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On Fair Pricing of Emission-Related Derivatives

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Abstract

The climate rescue is on the top of many agendas. In this context, emission trading schemes are considered as promising tools.The regulatory framework of an emission trading scheme introduces a market for emission allowances and creates need for risk management by appropriate financial contracts. In this work, we address logical principles underlying their valuation.

Keywords: environmental risk, emission derivatives.

1. Introduction

The generic principle of an emission trading scheme is based on the so-called cap and trade mechanism. In this framework, an authority allocates fully tradable credits among responsible installations. At pre-settled compliance dates, each source must have enough allowances to cover all its recorded emissions, or be subject to penalties.

A mandatory cap-and-trade system involves its participants in a risky business with an obvious need for risk management. That is, certificate trading is usually accompanied by an secondary market for emission-related futures including a fast-growing variety of their derivatives. Their pricing is addressed in this approach.

Our contribution focuses on a methodology between equilibrium and risk neutral approaches. Due to complexity of emission markets, risk neutral dynamics must be addressed in terms of explanatory variables, viewed as proxies of fundamental quantities. Thus, we utilize equilibrium analysis to explain the role of fundamentals for risk-neutral allowance price formation. The key issue thereby is a feedback relation between allowance prices and abatement activity. Namely, we demonstrate that any increase in allowance price causes market participants to enforce emission saving in order to sell their allowances. Hence, an increasing allowance price encourages a supply of certificates and lowers the probability of non-compliance which tends to bring down their prices. Apparently, the correct description of this feedback is the key to derivatives pricing. The

present work focuses on this issue. On this account, our contribution goes beyond any risk neutral approach to modeling of emission-related assets suggested in the existing literature to date.

2. Emission Markets

The literature on this subject is enormous: it encompass hundreds of books and papers. For this reason, we focus merely on those market models which are relevant in the present approach.

Economic theory of allowance trading can be traced back to [8] and [14] whose authors proposed a market model for the public good *environment* described by tradable permits.

Dynamic allowance trading is addressed in [7], [22], [16], [11], [17], [21], [13] and in the literature cited therein.

Empirical evidence from existing markets is discussed in [9]. This paper suggests economic implications and hints at several ways to model spot and futures allowance prices, whose detailed inter-relations are investigated in [23] and [24].

Econometric modeling is addressed in [1], where characteristic properties for financial time series are observed for prices of emission allowances from the mandatory European Scheme EU ETS. Furthermore, a Markov switch and AR-GARCH models are suggested. The work [15] considers also tail behavior and the heteroscedastic dynamics in the returns of emissions allowance prices.

Dynamic price equilibrium and optimal market design are investigated in [2].Based on this approach, [3] discusses the price formation for goods whose production is affected by emission regulations. In this setting, an equilibrium analysis confirms the existence of the so-called windfall profits (see [19]) and provides quantitative tools to analyze alternative market designs.

Pricing of options was addressed only recently. The paper [6] discusses an endogenous emission permit price dynamics within equilibrium setting and elaborates on valuation of European option on emission allowances. The paper [18] and the dissertation [25] deal with the the risk-neutral allowance price formation within EU ETS. Here, utilizing equilibrium properties, the price evolution is treated in terms of marginal abatement costs and optimal stochastic control. Also the work [5] is devoted to option pricing within EU ETS. The authors suppose that the drift of allowance spot prices is related to a hidden variable which describes the overall market position in allowance contracts and

make use of filtering techniques to derive option price formulas which reflect specific allowance banking regulations, valid in the EU ETS. Finally, the recent work [4] presents an approach where emission certificate futures are modeled in terms of deterministic time change applied to a certain class of interval-valued diffusion processes.

The present work brings aspects of risk-aversion into the research line followed in [18], [3], and [2], which we briefly sketch now.Within a stochastic model of emission market, a so-called *central planer* problem is introduced and discussed in [18]. Under additional assumptions, the authors formulate this problem in terms of continuous time stochastic optimization. Furthermore, they provide economic arguments justifying why optimal control solutions correspond to an equilibrium of the emission market. Interpreting the allowance certificate price as the marginal abatement costs, particular explicit solutions are discussed and yield a dynamic stochastic model for allowance price evolution. The work [2] starts from the opposite direction. In a discrete-time framework, the Radner equilibrium of emission market is introduced and constructed via a solution of the central planer problem. The work [3] yields an extension: in a slightly different setting, it is proven that any market equilibrium is reached by this methodology. Thus, results from [18], [2] and [3] show that a quantitative analysis of emission markets is tractable in terms of stochastic control theory.However, this connection is valid only if risk aversion is neglected, in other words, under the assumption that *each agent* possesses a *linear utility function*. Losing sight of risk aversion comes at the costs of unrealistic results. Among other singularities, it turns out that the equilibrium allowance price follows a martingale (with respect to objective measure!) with the consequence that allowance trading can be arbitrary, merely final position must be adjusted accordingly.

This work resolves all these problems. Starting from the no-arbitrage property which is satisfied in an equilibrium of a market with risk averse players, we show that the risk-neutral allowance price dynamics exhibits the above feedback property, which we formalize as a fixed point equation and discuss its solution. We show that for such a risk averse setting, our fixed point equation plays the same role as the central planer optimal control problem for the non-risk averse situation. Namely, it provides a methodology to describe the market equilibrium in terms of *aggregated quantities*. However, this description is valid only from the viewpoint of the so-called risk neutral dynamics, not suitable to discuss all interesting problems. Still, derivatives valuation is naturally addressed in and can be obtained in this setting.

3. Mathematical model

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t=0}^T)$ be a filtered probability space. Assume that $\mathcal{F}_0$ is deterministic and agree that all processes considered in this work are adapted to $(\mathcal{F}_t)_{t=0}^T$. Write $E_t(\cdot)$ and P_t to denote conditional expectation and conditional distribution with respect to $\mathcal{F}_t$. Consider a market with a finite number I of the agents confronted with emission

reduction.

Emission dynamics: For each agent $i \in I$, introduce the stochastic process $(E_t^i)_{t=0}^{T-1}$ with the interpretation that E_t^i describes the total pollution of the agent i which is emitted within the time interval $[t, t+1]$ in the case of the so-called business as usual scenario (where no abatement measure is applied). Although each agent is considered as potential producer, purely financial institutions are also covered with this approach by setting emission to zero $E_t^i = 0$ for $t = 0, \ldots, T-1$.

Abatement: Consider the opportunity to reduce emissions. Each agent i can decide at any time $t = 0, \ldots, T-1$ to reduce its emission within $[t, t+1]$ by ξ_t^i pollutant units. We suppose that each abatement level is possible ranging from no reduction to full reduction. Hence, we assume that $0 \le \xi_t^i \le E_t^i$ holds for all $t = 0, \ldots, T-1$.

Abatement costs: We assume that the cost of abatement is a random function of the reduced volume. The randomness is due to uncertainty in prices (of fuel) and is observable at the corresponding time. Thus, if the agent i decides at time $t = 0, \ldots, T-1$ on reduction of the own emission by $x \in [0, \infty[$ units, then it causes costs $C_t^i(x)$, with given

$$\left. \begin{array}{c} C_t^i : [0, \infty[\times \Omega \mapsto \mathbb{R} \text{ is } \mathcal{B}([0, \infty[) \otimes \mathcal{F}_t\text{-measurable} \\ \text{and for each } \omega \in \Omega, \ x \mapsto C_t^i(x)(\omega) \text{ is strictly} \\ \text{convex and continuous with } C(0) = 0. \end{array} \right\} \tag{3.1}$$

Since emission savings can not exceed the business as usual emission, the abatement activity $(\xi_t^i)_{t=0}^{T-1}$ is feasible if

$$0 \le \xi_t^i \le E_t^i, \qquad t = 0, \ldots, T-1. \tag{3.2}$$

Following abatement policy $(\xi_t^i)_{t=0}^{T-1}$, the agent i accumulates at the compliance date T the total terminal costs

$$\sum_{t=0}^{T-1} C_t^i(\xi_t^i). \tag{3.3}$$

Abatement volume: For later use, let us introduce for each $\omega \in \Omega$, $t = 0, \ldots, T-1$ and $a \in [0, \infty[$ the abatement volume $c_t^i(a)(\omega)$ as

$$c_t^i(a)(\omega) := \operatorname{argmin}\{C_t^i(x)(\omega) - ax \ : \ x \in [0, E_t^i(\omega)]\}, \tag{3.4}$$

which is well-defined, since under the assumptions (3.1) the minimum of the function $x \mapsto C_t^i(x)(\omega) - ax$ on $[0, E_t^i(\omega)]$ is attained at the unique point. The reader may imagine $c_t^i(a)(\omega)$ as the total reduction volume which is avialable within $[t, t+1]$ in the situation ω at a price which less or equal a (measured in currency unit per pollutant unit). A straight-forward proof shows that (3.1) ensures that

$$\begin{array}{c} [0, \infty[\mapsto \mathbb{R}, \ a \mapsto c_t^i(a)(\omega) \text{ is non-decreasing and} \\ \text{continuous for almost each } \omega \in \Omega \text{ and } t = 0, \ldots, T-1. \end{array} \tag{3.5}$$

For later use, we introduce the cumulative abatement volume function

$$c_t(a) := \sum_{i \in I} c_t^i(a), \qquad a \in [0, \infty[. \tag{3.6}$$

Obviously, $c_t(a)(\omega)$ stands for the total abatement in the market, which is available from all measures in the situation ω whose price is below or equal $a \in [0, \infty[$.

