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Market Stability Switches in a Continuous-Time Financial Market with Heterogeneous Beliefs *

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Abstract

By considering a financial market of fundamentalists and trend followers in which the price trend of the trend followers is formed as a weighted average of historical prices, we establish a continuous-time financial market model with time delay and examines the impact of time delay on market price dynamics. Conditions for the stability of the fundamental price in terms of agents' behavior parameters and time delay are obtained. In particular, it is found that an increase in time delay can not only destabilize the market price but also stabilize an otherwise unstable market price, leading to stability switching as delay increases. This interesting phenomena shed new light in understanding of mechanism on the market stability. When the fundamental price becomes unstable through Hopf bifurcations, sufficient conditions on the stability and global existence of the periodic solution are obtained.

Key words: Asset price, fundamentalists, trend followers, delay differential equations, stability, bifurcations.

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1 Introduction

Technical analysts or "chartists", who use various technical trading rules such as moving averages, attempt to forecast future prices by the study of patterns of past prices and other summary statistics about security trading. Basically, they believe that shifts in supply and demand can be detected in charts of market movements. Despite the efficient market hypothesis of financial markets in the academic finance literature (see Fama, 1970), the use of technical trading rules, such as moving average rules, still seems to be widespread amongst financial market practitioners (see Allen and Taylor, 1990; Taylor and Allen, 1992). This motivates recent studies on the impact of chartists on the market price behavior. Over the last two decades, heterogeneous agent models (HAMs) have been developed to explain various market phenomena and, as the main tool, the stability and bifurcation analysis has been widely used in HAMs. By incorporating heterogeneity and behavior of chartists and examining underlying deterministic models, HAMs have successfully explained the complicated role of chartists in market price behavior, market booms and crashes, and deviations of the market price from the fundamental price. Numerical simulations of the stochastic model based on the analytical analysis of the underlying deterministic model show some potentials of HAMs in generating the stylized facts (such as skewness, kurtosis, volatility clustering and fat tails of returns), and various power laws (such as the long memory in return volatility) observed in financial markets. We refer the reader to Hommes (2006), LeBaron (2006) and Chiarella, Dieci and He (2009) for surveys of the recent developments in this literature.

Most of the HAMs in the literature are in discrete-time rather than continuous-time setup. To examine the role of moving average rules in market stability theoretically, Chiarella, He and Hommes (2006) recently propose a discrete-time HAM in which demand for traded assets has both a fundamentalist and a chartist components. The chartist demand is governed by the difference between the current price and a moving average (MA). They show analytically and numerically that an increase in the lag length used in moving average can destabilize the market, leading to cyclic behavior of the market price around the fundamental price. The discrete-time setup facilities economic understanding and mathematical analysis, but it also faces some limitations when expectations of agents are formed in historical prices over different time periods. In particular, when dealing with MA rules in Chiarella, He and Hommes (2006), different lag lengths used in the MA rules lead to different dimensions of the system which need to be dealt with separately. Very often, an analytical analysis is difficult when the dimension of the system is higher. To overcome this difficulty, this paper extends the heterogeneous agent model of the financial market in Chiarella, He and Hommes (2006) from the discrete-time to a continuous-time framework. The financial market is consisting of a group of fundamentalists and a group of trend followers who use a weighted average of historical prices as price trend. The fundamentalists are assumed to buy (sell) the stock when its price is below (above) the fundamental price. The trend followers are assumed to react to buy-sell signals generated by the difference between the current price and the price trend. The model is described mathematically by a system of delay differential equations, which provides a systematic analysis on various moving average rules used in the discrete-time model in Chiarella, He and Hommes (2006).

Development of deterministic delay differential equation models to characterize fluctuation of commodity prices and cyclic economic behavior has a long history, see, for example, Haldane (1932), Kalecki (1935), Goodwin (1951), Larson (1964), Howroyd and Russell (1984) and Mackey (1989). The development further leads to the studies on the effect of policy lag on macroeconomic stability, see for example, Phillips (1954, 1957), Asada and Semmler (1995), Asada and Yoshida (2001) and Yoshida and Asada (2007). In particular, as indicated in Manfredi and Fanti (2004), an important class of delay economic model is that of distributed delay systems governed by Erlangian kernels, which are reducible to higher dimensional ordinary differential equation systems.

