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Simulation of Diversified Portfolios in a Continuous Financial Market

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Abstract. The paper analyzes the simulated long-term behavior of well diversified portfolios in continuous financial markets. It focuses on the equi-weighted index and the market portfolio. The paper illustrates that the equally weighted portfolio constitutes a good proxy of the growth optimal portfolio, which maximizes expected logarithmic utility. The multi-asset market models considered include the Black-Scholes model, the Heston model, the ARCH diffusion model, the geometric Ornstein-Uhlenbeck volatility model and a multi-asset version of the minimal market model. All these models are simulated exactly or almost exactly over an extremely long period of time to analyze the long term growth of the respective portfolios. The paper illustrates the robustness of the diversification phenomenon when approximating the growth optimal portfolio by the equi-weighted index. Significant outperformance in the long run of the market capitalization weighted portfolio by the equi-weighted index is documented for different market models. Under the multi-asset minimal market model the equi-weighted index outperforms remarkably the market portfolio. In this case the benchmarked market portfolio is a strict supermartingale, whereas the benchmarked equi-weighted index is a martingale. Equal value weighting overcomes the strict supermartingale property that the benchmarked market portfolio inherits from its strict supermartingale constituents under this model.

2000 *Mathematics Subject Classification:* 91B28, 62P05, 62P20

Key words and phrases: Growth optimal portfolio, Diversification Theorem, diversified portfolios, market portfolio, equi-weighted index, almost exact simulation, minimal market model.

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1 Introduction

Diversification is a concept that has been successfully applied in risk management for centuries. When constructing a *diversified portfolio* (DP) in the sense of this paper, the proportion of the value of the holding in each individual security, relative to the total portfolio value, converges sufficiently fast to zero as the number of securities increases. A prominent DP is the *equi-weighted index* (EWI). In the classical literature related to portfolio optimization and capital asset pricing the market portfolio or *market capitalization weighted index* (MCI), plays a significant role, see for instance Markowitz (1959), Lintner (1965), Sharpe (1964) and Merton (1973). This paper demonstrates via simulation that well diversified portfolios approximate the *growth optimal portfolio* (GOP), which is the portfolio that maximizes expected logarithmic utility, see Kelly (1956). They are likely to outperform the long run growth of the market portfolio. Asymptotic properties of diversified portfolios have been analyzed by Björk & Näslund (1998), Hofmann & Platen (2000), Guan, Liu & Chong (2004), Platen (2005) and Platen & Rendek (2010). A different notion of diversification than the one used in the current paper has been introduced in Fernholz (2002), see also Fernholz, Karatzas & Kardaras (2005).

The current paper considers DPs in a rather general setting under the benchmark approach, see Platen (2002, 2004) and Platen & Heath (2006). This approach is built upon the notion of the *numéraire portfolio* (NP) that when used as benchmark makes all benchmarked nonnegative securities supermartingales. The NP, which in the setting of this paper equals the GOP, is almost surely pathwise the best performing portfolio in the long run. It can be used in derivative pricing as numéraire and in portfolio optimization as benchmark, see Platen & Heath (2006). The numéraire portfolio appears in a range of literature including, for instance, Long (1990), Artzner (1997), Bajeux-Besnainou & Portait (1997), Karatzas & Shreve (1998), Kramkov & Schachermayer (1999), Becherer (2001), Platen (2002), Goll & Kallsen (2003) and Karatzas & Kardaras (2007).

In practice, it is difficult to construct the GOP in the real market since such a construction would need a valid model and accurate estimates of the parameters characterising the model. Unfortunately, it is not possible to estimate any drift parameter with sufficient significance to be useful in a sample-based Markowitz style portfolio optimization for a large financial market. This has been made clear by a recent study in DeMiguel, Garlappi & Uppal (2009). On the other hand, a *Diversification Theorem* derived in Platen (2005) shows under minor regularity assumptions that for a sequence of markets, a corresponding sequence of DPs is a sequence of approximate GOPs, see also Platen & Rendek (2010). The result allows one to approximate the GOP, for instance, by the *equi-weighted index* (EWI). It holds for continuous market dynamics where the benchmarked primary security accounts are local martingales. It covers also the case when these nonnegative local martingales are strict supermartingales. Nonnegative strict supermartingales show in the long run a systematic downward trend. This raises the practically important question, whether significantly better performance can be detected for the EWI when compared to the *market capitalization weighted index* (MCI) for a market model where the constituents are strict supermartingales? This paper demonstrates that this is indeed the case. It also shows the robustness of the diversification phenomenon. It simulates the EWI and MCI under various continuous financial market models. An important feature of this study is that these simulations are *exact* or at least *almost exact*. In an almost exact simulation the only numerical approximations performed are discrete time approximations of integrals with integrators of finite variation. This kind of numerical approximation can be handled by effective quadrature rules. The almost exact simulation avoids the typical problems that may arise in standard discrete time numerical approximations, see Kloeden & Platen (1999). These are often due to the non-Lipschitzness of coefficient functions and discrete time approximation, potentially causing long run numerical stability problems or even negative approximate values for positive price processes. By focusing on exact or at least almost exact approximations, the long-term simulation results of the study become extremely reliable. They give a realistic perspective on the closeness of the EWI to the GOP and show the divergence from the MCI under various models. When one constructs these portfolios from real market data, as in Platen & Rendek (2008) and Platen & Rendek (2010), then one observes effects very similar to those discussed in the current paper. Of particular importance is the realistic case where benchmarked primary security accounts are strict supermartingales, as will be explained below.

Another contribution of this paper is the exact and almost exact simulation of asset prices for a selection of market models. Along the lines of Platen & Rendek (2009), the paper simulates asset prices under the multi-asset Black-Scholes model, see Black & Scholes (1973); the multi-asset Heston model, see Heston (1993); the multi-asset ARCH-diffusion model, see Nelson (1990) and Frey (1997); the multi-asset geometric Ornstein-Uhlenbeck volatility model, see Wiggins (1987) and Bergomi (2004); and a multi-asset version of the minimal market model, see Platen (2001, 2002). The simulation study demonstrates that the convergence of the EWI towards the GOP appears to be remarkably robust. For the minimal market model, where the primary security accounts when expressed in units of the GOP are strict supermartingales, the EWI outperforms in the long run the MCI significantly. This can be explained by the fact that the benchmarked MCI, as the sum of strict supermartingales, is a strict supermartingale, whereas, the benchmarked GOP is constant.

2 Diversified Portfolios

The aim of the paper is to demonstrate the diversification phenomenon for particular market models with emphasis on the EWI and the MCI. The paper relies on a selection of continuous financial market models, which are typical in the continuous time finance literature.

Let be given a filtered probability space $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P)$, where $\underline{\mathcal{A}} = (\mathcal{A}_t)_{t\in[0,\infty)}$ is the filtration which models the evolution of information in the market and satisfies the usual conditions, see Protter (2004). For simplicity, the paper considers market models with continuous security prices. Traded uncertainty is modeled by the independent standard $(\underline{\mathcal{A}}, P)$-Wiener processes $W^k = \{W_t^k, t \in [0,\infty)\}$, $k \in \{1, 2, \ldots\}$.

Consider a sequence $(\mathcal{S}_{(d)}^C)_{d\in\{1,2,\ldots\}}$ of continuous financial market models (CFMs) where the dth market $\mathcal{S}_{(d)}^C$ is indexed by the number $d \in \{1, 2, \ldots\}$ of its risky assets. For given d, the corresponding CFM comprises $d+1$ primary security accounts, denoted by $S_{(d)}^0, S_{(d)}^1, \ldots, S_{(d)}^d$. These include a savings account $S_{(d)}^0 = \{S_{(d)}^0(t), t \in [0,\infty)\}$, which is the locally riskless primary security account of the domestic currency expressed by

$$S_{(d)}^0(t) = \exp\left\{\int_0^t r_s ds\right\} \tag{2.1}$$

for $t \in [0,\infty)$, where $r = \{r_t, t \in [0,\infty)\}$ denotes the finite adapted short rate process.

Fix for the moment $d \in \{1, 2, \ldots\}$. As shown in Platen & Heath (2006), in the dth CFM $\mathcal{S}_{(d)}^C$ there exists a unique GOP $S_{(d)}^{\delta_*} = \{S_{(d)}^{\delta_*}(t), t \in [0,\infty)\}$, which is a strictly positive portfolio that maximizes the expected log-utility from terminal wealth, see Kelly (1956). More precisely, it maximizes $E(\ln(S_{(d)}^{\delta}(T)))$ for any $T \in [0,\infty)$, over all strictly positive portfolios $S_{(d)}^{\delta}$ formed by the primary security accounts $S_{(d)}^0, \ldots, S_{(d)}^d$.

Define in the dth CFM the jth *benchmarked primary security account* process $\hat{S}_{(d)}^j = \{\hat{S}_{(d)}^j(t), t \in [0,\infty)\}$ by setting

$$\hat{S}_{(d)}^j(t) = \frac{S_{(d)}^j(t)}{S_{(d)}^{\delta_*}(t)} \tag{2.2}$$

for $t \in [0,\infty)$, $j \in \{0, 1, \ldots, d\}$. Assume that, as shown in Platen & Heath (2006), $\hat{S}_{(d)}^j(t)$ satisfies the stochastic differential equation (SDE)

$$d\hat{S}_{(d)}^j(t) = \hat{S}_{(d)}^j(t) \sum_{k=1}^{d} \sigma_{(d)}^{j,k}(t)\, dW_t^k \tag{2.3}$$

for $t \in [0,\infty)$ and predictable volatility processes $\sigma_{(d)}^{j,k}$, $j, k \in \{0, 1, \ldots, d\}$.