Allowance trading: Suppose that at any time $t = 0, \dots, T$, credits can be exchanged between agents by trading at the spot price A_t. Denote by ϑ_t^i the change at time t in allowance number, held by agent i. That is, given the allowance prices $(A_t)_{t=0}^T$, the position changes $(\vartheta_t^i)_{t=0}^T$ yield costs

$$\sum_{t=0}^{T} \vartheta_t^i A_t. \tag{3.7}$$

Penalty payment: The total pollution of the agent i can be expressed as a difference

$$\sum_{t=0}^{T-1} E_t^i - \sum_{t=0}^{T-1} \xi_t^i$$

of the cumulative business-as-usual emission less the entire reduction. As mentioned above, a penalty $\pi \in]0, \infty[$ is being paid at maturity T for each unit of pollutant, which is not covered by allowances. Considering the total change in the allowance position $\sum_{t=0}^{T} \vartheta_t^i$ effected by trading, the loss of the agent i resulting from potential penalty payment is

$$\pi \left(\sum_{t=0}^{T-1} (E_t^i - \xi_t^i - \vartheta_t^i) - \gamma^i - \vartheta_T^i \right)^+ \tag{3.8}$$

where

$$\gamma^i, \ i \in I \text{ are agents' initial allowance allocations.} \tag{3.9}$$

Remark: Our stylized scheme deals with stand-alone emission trading mechanism. In real world, cap and trade systems operate on multi-period scale, where unused allowances can be carried out (banked) into next period. Further period inter-connections may include a transfer of future allocation from the next into present period (borrowing) and, in the case of non-compliance, a withdrawal of an appropriate number of credits from next period allocation in addition to penalty payment. To complete the complexity, let us mention that different emission markets could be inter-connected by acceptance of foreign certificates in the national scheme. Emission trading in multi-period setting is addressed, among others, in [4] and [5]. Mathematically, it reduces to the specification of a more complex penalty mechanism than presented above. For this reason, we decided to focus on the stand-alone allowance market to analyze quantitative methods in the simplest situation before targeting multi-scale systems (such as second period of EU ETS).

Recording uncertainty: In what follows, we need to take into account also uncertainty in the emission recording. It is convenient to subtract these recording errors from the initial allocation. Hence we interpret γ^i as the credits, allocated to the agent i less emissions which become known with certainty only at time T. With this interpretation, γ_i stands for allowances effectively available for compliance and is modeled by an $\mathcal{F}_T$-measurable random variable. For later use, let us agree that the distribution of $\sum_{i \in I} \gamma^i$, conditioned on $\mathcal{F}_{T-1}$, possesses almost surely no point masses which implies that

$$P(\sum_{i \in I} \gamma^i = X) = 0 \quad \text{for each } \mathcal{F}_{T-1}\text{-measurable } X \tag{3.10}$$

Admissible policies: Since maximally possible reduction can not exceed emission we have (3.2). Let us define the space of feasible trading $\vartheta^i = (\vartheta_t^i)_{t=0}^T$ and abatement strategies $\xi^i = (\xi_t^i)_{t=0}^{T-1}$ of the agent $i \in I$ by

$$\mathcal{U}^i := \{(\vartheta^i, \xi^i) \,:\, 0 \le \xi_t^i \le E_t^i \quad t = 0, \dots, T-1\}. \tag{3.11}$$

Individual wealth: In view of (3.3), (3.7), and (3.8), the revenue of the agent i following admissible policy $(\vartheta^i, \xi^i) \in \mathcal{U}^i$ equals to

$$L^{A,i}(\vartheta^i, \xi^i) = -\sum_{t=0}^{T-1} (\vartheta_t^i A_t + C^i(\xi_t^i)) - \vartheta_T^i A_T - \pi(\sum_{t=0}^{T-1}(E_t^i - \xi_t^i - \vartheta_t^i) - \gamma^i - \vartheta_T^i)^+. \tag{3.12}$$

Risk aversion To face risk preferences, suppose that attitudes of the agents $i \in I$ are described by utility functions $U^i : \mathbb{R} \mapsto \mathbb{R}$, which are continuous, strictly increasing and concave. Consider the utility functional $u^i(X) = E(U^i(X))$, which is assumed to be defined for each random variable X where the expectation is finite or is $+\infty$. Given allowance price process $A = (A_t)_{t=0}^T$, the agent i behaves rationally maximizing $(\vartheta^i, \xi^i) \mapsto u^i(L^{A,i}(\vartheta^i, \xi^i))$ by an appropriate choice of the own policy $(\vartheta^{i*}, \xi^{i*})$.

Market equilibrium: Following standard apprehension, a realistic market state is described by the so-called equilibrium — a situation, where the allowance price, positions, and abatement measures are such that each agent is satisfied by the own policy and at the same time, natural restrictions are fulfilled. In our framework, an appropriate notion of equilibrium is given by

***Definition* 1.** The processes $A^* = (A_t^*)_{t=0}^T$ is called an equilibrium allowance price process, if for each $i \in I$ there is $(\vartheta^{*i}, \xi^{*i}) \in \mathcal{U}^i$ such that $u^i(L^{A^*,i}(\vartheta^{*i}, \xi^{*i}))$ is finite and

   (i) the cumulative changes in positions are in zero net supply

$$\sum_{i \in I} \vartheta_t^{*i} = 0, \text{ for all } t = 0, \dots, T. \tag{3.13}$$

   (ii) each agent $i \in I$ is satisfied by the own policy in the sense that

$$u^i \left(L^{A^*,i}(\vartheta^{*i}, \xi^{*i}) \right) \ge u^i \left(L^{A^*,i}(\vartheta^i, \xi^i) \right)$$
$$\text{for each } (\vartheta^i, \xi^i) \in \mathcal{U}^i \text{ where } u^i \left(L^{A^*,i}(\vartheta^i, \xi^i) \right) \text{ exists} \tag{3.14}$$

The existence of emission market equilibrium is addressed in [2] and [3], under the assumption of linear utility function and in a slightly different setting. However, although the equilibrium modeling in the spirit of these contributions is appropriate to investigate important questions of of optimal market design, it has little to offer to the problem of derivatives valuation. With the present approach, we intend to establish a reduced-form model which describes the evolution of emission-related assets from risk-neutral perspective. We obtain a realistic picture by incorporating the three essential assumptions into a risk neutral models. These assumptions are shown to be direct consequences of equilibrium situation:

a) There is no arbitrage, since in equilibrium, any profitable strategy would immediately be followed by all agents. This would instantaneously change prices and exhausts any arbitrage opportunity.
b) The allowance trading instantaneously triggers all abatement measures whose costs are below allowance price. The explanation here is that if agent possess a technology with lower reduction costs than the present allowance price, then it is optimal to immediately reduce own pollution and take profit from selling allowances.
c) There are merely two final outcomes for allowance price. Either the terminal allowance price drops to zero or it approaches the penalty level. The reason is that at maturity, price must vanish if there is an excess in allowances, whereas in the case of their shortage, the price will raise reaching penalty. We believe that in reality, an exact coincidences of allowance demand and supply occurs with zero probability and can be neglected.

Let us formalize the above assertions a), b) and c).

Proposition 1. Suppose that $(A_t^*)_{t=0}^T$ is an equilibrium allowance price and $(\xi_t^{i*})_{t=0}^{T-1}$ for $i \in I$ are corresponding equilibrium abatement policies.