Though there is a growing study on various market behavior, in our knowledge, using delay differential equations to model financial market behavior is relatively new. This paper aims to extend Chiarella, He and Hommes (2006) model in discrete-time to continuous-time with a time delay framework. This extension provides a uniform treatment on the moving average rules with different window length in discrete-time model. Different from the distributed delay of Erlangian kernel type used in economic modelling literature, the delay introduced in this paper is not 'reducible' in general. By focusing on the impact of the behavior of heterogeneous agents, the stabilizing role of the time delay is examined. Sufficient conditions for the stability of the fundamental price in terms of agents' behavior parameters and time delay are derived. Consistent with the results obtained in the discrete-time model in Chiarella, He and Hommes (2006), it is found that an increase in time delay can destabilize the market price, resulting in oscillatory market price characterized by a Hopf bifurcation. However, in contrast to the discrete-time model, it is also found that, depending on the behavior of the fundamentalists and trend followers, an increase in the time delay can also stabilize an otherwise unstable market price and such stability switching can happen many times. The stability switching is a very interesting and new phenomenon on price dynamics of the HAMs. The stabilising role of reducible distributed delay has been observed in economic modelling (see Manfredi and Fanti, 2004) and it is of interest to ascertain that this stability is preserved under non-reducible delay introduced in this paper. When the fundamental steady state becomes unstable, the market price displays cyclic behavior around the fundamental price characterized by Hopf bifurcations. We also examine the stability of the Hopf bifurcation and furthermore the global existence of periodic solutions bifurcating from the Hopf bifurcations.

The paper is organized as follows. We first introduce a deterministic HAM with two types of heterogeneous agents in a continuous time framework with time delay in Section 2. In Section 3, we first conduct a stability and bifurcation analysis of the delay differential equation model and then examine the stability of the periodic solution characterized by the Hopf bifurcation. In addition, we obtain some results on the global existence of periodic solutions resulting from the Hopf bifurcation. Section 4 concludes the paper. All the proofs of technical results are given in the appendices.

2 A Financial Market Model with Delay

Following the current HAM framework, see for example, Brock and Hommes (1998), Chiarella and He (2002, 2003) and, in particular, Chiarella, He and Hommes (2006) in discrete-time setup, this section proposes an asset pricing model in a continuous-time framework with two different types of heterogeneous traders, fundamentalists and trend followers, who trade according to fundamental analysis and technical analysis, respectively. The market price is arrived at via a market maker scenario in line with Beja and Goldman (1980), Day and Huang (1990) and Chiarella and He (2003).

Consider a market with a risky asset (such as stock market index) and let $P(t)$ denote the (cum dividend) price per share of the risky asset at time t. To focus on price dynamics, we follow Beja and Goldman (1980) and Day and Huang (1990) and motivate the demand functions of the two different types of traders by their trading rules directly, rather than deriving the demand functions from utility maximization of their portfolio investments with both risky and risk-free assets (as for example in Brock and Hommes, 1998 and Chiarella and He, 2003). The market population fractions[1] of fundamentalists and chartists are respectively α and $1-\alpha$, where $\alpha \in [0,1]$.

The fundamentalists trade based on their estimated fundamental price. They believe that the market price $P(t)$ is mean-reverting to the fundamental price $F(t)$ which is assumed to be a constant $F(t) = F$ for simplicity. We assume that the demand of the fundamentalists, $D_f(t)$ at time t, is proportional to the price deviation from the fundamental price, namely,

$$D_f(t) = \beta_f[F - P(t)], \tag{2.1}$$

where $\beta_f > 0$ is a constant parameter, measuring the mean-reverting of the market price to the fundamental price, which may be weighted by the risk aversion coefficient of the fundamentalists.

The chartists trade based on charting signals generated from historical prices. Given the well documented momentum trading strategy in empirical literature, see for example Hirshleifer (2001), we assume that the chartists are trend followers. They believe that the future market price follows a price trend $u(t)$. When the current price is above the trend, the trend followers believe the price will rise and they like to take a long position

[1]To simplify the analysis, we assume that the market fractions are constant parameters as in the market fraction model in He and Li (2008), rather than dependent variables based on some performance measure, as in Brock and Hommes (1998). An extension along this line, as in Hommes (1998) or Dieci et al (2006) in general, to allow investors switching between the two strategies is of interest and we leave this as future research.