For a benchmarked portfolio process $\hat{S}^{\delta}_{(d)} = \{\hat{S}^{\delta}_{(d)}(t), t \in [0,\infty)\}$, its value at time t is given by the sum

$$\hat{S}^{\delta}_{(d)}(t) = \sum_{j=0}^{d} \delta^j_t \hat{S}^j_{(d)}(t). \tag{2.4}$$

Here, the vector process of the predictable number of units invested $\boldsymbol{\delta} = \{\boldsymbol{\delta}_t = (\delta^0_t, \delta^1_t, \ldots, \delta^d_t)^\top, t \in \Re^+\}$ represents the corresponding strategy. Assume that the Itô integral

$$\int_0^t \delta^j_s d\hat{S}^j_{(d)}(s) \tag{2.5}$$

exists in a suitable sense, see Protter (2004), for all $j \in \{0, 1, \ldots, d\}$, $d \in \{1, 2, \ldots\}$ and $t \in [0,\infty)$. All portfolios that the paper considers are assumed to be self-financing, which is expressed via the SDE

$$d\hat{S}^{\delta}_{(d)}(t) = \sum_{j=0}^{d} \delta^j_t d\hat{S}^j_{(d)}(t) \tag{2.6}$$

for $t \in [0,\infty)$.

This study will focus on portfolios that remain strictly positive. Define the fraction $\pi^j_{\delta,t}$ of $\hat{S}^{\delta}_{(d)}(t)$ that is invested in the jth benchmarked primary security account $\hat{S}^j_{(d)}(t)$ at time t, which is given by the formula

$$\pi^j_{\delta,t} = \delta^j_t \frac{\hat{S}^j_{(d)}(t)}{\hat{S}^{\delta}_{(d)}(t)} \tag{2.7}$$

for $j \in \{0, 1, \ldots, d\}$. Note that fractions can be negative but always sum to one, such that

$$\sum_{j=0}^{d} \pi^j_{\delta,t} = 1 \tag{2.8}$$

for $t \in [0,\infty)$.

To give some theoretical background to the following simulations the following definition prepares a formulation of the Diversification Theorem derived in Platen (2005).

Definition 1 *For a sequence of CFMs* $(\mathcal{S}^C_{(d)})_{d\in\{1,2,\ldots\}}$ *a corresponding sequence* $(S^{\delta}_{(d)})_{d\in\{1,2,\ldots\}}$ *of strictly positive portfolio processes* $S^{\delta}_{(d)}$ *is called a sequence of* diversified portfolios *(DPs) if some constants* $K_1, K_2 \in (0,\infty)$ *and* $K_3 \in \{1, 2, \ldots\}$ *exist independently of* d*, such that for* $d \in \{K_3, K_3 + 1, \ldots\}$ *the inequality*

$$|\pi^j_{\delta,t}| \leq \frac{K_2}{d^{\frac{1}{2}+K_1}} \tag{2.9}$$

holds almost surely for all $j \in \{0, 1, \ldots, d\}$ *and* $t \in [0,\infty)$.

This means, for a DP the fractions decline faster than $d^{-\frac{1}{2}}$ for increasing d. Note that they do not need to be equal and can vary substantially. The equi-weighted index (EWI) is an example for a diversified portfolio. Note that a market capitalization weighted index (MCI) has fractions that may not always satisfy the bounds given in (2.9).

Assume that in a given CFM the primary security accounts are sufficiently different. Otherwise, one cannot expect any substantial diversification effect to emerge. Introduce $\hat{\sigma}^k_{(d)}(t)$ as the k-th total specific volatility, defined by the sum

$$\hat{\sigma}^k_{(d)}(t) = \sum_{j=0}^{d} |\sigma^{j,k}_{(d)}(t)| \tag{2.10}$$

for $k \in \{1, 2, \ldots\}$. The (j, k)th *specific volatility* $\sigma^{j,k}_{(d)}$, is the volatility of the jth benchmarked primary security account with respect to the kth source of trading uncertainty W^k, see (2.3).

To guarantee a sequence of markets with sufficiently different primary security accounts the following condition is imposed.

Assumption 1. *Assume a* regular sequence of CFMs, *which means that each of the independent sources of trading uncertainty influences only a restricted range of benchmarked primary security accounts such that*

$$E\left((\hat{\sigma}^k_{(d)}(t))^2\right) \le K_5 \tag{2.11}$$

for all $t \in [0, \infty)$*,* $d \in \{1, 2, \ldots\}$*,* $k \in \{1, 2, \ldots, d\}$*, with fixed constant* $K_5 \in (0, \infty)$*.*

To introduce some kind of a distance between an approximating portfolio $S^\delta_{(d)}$ and the GOP $S^{\delta_*}_{(d)}$ define the *tracking rate* $R^\delta_{(d)}(t)$ at time t by

$$R^\delta_{(d)}(t) = \sum_{k=1}^{d} \left(\sum_{j=0}^{d} \pi^j_{\delta,t} \sigma^{j,k}_{(d)}(t) \right)^2 \tag{2.12}$$

for $t \in [0, \infty)$. Obviously, the tracking rate vanishes when $S^\delta_{(d)}$ equals $S^{\delta_*}_{(d)}$. This is a consequence of the fact that the benchmarked GOP equals the constant one and, thus, has zero returns and zero volatility. We emphasize that a benchmarked portfolio with small tracking rate has a small volatility. This would be a necessary property of a portfolio that aims to approximate the GOP.

Definition 2 *For a sequence* $(\mathcal{S}^C_{(d)})_{d \in \{1,2,\ldots\}}$ *of CFMs a sequence of strictly positive portfolios* $(S^\delta_{(d)})_{d \in \{1,2,\ldots\}}$ *is called a* sequence of approximate *GOPs when the corresponding sequence of tracking rates vanishes in probability, that is, for each* $\varepsilon > 0$ *it holds*

$$\lim_{d \to \infty} P\left(R^\delta_{(d)}(t) > \varepsilon\right) = 0 \tag{2.13}$$

for all $t \in [0, \infty)$*.*

In reality one observes that well diversified equity portfolios have small volatilities and behave very similarly. This can be verified by the following *Diversification Theorem*:

Theorem 2.1 *For a regular sequence of CFMs, any sequence of DPs is a sequence of approximate GOPs. Moreover, for any* $d \in \{K_3, K_3+1, \ldots\}$ *and* $t \in [0, \infty)$*, the expected tracking rate of a given DP can be shown to satisfy the estimate*

$$E\left(R^{\delta}_{(d)}(t)\right) \leq \frac{(K_2)^2 K_5}{d^{2K_1}}. \tag{2.14}$$

Here, $K_1, K_2, K_5 \in (0, \infty)$ *and* $K_3 \in \{1, 2, \ldots\}$*.*

The proof of this result is an application of the Markov inequality and given in Platen (2005). This result is model independent and, therefore, very robust.

It is well-known that the GOP is in many ways the best performing portfolio. In particular, its long-term growth rate is almost surely greater or equal than the long-term growth rate of any other strictly positive portfolio, see Platen (2004). This means, if one observes proxies of the GOP over sufficiently long time periods, then one can most likely identify the better proxies by their larger long-term growth rates.

The aim of this paper is to analyze via simulation the long-term growth behavior of EWIs as proxies of the corresponding GOPs under various market models. It shall be demonstrated that a good approximation of the GOP can be asymptotically achieved by using the EWI, thus, avoiding any estimation of drift parameters for the calculation of the theoretical GOP fractions. This circumvents the seemingly unsolvable practical problem of estimating trends or risk premia, needed under the sample-based Markowitz (1959) portfolio optimization, as pointed out in DeMiguel, Garlappi & Uppal (2009).

To keep the presentation focused the paper only considers additionally to the simulation of the GOP the *market capitalization weighted index* (MCI) and the *equi-weighted index* (EWI), which are all self-financing portfolios. The simulations will be performed on an equi-distant time discretization $0 = t_0 < t_1 < t_2 < \ldots$ with $t_n = \Delta n$ for $\Delta > 0$, $n \in \{0, 1, 2, \ldots\}$.

For an MCI in the dth CFM the fraction of wealth invested in the jth primary security account at time t_n is given by the ratio

$$\pi^j_{\delta_{\mathrm{MCI}}, t_n} = \frac{\delta^j_{\mathrm{MCI}, t_n} \, S^j_{t_n}}{\sum_{i=1}^d \delta^i_{\mathrm{MCI}, t_n} \, S^i_{t_n}}, \tag{2.15}$$

for $j \in \{1, 2, \ldots, d\}$, where $\delta^j_{\mathrm{MCI}, t_n}$ denotes the number of units of the jth primary security account available in the market at time t_n. Note that in the market models to be considered, the GOP does not invest in the savings account and only in stocks. The interest rate will be set to zero for simplicity. An EWI is obtained in the dth CFM by setting all fractions equal at the beginning of each trading period. The jth fraction of an EWI is simply given by the ratio

$$\pi^j_{\delta_{\mathrm{EWI}},t_n} = \frac{1}{d} \tag{2.16}$$

for $j = \{1, 2, \ldots, d\}$.

Given the fractions π^j_{δ,t_n} of a strategy δ, the value of the portfolio $S^\delta_{t_n}$ at time t_n is recursively obtained by the relation

$$S^\delta_{t_n} = S^\delta_{t_{n-1}} \left(1 + \sum_{j=1}^{d} \pi^j_{\delta,t_{n-1}} \frac{S^j_{t_n} - S^j_{t_{n-1}}}{S^j_{t_{n-1}}} \right), \tag{2.17}$$

which is equivalent to

$$S^\delta_{t_n} = S^\delta_{t_{n-1}} + \sum_{j=1}^{d} \delta^j_{t_{n-1}} \left(S^j_{t_n} - S^j_{t_{n-1}} \right), \tag{2.18}$$

for $n \in \{1, 2, \ldots\}$ using (2.7).

The following simulation studies will illustrate the robustness of the diversification effect under various market models. For each market model the simulation of benchmarked primary security accounts will be performed exactly or at least almost exactly. This is highly important since the obtained accuracy allows to check the diversification phenomenon accurately over long periods of time.

For given market dynamics the study simulates for the long period of $T = 150$ years the benchmarked trajectories of $d = 1000$ primary security accounts, sampling 100 times per year. Since the interest rate is set to zero, the inverse of the benchmarked savings account provides the GOP when denominated in domestic currency, that is,

$$S^{\delta_*}_t = (\hat{S}^0_t)^{-1}. \tag{2.19}$$

The product of the GOP with the jth benchmarked primary security account yields the value of the jth primary security account denominated in domestic currency, that is,

$$S^j_t = \hat{S}^j_t \, S^{\delta_*}_t, \tag{2.20}$$

for $j \in \{0, 1, \ldots, d\}$.

