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Research Paper 279

June 2010

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ISSN 1441-8010

www.qfrc.uts.edu.au

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ABSTRACT. The context for this article is a continuous financial market consisting of a risk-free savings account and a single non-dividend-paying risky security. We present two concrete models for this market, in which strict local martingales play decisive roles. The first admits an equivalent risk-neutral probability measure under which the discounted price of the risky security is a strict local martingale, while the second model does not even admit an equivalent risk-neutral probability measure, since the putative density process for such a measure is itself a strict local martingale. We highlight a number of apparent anomalies associated with both models that may offend the sensibilities of the classically-educated reader. However, we also demonstrate that these issues are easily resolved if one thinks economically about the models in the right way. In particular, we argue that there is nothing inherently objectionable about either model.

1. INTRODUCTION

The setting for this article is a continuous financial market consisting of a single non-dividend-paying risky security and a risk-free money-market account. We consider two models for the price of the risky security, both of which are affected by the presence of strict local martingales. The first model admits an equivalent risk-neutral probability measure under which the discounted price of the risky security is a strict local martingale. In particular, this model satisfies the strong no-arbitrage requirement of “no free lunch with vanishing risk” (NFLVR) (see Delbaen and Schachermayer [1994b]). The second model, on the other hand, does not even admit an equivalent risk-neutral probability measure, since the density process associated with the putative equivalent risk-neutral probability measure is a strict local martingale. Nevertheless, we shall see that this model does possess a numéraire portfolio, which ensures that it satisfies the weaker no-arbitrage requirement of “no unbounded profit with bounded risk” (NUPBR) (see Karatzas and Kardaras [2007]).

Following the terminology introduced by Heston et al. [2007], we shall say that the first model is an example of a “stock price bubble,” while the second model is an example of a “bond price bubble.” Stock price bubbles have recently become a popular topic in the literature (see e.g. Cox and Hobson [2005], Ekström and Tysk [2009], Heston et al. [2007], Jarrow et al. [2007a,b] and Pal and Protter [2008]). Although such models remain firmly within the ambit of the risk-neutral approach to contingent claim pricing, they nevertheless exhibit a number of anomalies that have occupied the research articles cited above. For example, the risk-neutral forward price of the risky security is strictly less than its spot price grossed-up at the risk-free interest rate. Furthermore, the risk-neutral prices of European puts and calls on the risky security no longer obey the put-call parity relationship.

By contrast, bond price bubbles have received comparatively little attention in the literature—possibly due to a misguided concern about their arbitrage properties. The only obvious exception is the so-called “benchmark approach” of Platen and Heath [2006], which develops a systematic approach to the pricing and hedging of contingent claims in financial markets that

Date: April 6, 2010.

2000 Mathematics Subject Classification. Primary: 91B70; Secondary: 60G35, 60G44, 91B28.

Key words and phrases. Strict local martingales; arbitrage; Bessel processes; stock price bubbles; bond price bubbles; risk-neutral pricing; real-world pricing; hedging portfolios; replicating portfolios; put-call parity.

possess a numéraire portfolio, without necessarily admitting an equivalent risk-neutral probability measure. The resulting pricing methodology is referred to as “real-world pricing,” since it does not rely on any equivalent change of probability measure. At first blush, the real-world prices of contingent claims for the bond price bubble also appear to exhibit a number of anomalies. For example, the real-world price of a zero-coupon bond turns out to be strictly less than the discounted value of its principal. It is also possible to generate a risk-free profit from no initial investment, by constructing a portfolio whose value is bounded from below.

In this article we analyze the pricing and hedging of contingent claims written on the risky security, for both the stock price bubble and the bond price bubble. Our main objective is to demonstrate that the anomalies associated with strict local martingales are actually not so strange after all. In particular, if one thinks economically about the strict local martingales in our two models, then the apparently strange behaviour of contingent claim prices begins to seem quite natural.

The remainder of the paper is structured as follows: First Section 2 introduces the two models, both of which are based on a Bessel process of dimension three. Section 3 then derives pricing formulae and hedge ratios for a number of European contingent claims written on the the risky security, for the case of the stock price bubble. By analyzing the behaviour of these pricing formulae and hedge ratios, we argue that the anomalies associated with the claim prices for the stock price bubble are easily resolved. Section 4 then performs a similar analysis, for the case of the bond price bubble.

2. AN OVERVIEW OF THE TWO MODELS

As stated in the introduction, the subject of this article is a financial market consisting of a single non-dividend-paying risky security and a risk-free money-market account. We are concerned with the following two models for the price S of the risky security:

$$dS_t = -S_t^2 d\beta_t \tag{2.1a}$$

and

$$dS_t = \frac{1}{S_t} dt + d\beta_t, \tag{2.1b}$$

for all $t \in \mathbb{R}_+$. Here β is a standard Brownian motion residing on a filtered probability space $(\Omega, \mathcal{F}, \mathfrak{F}, \mathbf{P})$, whose filtration $\mathfrak{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$ satisfies the usual conditions of completeness and right-continuity. We also adopt the standard convention of assuming that the risk-free interest rate is zero, so that the value of the money-market account is given by $B = 1$.

Observe that the process specified by (2.1a) is by inspection a local martingale. The first of the two models presented above thus clearly satisfies the NFLVR condition, since $\mathbf{P}$ is itself the (unique) equivalent risk-neutral probability measure. In particular, the density process for the equivalent risk-neutral probability measure for that model is given by $Z = 1$. However, there is more to it than that, since the process described by (2.1a) is in fact the inverse of a Bessel process of dimension three, and is therefore a well-known example of a strict local martingale (see e.g. Revuz and Yor [1999], Exercise V.2.13, p. 194). This establishes the first model as an example of a stock price bubble.

In the case of the second model, (2.1b) expresses the price of the risky security as a Bessel process of dimension three. It then follows that S is the numéraire portfolio for that model (in the sense of Becherer [2001], Definition 4.1), since $B/S = 1/S$ and $S/S = 1$ are both non-negative local martingales, and consequently also supermartingales, by Fatou’s lemma. The model therefore satisfies the NUPBR condition, by virtue of Karatzas and Kardaras [2007], Theorem 4.12.¹ However, it does not satisfy the NFLVR condition, since the density process

¹Actually, this is not completely true, since Theorem 4.12 of Karatzas and Kardaras [2007] requires the value of the numéraire portfolio to be a “semimartingale up to infinity.” The Bessel process of dimension three does

for the putative equivalent risk-neutral probability measure may be expressed in terms of the numéraire portfolio as follows: $Z = (S_0/B_0)(B/S) = S_0/S$, which is a strict local martingale.² This establishes the second model as an example of a bond price bubble.

We acknowledge that the models presented by (2.1a) and (2.1b) have already been studied in detail by Delbaen and Schachermayer [1994a] and Delbaen and Schachermayer [1995], respectively. However, whereas the above-mentioned articles focus on questions of arbitrage, we wish to address the pricing and hedging of contingent claims. Our paper may therefore be regarded as a companion for those articles.

We begin our discussion of contingent claim pricing by introducing a finite maturity date $T > 0$ and a payoff function $h : (0, \infty) \rightarrow \mathbb{R}_+$, which satisfy the following two conditions:

- (i) h is continuous and of polynomial growth;
- (ii) $\mathbb{E}(Z_T h(S_T)) < \infty$; and
- (iii) $\mathbb{P}(h(S_T) > 0) > 0$.