(a) There exists a measure Q which equivalent to P such that $(A_t^*)_{t=0}^T$ follows a Q-martingale.
(b) For each $i \in I$ holds

$$\xi_t^{i*} = c_t^i(A_t^*), \quad t = 0, \ldots, T-1, \tag{3.15}$$

with abatement volume functions c_t^i, $t = 0, \ldots, T-1$ from (3.4).
(c) The terminal value of the allowance prices is given by

$$A_T^* = \pi 1_{\{\sum_{i \in I}(\sum_{t=0}^{T-1}(E_t^i - \xi_t^{i*}) - \gamma^i) \geq 0\}} \tag{3.16}$$

Before we proceed with the proof, let us emphasize that this result can serve a starting point for risk-neutral modeling. The above proposition states that in the equilibrium, allowance price process $(A_t^*)_{t=0}^T$ follows a martingale with respect to an equivalent measure $Q \sim P$ whose terminal value is

$$A_T^* = \pi 1_{\{\sum_{i \in I} \sum_{t=0}^T (E_t^i - \xi_t^*) - \gamma^i) \geq 0\}},$$

obviously depending on intermediate values $(A_t^*)_{t=0}^{T-1}$ through abatement volume function $\xi_t^{i*} = c_t^i(A_t^*)$ for $t = 0, \ldots, T-1$, $i \in I$. The surprising and far-reaching consequence is that, from risk-neutral perspective, only cumulative market quantities are relevant. To see this, define the overall allowance shortage

$$\mathcal{E}_T = \sum_{i \in I} (\sum_{t=0}^{T-1} E_t^i - \gamma^i) \tag{3.17}$$

which would appear in the market without any emission regulation. Further, recall from (3.4) and (3.6) the cumulative abatement functions to expresses the risk-neutral certificate price dynamics in terms of the following feedback equation

$$A_t = E_t^Q(\pi 1_{\{\mathcal{E}_T - \sum_{t=0}^{T-1} c_t(A_t^*) \geq 0\}}), \qquad t = 0, \ldots, T-1$$

Although individual market attributes and actions of the different agents seem to be irrelevant in this picture, the reader should notice that this picture appears only from the risk-neutral viewpoint. In line with standard aggregation theorems, the equilibrium market state heavily depends and is determined by market architecture, rules, risk attitudes and uncertainty. However, once equilibrium is reached and all arbitrage opportunities are exhausted, asset dynamics can be considered under risk neutral measure. With respect to this measure, market evolution appears as it was driven by cumulative quantities only.

With this, let us formulate the problem of the reduced-form modeling as

$$\left.\begin{array}{c} \text{given measure } Q \sim P, \text{ random variable } \mathcal{E}_T, \\ \text{and abatement volume functions } (c_t)_{t=0}^{T-1}, \\ \text{determine a } Q\text{-martingale } (A_t^*)_{t=0}^T \text{ with} \\ A_T^* = \pi 1_{\{\mathcal{E}_T - \sum_{t=0}^{T-1} c_t(A_t^*) \geq 0\}}. \end{array}\right\} \tag{3.18}$$

Note that this formulation serves a guideline to martingale modeling, since price-dependent abatement volume $c_t(a)$ can be estimated from market data whereas potential allowance shortage $\mathcal{E}_T$ can be modeled in terms of total allowance allocation and demand fluctuations on goods whose production causes the pollution. Finally, we shall emphasize a natural passage to continuous time.

$$\left.\begin{array}{c} \text{Given on a probability space } (\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \in [0,T]}) \\ \text{an equivalent measure } Q \sim P, \text{ random variable } \mathcal{E}_T, \\ \text{and a family of abatement functions } (c_t)_{t \in [0,T]}, \\ \text{determine a } Q\text{-martingale } (A_t^*)_{t \in [0,T]} \text{ with} \\ A_T^* = \pi 1_{\{\mathcal{E}_T - \int_0^T c_t(A_t^*)dt \geq 0\}}. \end{array}\right\} \tag{3.19}$$

Proof. (of the Proposition) (a) According to the first fundamental theorem of asset pricing (see [10]), it suffices to verify that, if $(A_t^*)_{t=0}^T$ is an equilibrium allowance price

process, then there is no arbitrage for allowance trading. Let us follow an indirect proof, supposing that $(\nu_t)_{t=0}^{T-1}$ is an allowance trading arbitrage, meaning that

$$\mathrm{P}(\sum_{t=0}^{T-1} \nu_t(A_{t+1} - A_t) \geq 0) = 1, \quad \mathrm{P}(\sum_{t=0}^{T-1} \nu_t(A_{t+1} - A_t) > 0) > 0. \qquad (3.20)$$

Now, we verify that in the presence of arbitrage no equilibrium can exist, since each agent i can change the own policy $(\vartheta^{*i}, \xi^{*i})$ to an improved strategy $(\tilde{\vartheta}^i, \xi^{*i})$ satisfying

$$u^i\left(L^{A^*,i}(\vartheta^{*i}, \xi^{*i})\right) < u^i\left(L^{A^*,i}(\tilde{\vartheta}^i, \xi^{*i})\right). \qquad (3.21)$$

The improvement is achieved by incorporating arbitrage $(\nu_t)_{t=0}^{T-1}$ into own allowance trading as follows

$$\tilde{\vartheta}^i_t := \vartheta^{*i}_t + (\nu^i_t - \nu^i_{t-1}) \quad \text{for all } t = 0, \dots, T,$$

with appropriate definitions $\nu_{-1} = \nu_T := 0$. Indeed, the revenue improvement from allowance trading is

$$-\sum_{t=0}^{T} \tilde{\vartheta}^i_t A_t = -\sum_{t=0}^{T} \vartheta^{*i}_t A_t + \sum_{t=0}^{T-1} \nu^i_t(A_{t+1} - A_t)$$

which we combine with (3.20) to see that there is no optimality, since

$$\mathrm{P}\left(L^{A,i}(\vartheta^{*i}, \xi^i) \leq L^{A,i}(\tilde{\vartheta}^i, \xi^i)\right) = 1, \qquad \mathrm{P}\left(L^{A,i}(\vartheta^{*i}, \xi^i) < L^{A,i}(\tilde{\vartheta}^i, \xi^i)\right) > 0$$

implies that

$$u^i(L^{A,i}(\vartheta^{*i}, \xi^i)) < u^i(L^{A,i}(\tilde{\vartheta}^i, \xi^i)).$$

(b) To prove (3.15), consider the bijection

$$\mathcal{U}^i \to \mathcal{U}^i, \qquad (\theta^i, \xi^i) \mapsto (\phi(\theta^i, \xi^i), \xi^i). \qquad (3.22)$$

where the transformed trading strategy $\vartheta^i = \phi(\theta^i, \xi^i)$ is given by

$$\vartheta^i_t = \theta^i_t - \xi^i_t, \qquad t = 1, \dots, T-1, \qquad \vartheta^i_T = \theta^i_T.$$

Obviously, $(\vartheta^{i*}, \xi^{i*})$ is a maximizer to the original problem

$$\mathcal{U}^i \to \mathbb{R}, \quad (\vartheta^i, \xi^i) \mapsto u^i(L^{A^*,i}(\vartheta^i, \xi^i))$$

if and only if $(\vartheta^{i*}, \xi^{i*}) = (\phi(\theta^{i*}, \xi^{i*}), \xi^{i*})$, where (θ^{i*}, ξ^{i*}) is a maximizer to the transformed problem

$$\mathcal{U}^i \to \mathbb{R}, \quad (\theta^i, \xi^i) \mapsto u^i(L^{A^*,i}(\phi(\theta^i, \xi^i), \xi^i)). \qquad (3.23)$$

The last line in the calculation

$$L^{A^*,i}(\phi(\theta^i, \xi^i), \xi^i) =$$

$$= -\sum_{t=0}^{T-1}(\theta_t^i - \xi_t^i)A_t^* - \sum_{t=0}^{T-1}C_t^i(\xi_t^i) - \pi(\sum_{t=0}^{T-1}(E_t^i - \xi_t^i - (\theta_t^i - \xi_t^i)) - \gamma^i - \theta_T^i)^+$$

$$= -\sum_{t=0}^{T-1}\theta_t^i A_t^* - \pi(\sum_{t=0}^{T-1}(E_t^i - \theta_t^i) - \gamma^i - \theta_T^i)^+ - \sum_{t=0}^{T-1}(C^i(\xi_t^i) - A_t^*\xi_t^i) \qquad (3.24)$$

shows that if (θ^{i*}, ξ^{i*}) is a maximizer to (3.23), then ξ^* must satisfy $\xi_t^{i*} := c_t^i(A_t^*)$ for $t = 0, \ldots, T - 1$, which proves (3.15).