In reality, the market capitalization of stocks is very different from each other. Statistical analysis of market data suggests that the size of companies and similarly the market capitalization of their stocks seem to be Pareto distributed, see Simon (1958). To generate realistic MCIs in the following simulations, the initial values S^j_0, $j \in \{0, 1, \ldots, d\}$, of primary security accounts follow here a Pareto distribution

$$F_j(x) = 1 - \left(\frac{x_0}{x} \right)^\lambda \tag{2.21}$$

with the parameters $\lambda = 1.1$ and $x_0 = \frac{\lambda - 1}{\lambda}$, see Simon (1958).

3 Multi-asset Black-Scholes Model

When using discrete time numerical schemes for the simulation of solutions of SDEs, as studied in Kloeden & Platen (1999), there may be issues arising when dealing with non-Lipschitz continuous drift or diffusion coefficients. Furthermore, problems concerning numerical stability may emerge for long-term simulations or even negative values could be simulated by standard discrete time approximations for strictly positive price processes, see Platen & Bruti-Liberati (2010). In the following sections *exact* and *almost exact* simulations of various market models are described. These avoid the above indicated numerical problems in scenario simulations when using standard discrete time approximations. For the following simulation studies market models have been selected where exact or almost exact simulations are possible. A contribution of this paper is, therefore, also the description of highly accurate scenario simulation methods for several frequently used market model classes in finance.

To begin with, recall first the simulation under the standard market model, which is the multi-asset version of the Black-Scholes model, see Black & Scholes (1973). Under this model the benchmarked primary security accounts satisfy the following vector SDE

$$d\hat{\boldsymbol{S}}_t = \sum_{k=1}^{d} \boldsymbol{B}^k \hat{\boldsymbol{S}}_t dW_t^k, \tag{3.1}$$

for $t \in [0, \infty)$. Here $\hat{\boldsymbol{S}} = \{\hat{\boldsymbol{S}}_t = (\hat{S}_t^1, \ldots, \hat{S}_t^d)^\top, t \in [0, \infty)\}$ is a vector of benchmarked primary security accounts, and $\boldsymbol{B}^k = [B^{k,i,j}]_{i,j=1}^d$ is a $d \times d$ diagonal volatility matrix, with elements

$$B^{k,i,j} = \begin{cases} b^{j,k} & \text{for} \quad i = j \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

for $k, i, j \in \{1, 2, \ldots, d\}$. Note that the benchmarked primary security accounts $\hat{S}^j$, $j \in \{1, 2, \ldots, d\}$, represent under this model martingales.

The multi-asset Black-Scholes model can be simulated exactly. The vector SDE (3.1) has an explicit solution. The jth benchmarked primary security account is represented by the exponential

$$\hat{S}_t^j = \hat{S}_0^j \exp\left\{-\frac{1}{2}\sum_{k=1}^{d}\left(b^{j,k}\right)^2 t + \sum_{k=1}^{d} b^{j,k} W_t^k\right\}. \tag{3.3}$$

Throughout all simulation studies the paper employs the equi-distant time discretization $0 = t_0 < t_1 < \ldots$, where $t_i = i\Delta$, $i \in \{0, 1, \ldots\}$. For simplicity, the simulation of benchmarked primary security accounts is performed for the case of independent benchmarked primary security accounts. However, for most of the market models the case of dependent benchmarked primary security accounts can

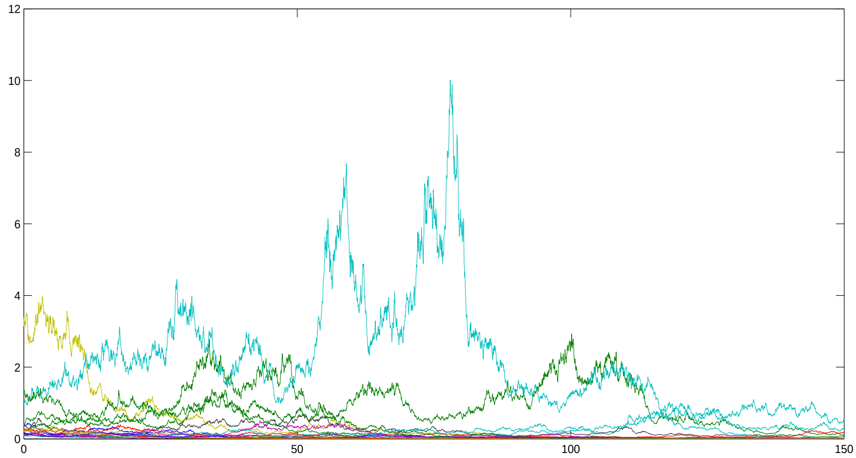

Figure 3.1: Simulated benchmarked primary security accounts under the Black-Scholes model

be handled analogously as shown in Platen & Rendek (2009). For the jth independent benchmarked primary security account one obtains at time t_{i+1} under the Black-Scholes model the exponential

$$\hat{S}^j_{t_{i+1}} = \hat{S}^j_0 \exp\left\{-\frac{1}{2}\left(b^{j,j}\right)^2 t_{i+1} + b^{j,j}\, W^j_{t_{i+1}}\right\}. \tag{3.4}$$

The phenomenon of diversification is now illustrated by simulating in a Black-Scholes market the GOP, the EWI and the MCI. This simulation can be exactly performed without generating any error by using the explicit expression described above. The study simulates $d = 1000$ benchmarked primary security accounts over 150 years, each with volatility $b^{j,j} = 0.2$ for $j \in \{1, \ldots, 1000\}$. The first 20 simulated trajectories are displayed in Fig. 3.1. Note that these benchmarked primary security accounts are modeled by martingales. As mentioned earlier, the independent initial values $\hat{S}^j_0$ are generated using a Pareto distribution. Consequently, there is great variety in the market capitalization of stocks.

Fig 3.2 shows the simulated trajectories of the GOP, the EWI and the MCI. In this case the EWI approximates rather well the GOP and the differences between both portfolio values are difficult to see, in particular, in the earlier parts of the trajectories. As one will see in Fig. 3.3 the MCI seems to be initially a reasonable proxy of the GOP, however, after some time it diverges from the GOP due to its lower long term performance. As can be seen in Fig. 3.1, a large total value for some stocks is present in the simulated market. The resulting large fractions of these stocks in the MCI are likely to distort its long run performance because the market portfolio appears to be not well diversified in the sense of this paper. The emerging fractions of the corresponding primary security accounts are probably too large to be acceptable as those of a DP under the conditions of the mentioned Diversification Theorem. This phenomenon becomes even clearer

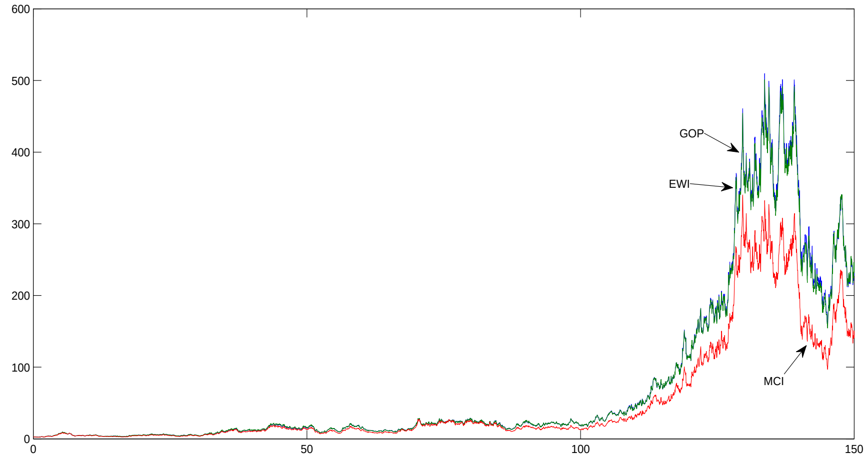

Figure 3.2: Simulated GOP, EWI and MCI under the Black-Scholes model

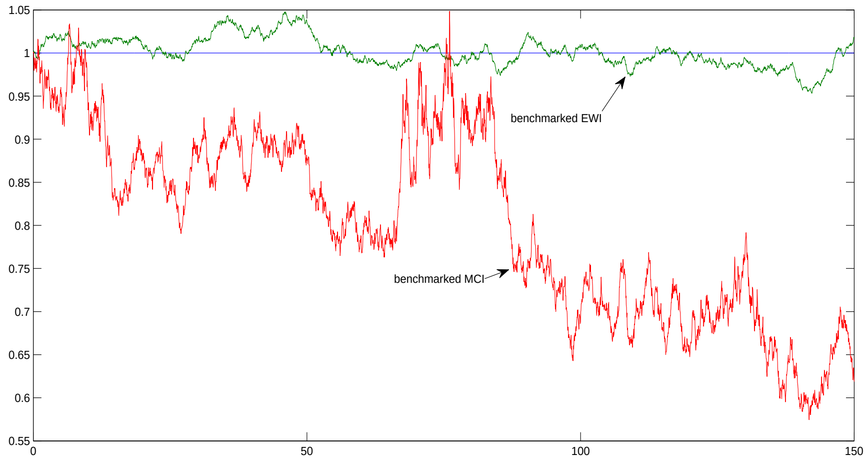

Figure 3.3: Simulated benchmarked GOP, EWI and MCI under the Black-Scholes model

in Fig. 3.3, which displays the constant benchmarked GOP, $\hat{S}_t^{\delta_*} = 1$, as well as, the benchmarked EWI and the benchmarked MCI. Note that initially, sometimes the benchmarked EWI and sometimes the benchmarked MCI performed better. However, in the long run the benchmarked EWI fluctuates around the benchmarked GOP, while the benchmarked MCI diverges downwards. Additionally, one notes in this figure that the benchmarked MCI has much larger volatility than the benchmarked EWI. This is typical also for the other market models considered and for the real market as well. The described simulation has been repeated many times for other scenarios. The better performance in the long run of the EWI over the MCI was similarly evident.