We now consider a European contingent claim on the risky security, with maturity T and payoff $h(S_T)$. We shall denote the value of this claim by the process $V^h = (V_t^h)_{t \in [0, T]}$, which is determined as follows:

$$V_t^h := \frac{1}{Z_t} \mathbb{E}(Z_T h(S_T) \mid \mathcal{F}_t), \quad (2.2)$$

for all $t \in [0, T]$. Observe that condition (ii) above ensures that $V_t^h < \infty$, for all $t \in [0, T]$, while condition (iii) implies that $V_t^h > 0$, for all $t \in [0, T]$. Also, note that since $V_T^h = h(S_T)$, it easily follows that the process $(Z_t V_t^h)_{t \in [0, T]}$ is a martingale.

We shall refer to (2.2) as the “real-world” price of the claim, since it does not require any transformation of probability measure.³ For the first model it is easily seen that (2.2) agrees with the risk-neutral price of the claim, since Z is a martingale. However, it is also clear that (2.2) remains well-defined for the second model. In that case there is no equivalent risk-neutral probability measure, since Z is a strict local martingale, and hence risk-neutral pricing is infeasible. In other words, real-world pricing offers a proper extension of the risk-neutral pricing concept.

An important observation for the two models under consideration is that

$$\frac{Z_T}{Z_t} = \begin{cases} 1 & \text{if the price of the risky security is modelled by (2.1a);} \\ \frac{S_t}{S_T} & \text{if the price of the risky security is modelled by (2.1b),} \end{cases}$$

for all $t \in [0, T]$. We may therefore exploit the Markov property of the price of the risky security, for both models, by writing $V_t^h = V^h(t, S_t)$, for all $t \in [0, T]$, where the “pricing function” $V^h : [0, T] \times (0, \infty) \rightarrow \mathbb{R}_+$ that appears in this expression may be computed as

not meet this requirement, since its sample paths diverge in the limit as time goes to infinity. Nevertheless, this provides no practical obstacle, since our applications only involve a finite time-horizon.

²Strictly speaking, we are abusing terminology by calling Z a density process in this instance, since most of the literature adopts the convention that density processes are positive martingales. In the case of the second model, it may therefore be more appropriate to describe Z as a “pre-density process” or a “candidate density process.” However, in the interest of maintaining a uniform terminology for discussing both models, we shall continue to refer to Z simply as a density process, with the understanding that the reader is sufficiently flexible on this point.

³The reader is directed to Section 9.1 of Platen and Heath [2006] for a more detailed account of the real-world approach to contingent claim pricing. Note that real-world prices are expressed there in terms of the numéraire portfolio, whereas (2.2) expresses them in terms of the density process. These two formulations are equivalent, by dint of the fact that the density process is proportional to the numéraire-denominated money-market account.

follows:

$$V^h(t, S) := \begin{cases} \mathbb{E}_{t,S}(h(S_T)) & \text{if the price of the risky security is modelled by (2.1a);} \\ S\mathbb{E}_{t,S}\left(\frac{h(S_T)}{S_T}\right) & \text{if the price of the risky security is modelled by (2.1b),} \end{cases} \quad (2.3)$$

for all $(t, S) \in [0, T] \times (0, \infty)$.⁴ The following proposition will allow us to obtain convenient expressions for the pricing functions of a number of standard European contingent claims in Sections 3 and 4:

Proposition 2.1. *A Bessel process ρ of dimension three has the following (truncated) moments:*

$$\mathbb{E}_x\left(\mathbf{1}_{\{\rho_t \leq z\}} \frac{1}{\rho_t}\right) = \frac{1}{x} \left(\Phi\left(-\frac{x-z}{\sqrt{t}}\right) - \Phi\left(\frac{x+z}{\sqrt{t}}\right) + \Phi\left(\frac{x}{\sqrt{t}}\right) - \Phi\left(-\frac{x}{\sqrt{t}}\right) \right); \quad (2.4)$$

$$\mathbb{E}_x\left(\frac{1}{\rho_t}\right) = \frac{1}{x} \left(\Phi\left(\frac{x}{\sqrt{t}}\right) - \Phi\left(-\frac{x}{\sqrt{t}}\right) \right); \quad (2.5)$$

$$\mathbb{E}_x\left(\mathbf{1}_{\{\rho_t > z\}} \frac{1}{\rho_t}\right) = \frac{1}{x} \left(\Phi\left(\frac{x+z}{\sqrt{t}}\right) - \Phi\left(-\frac{x-z}{\sqrt{t}}\right) \right); \quad (2.6)$$

$$\mathbb{P}_x(\rho_t \leq z) = \Phi\left(\frac{x+z}{\sqrt{t}}\right) - \Phi\left(\frac{x-z}{\sqrt{t}}\right) + \frac{\sqrt{t}}{x} \left(\phi\left(\frac{x+z}{\sqrt{t}}\right) - \phi\left(\frac{x-z}{\sqrt{t}}\right) \right); \quad (2.7)$$

and

$$\mathbb{P}_x(\rho_t > z) = \Phi\left(-\frac{x+z}{\sqrt{t}}\right) + \Phi\left(\frac{x-z}{\sqrt{t}}\right) - \frac{\sqrt{t}}{x} \left(\phi\left(\frac{x+z}{\sqrt{t}}\right) - \phi\left(\frac{x-z}{\sqrt{t}}\right) \right), \quad (2.8)$$

for all $t > 0$, $x, y \in (0, \infty)$ and $z > 0$.⁵ In these expressions Φ denotes the cumulative distribution function of a standard normal random variable, while ϕ is the associated density function.⁶

Proof. To begin with, Revuz and Yor [1999], Proposition VI.3.1, p. 251 provides the following expression for the transition density of ρ :

$$q(t, x, y) := \frac{\partial}{\partial y} \mathbb{P}_x(\rho_t \leq y) = \frac{x}{t} \left(\frac{y}{x} \right)^{\frac{3}{2}} I_{\frac{1}{2}}\left(\frac{xy}{t}\right) \exp\left(-\frac{x^2 + y^2}{2t}\right),$$

for all $t > 0$ and $x, y \in (0, \infty)$. The function $I_{1/2}$ that appears above is the modified Bessel function of the first kind with index one half (see e.g. Abramowitz and Stegun [1972], Chapter 9). It satisfies the following identity:

$$I_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sinh z = \frac{1}{\sqrt{2\pi z}} (e^z - e^{-z}),$$

⁴As usual, $\mathbb{E}_{t,S}$ denotes the expected value operator with respect to the probability measure $\mathbb{P}_{t,S}$, under which the price of the risky security at time $t \in \mathbb{R}_+$ is $S \in (0, \infty)$. Of course, we are abusing notation slightly by using S to denote a particular value for the price of the risky security, as well the price process for this asset.

⁵In this case $\mathbb{E}_x$ should be understood as the expected value operator with respect to the probability measure $\mathbb{P}_x$, under which $\rho_0 = x$, for all $x \in (0, \infty)$.