(c) This assertion is proven by argumentation identical to that given in [2]. $\qquad \square$

4. Reduced-form modeling

In what follows, we propose a solution to the problem of risk-neutral allowance price modeling (3.18). Below, we prove that under the assumptions given above (in particular (3.10) is essential), the problem (3.18) possess a solution. Moreover, we show how to obtain the martingale $(A_t^*)_{t=0}^T$.

It turns out that the martingale closed by $\mathcal{E}_T$ plays a crucial role, so we introduce

$$\mathcal{E}_t = E^Q(\mathcal{E}_T \mid \mathcal{F}_t), \qquad t = 0, \ldots, T.$$

For later use, let us also define its increments as

$$\varepsilon_t = \mathcal{E}_t - \mathcal{E}_{t-1} \qquad t = 1, \ldots, T.$$

Following the intuition that the equilibrium allowance price should be uniquely determined by the present time, and the general market situation, we express a candidate for allowance price as

$$A_t^*(\omega) = \alpha_t(G_t(\omega))(\omega) \quad \omega \in \Omega, \ t = 0, \ldots, T, \qquad (4.1)$$

with hypothetic functionals

$$\alpha_t : \mathbb{R} \times \Omega \to [0, \pi], \quad \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}_t\text{-measurable, for } t = 0, \ldots T \qquad (4.2)$$

applied to

$$G_t = \mathcal{E}_t - \sum_{s=0}^{t-1} c_t(A_s^*), \quad t = 0, \ldots, T. \qquad (4.3)$$

According to (3.18), this approach yields an obvious definition for α_T:

$$\alpha_T(g)(\omega) = \pi 1_{[0,\infty[}(g), \quad \omega \in \Omega, \ g \in \mathbb{R}. \qquad (4.4)$$

Note that, given functionals (4.2), the price process $(A_t^*)_{t=0}^T$ is indeed well-defined by recursive application of (4.3) and (4.1):

$$
\begin{aligned}
A_t^*(\omega) &:= \alpha_t(G_t(\omega))(\omega), & (4.5) \\
G_{t+1}(\omega) &:= G_t(\omega) - c_t(A_t^*(\omega))(\omega) + \varepsilon_{t+1}(\omega), & (4.6) \\
& \text{started at } G_0 := \mathcal{E}_0. & (4.7)
\end{aligned}
$$

Generated by this recursion, the process $(A_t^*)_{t=0}^T$ follows a martingale, if for all $t = 0, \ldots T-1$, for almost all $\omega \in \Omega$ holds

$$
\alpha_t(g)(\omega) = \mathbb{E}_t^{\mathrm{Q}}(\alpha_{t+1}(g - c_t(\alpha_t(g)(\omega))(\omega) + \varepsilon_{t+1}))(\omega), \quad \text{for all } g \in \mathbb{R},\ \omega \in \Omega.
$$

Indeed, we have

$$
\begin{aligned}
\mathbb{E}_t^{\mathrm{Q}}(A_{t+1}^*)(\omega) &= \mathbb{E}_t^{\mathrm{Q}}(\alpha_{t+1}(G_{t+1}))(\omega) \\
&= \int_\Omega \alpha_{t+1}(G_t(\omega') - c_t(A_t^*(\omega'))(\omega') + \varepsilon_{t+1}(\omega'))(\omega') \mathrm{Q}_t(d\omega')(\omega) \\
&= \int_\Omega \alpha_{t+1}(G_t(\omega) - c_t(A_t^*(\omega))(\omega) + \varepsilon_{t+1}(\omega'))(\omega) \mathrm{Q}_t(d\omega')(\omega) \\
&= \mathbb{E}_t^{\mathrm{Q}}(\alpha_{t+1}(G_t(\omega) - c_t(\alpha_t(G_t(\omega))(\omega))(\omega) + \varepsilon_{t+1}))(\omega) \\
&= \alpha_t(G_t(\omega))(\omega) = A_t^*(\omega)
\end{aligned}
$$

In other words, it is sufficient to ensure that

$$
\begin{aligned}
& \text{for each } g \in \mathbb{R},\ \alpha_t(g)(\omega) \text{ solves} \\
a &= \mathbb{E}_t^{\mathrm{Q}}(\alpha_{t+1}(g - c(a) + \varepsilon_{t+1}))(\omega) \\
& \text{for almost all } \omega \in \Omega.
\end{aligned} \tag{4.8}
$$

In the reminder of this section, we show that the functionals (4.2) are recursively obtained as the unique solution to (4.8), starting with α_T from (4.4). First, let us prepare an auxiliary result dealing with the solution to (4.8) where no conditional information needs to be considered.

***Lemma* 1.** Given

$$
\begin{aligned}
c &: \mathbb{R} \to \mathbb{R}, \quad \text{non-decreasing, continuous} & (4.9) \\
\alpha_1 &: \mathbb{R} \times \Omega \to [0, \pi], \quad \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}\text{-measurable}, & (4.10) \\
g &\mapsto \alpha_1(g)(\omega) \quad \text{non-decreasing for almost all } \omega \in \Omega, & (4.11)
\end{aligned}
$$

suppose that the random variable ε satisfies

$$
\begin{aligned}
\mathbb{R} \to [0, \pi], \quad x \mapsto \mathbb{E}^Q(\alpha_1(x + \varepsilon)) = \int_\Omega \alpha_1(x + \varepsilon(\omega'))(\omega') \mathrm{Q}(d\omega') \\
\text{is continuous.}
\end{aligned} \tag{4.12}
$$

For each $g \in \mathbb{R}$ introduce the function $f^g : [0, \pi] \to \mathbb{R}$ given by

$$
\begin{aligned}
f^g(a) &:= a - \mathbb{E}^Q(\alpha_1(g - c(a) + \varepsilon)) \\
&= a - \int_\Omega \alpha_1(g - c(a) + \varepsilon(\omega'))(\omega')Q(d\omega'), \quad a \in [0, \pi],
\end{aligned}
\tag{4.13}
$$

then following assertions hold

(i) For each $g \in \mathbb{R}$ there exists a unique $\alpha_0(g) \in [0, \pi]$ with $f^g(\alpha_0(g)) = 0$.

(ii) The root $\alpha_0(g)$ of f^g is obtained as a limit $\alpha_0(g) = \lim_{n \to \infty} a_n^g$ in the standard bisection method

$$
a_n^g = \frac{1}{2}(\overline{a}_n^g + \underline{a}_n^g), \quad
\begin{array}{ll}
\overline{a}_{n+1}^g = a_n^g, & \underline{a}_{n+1}^g := \underline{a}_n^g \quad \text{if } f^g(a_n^g) \geq 0 \\
\overline{a}_{n+1}^g = \overline{a}_n^g, & \underline{a}_{n+1}^g := a_n^g \quad \text{if } f^g(a_n^g) < 0
\end{array}
\tag{4.14}
$$

started at $\underline{a}_0^g := 0$, $\overline{a}_0^g := \pi$.

(iii) The mapping $\mathbb{R} \to [0, \pi]$, $g \mapsto \alpha_0(g)$ is non-decreasing and continuous.

Proof. (i) For each $g \in \mathbb{R}_+$, the function f^g is continuous due to (4.12) and to the continuity (4.9) of c. Thus the existence of a root follows from the intermediate value theorem because of

$$
f^g(0) \leq 0, \qquad f^g(\pi) \geq 0.
\tag{4.15}
$$

The uniqueness of the root is ensured by the strict monotonic increase of f^g. To verify this, observe that (4.11) and (4.9) imply that the subtrahend

$$
a \mapsto \int_\Omega \alpha_1(g - c(a) + \varepsilon(\omega'))(\omega')Q(d\omega')
$$

in (4.13) is non-increasing, whereas the minuend $a \mapsto a$ is strictly increasing in a.

(ii) The bisection algorithm is properly initialized because of (4.15), standard arguments ensure its convergence to the root.

(iii) To show the monotonic increase of $g \mapsto \alpha_0(g)$, suppose that $g' < g$, then (4.11) ensures that for each $a \in [0, \pi]$

$$
\int_\Omega \alpha_1(g' - c(a) + \varepsilon(\omega'))(\omega')Q(d\omega') \leq \int_\Omega \alpha_1(g - c(a) + \varepsilon(\omega'))(\omega')Q(d\omega')
$$

giving $f^{g'}(a) \geq f^g(a)$ for all $a \in [0, \pi]$, which implies that $\alpha_0(g') \leq \alpha_0(g)$.