4 Multi-asset Heston Model

The multi-asset version of the Heston model, see Heston (1993), can be described by a set of two vector SDEs in the form

$$d\hat{\boldsymbol{S}}_t = \text{diag}\left(\sqrt{\boldsymbol{V}_t}\right)\text{diag}\left(\hat{\boldsymbol{S}}_t\right)\left(\boldsymbol{A}d\tilde{\boldsymbol{W}}_t^1 + \boldsymbol{B}d\tilde{\boldsymbol{W}}_t^2\right), \tag{4.1}$$

$$d\boldsymbol{V}_t = (\boldsymbol{a} - \boldsymbol{E}\boldsymbol{V}_t)\,dt + \boldsymbol{F}\text{diag}\left(\sqrt{\boldsymbol{V}_t}\right)d\tilde{\boldsymbol{W}}_t^1, \tag{4.2}$$

for $t \in [0,\infty)$. Here $\hat{\boldsymbol{S}} = \{\hat{\boldsymbol{S}}_t = (\hat{S}_t^1, \dots, \hat{S}_t^d)^\top, t \in [0,\infty)\}$ is a vector of benchmarked primary security accounts, which are local martingales but not necessarily martingales or strict supermartingales, see Platen & Heath (2006). Moreover, $\tilde{\boldsymbol{W}}^1 = \{\tilde{\boldsymbol{W}}_t^1 = (\tilde{W}_t^{1,1}, \tilde{W}_t^{1,2}, \dots, \tilde{W}_t^{1,d})^\top, t \in [0,\infty)\}$ and $\tilde{\boldsymbol{W}}^2 = \{\tilde{\boldsymbol{W}}_t^2 = (\tilde{W}_t^{2,1}, \tilde{W}_t^{2,2}, \dots, \tilde{W}_t^{2,d})^\top, t \in [0,\infty)\}$ are independent vectors of correlated Wiener processes. That is, one has

$$\tilde{\boldsymbol{W}}_t^k = \boldsymbol{C}^k \boldsymbol{W}_t^k, \tag{4.3}$$

where $\boldsymbol{C}^k = [C^{k,i,j}]_{i,j=1}^d$, and with $\boldsymbol{W}^k = \{\boldsymbol{W}_t^k = (W_t^{k,1}, W_t^{k,2}, \dots, W_t^{k,d})^\top, t \in [0,\infty)\}$ for $k \in \{1,2\}$, denoting again a vector of independent Wiener processes.

Additionally, $\boldsymbol{A} = [A^{i,j}]_{i,j=1}^d$ is a diagonal matrix with elements

$$A^{i,j} = \begin{cases} \varrho_i & \text{for} \quad i = j \\ 0 & \text{otherwise,} \end{cases} \tag{4.4}$$

and $\boldsymbol{B} = [B^{i,j}]_{i,j=1}^d$ is a diagonal matrix with elements

$$B^{i,j} = \begin{cases} \sqrt{1-\varrho_i^2} & \text{for} \quad i = j \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

Moreover, $\boldsymbol{V} = \{\boldsymbol{V}_t = (V_t^1, V_t^2, \dots, V_t^d)^\top, t \in [0,\infty)\}$ is a vector of squared volatilities, $\boldsymbol{a} = (a_1, a_2, \dots, a_d)^\top$; and $\boldsymbol{E} = [E^{i,j}]_{i,j=1}^d$ is a diagonal matrix with elements

$$E^{i,j} = \begin{cases} \kappa_i & \text{for} \quad i = j \\ 0 & \text{otherwise,} \end{cases} \tag{4.6}$$

and $\boldsymbol{F} = [F^{i,j}]_{i,j=1}^d$ is a diagonal matrix with elements

$$F^{i,j} = \begin{cases} \gamma_i & \text{for} \quad i = j \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

One method for the exact simulation of the Heston model has been suggested in Broadie & Kaya (2006). The study in the current paper uses a slightly different, possibly more convenient, almost exact simulation for the benchmarked primary security accounts under the Heston model. The method involves exact simulation of the squared volatility processes and some almost exact simulation of the independent benchmarked primary security accounts. The latter simulation is conditional on the exactly simulated trajectories of the squared volatilities, see Platen & Rendek (2009).

One obtains the value of the jth squared volatility $V^j_{t_{i+1}}$ at time t_{i+1}, $i \in \{0, 1, \ldots\}$, by sampling directly from the noncentral chi-square distribution $\chi'^2_{\nu_j}(\lambda_j)$ with ν_j degrees of freedom and noncentrality parameter λ_j, that is,

$$V^j_{t_{i+1}} = \frac{\gamma_j^2 \left(1 - \exp\{-\kappa_j \Delta\}\right)}{4\kappa_j} \chi'^2_{\nu_j} \left(\frac{4\kappa_j e^{-\kappa_j \Delta}}{\gamma_j^2 \left(1 - e^{-\kappa_j \Delta}\right)} V^j_{t_i} \right), \tag{4.8}$$

where $\nu_j = \frac{4a_j}{\gamma_j^2}$. Details on sampling from noncentral chi-square distributions can be found, for instance, in Glasserman (2004). The resulting simulation method for the jth squared volatility V^j is exact.

It remains to describe the almost exact simulation of the vector of the logarithms of benchmarked assets. By following first Broadie & Kaya (2006), the jth value $X^j_{t_{i+1}} = \ln\left(\hat{S}^j_{t_{i+1}}\right)$ at time t_{i+1} can be represented in the form

$$\begin{aligned} X^j_{t_{i+1}} &= X^j_{t_i} + \frac{\varrho_j}{\gamma_j}\left(V^j_{t_{i+1}} - V^j_{t_i} - a_j \Delta\right) + \left(\frac{\varrho_j \kappa_j}{\gamma_j} - \frac{1}{2}\right) \int_{t_i}^{t_{i+1}} V^j_u du \\ &\quad + \sqrt{1 - \varrho_j^2} \int_{t_i}^{t_{i+1}} \sqrt{V^j_u} dW^{2,j}_u. \end{aligned} \tag{4.9}$$

Furthermore, the distribution of the integral increment

$$\int_{t_i}^{t_{i+1}} \sqrt{V^j_u} dW^{2,j}_u, \tag{4.10}$$

given the path of V^j, is conditionally Gaussian with mean zero and variance $\int_{t_i}^{t_{i+1}} V^j_u \, du$, since V^j is independent of the Brownian motion $W^{2,j}$ for all $j \in \{1, 2, \ldots, d\}$. Moreover, one needs to evaluate the variance $\int_{t_i}^{t_{i+1}} V^j_u du$ conditioned on the path of the process V^j. The proposed almost exact simulation method uses as an approximation via the trapezoidal rule

$$\int_{t_i}^{t_{i+1}} V^j_u du \approx \frac{\Delta}{2}\left(V^j_{t_i} + V^j_{t_{i+1}}\right). \tag{4.11}$$

It is well-known that this quadrature rule generates by its symmetry excellent approximations, see Kloeden & Platen (1999). Consequently, one has the approximate conditionally Gaussian random variable

$$\int_{t_i}^{t_{i+1}} \sqrt{V^j_u} dW^{2,j}_u \approx \mathcal{N}\left(0, \frac{\Delta}{2}\left(V^j_{t_i} + V^j_{t_{i+1}}\right)\right) \tag{4.12}$$

to compute. This approximation can be achieved with practically negligible error by using a sufficiently small time step size. Since it is here the aim to illustrate

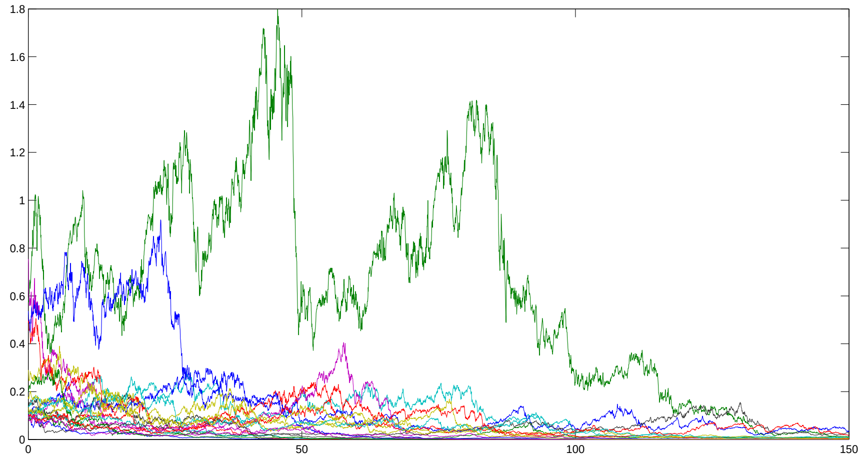

Figure 4.1: Simulated benchmarked primary security accounts under the Heston model

the diversification effect, the paper omits any particular error analysis. The above approximation converges in distribution as the time step size decreases, as can be deduced from (4.12). For the multi-asset Heston model this results in an efficient, almost exact simulation technique. Alternatively, one could have used the Broadie & Kaya (2006) exact simulation method, which leads from the perspective of this paper to the same conclusions.

Benchmarked primary security accounts are simulated under the multi-asset Heston model. Fig. 4.1 displays the first 20 simulated benchmarked primary security accounts. These benchmarked primary security accounts are nonnegative local martingales and, thus, supermartingales. The parameters in (4.8) are chosen according to market data, see Gatheral (2006). The jth squared volatility process is simulated according to (4.8) for the initial value $V_0^j = 0.0174$, $a_j = 0.0469$, $\kappa_j = 1.3253$, $\gamma_j = 0.3877$, $j \in \{1, \ldots, 1000\}$. The correlation parameter is here set to $\varrho_j = -0.7165$ in order to reflect the typically observed leverage effect, see Black (1976). The initial values $\hat{S}_0^j$ are again generated from the Pareto distribution mentioned earlier. For illustration, Fig. 4.2 displays a typical trajectory of the squared volatility process V^j under the Heston model, simulated exactly according to the formula (4.8).

Fig 4.3 exhibits the simulated GOP, EWI and MCI under the Heston model. Also here the EWI provides an excellent proxy for the GOP and it is difficult to distinguish both trajectories. The MCI, however, does not perform as well as the EWI. Finally, Fig. 4.4 illustrates the constant benchmarked GOP $\hat{S}_t^{\delta_*} = 1$, the benchmarked MCI and the benchmarked EWI. This plot shows the closeness of the benchmarked EWI to the benchmarked GOP. Note that, the benchmarked MCI has a larger volatility than the benchmarked EWI and fluctuates in the long run significantly.