⁶In other words,

$$\Phi(z) := \int_{-\infty}^z \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\zeta^2/2}}_{\phi(\zeta)} d\zeta,$$

for all $z \in \mathbb{R}$. Also, recall that the mapping

$$\mathbb{R} \ni \zeta \mapsto \frac{1}{\sigma} \phi\left(\frac{\zeta - \mu}{\sigma}\right)$$

is the probability density function of a normal random variable with mean $\mu \in \mathbb{R}$ and standard deviation $\sigma > 0$.

for all $z \in \mathbb{R}$, according to Lebedev [1972], Equation 5.8.5, p. 112. By combining these two expressions, we obtain

$$q(t, x, y) = \frac{y}{x} \left(\frac{1}{\sqrt{t}} \phi \left(\frac{y-x}{\sqrt{t}} \right) - \frac{1}{\sqrt{t}} \phi \left(\frac{y+x}{\sqrt{t}} \right) \right),$$

for all $t > 0$ and $x, y \in (0, \infty)$. Now, fix $t > 0$, $x \in (0, \infty)$ and $z > 0$, and observe that

$$\mathbb{E}_x \left(\mathbf{1}_{\{\rho_t \leq z\}} \frac{1}{\rho_t} \right) = \int_0^z \frac{1}{y} q(t, x, y) dy = \frac{1}{x} \left(\int_0^z \frac{1}{\sqrt{t}} \phi \left(\frac{y-x}{\sqrt{t}} \right) dy - \int_0^z \frac{1}{\sqrt{t}} \phi \left(\frac{y+x}{\sqrt{t}} \right) dy \right).$$

Since the final expression above involves the difference between the truncated zeroth moments of two normal random variables, we obtain (2.4) from Jawitz [2004], Table 1, for example. We may then derive (2.5) from (2.4), by an application of the monotone convergence theorem, while (2.6) follows directly from (2.4) and (2.5). Next, we see that

$$\mathbb{P}_x(\rho_t \leq z) = \int_0^z q(t, x, y) dy = \frac{1}{x} \left(\int_0^z \frac{y}{\sqrt{t}} \phi \left(\frac{y-x}{\sqrt{t}} \right) dy - \int_0^z \frac{y}{\sqrt{t}} \phi \left(\frac{y+x}{\sqrt{t}} \right) dy \right).$$

In this case we recognize that the final expression above involves the difference between the truncated first moments of two normal random variables. Once again, we may look these up in Jawitz [2004], Table 1, to get (2.7). Finally, (2.8) is obtained directly from (2.7). $\square$

We turn our attention now to the question of how to hedge the contingent claim introduced above, for the two models under consideration. To begin with, throughout this article the phrase “viable portfolio” should be understood to mean a self-financing portfolio consisting of holdings in the risky security and the money-market account, whose value is strictly positive over $[0, T)$ and non-negative at time T .⁷ Such a portfolio may be presented by a “trading strategy” $\pi = (\pi_t)_{t \in [0, T]}$, which specifies the fraction of its wealth invested in the risky security. Given an initial endowment $x > 0$ and a trading strategy $\pi = (\pi_t)_{t \in [0, T]}$ for a viable portfolio, the associated “wealth process” $W^{x, \pi} = (W_t^{x, \pi})_{t \in [0, T]}$ is easily seen to satisfy one of the following two stochastic differential equations:

$$dW_t^{x, \pi} = -\pi_t S_t W_t^{x, \pi} d\beta_t \tag{2.9a}$$

or

$$dW_t^{x, \pi} = \frac{\pi_t}{S_t^2} W_t^{x, \pi} dt + \frac{\pi_t}{S_t} W_t^{x, \pi} d\beta_t, \tag{2.9b}$$

for all $t \in [0, T)$, depending on whether the price of the risky security is modelled by (2.1a) or by (2.1b).⁸

As an aside, observe that (2.9) suggests the following stochastic exponential representation for the wealth of a viable portfolio with initial endowment $x > 0$ and trading strategy $\pi = (\pi_t)_{t \in [0, T]}$:

$$W_t^{x, \pi} = x \mathcal{E} \left(\int_0^t \pi_s dX_s \right)_t,$$

for all $t \in [0, T]$. The process X in the expression above is given by

$$dX_t = -S_t d\beta_t \tag{2.10a}$$

⁷Strictly speaking, the last requirement is redundant, since all self-financing portfolios in the financial market introduced above have continuous sample paths.

⁸Of course, π should belong to an appropriate class of processes in order for the stochastic integrals in (2.9) to be well-defined. For example, we could stipulate that all trading strategies should be progressively measurable.

or

$$dX_t = \frac{1}{S_t^2} dt + \frac{1}{S_t} d\beta_t, \quad (2.10b)$$

for all $t \in \mathbb{R}_+$, depending on whether the price of the risky security is determined by (2.1a) or by (2.1b). The requirement that the portfolio wealth is strictly positive over $[0, T)$ then translates into the following concrete condition:

$$\int_0^t \pi_s^2 d\langle X \rangle_s < \infty,$$

for all $t \in [0, T)$. Moreover, we obtain the following correspondence between events:

$$\left\{ W_T^{x, \pi} = 0 \right\} = \left\{ \int_0^T \pi_s^2 d\langle X \rangle_s = \infty \right\},$$

by an application of the law of large numbers for local martingales (see e.g. Revuz and Yor [1999], Exercise V.1.16, pp. 186–187).

It will be useful for us to identify two notions of hedging, for the contingent claim introduced earlier. The first of these corresponds to the situation when the value of a viable portfolio precisely matches the payoff of the claim at its maturity, while the second describes the situation when the value of a viable portfolio precisely matches the real-world price of the claim at all times up to its maturity:

Definition 2.2. Consider a viable portfolio, which is determined by an initial endowment $x > 0$ and a trading strategy $\pi = (\pi_t)_{t \in [0, T)}$.

- (i) The portfolio is said to “hedge the claim” if and only if $W_T^{x, \pi} = h(S_T)$.
- (ii) The portfolio is said to “replicate the claim” if and only if $W_t^{x, \pi} = V_t^h$, for all $t \in [0, T]$.

It is immediately evident from the definition above that any viable portfolio that replicates the claim must also hedge it. To make the relationship between hedging and replication more explicit, suppose that the initial endowment $x^{\text{hed}} > 0$ and trading strategy $\pi^{\text{hed}} = (\pi_t^{\text{hed}})_{t \in [0, T)}$ specify a viable portfolio that hedges the claim. Similarly, let the initial endowment $x^{\text{rep}} > 0$ and trading strategy $\pi^{\text{rep}} = (\pi_t^{\text{rep}})_{t \in [0, T)}$ specify a viable portfolio that replicates the claim. We begin by observing that the process $(Z_t W_t^{x^{\text{rep}}, \pi^{\text{rep}}})_{t \in [0, T]}$ is a martingale, since

$$Z_t W_t^{x^{\text{rep}}, \pi^{\text{rep}}} = Z_t V_t^h = \mathbb{E}(Z_T V_T^h | \mathcal{F}_t) = \mathbb{E}(Z_T W_T^{x^{\text{rep}}, \pi^{\text{rep}}} | \mathcal{F}_t),$$

for all $t \in [0, T]$, by virtue of the already established fact that $(Z_t V_t^h)_{t \in [0, T]}$ is a martingale.