Now, let us turn to the continuity. If $\alpha_0(g) \in [0, \pi[$, then there exist $\delta > 0$ with $\alpha_0(g) + \delta \leq \pi$. Due to the strict monotonic increase of f^g, we obtain $0 < f^g(\alpha_0(g) + \delta)$. If $(g_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+$ is a sequence with $\lim_{n \to \infty} g_n = g$ then according to (4.12)

$$
\lim_{n \to \infty} f^{g_n}(\alpha_0(g) + \delta) = f^g(\alpha_0(g) + \delta) > 0.
\tag{4.16}
$$

Hence, there exists $N \in \mathbb{N}$ such that $f^{g_n}(\alpha_0(g) + \delta) > 0$ holds for all $n \geq N$. Thus, we obtain

$$\alpha_0(g) \in [0, \pi[\quad \Longrightarrow \limsup_{n \to \infty} \alpha_0(g_n) \leq \alpha_0(g) + \delta \qquad \text{if } \alpha_0(g) + \delta \leq \pi. \qquad (4.17)$$

Since $\delta > 0$ is arbitrary small and $0 \leq \alpha_0(g) \leq \pi$ due to (i), this implication shows that $\alpha_0(\cdot)$ is continuous on each point g with $\alpha_0(g) = 0$. Similar argumentation yields

$$\alpha_0(g) \in]0, \pi] \quad \Longrightarrow \liminf_{n \to \infty} \alpha_0(g_n) \geq \alpha_0(g) - \delta \qquad \text{if } \alpha_0(g) - \delta \geq 0. \qquad (4.18)$$

Again, since $\delta > 0$ is arbitrary, we obtain the continuity of $\alpha_0(\cdot)$ on each point g with $\alpha_0(g) = \pi$. If $\alpha_0(g) \in]0, \pi[$ the continuity of $\alpha_0(\cdot)$ on g follows by the combination of (4.17) and (4.18). $\qquad \square$

Let us now turn to the conditioned version of the Lemma 1. Supposing the existence of the regular $\mathcal{F}_t$-conditioned distribution Q_t, the proof copies the arguments of the previous lemma with appropriate notational changes due to conditioning on the event $\omega \in \Omega$. However, a useful insight is that the approximating points a_n^g, $n = 0, 1, 2, \ldots$ of the bisection algorithm turn out to be dependent on $g \in \mathbb{R}$ and $\omega \in \Omega$ in a $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}_t$-measurable way, which shows that the functional under discussion $(g, \omega) \mapsto \alpha_t(g)(\omega)$, is also $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}_t$-measurable being the limit of the sequence $((g, \omega) \mapsto a_n^{g,\omega})_{n=0}^{\infty}$ of measurable functions.

Lemma 2. Suppose that for $t \in \{0, \ldots, T-1\}$

$$c : \mathbb{R} \times \Omega \to \mathbb{R}, \quad \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}\text{-measurable such that} \qquad (4.19)$$
$$a \mapsto c_t(a)(\omega) \text{ is non-decreasing, continuous} \qquad (4.20)$$
$$\alpha_{t+1} : \mathbb{R} \times \Omega \to [0, \pi], \quad \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}\text{-measurable such that} \qquad (4.21)$$
$$g \mapsto \alpha_{t+1}(g)(\omega) \text{ is non-decreasing for all } \omega \in \Omega. \qquad (4.22)$$

Given a regular version Q_t of the $\mathcal{F}_t$ conditioned distribution Q, assume that the random variable ε_{t+1} satisfies

$$\mathbb{R} \to [0, \pi], \quad x \mapsto \int_{\Omega} \alpha_{t+1}(x + \varepsilon_{t+1}(\omega'))(\omega') Q_t(d\omega')(\omega)$$
$$\text{is continuous for each } \omega \in \Omega \qquad (4.23)$$

then the following assertions hold:

(i) There exists a unique $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}_t$-measurable $[0, \pi]$-valued α_t satisfying

$$\alpha_t(g)(\omega) = \mathbb{E}_t^Q(\alpha_{t+1}(g - c_t(\alpha_t(g)) + \varepsilon_{t+1}))(\omega)$$
$$\text{for all } g \in \mathbb{R}, \text{ for almost all } \omega \in \Omega. \qquad (4.24)$$

(ii) The mapping $\mathbb{R} \to [0, \pi]$, $g \mapsto \alpha_t(g)(\omega)$ is non-decreasing and continuous for all $\omega \in \Omega$.

Proof. (i) As in the proof of the Lemma 1, we obtain the unique root $\alpha_t(g)(\omega)$ of the function

$$f^{g,\omega}(a) := a - \int_\Omega \alpha_{t+1}(g - c_t(a)(\omega) + \varepsilon_{t+1}(\omega'))(\omega')Q_t(d\omega')(\omega), \quad a \in [0,\pi].$$

by the bisection method

$$a_n^{g,\omega} = \frac{1}{2}(\overline{a}_n^{g,\omega} + \underline{a}_n^{g,\omega}), \quad \begin{matrix} \overline{a}_{n+1}^{g,\omega} = a_n^{g\omega}, & \underline{a}_{n+1}^{g,\omega} := \underline{a}_n^{g,\omega} & \text{if } f^{g,\omega}(a_n^{g,\omega}) \geq 0 \\ \overline{a}_{n+1}^g = \overline{a}_n^{g,\omega}, & \underline{a}_{n+1}^{g,\omega} := a_n^g & \text{if } f^{g,\omega}(a_n^{g,\omega}) < 0 \end{matrix}$$

started at $\underline{a}_0^{g,\omega} := 0$, $\overline{a}_0^{g,\omega} := \pi$. Since

$$(g,\omega,a) \mapsto f^{g,\omega}(a) \quad \text{is } \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}_t \otimes \mathcal{B}([0,\pi])\text{-measurable},$$

each bisection point $(g,\omega) \mapsto a_n^{g,\omega}$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}_t$-measurable, which shows that for $n \to \infty$, the pointwise limit $(g,\omega) \mapsto \alpha_t(g,\omega)$ of the bisection sequence is also $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}_t$-measurable. By construction, the equality

$$\alpha_t(g)(\omega) = \int_\Omega \alpha_{t+1}(g - c_t(\alpha_t(g)(\omega))(\omega) + \varepsilon_{t+1}(\omega'))(\omega')Q_t(d\omega')(\omega)$$

holds for all $g \in \mathbb{R}$ and $\omega \in \Omega$, whose right-hand side is nothing but the right-hand side of (4.24) for each $g \in \mathbb{R}$.
(ii) The proof is obtained from (iii) of the previous lemma by replacing $\alpha_1(\cdot)$, $\alpha_0(\cdot)$, $c(\cdot)$ and $Q(d\omega')$ by $\alpha_{t+1}(\cdot)(\omega), \alpha_t(\cdot)(\omega)$, $c_t(\cdot)(\omega)$ and $Q_t(d\omega')(\omega)$ with appropriate notational adaptations according to the conditioning on ω. $\quad\square$

Finally, we address a solution to (3.18) in the last point of the following

Proposition 2. Consider $\mathcal{E}_T$ under the model assumption (3.10) and the cumulative abatement volume functions from (3.6) under t (3.1) and (3.4).

i) Given measure $Q \sim P$ there exist functionals

$$\alpha_t : \mathbb{R} \times \Omega \to [0,\pi], \quad \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}_t\text{-measurable, for } t = 0,\ldots T \tag{4.25}$$

which fulfill for all $g \in \mathbb{R}$

$$\alpha_T(g) = \pi 1_{[0,\infty[}(g), \tag{4.26}$$

$$\alpha_t(g) = \mathbb{E}_t^Q(\alpha_{t+1}(g - c_t(\alpha_t(g)) + \varepsilon_{t+1})), \quad t = 0,\ldots,T-1. \tag{4.27}$$

ii) There exists a Q–martingale $(A_t^*)_{t=0}^T$ which satisfies

$$A_T^* = \pi 1_{\{\mathcal{E}_T - \sum_{t=0}^{T-1} c_t(A_t^*) \geq 0\}}. \tag{4.28}$$