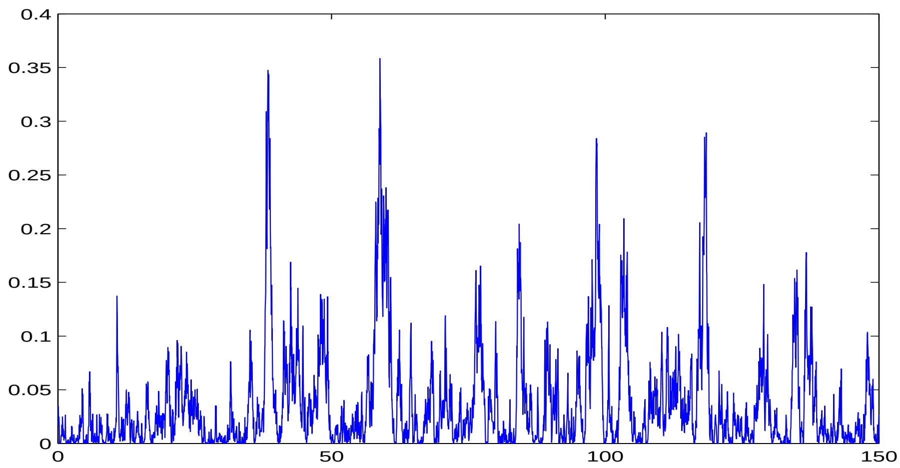

Figure 4.2: Simulated squared volatility under the Heston model

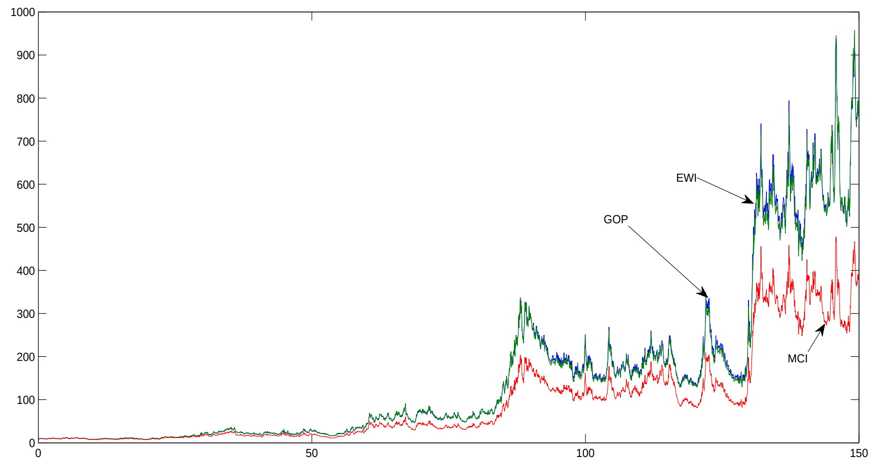

Figure 4.3: Simulated GOP, EWI and MCI under the Heston model

5 Multi-asset ARCH-diffusion Model

Some continuous time limits of popular time series models in finance, including several ARCH and GARCH models, can be captured by the multi-dimensional ARCH-diffusion model that is considered below. The class of ARCH and GARCH time series models was initiated in Engle (1982). The ARCH-diffusion model is obtained as a continuous time limit of the innovation process of the GARCH$(1,1)$ and NGARCH$(1,1)$ models, see Nelson (1990) and Frey (1997). The ARCH-diffusion model can be described by the following set of two vector SDEs

$$d\hat{\boldsymbol{S}}_t = \text{diag}\left(\sqrt{\boldsymbol{V}_t}\right)\text{diag}\left(\hat{\boldsymbol{S}}_t\right)\left(\boldsymbol{A}d\tilde{\boldsymbol{W}}_t^1 + \boldsymbol{B}d\tilde{\boldsymbol{W}}_t^2\right), \quad (5.1)$$

$$d\boldsymbol{V}_t = (\boldsymbol{a} - \boldsymbol{E}\boldsymbol{V}_t)\,dt + \boldsymbol{F}\text{diag}\left(\boldsymbol{V}_t\right)d\tilde{\boldsymbol{W}}_t^1, \quad (5.2)$$

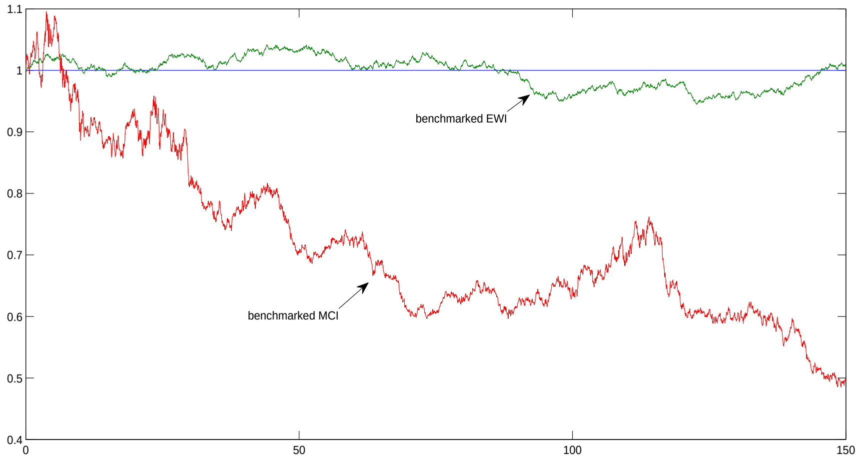

Figure 4.4: Simulated benchmarked GOP, EWI and MCI under the Heston model

for $t \in [0, \infty)$. Here $\hat{\boldsymbol{S}} = \{\hat{\boldsymbol{S}}_t = (\hat{S}_t^1, \ldots, \hat{S}_t^d)^\top, t \in [0, \infty)\}$ denotes again a vector of benchmarked primary security accounts, where one knows that these are local martingales and, thus, supermartingales. Furthermore, $\tilde{\boldsymbol{W}}^1 = \{\tilde{\boldsymbol{W}}_t^1 = (\tilde{W}_t^{1,1}, \tilde{W}_t^{1,2}, \ldots, \tilde{W}_t^{1,d})^\top, t \in [0, \infty)\}$ and $\tilde{\boldsymbol{W}}^2 = \{\tilde{\boldsymbol{W}}_t^2 = (\tilde{W}_t^{2,1}, \tilde{W}_t^{2,2}, \ldots, \tilde{W}_t^{2,d})^\top,$ $t \in [0, \infty)\}$ are independent vectors of correlated Wiener processes. Additionally, $\boldsymbol{A} = [A^{i,j}]_{i,j=1}^d$ is a diagonal matrix with elements as in (4.4), and $\boldsymbol{B} = [B^{i,j}]_{i,j=1}^d$ is a diagonal matrix with elements given in (4.5). Moreover, $\boldsymbol{V} = \{\boldsymbol{V}_t = (V_t^1, V_t^2, \ldots, V_t^d)^\top, t \in [0, \infty)\}$ is a vector of squared volatilities, $\boldsymbol{a} = (a_1, a_2, \ldots, a_d)^\top$ is a vector; $\boldsymbol{E} = [E^{i,j}]_{i,j=1}^d$ is a diagonal matrix with elements as in (4.6); and $\boldsymbol{F} = [F^{i,j}]_{i,j=1}^d$ is a diagonal matrix with elements as in (4.7).

In the given case one can simulate the jth squared volatility process V^j, $j \in \{1, 2, \ldots\}$, almost exactly by approximating the time integral via the trapezoidal rule in the following exact representation

$$\begin{aligned} V_{t_{i+1}}^j &= \exp\left\{\left(-\kappa_j - \frac{1}{2}\gamma_j^2\right) t_{i+1} + \gamma_j W_{t_{i+1}}^{1,j}\right\} \\ &\quad \times \left(V_{t_0}^j + a_j \sum_{k=0}^{i} \int_{t_k}^{t_{k+1}} \exp\left\{\left(\kappa_j + \frac{1}{2}\gamma_j^2\right) s - \gamma_j W_s^{1,j}\right\} ds\right). \end{aligned} \tag{5.3}$$

This yields the approximation

$$\begin{aligned} V_{t_{i+1}}^{j,\Delta} &= \exp\left\{\left(-\kappa_j - \frac{1}{2}\gamma_j^2\right) t_{i+1} + \gamma_j W_{t_{i+1}}^{1,j}\right\} \\ &\quad \times \left(V_{t_0}^j + a_j \frac{\Delta}{2} \sum_{k=0}^{i} \left[\exp\left\{\left(\kappa_j + \frac{1}{2}\gamma_j^2\right) t_k - \gamma_j W_{t_k}^{1,j}\right\}\right.\right. \\ &\quad \left.\left. + \exp\left\{\left(\kappa_j + \frac{1}{2}\gamma_j^2\right) t_{k+1} - \gamma_j W_{t_{k+1}}^{1,j}\right\}\right]\right) \end{aligned} \tag{5.4}$$

for $i \in \{0, 1, \ldots\}$.