Next, we note that the process $(Z_t W_t^{x^{\text{hed}}, \pi^{\text{hed}}})_{t \in [0, T]}$ is a local martingale. In the case of the stock price bubble, this is an easy consequence of (2.9a) and the fact that $Z = 1$. In the case of the bond price bubble, the stochastic integration by parts formula yields

$$d(Z W^{x^{\text{hed}}, \pi^{\text{hed}}})_t = -\left(1 - \pi_t^{\text{hed}}\right) \frac{1}{S_t} (Z W^{x^{\text{hed}}, \pi^{\text{hed}}})_t d\beta_t,$$

for all $t \in [0, T)$, with the help (2.9b) and the fact that $Z = S_0/S$. We may therefore deduce that $(Z_t W_t^{x^{\text{hed}}, \pi^{\text{hed}}})_{t \in [0, T]}$ is in fact a supermartingale, for both models, by an application of Fatou’s lemma. Finally, putting all of the above together, we obtain

$$Z_t W_t^{x^{\text{rep}}, \pi^{\text{rep}}} = \mathbb{E}(Z_T W_T^{x^{\text{rep}}, \pi^{\text{rep}}} | \mathcal{F}_t) = \mathbb{E}(Z_T W_T^{x^{\text{hed}}, \pi^{\text{hed}}} | \mathcal{F}_t) \leq Z_t W_t^{x^{\text{hed}}, \pi^{\text{hed}}},$$

for all $t \in [0, T]$. In particular, since $Z > 0$ for both the stock price bubble and the bond price bubble, we must have

$$W_t^{x^{\text{rep}}, \pi^{\text{rep}}} \leq W_t^{x^{\text{hed}}, \pi^{\text{hed}}}, \quad (2.11)$$

for all $t \in [0, T]$, irrespective of which model is chosen.

The importance of (2.11) lies in the justification it provides for the real-world pricing formula (2.2). In particular, since the value of the replicating portfolio—if it exists—corresponds with the real-world price of the claim, we see that the real-world price represents the value of the cheapest possible hedging portfolio. This raises the following two important questions: Does the replicating portfolio necessarily exist, and can we characterize it? These questions are addressed by the next proposition, for which the crucial ingredient is a converse for the Feynman-Kac Theorem, due to Janson and Tysk [2006]:

Proposition 2.3. *Define the functions $x^{\text{rep}} : (0, \infty) \rightarrow (0, \infty)$ and $\pi^{\text{rep}} : [0, T) \times (0, \infty) \rightarrow \mathbb{R}_+$ as follows:*

$$x^{\text{rep}}(S) := \begin{cases} E_{0,S}(h(S_T)) & \text{if the price of the risky security is modelled by (2.1a);} \\ SE_{0,S}\left(\frac{h(S_T)}{S_T}\right) & \text{if the price of the risky security is modelled by (2.1b),} \end{cases} \quad (2.12)$$

for all $S \in (0, \infty)$, and

$$\pi^{\text{rep}}(t, S) := \frac{S}{V^h(t, S)} \frac{\partial V^h}{\partial S}(t, S), \quad (2.13)$$

for all $(t, S) \in [0, T) \times (0, \infty)$, respectively. In particular, (2.13) is well-defined, since the pricing function (2.3) is both strictly positive and differentiable. The replicating portfolio for the contingent claim under consideration is then determined by the initial endowment $x^{\text{rep}} = x^{\text{rep}}(S_0)$ and the trading strategy $\pi^{\text{rep}} = (\pi_t^{\text{rep}})_{t \in [0, T)}$, with $\pi_t^{\text{rep}} = \pi^{\text{rep}}(t, S_t)$, for all $t \in [0, T)$.

Proof. We have already observed that by imposing condition (iii) on the payoff function, we guarantee that the pricing function is strictly positive, for both the stock price bubble and the bond price bubble. So to establish that the expression in (2.13) is well-defined, we need only demonstrate that the pricing function of the claim is differentiable. We shall now analyze the the stock price bubble and the bond price bubble separately:

(i) Suppose the price of the risky security is modelled by (2.1a). It then follows from Janson and Tysk [2006], Theorem 6.1 that the pricing function (2.3) is continuously differentiable with respect to the temporal variable and twice continuously differentiable with respect to the spatial variable, and that it satisfies the following partial differential equation:

$$\frac{\partial V^h}{\partial t}(t, S) + \frac{1}{2} S^4 \frac{\partial^2 V^h}{\partial S^2}(t, S) = 0, \quad (2.14)$$

for all $(t, S) \in [0, T) \times (0, \infty)$.⁹ The differentiability of the pricing function ensures that the trading strategy (2.13) is well-defined, while Itô's formula combined with (2.14) yields

$$\begin{aligned} dV^h(t, S_t) &= \left(\frac{\partial V^h}{\partial t}(t, S_t) + \frac{1}{2} S_t^4 \frac{\partial^2 V^h}{\partial S^2}(t, S_t) \right) dt - S_t^2 \frac{\partial V^h}{\partial S}(t, S_t) d\beta_t \\ &= -S_t^2 \frac{\partial V^h}{\partial S}(t, S_t) d\beta_t \\ &= -\pi^{\text{rep}}(t, S_t) S_t V^h(t, S_t) d\beta_t, \end{aligned} \quad (2.15)$$

for all $t \in [0, T)$. Finally, by comparing (2.15) with (2.9a), we see that $W_t^{x^{\text{rep}}, \pi^{\text{rep}}} = V^h(t, S_t)$, for all $t \in [0, T]$, by virtue of the fact that $x^{\text{rep}}(S_0) = V^h(0, S_0)$.

(ii) Suppose the price of the risky security is modelled by (2.1b), and consider the function $\widehat{V}^h : [0, T] \times (0, \infty) \rightarrow \mathbb{R}_+$, given by

$$\widehat{V}^h(t, S) := \frac{V^h(t, S)}{S}, \quad (2.16)$$

⁹Note that Theorem 6.1 of Janson and Tysk [2006] requires a continuous payoff function of polynomial growth. This explains why we imposed condition (i), when we introduced the payoff function for the claim.

for all $(t, S) \in [0, T] \times (0, \infty)$. It then follows from Janson and Tysk [2006], Theorem 6.1 and (2.3) that the function defined by (2.16) is continuously differentiable with respect to the temporal variable and twice continuously differentiable with respect to the spatial variable, and that it satisfies the following partial differential equation:

$$\frac{\partial \widehat{V}^h}{\partial t}(t, S) + \frac{1}{S} \frac{\partial \widehat{V}^h}{\partial S}(t, S) + \frac{1}{2} \frac{\partial^2 \widehat{V}^h}{\partial S^2}(t, S) = 0, \quad (2.17)$$

for all $(t, S) \in [0, T] \times (0, \infty)$. In particular, the differentiability of this function implies that the pricing function is differentiable, from which it follows that the trading strategy (2.13) is well-defined. Next, by combining (2.17) with (2.16), we obtain the following partial differential equation for the pricing function itself:

$$\frac{\partial V^h}{\partial t}(t, S) + \frac{1}{S} \frac{\partial V^h}{\partial S}(t, S) + \frac{1}{2} \frac{\partial^2 V^h}{\partial S^2}(t, S) - \frac{1}{S} \frac{\partial V^h}{\partial S}(t, S) = 0, \quad (2.18)$$