Proof. (i) In this proof, we repeatedly make use of Lemma 2. Let us start with $t = T - 1$ and verify that the assumptions of this lemma are satisfied. Due to continuity (3.5) of the abatement function we have (4.9). The properties (4.21) and (4.22) hold for $t = T - 1$ by definition (4.26). To show (4.23), we utilize the specific form of α_T

$$x \mapsto \int_\Omega \alpha_T(x + \varepsilon_T(\omega'))(\omega')Q_{T-1}(d\omega')(\omega) = Q_{T-1}(x + \varepsilon_T \geq 0)(\omega). \qquad (4.29)$$

Note that due to (3.10) there are almost surely no point masses in the distribution of

$$\varepsilon_T = \sum_{i \in I} \gamma^i - E^Q_{T-1}(\sum_{i \in I} \gamma^i)$$

conditioned on $\mathcal{F}_{T-1}$ (with respect Q, since $Q \sim P$). That is, (4.29) is continuous for each $\omega \in \Omega$, as required in (4.23). Hence, (i) of Lemma 2 yields functional α_{T-1} satisfying (4.27) (with $t = T - 1$), as required. To proceed by induction, we emphasize that (ii) of Lemma 2 ensures that $g \mapsto \alpha_{T-1}(g)(\omega)$ is non-decreasing and continuous for all $\omega \in \Omega$. That is, for the next step $t = T - 2$ the assumption (4.22) on α_{T-1} is automatically satisfied. Moreover (4.23) follows now due to the continuity of $g \mapsto \alpha_{T-1}(g)(\omega)$ from the pointwise convergence

$$\lim_{n \to \infty} \alpha_{T-1}(x_n + \varepsilon_{T-1}(\omega'))(\omega') = \alpha_{T-1}(x + \varepsilon_{T-1}(\omega'))(\omega') \quad \text{for all } \omega' \in \Omega,$$

dominated by π, which holds for each $(x_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ with $\lim_{n \to \infty} x_n = x$. That is, all assumptions of the Lemma 2 are also fulfilled for $t = T - 2$. Proceeding recursively for $t = T - 2, \ldots, 0$, we obtain $(\alpha_t)_{t=0}^T$ with (4.25), (4.26), and (4.27).
(ii) As suggested by (4.5) – (4.7), we define for all $\omega \in \Omega$

$$\begin{aligned}
A^*_t(\omega) &:= \alpha_t(G_t(\omega))(\omega), \\
G_{t+1}(\omega) &:= G_t(\omega) - c_t(A^*_t(\omega))(\omega) + \varepsilon_{t+1}(\omega), \\
&\quad \text{started at } G_0 := \mathcal{E}_0.
\end{aligned}$$

The process $(A^*_t)_{t=0}^T$ generated in this way obeys the terminal condition (4.28), in view of (4.26). To show the Q-martingale property of $(A^*_t)_{t=0}^T$, we calculate for $t = 0, \ldots T - 1$

$$E^Q_t(A^*_{t+1})(\omega) = \mathbb{E}^Q_t(\alpha_{t+1}(G_{t+1}))(\omega) = \mathbb{E}^Q_t(\alpha_{t+1}(G_t(\omega) - c_t(A^*_t(\omega))(\omega) + \varepsilon_{t+1}))(\omega)$$

$$= \mathbb{E}^Q_t(\alpha_{t+1}(G_t(\omega) - c_t(\alpha_t(G_t(\omega))(\omega))(\omega) + \varepsilon_{t+1}))(\omega) = \alpha_t(G_{t+1}(\omega))(\omega) = A^*_t(\omega)$$

for almost all $\omega \in \Omega$, where second last equality follows from (4.27). $\qquad \square$

5. Applications

Let us elaborate on the computational feasibility of our reduced-form modeling. For illustration purposes, we focus on the simplest case martingales with independent increments

and deterministic abatement functions. We assume that

$$\varepsilon_{t+1} \text{ and } \mathcal{F}_t \text{ are independent under } Q \text{ for all } t = 0, \ldots, T-1. \tag{5.1}$$

$$c_t : [0, \infty[\rightarrow \mathbb{R} \text{ is deterministic and time constant } (c_t = c)_{t=0}^{T-1} \tag{5.2}$$

Under this assumption, the randomness enters the allowance price through the present up-to-day emissions only. More precisely, (5.1) ensures that

$$\omega \mapsto \alpha_t(g)(\omega) = \alpha_t(g) \quad \text{is constant on } \Omega. \tag{5.3}$$

Let us verify this assertion. For $t = T$, (5.3) holds by definition (4.26). For $t = T-1, \ldots, 1$ we proceed inductively as follows: by construction, $\alpha_t(g)(\omega)$ is the unique solution a to

$$
\begin{aligned}
a &= \int_\Omega \alpha_{t+1}(g - c_t(a) + \varepsilon_{t+1}(\omega'))(\omega')Q_t(d\omega')(\omega) \\
&= \int_\Omega \alpha_{t+1}(g - c_t(a) + \varepsilon_{t+1}(\omega'))Q(d\omega') \tag{5.4}
\end{aligned}
$$

where in the last equality we have utilized $Q_t = Q$ due to the independence (5.1) and that $\alpha_{t+1}(g)$ does not depend on ω by the the induction assumption. Obviously, fixed point $\alpha_t(g)(\omega) := a$ from (5.4) also does not depend on ω.

For numerical calculation, we rely on one-dimensional least-square Monte-Carlo method which is applicable in our case of martingales with independent increments. Although this setting is relatively restrictive, it covers a sufficiently rich martingales class. For instance, important cases of information shocks leading to allowance price jumps can be easily addressed under this approach when $(\mathcal{E}_t)_{t=0}^T$ is modeled as an appropriately sampled centered Poisson process. In this case, fixed point equations can be treated analytically. We do not follow this path in favor of numerical methods, which deserve a particular attention due to complexity of emission markets. In particular, extensions of Monte Carlo methods to the multidimensional setting (see [20]) seem to be appropriate. A preliminary analysis shows that assuming the existence of a global Markovian state process allows to weaken independence to conditional independence, which leads to multidimensional Monte-Carlo in the sense of [20], since the state process gives additional dimensions.

We now focus on computational aspects. From (5.3) it follows that $\alpha_t(G_t)$ is a $\sigma(G_t)$–measurable random variable. Thus, in the equality (4.27), the condition $\mathcal{F}_t$ can be replaced by the condition $\sigma(G_t)$:

$$\alpha_t(G_t) = \mathbb{E}^Q(\alpha_{t+1}(G_t - c_t(\alpha_t(G_t)) + \varepsilon_{t+1}) \,|\, \sigma(G_t)). \tag{5.5}$$

We shall treat this relation as a fixed point equation for Borel measurable function α_t and attempt to obtain a solution in the limit $\alpha_t = \lim_{n\to\infty} \alpha_t^n$ of iterations

$$\alpha_t^{n+1}(G_t) = E^Q(\alpha_{t+1}(G_t - c_t(\alpha_t^n(G_t)) + \varepsilon_{t+1}) \,|\, \sigma(G_t)) \qquad n \in \mathbb{N} \tag{5.6}$$

started at $\alpha_t^0 = \alpha_{t+1}$. (Note that, given α_{t+1} and α_t^n, the equation (5.6) indeed defines a Borel function α_t^{n+1} by the factorization of the $\sigma(G_t)$-measurable random variable on the right of (5.6)). For numerical calculation of conditional expectations, we suggest to use the least-square Monte-Carlo method.