The following describes the almost exact simulation of the logarithms of stocks $X^j_{t_{i+1}} = \ln\left(\hat{S}^j_{t_{i+1}}\right)$, $j \in \{1, 2, \ldots, d\}$. One can represent the value of $X^j_{t_{i+1}}$ at time t_{i+1} as

$$\begin{aligned} X^j_{t_{i+1}} &= X^j_{t_i} - \frac{1}{2}\int_{t_i}^{t_{i+1}} V^j_u du + \frac{2\varrho_j}{\gamma_j}\left(\sqrt{V^j_{t_{i+1}}} - \sqrt{V^j_{t_i}}\right) \\ &\quad -\frac{2\varrho_j}{\gamma_j}\int_{t_i}^{t_{i+1}}\left(\frac{a_j}{2\sqrt{V^j_u}} - \left(\frac{\kappa_j}{2} + \frac{\gamma_j^2}{8}\right)\sqrt{V^j_u}\right) du \\ &\quad +\sqrt{1-\varrho_j^2}\int_{t_i}^{t_{i+1}}\sqrt{V^j_u}\, dW^{2,j}_u. \end{aligned} \tag{5.5}$$

Furthermore, the distribution of

$$\int_{t_i}^{t_{i+1}}\sqrt{V^j_u}\, dW^{2,j}_u, \tag{5.6}$$

conditioned on the path of V^j, is conditionally Gaussian with mean zero and variance $\int_{t_i}^{t_{i+1}} V^j_u du$ because V^j is independent of the Brownian motion $W^{2,j}$ for all $j \in \{1, 2, \ldots, d\}$. Moreover, it is possible to approximate $\int_{t_i}^{t_{i+1}} V^j_u du$ given the path of the process V^j. One can use here again the trapezoidal approximation

$$\int_{t_i}^{t_{i+1}} V^j_u du \approx \frac{\Delta}{2}\left(V^j_{t_i} + V^j_{t_{i+1}}\right) \tag{5.7}$$

to obtain the conditionally Gaussian integral increment

$$\int_{t_i}^{t_{i+1}}\sqrt{V^j_u}\, dW^{2,j}_u \approx \mathcal{N}\left(0, \frac{\Delta}{2}\left(V^j_{t_i} + V^j_{t_{i+1}}\right)\right). \tag{5.8}$$

Similarly, it is possible to approximate the second integral on the right hand side of (5.5) given in the form

$$\begin{aligned} &\int_{t_i}^{t_{i+1}}\left(\frac{a_j}{2\sqrt{V^j_u}} - \left(\frac{\kappa_j}{2} + \frac{\gamma_j^2}{8}\right)\sqrt{V^j_u}\right) du \approx \\ &\frac{\Delta}{2}\left(\frac{a_j}{2\sqrt{V^j_{t_i}}} - \left(\frac{\kappa_j}{2} + \frac{\gamma_j^2}{8}\right)\sqrt{V^j_{t_i}} + \frac{a_j}{2\sqrt{V^j_{t_{i+1}}}} - \left(\frac{\kappa_j}{2} + \frac{\gamma_j^2}{8}\right)\sqrt{V^j_{t_{i+1}}}\right). \end{aligned} \tag{5.9}$$

This approximation can be achieved with high accuracy when the time step size is small by using again the trapezoidal quadrature formula. In this manner one obtains an efficient almost exact simulation technique for the multi-asset ARCH-diffusion model, which converges in distribution as the time step size decreases.

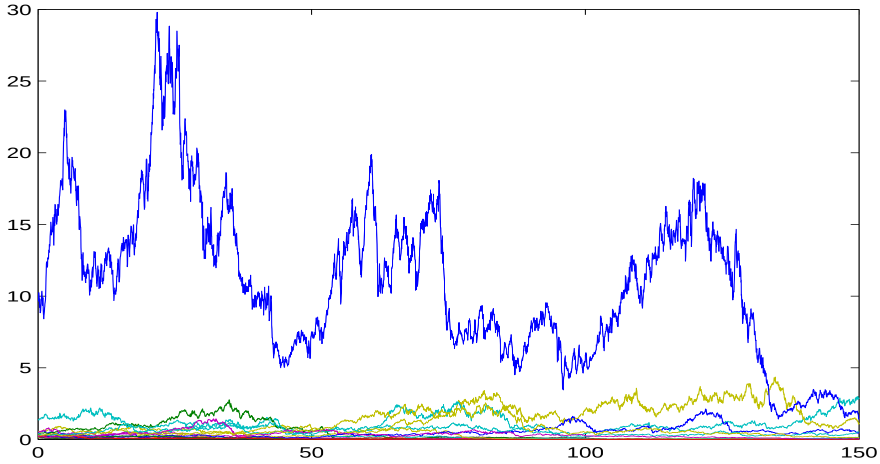

Figure 5.1: Simulated benchmarked primary security accounts under the ARCH-diffusion model

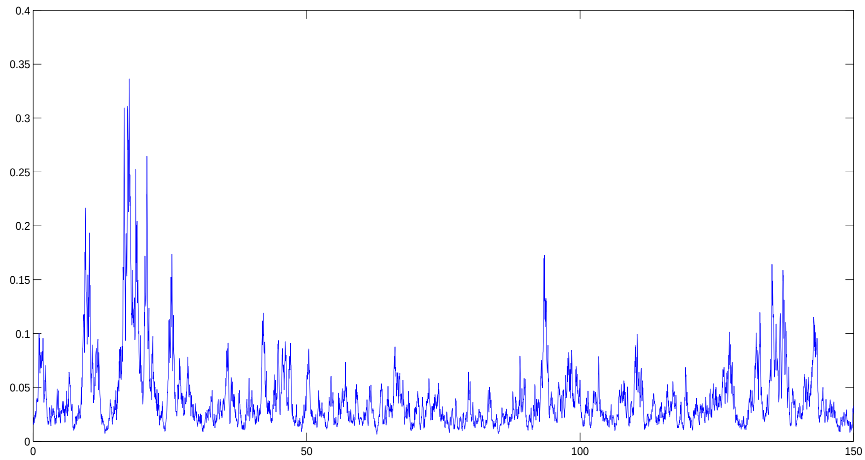

Figure 5.2: Simulated squared volatility under the ARCH-diffusion model

Now, benchmarked primary security accounts are simulated as multi-dimensional ARCH diffusions. For simplicity, the same squared volatility process is used for all benchmarked primary security accounts, where $a = 0.0469$, $\kappa = 1.3253$, $\gamma = 1$ and $V_0 = 0.0174$, similar as for the Heston squared volatility model of the previous section. Furthermore, the driving noise of each of the benchmarked asset prices is correlated with $\varrho = -0.7165$ to the noise that drives the squared volatility process. The extra Wiener processes that drive the benchmarked asset prices are independent from each other. Fig. 5.1 shows the first 20 simulated benchmarked risky primary security accounts with Pareto distributed initial values. A typical trajectory of the squared volatility under the ARCH-diffusion model is displayed in Fig. 5.2.

Fig. 5.3 shows the constant benchmarked GOP, the benchmarked EWI and the

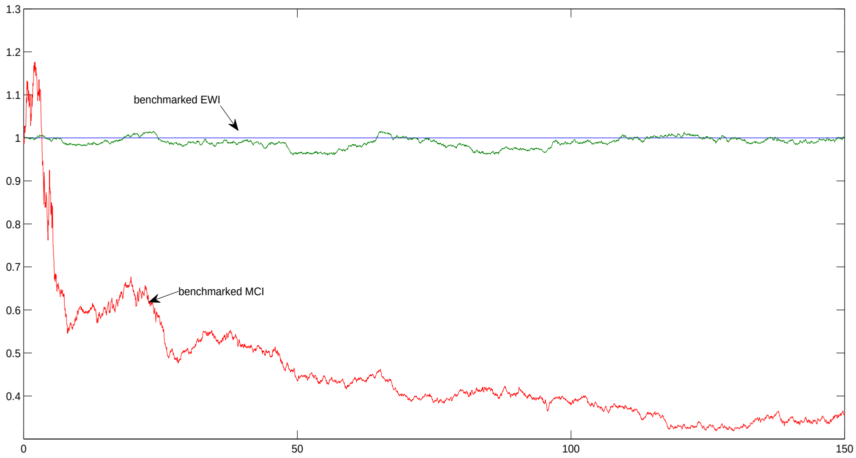

Figure 5.3: Simulated benchmarked GOP, EWI and MCI under the ARCH-diffusion model

benchmarked MCI. Also here the EWI appears to be an excellent proxy of the GOP, while the MCI diverges from the GOP in the long run. Also here one notes the smaller volatility of the benchmarked EWI compared to the benchmarked MCI.

6 Geometric Ornstein-Uhlenbeck Volatility Model

One can generate benchmarked asset prices also by using as stochastic volatility a geometric Ornstein-Uhlenbeck process, as proposed in Bergomi (2004) for modeling volatility derivatives. The geometric Ornstein-Uhlenbeck volatility model can be described by the following two vector SDEs

$$d\hat{\boldsymbol{S}}_t = \operatorname{diag}\left(\exp\{\boldsymbol{V}_t\}\right)\operatorname{diag}\left(\hat{\boldsymbol{S}}_t\right)\left(\boldsymbol{A}d\tilde{\boldsymbol{W}}^1_t + \boldsymbol{B}d\tilde{\boldsymbol{W}}^2_t\right), \quad (6.1)$$

$$d\boldsymbol{V}_t = \left(\boldsymbol{a} - \boldsymbol{E}\boldsymbol{V}_t\right)dt + \boldsymbol{F}d\tilde{\boldsymbol{W}}^1_t, \quad (6.2)$$

for $t \in [0,\infty)$. Here $\hat{\boldsymbol{S}} = \{\hat{\boldsymbol{S}}_t = (\hat{S}^1_t,\dots,\hat{S}^d_t)^\top, t \in [0,\infty)\}$ denotes again a vector of benchmarked primary security accounts. Furthermore, $\tilde{\boldsymbol{W}}^1 = \{\tilde{\boldsymbol{W}}^1_t = (\tilde{W}^{1,1}_t, \tilde{W}^{1,2}_t,\dots,\tilde{W}^{1,d}_t)^\top, t \in [0,\infty)\}$ and $\tilde{\boldsymbol{W}}^2 = \{\tilde{\boldsymbol{W}}^2_t = (\tilde{W}^{2,1}_t, \tilde{W}^{2,2}_t,\dots,\tilde{W}^{2,d}_t)^\top,$ $t \in [0,\infty)\}$ are independent vectors of correlated Wiener processes. Additionally, $\boldsymbol{A} = [A^{i,j}]^d_{i,j=1}$ is a diagonal matrix with elements as in (4.4), and $\boldsymbol{B} = [B^{i,j}]^d_{i,j=1}$ is a diagonal matrix with elements (4.5). Moreover, the elementwise exponential

$$\exp\{\boldsymbol{V}\} = \left\{\exp\{\boldsymbol{V}_t\} = \left(\exp\{V^1_t\},\exp\{V^2_t\},\dots,\exp\{V^d_t\}\right)^\top, t \in [0,\infty)\right\} \quad (6.3)$$

represents a vector of volatilities, whose elements are correlated exponents of the Ornstein-Uhlenbeck processes. Additionally $\boldsymbol{a} = (a_1, a_2, \dots, a_d)^\top$ is a vector;

$\boldsymbol{E} = [E^{i,j}]_{i,j=1}^d$ is a diagonal matrix with elements as in (4.6); and $\boldsymbol{F} = [F^{i,j}]_{i,j=1}^d$ is a diagonal matrix with elements as in (4.7).