for all $(t, S) \in [0, T] \times (0, \infty)$. By combining Itô's formula with (2.18), we then get

$$\begin{aligned} dV^h(t, S_t) &= \left(\frac{\partial V^h}{\partial t}(t, S_t) + \frac{1}{S_t} \frac{\partial V^h}{\partial S}(t, S_t) + \frac{1}{2} \frac{\partial^2 V^h}{\partial S^2}(t, S_t) \right) dt + \frac{\partial V^h}{\partial S}(t, S_t) d\beta_t \\ &= \frac{1}{S_t} \frac{\partial V^h}{\partial S}(t, S_t) dt + \frac{\partial V^h}{\partial S}(t, S_t) d\beta_t \\ &= \frac{\pi^{\text{rep}}(t, S_t)}{S_t^2} V^h(t, S_t) dt + \frac{\pi^{\text{rep}}(t, S_t)}{S_t} V^h(t, S_t) d\beta_t, \end{aligned} \quad (2.19)$$

for all $t \in [0, T]$. As before, by comparing (2.19) with (2.9b), we see that $W_t^{x^{\text{rep}}, \pi^{\text{rep}}} = V^h(t, S_t)$, for all $t \in [0, T]$, since $x^{\text{rep}}(S_0) = V^h(0, S_0)$. $\square$

Together, (2.11) and Proposition 2.3 establish the canonical nature of the real-world pricing formula: Given any payoff function satisfying conditions (i)–(iii) above, the real-world price of the corresponding claim provides the minimal cost of hedging it. Moreover, we see that the trading strategy (2.13) for the replicating portfolio may be expressed in terms of the “delta” of the claim. The latter is in turn completely determined by a function $\Delta^h : [0, T] \times (0, \infty) \rightarrow \mathbb{R}$, which is defined by

$$\Delta^h(t, S) := \frac{\partial V^h}{\partial S}(t, S),$$

for all $(t, S) \in [0, T] \times (0, \infty)$.

3. THE STOCK PRICE BUBBLE

In this section we focus on the situation when the price of the risky security is modelled by (2.1a). Since its price is then an inverted Bessel process of dimension three, we may use Proposition 2.1 to derive the pricing functions and deltas for a number of standard European claims on the risky security, all of which share a common maturity date $T > 0$ and strike $K > 0$ (when appropriate).

We begin by considering a zero-coupon bond with a face-value of one dollar.¹⁰ As one would expect, this instrument is trivial in the case of the stock price bubble. In particular, its delta is uniformly zero, so that its replicating portfolio consists simply of an initial endowment—equal to (the discounted value of) its principal—invested in the money-market account:

Example 3.1. The pricing function $Z : [0, T] \times (0, \infty) \rightarrow (0, \infty)$ for a zero-coupon bond is given by

$$Z(t, S) := \mathbb{E}_{t, S}(1) = 1, \quad (3.1)$$

¹⁰Here a “dollar” should be interpreted as a generic unit of currency—we assume that the values of all instruments are denominated in “dollars.”

FIGURE 3.1. The pricing function and the delta of a prepaid forward on the risky security ($T = 1$).

for all $(t, S) \in [0, T) \times (0, \infty)$. The associated replicating portfolio is determined by the delta $\Delta^Z : [0, T) \times (0, \infty) \rightarrow \mathbb{R}$ of the contract, which is given by

$$\Delta^Z(t, S) := \frac{\partial Z}{\partial S}(t, S) = 0, \quad (3.2)$$

for all $(t, S) \in [0, T) \times (0, \infty)$.

Next, we examine a prepaid forward contract on the risky security.¹¹ Its pricing function and delta are presented below, and are plotted in Figure 3.1:

Example 3.2. The pricing function $F : [0, T) \times (0, \infty) \rightarrow (0, \infty)$ for a prepaid forward on the risky security is given by

$$F(t, S) := E_{t,S}(S_T) = S \left(1 - 2\Phi \left(-\frac{1/S}{\sqrt{T-t}} \right) \right), \quad (3.3)$$

for all $(t, S) \in [0, T) \times (0, \infty)$, by an application of (2.5). The associated replicating portfolio is determined by the delta $\Delta^F : [0, T) \times (0, \infty) \rightarrow \mathbb{R}$ of the contract, which is given by

$$\Delta^F(t, S) := \frac{\partial F}{\partial S}(t, S) = 1 - 2\Phi \left(-\frac{1/S}{\sqrt{T-t}} \right) - \frac{2}{S\sqrt{T-t}} \phi \left(\frac{1/S}{\sqrt{T-t}} \right), \quad (3.4)$$

for all $(t, S) \in [0, T) \times (0, \infty)$.

We immediately notice from Figure 3.1 that the value of the prepaid forward contract is strictly less than the price of the risky security itself, at all times prior to maturity. In particular, we have

$$S - F(t, S) = \underbrace{2S\Phi \left(-\frac{1/S}{\sqrt{T-t}} \right)}_{\gamma(t, S)} > 0, \quad (3.5)$$

for all $(t, S) \in [0, T) \times (0, \infty)$. The process $(\gamma(t, S_t))_{t \in \mathbb{R}_+}$ introduced above corresponds with what Elworthy et al. [1999] refer to as the “default” of a strict local martingale (in this case the price of the risky security).

¹¹The principal difference between prepaid forwards and conventional forward contracts is that the latter are settled at maturity, while the purchaser of a prepaid forward pays an up-front premium for subsequent delivery of the underlying asset when the contract matures. In other words, a prepaid forward is simply a European call with a strike price of zero.

Another interesting observation is that the price of the prepaid forward is bounded from above at any fixed time $t \in [0, T)$, since (3.3) yields

$$\lim_{S \uparrow \infty} F(t, S) = \frac{2}{\sqrt{2\pi(T-t)}}. \quad (3.6)$$

This explains why one cannot hope to exploit any arbitrage opportunity by simultaneously purchasing the prepaid forward and short-selling the underlying risky security—the value of the resulting portfolio is not bounded from below. It also explains why, for large values of the risky security price, the portfolio that replicates the prepaid forward is almost completely invested in the money-market account, as evidenced by the surface plot of its delta in Figure 3.1. In particular, (3.6) ensures that the delta of the prepaid forward is zero in the limit, as the price of the risky security increases to infinity.

Let us now attempt to explain the intuition behind the fact that the prepaid forward is worth less than its underlying risky security. To begin with, consider for a moment a model that does admit an equivalent risk-neutral probability measure, under which the (discounted) price of the risky security is in fact a martingale. In such a case the risk-neutral value of the prepaid forward would exactly match the market price of the risky security. This phenomenon is often interpreted to mean that risk-neutral valuation allows one to retrieve the price the risky security itself, by computing its discounted expected future value. By this line of reasoning the stock price bubble seems incoherent, since it appears that there are now two prices for the risky security: a market price and a model price.