To explain the principle of the least square Monte-Carlo approach (see [12] and [20]) in more detail, we abstract from the concrete situation (5.6) and consider

$$\varphi(G) = \mathbb{E}^Q(\phi(G, \varepsilon) \,|\, \sigma(G)),$$

where ε, G are $\mathbb{R}$-valued and independent with respect to Q and ϕ is a bounded Borel function on $\mathbb{R}^2$. Under this assumptions, the function φ is obtained as $\varphi(g) = \int_{\mathbb{R}} \phi(g, e) Q^\varepsilon(de)$ for Q^G-almost all $g \in \mathbb{R}$, where Q^ε, Q^G are image measures of Q under ε and G respectively. An equivalent condition defining φ is the orthogonality

$$
\begin{aligned}
&\text{determine } \varphi \in L^2(\mathbb{R}, \mu) \text{ such that for all } \psi \in \Psi \\
&\int_{\mathbb{R}^2} (\varphi(g) - \phi(g, e)) \psi(g) (Q^\varepsilon \otimes \mu)(de, dg) = 0,
\end{aligned}
\tag{5.7}
$$

where μ is a measure which is equivalent to Q^G and Ψ stands for a set of functions square integrable with respect to μ whose linear space is dense in $L^2(\mathbb{R}, \mu)$. The idea of the least square Monte Carlo method is to relax, for computational tractability, the principle (5.7) to

$$
\begin{aligned}
&\text{determine } \varphi \in \mathrm{lin}\Psi \text{ such that for all } \psi \in \Psi \\
&\sum_{k=1}^{K} (\varphi(g_k) - \phi(g_k, e_k)) \psi(g_k) = 0,
\end{aligned}
\tag{5.8}
$$

with a finite set of basis functions

$$\Psi = \{\psi_j \,:\, j = 1, \ldots, J\}$$

and an appropriate sample

$$S := (e_k, g_k)_{k=1}^{K} \subset \mathbb{R}^2,$$

chosen such that the combination $\frac{1}{K} \sum_{k=1}^{K} \delta_{(e_k, g_k)}$ of the Dirac measures approximates the distribution $Q^\varepsilon \otimes \mu$. (For instance, S being realizations of $K \in \mathbb{N}$ independent $Q^\varepsilon \otimes \mu$-distributed random variables). The solution to the weakened problem (5.8) is given in terms of

$$
\begin{aligned}
&\text{realizations } \phi(S) = (\phi(e_k, g_k))_{k=1}^{K} \text{ of } \phi \text{ on the sample } S \\
&\text{realizations } M = (\psi_j(g_k))_{k=1, j=1}^{K, J} \text{ of basis functions on } S
\end{aligned}
\tag{5.9}
$$

as follows:

$$
\begin{aligned}
&\text{if } q = (q_j)_{j=1}^{J} \text{ fulfills } M^\top M q = M^\top \phi(S), \\
&\text{then } (5.8) \text{ is solved by } \varphi = \sum_{j=1}^{J} q_j \psi_j.
\end{aligned}
$$

We now formulate an algorithm for the approximate calculation of (5.5) in which the conditional expectation is replaced by the least-square Monte-Carlo projection. To ease notation, let us suppose that $(\varepsilon_t)_{t=1}^{T}$ are identically distributed (in addition to independence (5.1)).

Allowance prices via Monte Carlo method

1. *Initialization:* Given sample $S = (e_k, g_k)_{k=1}^K \subset \mathbb{R}^2$ describing the distribution of $Q^{\varepsilon_1} \otimes \mu$ and a set of basis functions $\Psi = (\psi_i)_{j=1}^J$ on $\mathbb{R}$, define M as in (5.9). Set $\alpha_T(g) = 1_{[0,\infty[}(g)$ for all $g \in \mathbb{R}$, and proceed in the next step with $t := T - 1$.
2. *Iteration:* Define $\alpha_t^0 = \alpha_t$, and proceed in the next step with $n := 0$.

   2a) Calculate $\phi^{n+1}(S) := (\alpha_{t+1}(g_k - c_t(\alpha_t^n(g_k)) + e_k))_{k=1}^K$

   2b) Determine a solution $q^{n+1} \in \mathbb{R}^J$ to $M^\top M q^{n+1} = M^\top \phi^{n+1}(S)$.

   2c) Define $\alpha_t^{n+1} := \sum_{j=1}^J q_j^{n+1} \psi_j$.

   2d) If $\max_{k=1}^K |\alpha_t^{n+1}(g_k) - \alpha_t^n(g_k)| \geq \varepsilon$, then put $n := n+1$ and continue with the step 2a).
   If $\max_{k=1}^K |\alpha_t^{n+1}(g_k) - \alpha_t^n(g_k)| < \varepsilon$ then set $t := t - 1$. If $t > 1$, go to the step 2, otherwise finish.

Example To illustrate allowance prices calculation via Monte Carlo method, we consider the following numerical example. Suppose that the penalty is set at $\pi = 100$ and that the martingale increments $(\varepsilon_t)_{t=1}^T$ are independent, identically normally distributed. Note that by an appropriate choice of the emission measurement scale, the standard deviation can always be normalized, thus we have assumed that each ε_t is $\mathcal{N}(0.5, 1)$-distributed. Further, consider the basis consisting of piecewise linear hut functions

$$\psi_j : \mathbb{R} \to [0, 1], \quad x \mapsto (1 - |z_j - x|/h)^+ \quad \text{for } x \in \mathbb{R}, \ j = 1, \dots, J$$

where the peaks $z_1 = -(J-1) * h/2, \dots, z_J = (J-1) * h/2$ are chosen to be equidistant with the distance $h > 0$. For numerical illustration, we set $J = 16$ and $h := 1$. Further, the sample $S = (e_k, g_k)_{k=1}^K$ for the Monte-Carlo Method is generated with $K = 1000$ outcomes. For $(e_k)_{k=1}^K$, we followed a natural choice taking realizations of K independent $\mathcal{N}(0.5, 1)$-distributed random variables. However, since the distribution of G_t is not known in advance, an appropriate candidate for μ seems to be the uniform distribution concentrated on the interval which is relevant for calculations. That is, the outcomes $(g_k)_{k=1}^K$ are constructed by equidistant sampling of $[z_1, z_J]$, ranging from $g_1 = z_1 = -7.5$ to $g_K = z_J = 7.5$. For the cumulative volume function $c : \mathbb{R} \to \mathbb{R}, a \mapsto 0.1\sqrt{(a)^+}$ we observed a fast and stable convergence which gave reasonable outcome within few iterations. The resulting functions $(\alpha_t)_{t=T-1}^{T-6}$ are depicted in the Figure 1.

Let us outline a valuation procedure for a European call on emission allowance price.

Valuation of European call via Monte Carlo method

1. Given basis functions $\Psi = (\psi_j)_{j=1}^J$ and sample $S = (e_k, g_k)_{k=1}^K \subset \mathbb{R}^2$ which approximates $Q^{\varepsilon_1} \otimes \mu$, determine $(\alpha_t)_{t=T}^0$ in terms of basis coefficients using the above least-square Monte Carlo algorithm.
2. Given maturity time $\tau \in \{1, \dots, T\}$ of the European call, determine its payoff $f_\tau^\tau := (\alpha_\tau - K)^+$. Calculate least-square projections recursively processing for $u = \tau, \dots, t$ as follows

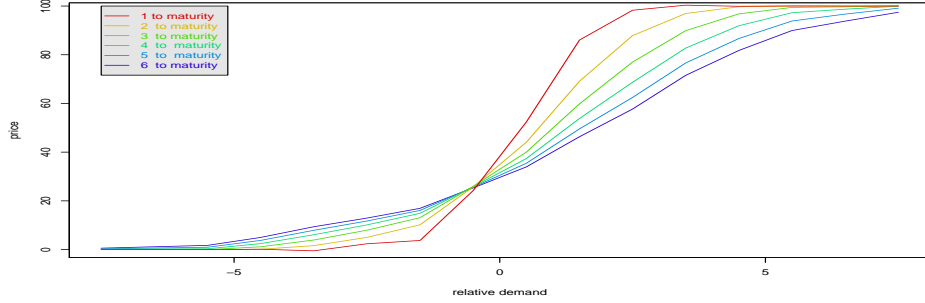   a) put $\phi(S) = (f_u^\tau(g_k - c_u(\alpha_u(g_k)) + e_k))_{k=1}^K$

Figure 1. The functions α_t for $t = T-1, \ldots, T-6$, from the least squares Monte Carlo method.

    b) obtain q as solution to $M^\top M q = M\phi$

    c) set $f_{u-1}^\tau = \sum_{j=1}^J q_j \psi_j$

    d) if $u - 1 = t$ finish, else set $u := u - 1$ and go to a).

3. Given recent allowance price a, calculate the state variable g as solution to $a = \alpha_t(g)$
4. Plug in the state variable g and into function $f^\tau(t, \cdot)$ to obtain the price of the European call as as $f^\tau(t, g)$.