of the risky asset; otherwise, the trend followers will take a short position. We therefore assume that the demand of the chartists is given by

$$D_c(t) = g\big(P(t) - u(t)\big), \tag{2.2}$$

where the demand function g satisfies:

$$g'(x) > 0, \qquad g'(0) = \beta_c > 0, \qquad xg''(x) < 0 \quad \text{for } x \neq 0. \tag{2.3}$$

The S-shaped demand function g capturing the trend following behavior is well documented in the HAM literature (see Chiarella, Dieci and He, 2009), where the parameter β_c represents the extrapolation rate of the trend followers on the future price trend when the price deviation from the trend is small. In the following discussion, we let $g(x) = \tanh(\beta_c x)$, which satisfies the conditions in (2.3).

Among various price trends used in practice, weighted moving average rules are the most popular ones. In this paper, we assume that the price trend $u(t)$ of the trend followers at time t is measured by an exponentially decayed weighted average of historical prices over a time interval $[t-\tau, t]$ with time delay $\tau > 0$, namely,

$$u(t) = \frac{1}{A}\int_{t-\tau}^{t} e^{-k(t-s)}P(s)ds, \qquad A = \frac{1-e^{-k\tau}}{k}, \tag{2.4}$$

where $k > 0$ measures the decaying rate of the weights on the historical prices and A is a normalization constant. Note that the distribution delay used in (2.4) is not reducible for $0 < \tau < \infty$. Equation (2.4) implies that, when forming the price trend, the trend followers believe the more recent prices contain more information about the future price movements so that the weights associated to the historical prices decay exponentially with a decay rate k. In particular, when $k \to 0$, the price trend $u(t)$ in equation (2.4) is simply given by the standard moving average, that is,

$$u(t) = \frac{1}{\tau}\int_{t-\tau}^{t} P(s)ds. \tag{2.5}$$

When $k \to \infty$, all the weights go to the current price so that $u(t) \to P(t)$.

In general, for $0 < k < \infty$, equation (2.4) can be expressed as a differential equation with time delay τ

$$\frac{du(t)}{dt} = \frac{k}{1-e^{-k\tau}}\big[P(t) - e^{-k\tau}P(t-\tau) - (1-e^{-k\tau})u(t)\big]. \tag{2.6}$$

In particular, when $\tau \to 0$, $u(t) \to P(t)$, implying that the trend followers regard the current price as their price trend; when $\tau \to \infty$, the distributed delay becomes reducible and hence the trend followers use all the historical prices to form the price trend

$$u(t) = \frac{1}{k}\int_{-\infty}^{t} e^{-k(t-s)}P(s)ds. \tag{2.7}$$

Consequently, equation (2.6) becomes an ordinary differential equation

$$\frac{du(t)}{dt} = \frac{1}{k}\big[P(t) - u(t)\big]. \tag{2.8}$$

Assume a net zero supply of the risky asset. Then the aggregate market excess demand for the risky asset, weighted by the population weights of the fundamentalists and trend followers, is given by $\alpha D_f(t) + (1-\alpha)D_c(t)$. Following Beja and Goldman (1980), Day and Huang (1990) and Chiarella, He and Hommes (2006), we assume that the market price $P(t)$ at time t is set via a market maker mechanism and is adjusted according to the aggregate excess demand, that is

$$\frac{dP(t)}{dt} = \mu\Big[\alpha D_f(t) + (1-\alpha)D_c(t)\Big], \tag{2.9}$$

where $\mu > 0$ represents the speed of the price adjustment by the market maker.

Based on (2.9) and the above analysis, the market price of the risky asset is determined according to a delay differential system

$$\begin{cases} \dfrac{dP(t)}{dt} = \mu\Big[\alpha\beta_f\big(F - P(t)\big) + (1-\alpha)\tanh\Big(\beta_c\big(P(t) - u(t)\big)\Big)\Big], \\ \dfrac{du(t)}{dt} = \dfrac{k}{1-e^{-k\tau}}\Big[P(t) - e^{-k\tau}P(t-\tau) - (1-e^{-k\tau})u(t)\Big]. \end{cases} \tag{2.10}$$

In the following sections, we examine the stability and bifurcation induced fluctuation of the market price of the system (2.10).