To resolve the conundrum above, we should first point out that the prices of the underlying primary securities (i.e. the risky security and the money-market account) are completely exogenous in the modelling framework considered here—they are what they are, and they are not subject to any valuation principle. The fact that the risk-neutral value of the prepaid forward agrees with the market price of the risky security, in the case when the (discounted) price of the risky security is a martingale under the equivalent risk-neutral probability measure, is simply a curiosity induced by the properties of martingales. In truth, the risky security and the prepaid forward contract written on it are fundamentally different instruments. To make this point clear, observe that the prepaid forward may also be regarded as a European call on the risky security with a strike price of zero. By the same token, the risky security may be regarded as an American call on itself, also with a strike price of zero. Seen in this light, it is not strange that the value of the prepaid forward should be less than the price of the risky security—the difference is simply an early exercise premium! Moreover, in the case of the stock price bubble—when the price of the risky security is expected to decrease over time—this early exercise premium becomes significant. In this way we would argue that the inequality (3.5) is in fact quite natural.

The next instrument we consider is a European call on the risky security. Its pricing function and delta are presented below, and are plotted in Figure 3.2:

Example 3.3. The pricing function $C : [0, T) \times (0, \infty) \rightarrow (0, \infty)$ for a European call on the risky security is given by

$$\begin{aligned} C(t, S) &:= E_{t,S}((S_T - K)^+) = E_{t,S}(\mathbf{1}_{\{S_T > K\}} S_T) - K P_{t,S}(S_T > K) \\ &= (S - K) \Phi\left(-\frac{1/S - 1/K}{\sqrt{T-t}}\right) + (S + K) \Phi\left(-\frac{1/S + 1/K}{\sqrt{T-t}}\right) - 2S \Phi\left(-\frac{1/S}{\sqrt{T-t}}\right) \\ &\quad + KS\sqrt{T-t} \left(\phi\left(\frac{1/S - 1/K}{\sqrt{T-t}}\right) - \phi\left(\frac{1/S + 1/K}{\sqrt{T-t}}\right) \right), \end{aligned} \quad (3.7)$$

FIGURE 3.2. The pricing function and the delta of a call on the risky security ($T = 1$ and $K = 1$).

for all $(t, S) \in [0, T] \times (0, \infty)$, by an application of (2.4) and (2.7). The associated replicating portfolio is determined by the delta $\Delta^C : [0, T] \times (0, \infty) \rightarrow \mathbb{R}$ of the call, which is given by

$$\begin{aligned} \Delta^C(t, S) := \frac{\partial C}{\partial S}(t, S) &= \Phi\left(-\frac{1/S - 1/K}{\sqrt{T-t}}\right) + \Phi\left(-\frac{1/S + 1/K}{\sqrt{T-t}}\right) \\ &+ \frac{S-K}{S^2\sqrt{T-t}}\phi\left(\frac{1/S - 1/K}{\sqrt{T-t}}\right) + \frac{S+K}{S^2\sqrt{T-t}}\phi\left(\frac{1/S + 1/K}{\sqrt{T-t}}\right) - 2\Phi\left(-\frac{1/S}{\sqrt{T-t}}\right) \\ &- \frac{2}{S\sqrt{T-t}}\phi\left(\frac{1/S}{\sqrt{T-t}}\right) - \frac{S-K-KS^2(T-t)}{S^2\sqrt{T-t}}\phi\left(\frac{1/S - 1/K}{\sqrt{T-t}}\right) \\ &- \frac{S+K+KS^2(T-t)}{S^2\sqrt{T-t}}\phi\left(\frac{1/S + 1/K}{\sqrt{T-t}}\right), \end{aligned} \quad (3.8)$$

for all $(t, S) \in [0, T] \times (0, \infty)$.

It is evident from Figure 3.2 that the call option exhibits similar anomalies to the prepaid forward contract. In particular, for the price of the call we obtain the following limit from (3.7):

$$\lim_{S \uparrow \infty} C(t, S) = \frac{2}{\sqrt{2\pi(T-t)}} - K \left(2\Phi\left(\frac{1/K}{\sqrt{T-t}}\right) - 1 \right), \quad (3.9)$$

for all $t \in [0, T)$. Once again, this implies that the call price is bounded from above at any fixed time prior to maturity. In addition, it establishes that the pricing function for the call does not preserve the convexity of its payoff function, with respect to the price of the underlying risky security. This non-convexity is easily observed in the surface plot of the call delta in Figure 3.2, where we see that the second derivative of the price of the call with respect to the price of the risky security is negative, for large enough prices of the risky security.

Another easy consequence of (3.9) is that the slope of the call price with respect to the price of the risky security must tend to zero asymptotically, at any fixed time before maturity, for large values of the risky security price. In other words, the call delta should decrease to zero, when the price of the risky security becomes large, which implies that the replicating portfolio for the call should ultimately become fully invested in the money-market account. Once again, this phenomenon is illustrated by Figure 3.2.

The final claim considered in this section is a European put on the risky security. Expressions for its pricing function and delta are obtained below, and are plotted in Figure 3.3:

FIGURE 3.3. The pricing function and the delta of a put on the risky security ($T = 1$ and $K = 1$).

Example 3.4. The pricing function $P : [0, T] \times (0, \infty) \rightarrow (0, \infty)$ for a European put on the risky security is given by

$$\begin{aligned} P(t, S) &:= E_{t,S}((K - S_T)^+) = KP_{t,S}(S_T \leq K) - E_{t,S}(\mathbf{1}_{\{S_T \leq K\}} S_T) \\ &= (S + K)\Phi\left(-\frac{1/S + 1/K}{\sqrt{T-t}}\right) - (S - K)\Phi\left(\frac{1/S - 1/K}{\sqrt{T-t}}\right) \\ &\quad + KS\sqrt{T-t}\left(\phi\left(\frac{1/S - 1/K}{\sqrt{T-t}}\right) - \phi\left(\frac{1/S + 1/K}{\sqrt{T-t}}\right)\right), \end{aligned} \quad (3.10)$$

for all $(t, S) \in [0, T] \times (0, \infty)$, by an application of (2.4) and (2.7). The associated replicating portfolio is determined by the delta $\Delta^P : [0, T] \times (0, \infty) \rightarrow \mathbb{R}$ of the put, which is given by

$$\begin{aligned} \Delta^P(t, S) &:= \frac{\partial P}{\partial S}(t, S) = \Phi\left(-\frac{1/S + 1/K}{\sqrt{T-t}}\right) - \Phi\left(\frac{1/S - 1/K}{\sqrt{T-t}}\right) \\ &\quad + \frac{S + K}{S^2\sqrt{T-t}}\phi\left(\frac{1/S + 1/K}{\sqrt{T-t}}\right) + \frac{S - K}{S^2\sqrt{T-t}}\phi\left(\frac{1/S - 1/K}{\sqrt{T-t}}\right) \\ &\quad - \frac{S - K - KS^2(T-t)}{S^2\sqrt{T-t}}\phi\left(\frac{1/S - 1/K}{\sqrt{T-t}}\right) - \frac{S + K + KS^2(T-t)}{S^2\sqrt{T-t}}\phi\left(\frac{1/S + 1/K}{\sqrt{T-t}}\right), \end{aligned} \quad (3.11)$$

for all $(t, S) \in [0, T] \times (0, \infty)$.

It is immediately evident from Figure 3.3 that the behaviour of the put is much more conventional than that of the call. In particular, we see that the pricing function for the put preserves the convexity of its payoff function, with respect to the price of the risky security, at all times prior to maturity. Furthermore, the delta of the put indicates that the replicating strategy for the claim converges to a short position in one share of the risky security, if the price of this security is very low, but has almost no exposure to the risky security, if its price is high. Once again, this corresponds with the “normal” behaviour of European puts under the assumptions of the Black-Scholes model, for example.