As before, one can simulate independent benchmarked primary security accounts. The value of the jth volatility, $\exp\left\{V_{t_i}^j\right\}$, is obtained at time t_i, $i \in \{0, 1, \dots\}$, by the following exact expression

$$\exp\{V_{t_i}^j\} = \exp\left\{V_0^j e^{-\kappa_j t_i} + \frac{a_j}{\kappa_j}\left(1 - e^{-\kappa_j t_i}\right) + \gamma_j e^{-\kappa_j t_i} \sum_{k=1}^{i} \int_{t_{k-1}}^{t_k} e^{\kappa_j s} dW_s^{1,j}\right\}, \tag{6.4}$$

where

$$\int_{t_{k-1}}^{t_k} e^{\kappa_j s} dW_s^{1,j} \sim \mathcal{N}\left(0, \frac{1}{2\kappa}\left(e^{2\kappa_j t_k} - e^{2\kappa_j t_{k-1}}\right)\right). \tag{6.5}$$

Note, the resulting simulation method for generating $\exp\{V_{t_i}^j\}$ is exact.

The following describes the almost exact simulation of $X_{t_{i+1}}^j = \ln\left(\hat{S}_{t_{i+1}}^j\right)$. One can represent the jth value of $X_{t_{i+1}}^j$ at time t_{i+1} as

$$\begin{aligned} X_{t_{i+1}}^j \quad = \quad & X_{t_i}^j - \frac{1}{2}\int_{t_i}^{t_{i+1}} e^{2V_u^j} du + \frac{\varrho_j}{\gamma_j}\left(e^{V_{t_{i+1}}^j} - e^{V_{t_i}^j}\right) \\ & -\frac{\varrho_j}{\gamma_j}\int_{t_i}^{t_{i+1}} \left(\left(a_j + \frac{\gamma_j^2}{2}\right) e^{V_u^j} - \kappa_j V_u^j e^{V_u^j}\right) du \\ & +\sqrt{1 - \varrho_j^2}\int_{t_i}^{t_{i+1}} e^{V_u^j} dW_u^{2,j}. \end{aligned} \tag{6.6}$$

Furthermore, the distribution of

$$\int_{t_i}^{t_{i+1}} e^{V_u^j} dW_u^{2,j}, \tag{6.7}$$

conditioned on the path of V^j, is conditionally Gaussian with mean zero and variance $\int_{t_i}^{t_{i+1}} e^{2V_u^j}\ du$, because V^j is independent of the Brownian motion $W^{2,j}$ for all $j \in \{1, 2, \dots, d\}$. Moreover, it is possible to approximate $\int_{t_i}^{t_{i+1}} e^{2V_u^j} du$ given the path of the process V^j. One can use here again the trapezoidal approximation

$$\int_{t_i}^{t_{i+1}} e^{2V_u^j} du \approx \frac{\Delta}{2}\left(e^{2V_{t_i}^j} + e^{2V_{t_{i+1}}^j}\right) \tag{6.8}$$

to obtain the conditionally Gaussian random variable

$$\int_{t_i}^{t_{i+1}} e^{V_u^j} dW_u^{2,j} \approx \mathcal{N}\left(0, \frac{\Delta}{2}\left(e^{2V_{t_i}^j} + e^{2V_{t_{i+1}}^j}\right)\right). \tag{6.9}$$

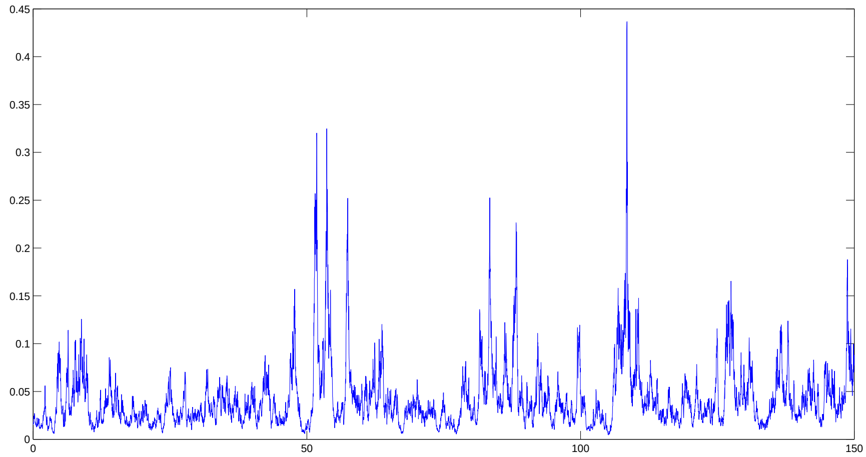

Figure 6.1: Simulated squared volatility under the geometric Ornstein-Uhlenbeck volatility model

Similarly, it is possible to approximate the second integral on the right hand side of (6.6) in the form

$$\int_{t_i}^{t_{i+1}} \left(\left(a_j + \frac{\gamma_j^2}{2} \right) e^{V_u^j} - \kappa_j V_u^j e^{V_u^j} \right) du \approx \tag{6.10}$$
$$\frac{\Delta}{2} \left(\left(a_j + \frac{\gamma_j^2}{2} \right) e^{V_{t_i}^j} - \kappa_j V_{t_i}^j e^{V_{t_i}^j} + \left(a_j + \frac{\gamma_j^2}{2} \right) e^{V_{t_{i+1}}^j} - \kappa_j V_{t_{i+1}}^j e^{V_{t_{i+1}}^j} \right).$$

Also this approximation can be achieved with any desired accuracy by choosing a small enough time step size, where the approximation converges in distribution. In this manner one obtains an efficient almost exact simulation method by conditioning for the multi-asset geometric Ornstein-Uhlenbeck volatility model.

Now, one can simulate the benchmarked primary security accounts from a multidimensional geometric Ornstein-Uhlenbeck volatility model of the above form. The squared volatility process e^{2V^j} is calibrated similarly to the Heston squared volatility process. Here one sets $a_j = -2.2143$, $\kappa_j = 1.3253$, $\gamma_j = 0.52$ and $V_0^j = -2.0257$, $j \in \{1, \dots, 1000\}$. The driving noise of each of the benchmarked asset prices is assumed to be correlated with $\varrho_j = -0.7165$ to the corresponding squared volatility process. The extra noise sources driving the benchmarked primary security account process are assumed to be independent from the others. A typical trajectory of the squared volatility under the geometric Ornstein-Uhlenbeck volatility model is displayed in Fig. 6.1.

Fig. 6.2 exhibits the constant benchmarked GOP, the benchmarked EWI and the benchmarked MCI. Also here the EWI appears to be a very good proxy of the GOP. As before, the benchmarked MCI diverges from the benchmarked GOP in the long run and has a larger volatility than the benchmarked EWI.

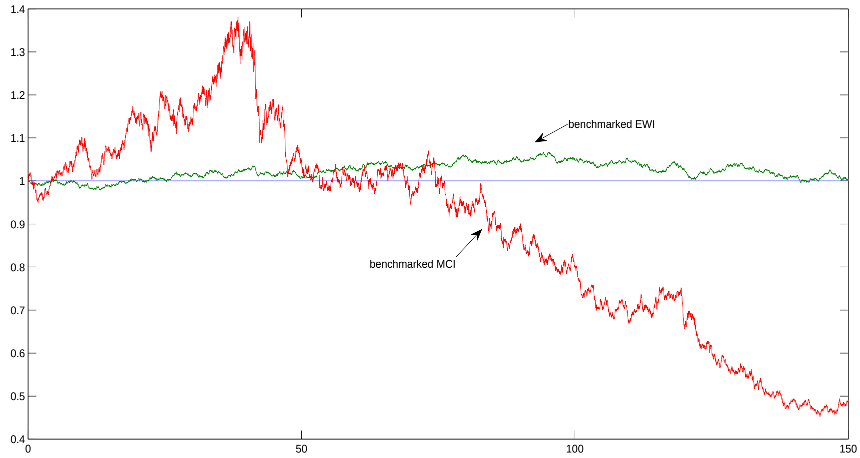

Figure 6.2: Simulated benchmarked GOP, EWI and MCI under the geometric Ornstein-Uhlenbeck volatility model

7 Stylized Minimal Market Model

The previous models can be interpreted as rather direct generalizations of the Black-Scholes model, obtained by introducing some stochastic volatility process. In most versions of these models, when applied in practice, the benchmarked primary security accounts are rather close to martingales. Consider now the stylized minimal market model (MMM), which is similar to the version of the MMM described in Platen (2001) and Platen & Heath (2006). This model generates by its nature strict supermartingales as benchmarked primary security accounts out of scalar diffusion processes. Each benchmarked primary security account is the inverse of a time transformed squared Bessel process of dimension four and, thus, a strict supermartingale, see Revuz & Yor (1999). The MMM models the jth benchmarked primary security account by the expression

$$\hat{S}_t^j = \frac{1}{Y_t^j \alpha_t^j}, \tag{7.1}$$

where $\alpha_t^j = \alpha_0^j \exp\{\eta^j t\}$, $j \in \{1,\ldots,d\}$. Here η^j is the jth net growth rate for $j \in \{1,\ldots,d\}$, and Y_t^j is the time t value of the square root process Y^j, which satisfies the SDE

$$dY_t^j = \left(1 - \eta^j Y_t^j\right) dt + \sqrt{Y_t^j}\, d\tilde{W}_t^j \tag{7.2}$$

for $t \in [0,\infty)$, where $Y_0^j = \frac{1}{\eta^j}$ for $j \in \{1,\ldots,d\}$. Here $\tilde{\boldsymbol{W}} = \{\tilde{\boldsymbol{W}}_t = (\tilde{W}_t^1, \tilde{W}_t^2, \ldots, \tilde{W}_t^d)^\top, t \in [0,\infty)\}$ is a vector of correlated Wiener processes.

