(E.15)}$$

Analogously, the default-free bond price can be expressed as

$$P(t,T,V) = \exp\Big[-\int_t^T f(0,s)ds - \sum_{i=1}^{3} \Big(\int_t^T \int_0^t \tilde{\sigma}_i^{f*}(u,s,V)duds + \int_t^T \int_0^t \tilde{\sigma}_i^{f}(u,s,V)d\tilde{W}_i(u)ds \Big) \Big]. \tag{E.16}$$

As in (E.10), we define a new variable $\bar{I}$ such that

$$\bar{I} = \int_0^t \int_t^T \tilde{\sigma}_i^{f*}(u,s,V)dsdu + \int_0^t \int_t^T \tilde{\sigma}_i^{f}(u,s,V)dsd\tilde{W}_i(u), \tag{E.17}$$

which can be reduced to

$$\bar{I} = \beta_f(t,T)[r(t,V) - f(0,t)] + \frac{1}{2}\beta_f^2(t,T) \int_0^t \tilde{\sigma}_i^{2f}(u,t,V)du. \tag{E.18}$$

Substituting (E.18) into (E.16) and making use of the definition of the state variable $\eta_1(t,V)$ then yields (3.9). ♦

F Proof of Proposition 3.4

From definition (2.4) we have that

$$f^d(t,T,V) = -\frac{\partial}{\partial T} \ln P^d(t,T), \quad \text{for all} \quad t \in [0,T]. \tag{F.1}$$

From (F.1) and by using the dynamics of the defaultable bond price, recall (3.10), which can be written as

$$P^d(t,T) = \exp\Big[-\int_t^T f^d(0,y)dy - \zeta(t,T) - \frac{1}{2}\beta_f^2(t,T)\eta_1(t,V) - \frac{1}{2}\beta_\lambda^2(t,T)\eta_2(t,V) \tag{F.2}$$
$$-\mathfrak{a}(t,T)\eta_3(t,V) - \big[\beta_f(t,T) + \beta_\lambda(t,T)\big]S_3(t,V) - \beta_f(t,T)r(t,V) - \beta_\lambda(t,T)c(t,V)\Big],$$

we have that (set $f^d(0,T,V_0) = f^d(0,T)$)

$$\begin{aligned} f^d(t,T,V) - f^d(0,T) = & -e^{-\kappa_f(T-t)}f(0,t) - e^{-\kappa_\lambda(T-t)}\lambda(0,t) \\ & + e^{-\kappa_f(T-t)}\beta_f(t,T)\eta_1(t,V) + e^{-\kappa_\lambda(T-t)}\beta_\lambda(t,T)\eta_2(t,V) \\ & + [e^{-\kappa_f(T-t)}\beta_\lambda(t,T) + e^{-\kappa_\lambda(T-t)}\beta_f(t,T)]\eta_3(t,V) \\ & + [e^{-\kappa_f(T-t)} + e^{-\kappa_\lambda(T-t)}]S_3(t,V) + e^{-\kappa_f(T-t)}r(t,V) + e^{-\kappa_\lambda(T-t)}c(t,V). \end{aligned} \tag{F.3}$$

By using the deterministic functions $a_i(t,T)$ of Definition 3.2 and the term (3.13), that includes the information of the initial term structure of the forward rates and credit spread, equation (F.3) can be expressed as

$$f^d(t,T,V) - \widetilde{f^d}(0,t;0,T) = \mathcal{A}\mathcal{X}^\top \tag{F.4}$$

where the scalar matrices $\mathcal{A} = [a_i(t,T)], i = 1, 2, \dots, 6$, and $\mathcal{X} = [r(t,V)c(t,V)\eta_1(t,V)\eta_2(t,V)\eta_3(t,V)S_3(t,V)]$. Since $a_i(t,T)$ are deterministic functions, the value of $f^d(t,T,V)$ can be expressed as a linear combination of the six state variables $r(t,V)$, $c(t,V)$, $S_3(t,V)$ and $\eta_i(t,V)$ for $i = 1, 2, 3$. Then equation (F.4) can be used to express the state variables as a linear combination of a finite set of six forward rates.

Let $T = t + \tau$ and denote $\Delta f^d_\tau(t,V) := f^d(t, t+\tau, V) - \widetilde{f}^d(0,t;0,t+\tau)$. Then by fixing six tenors $0 \leq \tau_1 \leq ... \leq \tau_6$ and setting $\tau = \tau_i$, for $i = 1, 2, \dots, 6$, (F.4) yields the system

$$\begin{bmatrix} \Delta f^d_{\tau_1}(t,V) \\ \Delta f^d_{\tau_2}(t,V) \\ \vdots \\ \Delta f^d_{\tau_6}(t,V) \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{16} \\ a_{21} & a_{22} & \dots & a_{26} \\ \vdots & \vdots & \ddots & \\ a_{61} & a_{62} & \dots & a_{66} \end{bmatrix}}_{A(t)} \begin{bmatrix} r(t,V) \\ c(t,V) \\ \eta_1(t,V) \\ \eta_2(t,V) \\ \eta_3(t,V) \\ S_3(t,V) \end{bmatrix}, \tag{F.5}$$

where $a_{jm} = a_m(t, T_j)$, see Definition 3.2. Assume that the determinant of matrix $A(t)$ exists and is not equal to zero i.e, $detA(t) \neq 0$, the system of equations (F.5) is invertible. If the matrix $A(t)$ is invertible, then the corresponding HJM model admits an affine Markovian realization in terms of the forward rates $f^d(t, t+\tau_i, V)$ for $i = 1, 2, \dots, 6$. We can then write the state variables as linear combinations of forward rates $f^d_{\tau_1}(t,V)$, $\dots$, $f^d_{\tau_6}(t,V)$ in the form,

$$\mathcal{X}^\top = \begin{bmatrix} r(t,V) \\ c(t,V) \\ \eta_1(t,V) \\ \eta_2(t,V) \\ \eta_3(t,V) \\ S_3(t,V) \end{bmatrix} = A(t)^{-1} \begin{bmatrix} \Delta f^d_{\tau_1}(t,V) \\ \Delta f^d_{\tau_2}(t,V) \\ \vdots \\ \Delta f^d_{\tau_6}(t,V) \end{bmatrix}. \tag{F.6}$$

By substituting the expressions (F.6) for the state variables into the equation (F.4) and collecting like terms the equation (3.12) is obtained. ♦

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