Finally, we turn our attention to the question of put-call parity. It is already well-known that put-call parity fails for the stock price bubble, and we may confirm this explicitly, with the help

of (3.7) and (3.10):

$$C(t, S) - P(t, S) = S \left(2\Phi \left(\frac{1/S}{\sqrt{T-t}} \right) - 1 \right) - K < S - K, \quad (3.12)$$

for all $(t, S) \in [0, T) \times (0, \infty)$. There are two ways of interpreting the above inequality: Either the model is pathological, or else the put-call parity relationship is misspecified. We shall argue for the latter interpretation. To wit, note that put-call parity asserts an equivalence between a portfolio containing a long European call and a short European put, on the one hand, and a long position in the risky security combined with a short position in the money-market account, on the other hand. Since the latter portfolio offers investors the possibility of taking advantage of favourable interim movements in the price of the risky security, while the portfolio of European options does not, the inequality (3.12) simply expresses a type of early exercise premium associated with holding the underlying securities instead of the European options. In the case of the stock price bubble, where the price of the risky security is expected to decline over time, the flexibility associated with holding the underlying securities becomes valuable.

A properly specified put-call relationship should only involve instruments that are European by nature, to avoid the type of phenomenon described above. We therefore propose the following as the correct formulation:

$$C(t, S) - P(t, S) = F(t, S) - KZ(t, S), \quad (3.13)$$

for all $(t, S) \in [0, T) \times (0, \infty)$. Note that this identity is easily verified for the stock price bubble, using the pricing functions presented in this section. Moreover, in Section 4 we shall see that (3.13) continues to hold for the case of the bond price bubble. It is therefore, in some sense, the fundamental statement of put-call parity.

4. THE BOND PRICE BUBBLE

This section examines the bond price bubble, when the price of the risky security is modelled by (2.1b). Since its price is a Bessel process of dimension three, in this case, we may once again apply Proposition 2.1 to derive the pricing functions and deltas for a number of European claims on the risky security, all of which share a common maturity date $T > 0$ and strike $K > 0$ (when appropriate).

The first instrument we consider is a zero-coupon bond with a face-value of one dollar, which we see is no longer trivial. In fact, we make the rather startling discovery that such a claim is actually an equity derivative, in the setting of the bond price bubble. Its pricing function and delta are presented below, and are plotted in Figure 4.1:

Example 4.1. The pricing function $Z : [0, T) \times (0, \infty) \rightarrow (0, \infty)$ for a zero-coupon bond is given by

$$Z(t, S) := S E_{t,S} \left(\frac{1}{S_T} \right) = 1 - 2\Phi \left(-\frac{S}{\sqrt{T-t}} \right), \quad (4.1)$$

for all $(t, S) \in [0, T) \times (0, \infty)$, by an application of (2.5). The associated replicating portfolio is determined by the delta $\Delta^Z : [0, T) \times (0, \infty) \rightarrow \mathbb{R}$ of the zero-coupon bond, which is given by

$$\Delta^Z(t, S) := \frac{\partial Z}{\partial S}(t, S) = \frac{2}{\sqrt{T-t}} \phi \left(\frac{S}{\sqrt{T-t}} \right), \quad (4.2)$$

for all $(t, S) \in [0, T) \times (0, \infty)$.

Figure 4.1 reveals that the price of the bond is only significantly less than its (discounted) principal—which would have been its price if an equivalent risk-neutral probability measure existed—when the price of the risky security is low. The reason for this can be found in (2.1b),

FIGURE 4.1. The pricing function and the delta of a zero-coupon bond ($T = 1$).

which reveals that when the price of the risky security is small, its drift rate explodes, resulting in a strong repulsion from the origin. Under these circumstances the risky security is an extremely attractive investment, since it generates a positive return over a short period of time, with a high degree of certainty. By comparison, the zero-coupon bond is relatively unattractive under these conditions, unless it trades at a price substantially lower than its discounted face-value. This explains why the price of the zero-coupon bond vanishes as the price of the risky security approaches zero.

It is also interesting to analyze the zero-coupon bond from the perspective of its replicating strategy. The surface plot of the delta of this instrument in Figure 4.1 indicates that, at any fixed time before maturity, its replicating portfolio becomes progressively more heavily invested in the risky security as the price of that asset decreases. Once again, this simply takes advantage of the strong growth in the price of the risky security when it is low. In particular, (4.2) yields

$$\lim_{S \downarrow 0} \Delta^Z(t, S) = \frac{2}{\sqrt{2\pi(T-t)}},$$

for all $t \in [0, T)$. An interesting consequence of this is that when maturity is imminent and the price of the risky security is close to zero, the certain principal payment of the zero-coupon bond can be hedged by purchasing an arbitrarily large number of units of the risky security. In other words, over short periods the price of the risky security exhibits growth that is almost deterministic, if its initial value is close to zero. This can be exploited to produce a non-random payoff at very low cost.

We have already pointed out that the bond price bubble does not admit an equivalent risk-neutral probability measure, from which it follows that this model does not satisfy the NFLVR condition, by the results of Delbaen and Schachermayer [1994b]. As we shall now demonstrate, Example 4.1 allows us to establish the failure of NFLVR explicitly. To do so, consider a portfolio comprising a long position in the zero-coupon bond, which is funded by borrowing $Z(0, S_0)$ from the money-market account. The initial value of this portfolio is obviously zero, but its payoff at maturity is

$$Z(T, S_T) - Z(0, S_0) = 2\Phi\left(-\frac{S_0}{\sqrt{T}}\right) > 0,$$

by (4.1). Moreover, since $Z(t, S_t) > 0$, for all $t \in [0, T]$, it follows that the value of this portfolio is uniformly bounded from below by $-Z(0, S_0)$. In other words, the portfolio described above violates what Kabanov and Stricker [2004] refer to as the “true no-arbitrage property,” which is in fact a weaker condition than NFLVR.

FIGURE 4.2. The pricing function and the delta of a call on the risky security ($T = 1$ and $K = 1$).

In Section 3 we saw that a prepaid forward contract on the risky security is non-trivial, in the case of the stock price bubble. The opposite is true for the bond price bubble, as demonstrated below:

Example 4.2. The pricing function $F : [0, T) \times (0, \infty) \rightarrow (0, \infty)$ for a prepaid forward on the risky security is given by

$$F(t, S) := SE_{t,S}\left(\frac{S_T}{S_T}\right) = S, \quad (4.3)$$

for all $(t, S) \in [0, T) \times (0, \infty)$. The associated replicating portfolio is determined by the delta $\Delta^F : [0, T) \times (0, \infty) \rightarrow \mathbb{R}$ of the contract, which is given by

$$\Delta^F(t, S) := \frac{\partial F}{\partial S}(t, S) = 1, \quad (4.4)$$

for all $(t, S) \in [0, T) \times (0, \infty)$.