Note that under the MMM, $S^j(\varphi^j(t)) = Y_t^j \alpha_t^j$ is a squared Bessel process of dimension four in the, so called, φ^j-time. That is, one has the SDE

$$dS^j(\varphi^j(t)) = 4d\varphi^j(t) + 2\sqrt{S^j(\varphi^j(t))}\, d\bar{W}^j(\varphi^j(t)) \tag{7.3}$$

for $t \in [0, \infty)$, where

$$\varphi^j(t) = \frac{\alpha_0^j}{4\eta^j} \left(\exp\{\eta^j t\} - 1\right) \tag{7.4}$$

and

$$d\bar{W}^j(\varphi^j(t)) = d\tilde{W}_t^j \sqrt{\frac{d\varphi^j(t)}{dt}}. \tag{7.5}$$

The inverse $\hat{S}^j(\varphi^j(t))$ of the jth squared Bessel process of dimension four in φ^j-time satisfies the SDE

$$d\hat{S}^j(\varphi^j(t)) = -2 \left(\hat{S}^j(\varphi^j(t))\right)^{\frac{3}{2}} d\bar{W}^j(\varphi^j(t)) \tag{7.6}$$

for $t \in [0, \infty)$. Note that, $\hat{S}^j$ is a local martingale. More precisely, it can be shown that $\hat{S}^j$ is a nonnegative strict local martingale and, thus, a strict supermartingale, see Revuz & Yor (1999).

It remains to explain the exact simulation of benchmarked primary security accounts under the MMM. Given the time discretization $0 < t_0 < t_1 < \ldots$, where $t_i = i\Delta$, $i \in \{0, 1, \ldots\}$, one first generates by (7.4) for $j \in \{0, 1, \ldots, d\}$ the φ^j-time at the physical time t_i.

The next step is to simulate four independent Wiener processes $\bar{W}^{k,j}$, $k \in \{1, 2, 3, 4\}$, in φ^j-time. This can be achieved by calculating

$$\bar{W}_{t_{i+1}}^{k,j} = \bar{W}_{t_i}^{k,j} + \sqrt{\varphi^j(t_{i+1}) - \varphi^j(t_i)} Z_{i+1}^k, \tag{7.7}$$

where $Z_{i+1}^k \sim \mathcal{N}(0, 1)$ is a standard Gaussian random variable. Here $k \in \{1, 2, 3, 4\}$, $j \in \{0, 1, \ldots, d\}$ and $i \in \{0, 1, \ldots\}$.

Then the jth benchmarked primary security account at time t_{i+1} is obtained by the expression

$$\hat{S}_{t_{i+1}}^j = \hat{S}^j\left(\varphi^j(t_{i+1})\right) = \frac{1}{\sum_{k=1}^4 \left(w^k + \bar{W}_{t_{i+1}}^{k,j}\right)^2}, \tag{7.8}$$

for $i \in \{0, 1, \ldots\}$, where $\hat{S}_0^j = \sum_{k=1}^4 (w^k)^2$.

Now, one can simulate the independent benchmarked risky primary security accounts according to the MMM dynamics. The simulation uses the net growth rate $\eta^j = 0.09$ and the scaling parameter $\alpha_0^j = 0.05$. Fig. 7.1 plots the first 20 simulated benchmarked primary security accounts. These processes are strict supermartingales and one clearly observes their systematic long run downward trend. Remarkable are the extreme values of benchmarked asset prices that typically appear from time to time under the MMM. The jth volatility equals under the MMM at time t_i the expression $\hat{S}_{t_i}^j \alpha_{t_i}^j$. Fig. 7.2 plots a typical path of squared volatility under the MMM.

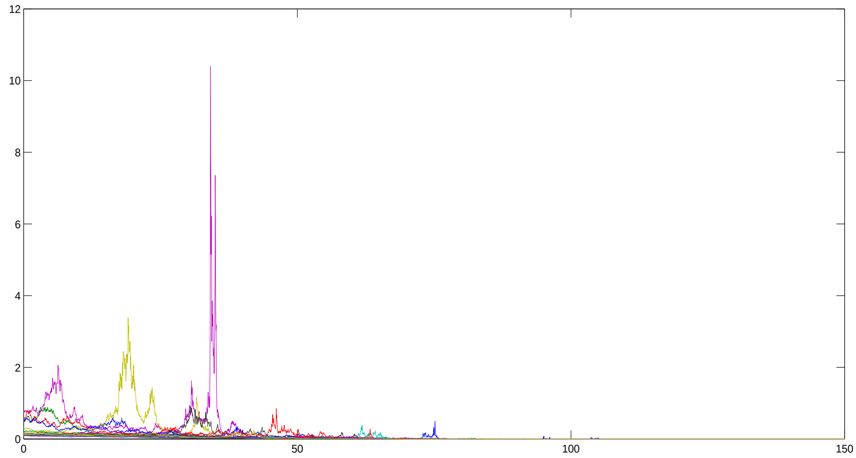

Figure 7.1: Simulated benchmarked primary security accounts under the MMM

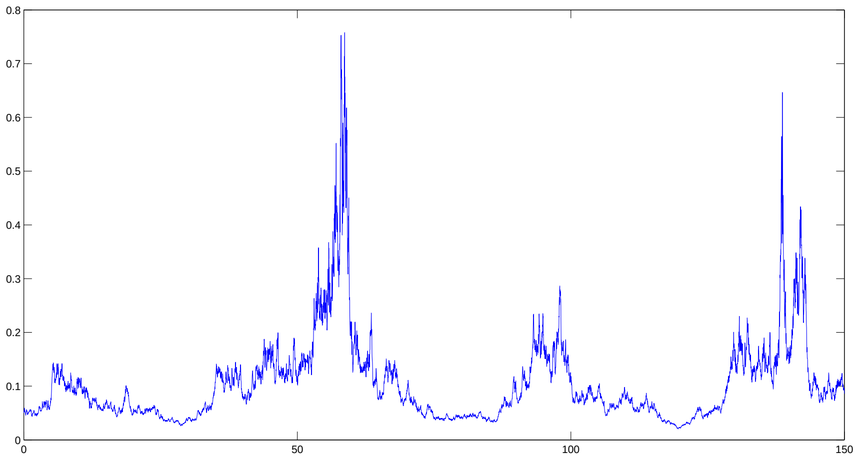

Figure 7.2: Simulated squared volatility under the MMM

The constant benchmarked GOP, benchmarked EWI and benchmarked MCI are shown in Fig. 7.4. The EWI represents a good proxy for the GOP. In this case the benchmarked MCI does by far not reach the long run performance of the benchmarked EWI. One observes a qualitative difference between these two processes. This difference is explained by the fact that the benchmarked MCI, as the sum of strict supermartingales, is a strict supermartingale. It therefore exhibits a significant long run downward trending mean. On the other hand, the return process of the EWI, by keeping the fractions constant, becomes a martingale with asymptotically vanishing fluctuations as $d \to \infty$. Consequently, the return process for the benchmarked EWI tends for $d \to \infty$ to zero, which makes its benchmarked value to a constant. Equal value weighting can be interpreted as a form of hedging. It turns out that one needs to hedge to avoid the strict supermartingale property of the constituents to be inherited by the resulting benchmarked portfolio. It is not

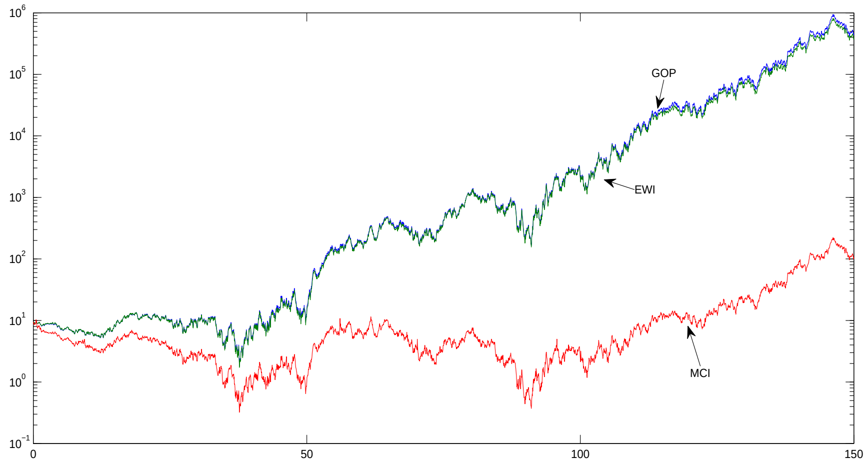

Figure 7.3: Simulated GOP, EWI and MCI under the MMM model in log-scale

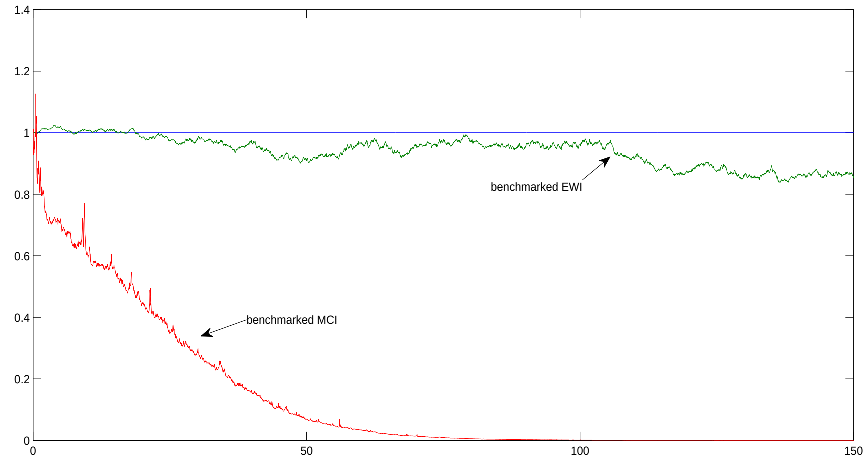

Figure 7.4: Simulated benchmarked GOP, EWI and MCI under the MMM model

enough to diversify over the constituents as happens in the MCI. The exposure to each source of uncertainty needs to be controlled by some kind of hedging. In this paper the hedging is facilitated by equal value weighting.

The important observation of this paper is that if the strict supermartingale property of benchmarked primary security accounts is what needs to be modeled to reflect realistically the real market dynamics, then portfolio management should focus on well diversified portfolios that when benchmarked form martingales. These portfolios when hedged in an appropriate form should perform in the long run better than market capitalization weighted portfolios.

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