Let us finalize this section by sketching core ideas on continuous time modeling. Our analysis shows that the risk-neutral allowance price evolution $(A_t)_{t=0}^T$ must be described by a martingale whose terminal value is digital and depends on the intermediate values, (see 3.19). Suppose that the compliance period is given by an interval $[0, T]$, such that all relevant random evolutions are described by adapted stochastic processes on

$$\begin{aligned}\text{filtered probability space } (\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in[0,T]}) \\ \text{equipped with probability measure } Q \sim P,\end{aligned} \tag{5.10}$$

where Q represents the spot martingale measure. Given a random variable $\mathcal{E}_T$ and appropriate non-decreasing and continuous abatement functions $c_t : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ indexed by $t \in [0, T]$, we follow an analogy to discrete time and consider solutions $(A_t)_{t\in[0,T]}$ to

$$A_t = \pi E^Q (1_{\{\mathcal{E}_T - \int_0^T c_t(A_s)ds\} \geq 0} \,|\, \mathcal{F}_t) \qquad t \in [0, T]. \tag{5.11}$$

Our results from discrete-time setting suggest that if

$$\left.\begin{aligned}\text{the increments of the martingale } (\mathcal{E}_t = E^Q(\mathcal{E}_T \,|\, \mathcal{F}_t))_{t\in[0,T]} \text{ are} \\ \text{independent and the abatement functions } c_t : \mathbb{R}_+ \times \to \mathbb{R}_+ \\ \text{are deterministic and time constant } (c_t = c)_{t\in[0,T]}\end{aligned}\right\} \tag{5.12}$$

then a solution to (5.11) should be expected in the functional form

$$A_t = \alpha(t, G_t), \qquad t \in [0, T]$$

with an appropriate deterministic function

$$\alpha : [0, T] \times \mathbb{R} \mapsto \mathbb{R}, \qquad (t, g) \mapsto \alpha(t, g) \tag{5.13}$$

and a state process $(G_t)_{t \in [0,T]}$ given by

$$G_t = \mathcal{E}_t - \int_0^t c_s(A_s)ds, \qquad t \in [0, T].$$

To illustrate how such approach allows to guess a solution, assume that

$$(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \in [0,T]}) \text{ supports the process } (W_t, \mathcal{F}_t)_{t \in [0,T]} \tag{5.14}$$
$$\text{of Brownian motion with respect to } Q \sim P.$$

Furthermore, we respond to (5.12) supposing that

$$d\mathcal{E}_t = \sigma_t dW_t \text{ with pre-specified deterministic } (\sigma_t)_{t \in [0,T]} \text{ and}$$
$$\text{continuous and non-decreasing abatement functions } (c_t = c)_{t \in [0,T]}.$$

To ensure the martingale property of $(A_t = \alpha(t, G_t))_{t \in [0,T]}$, apply the Ito formula

$$dA_t = d\alpha(t, G_t) = \partial_{(1,0)}\alpha(t, G_t)dt + \partial_{(0,1)}\alpha(t, G_t)dG_t + \frac{1}{2}\partial_{(0,2)}\alpha(t, G_t)d[G]_t$$

$$= \partial_{(1,0)}\alpha(t, G_t)dt - \partial_{(0,1)}\alpha(t, G_t)c(\alpha(t, G_t))dt + \frac{1}{2}\partial_{(0,2)}\alpha(t, G_t)\sigma_t^2 dt$$

$$+ \partial_{(0,1)}\alpha(t, G_t)\sigma_t dW_t$$

and claim the function α as a solution on $[0, T[\times \mathbb{R}$ to

$$\partial_{(1,0)}\alpha(t, g) - \partial_{(0,1)}\alpha(t, g)c(\alpha(t, g)) + \frac{1}{2}\partial_{(0,2)}\alpha(t, g)\sigma_t^2 = 0 \tag{5.15}$$

with boundary condition

$$\alpha(T, g) = \pi 1_{[0,\infty[}(g) \text{ for all } g \in \mathbb{R} \tag{5.16}$$

justified by the digital terminal allowance price. Having obtained α in this way, we construct the state process as the solution to stochastic differential equation

$$dG_t = d\mathcal{E}_t - c(\alpha(t, G_t))dt, \qquad G_0 = \mathcal{E}_0 \tag{5.17}$$

and then determine

$$A_t := \alpha(t, G_t), \qquad t \in [0, T]. \tag{5.18}$$

Finally, this process must be verified to solve (5.11).

6. Conclusion

This article explains logical principles underlying risk-neutral modeling of emission certificate price evolution. We show that within a realistic situation of risk averse market players, there is no connection between social optimality and market equilibrium, but there is a useful feedback relation characterizing risk neutral allowance price dynamics. Expressing this result in terms of fixed point equations on the level of martingales, we address the existence of its solution and elaborate on its algorithmic tractability. Furthermore, we suggest an extension of these concepts to continuous time and show that promising results can be obtained using diffusion processes. Here, emission allowances and their options can be described in terms of standard partial differential equations. Although option pricing in this framework seems to be appealing, we believe that it is not superior to our Monte-Carlo method since the latter can be used in high dimensions and, more importantly, in the presence of jumps in the martingale $(\mathcal{E}_t)_{t=0}^T$. This is particularly important to describe price shocks, which may result from possible discontinuities in the information flow.

References

[1] E. Benz and S. Trueck. Modeling the price dynamics of co2 emission allowances. Technical report, Bonn Graduate School of Economics, 2008.

[2] R. Carmona, F. Fehr, and J. Hinz. Optimal stochastic control and carbon price formation. *SIAM Journal on Control and Optimization*, 2009.

[3] R. Carmona, F. Fehr, J. Hinz, and A. Porchet. Market designs for emissions trading schemes. *proprint*, 2009.

[4] R. Carmona and J. Hinz. Risk neutral modeling of emission allowance prices and option valuation. Technical report, Princeton University, 2009.

[5] U. Cetin and M. Verschuere. Pricing and hedging in carbon emissions markets. Technical report, London School of Economics, 2008.

[6] M Chesney and L. Taschini. The endogenous price dynamics of the emission allowances: An application to co2 option pricing. Technical report, 2008.

[7] M. Cronshaw and J. B. Kruse. Regulated firms in pollution permit markets with banking. *Journal of Regulatory Economics*, 9(2):179–89, 1996.

[8] J. H. Dales. *Pollution, Property and Prices*. University of Toronto Press, Toronto, 1968.

[9] G. Daskalakis, D. Psychoyios, and Markellos R. N. Modeling co2 emission allowance prices and derivatives: Evidence from the eex. *Preprint*, 2008.

[10] Yu. M. Kabanov and Ch. Stricker. A teachers' note on no-arbitrage criteria. *Lecture Notes in Mathematics 1775*, pages 149–152, 2001.

[11] P. Leiby and J. Rubin. Intertemporal permit trading for the control of greenhouse gas emissions. *Environmental and Resource Economics*, 19(3):229–256, 2001.

[12] F. Longstaff and E. Schwartz. Valuing american options by simulation: a simple least-squares approach. *Review of Financial Studies*, (1):113–147, 2001.

[13] A. Maeda. Impact of banking and forward contracts on tradable permit markets. *Environmental Economics and Policy Studies*, 6(2):81–102, 2004.

[14] W. D. Montgomery. Markets in licenses and efficient pollution control programs. *Journal of Economic Theory*, 5(3):395–418, 1972.

[15] M.S. Paolella and L. Taschini. An econometric analysis of emissions trading allowances. *Journal of Banking and Finance*, 2008.

[16] J. Rubin. A model of intertemporal emission trading, banking and borrowing. *Journal of Environmental Economics and Management*, 31(3):269–286, 1996.

[17] S. M. Schennach. The economics of pollution permit banking in the context of title iv of the 1990 clean air act amendments. *Journal of Environmental Economics and Management*, 40(3):189–21, 2000.

[18] J. Seifert, M. Uhrig-Homburg, and M. Wagner. Dynamic behavior of carbon spot prices .theory and empirical evidence. *Preprint*, 2006.

[19] J. Sijm, K. Neuhoff, and Y. Chen. Co2 cost pass-through and windfall profits in the power. *Climate Policy*, 6:49–72, 2006.

[20] L. Stentoft. Convergence of the least squares monte carlo approach to american option valuation. *Management Science*, 50(9):576–611, 2004.

[21] B. Stevens and A. Rose. a dynamic analysis of the marketable permits approach to global warming policy: a comparison of spatial and temporal flexibility. *Journal of Environmental Economics and Management*, 44:45–69, 2002.

[22] T. Tietenberg. *Emissions Trading: An Exercise in Reforming Pollution Policy.* Resources for the Future, Boston, 1985.

[23] M. Uhrig-Homburg and M. Wagner. Derivatives instruments in the EU emissions trading scheme, and early market perspective. Technical report, University of Karlsruhe, 2007.

[24] M. Uhrig-Homburg and M. Wagner. Futures price dynamics of cO2 emissions certificates: an empirical analysis. Technical report, University of Karlsruhe, 2007.

[25] M. Wagner. *co2-Emissionszertifikate, Preismodellierung und Derivatebewertung.* PhD thesis, Universität Karlsruhe., 2006.