The next instrument we consider is a European call on the risky security. Its pricing function and delta are presented below, and are plotted in Figure 4.2:

Example 4.3. The pricing function $C : [0, T) \times (0, \infty) \rightarrow (0, \infty)$ for a European call on the risky security is given by

$$\begin{aligned} C(t, S) &:= SE_{t,S}\left(\frac{(S_T - K)^+}{S_T}\right) = SP_{t,S}(S_T > K) - KSE_{t,S}\left(\mathbf{1}_{\{S_T > K\}} \frac{1}{S_T}\right) \\ &= (S - K)\Phi\left(\frac{S - K}{\sqrt{T - t}}\right) + (S + K)\Phi\left(-\frac{S + K}{\sqrt{T - t}}\right) \\ &\quad + \sqrt{T - t}\left(\phi\left(\frac{S - K}{\sqrt{T - t}}\right) - \phi\left(\frac{S + K}{\sqrt{T - t}}\right)\right), \end{aligned} \quad (4.5)$$

for all $(t, S) \in [0, T) \times (0, \infty)$, by an application of (2.6) and (2.8). The associated replicating portfolio is determined by the delta $\Delta^C : [0, T) \times (0, \infty) \rightarrow \mathbb{R}$ of the call, which is given by

$$\Delta^C(t, S) := \frac{\partial C}{\partial S}(t, S) = \Phi\left(\frac{S - K}{\sqrt{T - t}}\right) + \Phi\left(-\frac{S + K}{\sqrt{T - t}}\right), \quad (4.6)$$

for all $(t, S) \in [0, T) \times (0, \infty)$.

By inspecting Figure 4.2 we see that the price of the call and its delta behave quite conventionally, when compared with European calls in the Black-Scholes model, for example. In particular, the call price preserves the convexity of its payoff function, with respect to the price of the underlying risky security, at all times prior to maturity. Also, contrary to what we observed for the case of the stock price bubble, we see that the call price is an unbounded function of the price of the risky security, at any fixed time prior to maturity.

The delta of the call behaves as one would expect, for large values of risky security price, by converging to one. In other words, the replicating portfolio for the call is almost completely invested in the risky security when the contract is deep in-the-money. However, the behaviour of the call delta is a little unusual when the price of the risky security is very low. In particular, we obtain the following from (4.6):

$$\lim_{S \downarrow 0} \Delta^C(t, S) = 2\Phi\left(-\frac{K}{\sqrt{T-t}}\right),$$

for all $t \in [0, T)$. In other words, at any fixed time before to maturity, the replicating portfolio for the call always holds at least a minimal number of units of the risky security. Once again, this phenomenon is explained by the strong growth of the price of the risky security when it is small. By comparison, under the assumptions of the Black-Scholes model, for example, the price of the risky security does not exhibit such growth behaviour near the origin, and the delta of a European call converges to zero as the price of the underlying asset becomes very small.

To complete this section, we finally consider a European put on the risky security. Its pricing function and delta are presented below, and are plotted in Figure 4.3:

Example 4.4. The pricing function $P : [0, T) \times (0, \infty) \rightarrow (0, \infty)$ for a European put on the risky security is given by

$$\begin{aligned} P(t, S) &:= SE_{t,S}\left(\frac{(K - S_T)^+}{S_T}\right) = KSE_{t,S}\left(\mathbf{1}_{\{S_T \leq K\}} \frac{1}{S_T}\right) - SP_{t,S}(S_T \leq K) \\ &= (S - K)\Phi\left(\frac{S - K}{\sqrt{T-t}}\right) - (S + K)\Phi\left(\frac{S + K}{\sqrt{T-t}}\right) + 2K\Phi\left(\frac{S}{\sqrt{T-t}}\right) \\ &\quad + \sqrt{T-t}\left(\phi\left(\frac{S - K}{\sqrt{T-t}}\right) - \phi\left(\frac{S + K}{\sqrt{T-t}}\right)\right), \end{aligned} \quad (4.7)$$

for all $(t, S) \in [0, T) \times (0, \infty)$, by an application of (2.6) and (2.8). The associated replicating portfolio is determined by the delta $\Delta^P : [0, T) \times (0, \infty) \rightarrow \mathbb{R}$ of the put, which is given by

$$\Delta^P(t, S) := \frac{\partial P}{\partial S}(t, S) = \Phi\left(\frac{S - K}{\sqrt{T-t}}\right) - \Phi\left(\frac{S + K}{\sqrt{T-t}}\right) + \frac{2K}{\sqrt{T-t}}\phi\left(\frac{S}{\sqrt{T-t}}\right), \quad (4.8)$$

for all $(t, S) \in [0, T) \times (0, \infty)$.

Figure 4.3 reveals that the put option shares some of the strange features of the zero-coupon bond, discussed earlier in this section. In particular, we observe that the put price converges to zero as the price of the risky security decreases, for any fixed time before maturity. Once again, there is a sound economic explanation for this phenomenon, which takes into account the behaviour of the risky security price near zero. In detail, since the drift term in (2.1b) explodes near the origin, an almost certain profit can be earned in a short time by investing in the risky security when its price is very low. Due to its bounded payoff structure, the put therefore becomes unattractive relative to an investment in the risky security, when the price of the latter is very low.

Another feature of the put, which is clearly visible from the surface plot of its pricing function in Figure 4.3, is the fact that its price does not preserve the convexity of its payoff, with respect

FIGURE 4.3. The pricing function and the delta of a put on the risky security ($T = 1$ and $K = 1$).

to the underlying risky security price, at any fixed time prior to maturity. This non-convexity is also observable in the surface plot of the put delta, where we see that the second derivative of the put price with respect to the price of the risky security is negative, for small enough prices of the risky security.

In addition to the above, the expression (4.8) for the delta of the put reveals the following anomaly as well:

$$\begin{aligned} \lim_{S \downarrow 0} \Delta^P(t, S) &= 1 - 2\Phi\left(\frac{K}{\sqrt{T-t}}\right) + \frac{2K}{\sqrt{2\pi(T-t)}} = 1 - 2\left(\frac{1}{2} + \int_0^{\frac{K}{\sqrt{T-t}}} \phi(\zeta) d\zeta\right) + \frac{2K}{\sqrt{2\pi(T-t)}} \\ &= 2 \int_0^{\frac{K}{\sqrt{T-t}}} \frac{1}{\sqrt{2\pi}} d\zeta - 2 \int_0^{\frac{K}{\sqrt{T-t}}} \phi(\zeta) d\zeta > 0, \end{aligned}$$

for all $t \in [0, T)$, by virtue of the fact that $\phi(\zeta) \leq \phi(0) = \frac{1}{\sqrt{2\pi}}$, for all $\zeta \in \mathbb{R}$. This tells us that the replicating portfolio for the put option contains a long position in the risky security, if the price of the latter is very low. In other words, a long position in the risky security may be required to hedge a contract whose payoff is negatively related to the price of the risky security at maturity! In fact, as we see from the surface plot of the put delta in Figure 4.3, when the price of the risky security is very low and maturity is imminent, then a very long position in the risky security may be required to replicate the put. The explanation for this apparent contradiction is to be found, once more, in the strong growth of the risky security price near the origin.

Finally, it is easily verified that the pricing functions derived in this section are all related via the put-call parity relationship (3.13) proposed in Section 3. The fact that this identity holds for the stock price bubble and the bond price bubble (as well as for models where all local martingales are martingales) supports our claim that it is the proper formulation of put-call parity.

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