3 Price Stability and Bifurcation

It is easy to see that $(\bar{P}, \bar{u}) = (F, F)$ is an equilibrium point of (2.10) where the equilibrium steady state price is given by the constant fundamental price. We therefore call $(\bar{P}, \bar{u}) = (F, F)$ the *fundamental steady state*. In this section, we study the dynamics of the deterministic model (2.10), including the stability of the fundamental steady state, bifurcation, and global existence of the periodic solutions resulted from the Hopf bifurcation. In general, the dynamics depend on the behavior of the fundamentalists, the trend followers, and the market maker. As we known (see, for example, Hale and Kocak, 1991; Gopalsamy, 1992; Kuang 1993) that the stability is characterized by the eigenvalues of the characteristic equation of the system at the steady state. Because of the time delay, the eigenvalue analysis can be very complicated in general.

We first consider two special cases $\alpha = 1$ and $\alpha = 0$, that is the market consists of either the fundamentalists or the trend followers only. For $\alpha = 1$, the system (2.10) is reduced to

$$\frac{dP(t)}{dt} = \mu\alpha\beta_f(F - P(t)).$$

Because of $\mu\alpha\beta_f > 0$, it represents a mean-reverting process of the market price to the fundamental price. Hence the price converges to its fundamental value eventually and therefore the fundamental steady state is globally asymptotically stable. This exhibits the stabilizing role of the fundamentalists. For $\alpha = 0$, the market consists of the trend followers only and, consequently, the system (2.10) reduces to

$$\begin{cases} \dfrac{dP(t)}{dt} = \mu \tanh\big(\beta_c(P(t) - u(t))\big), \\ \dfrac{du(t)}{dt} = \dfrac{k}{1 - e^{-k\tau}}\big[P(t) - e^{-k\tau}P(t-\tau) - (1 - e^{-k\tau})u(t)\big]. \end{cases} \tag{3.1}$$

It is easy to see that any point $(\bar{P}, \bar{u})$ along the line $\bar{P} = \bar{u}$ is an equilibrium of the system (3.1). This means that the system has infinite many steady states. Near the line, the solution of (3.1) converges to a point on the line $\bar{P} = \bar{u}$ and the solutions with different initial values converge to different equilibria on the line. This implies that the line $\bar{P} = \bar{u}$ is locally attractive. This property is illustrated in Fig. 1, in which trajectories with different initial values converge to different points along the line $P = u$.

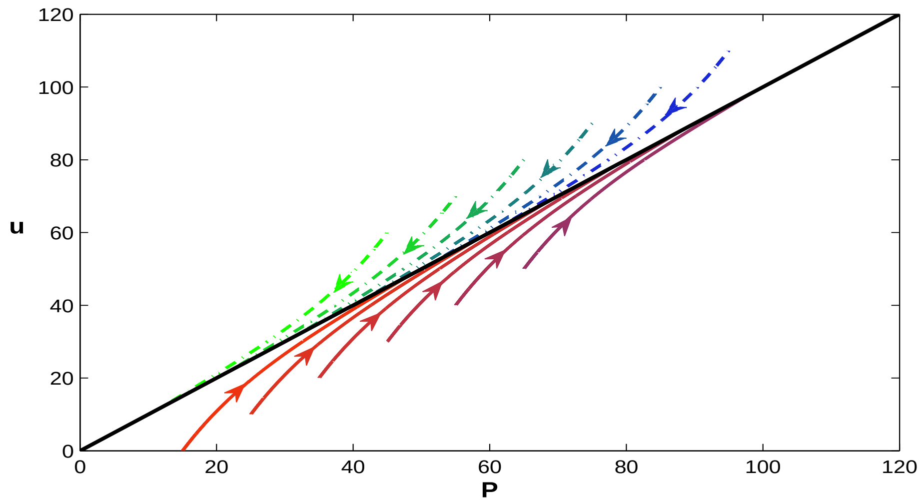


Figure 1: The local attractor line $\bar{P} = \bar{u}$ when $\alpha = 0$. Here $\mu = 1$, $\beta_c = 0.8$, $k = 0.1$ and $\tau = 0.1$.

3.1 Stability and Bifurcation Analysis

In the following, we consider $0 < \alpha < 1$ and denote

$$\gamma_f = \mu\alpha\beta_f \ (> 0), \qquad \gamma_c = \mu(1-\alpha)\beta_c \ (> 0).$$

The parameters γ_f and γ_c capture the activities of the fundamentalists and the trend followers, β_f and β_c, weighted by their market population fractions, α and $1-\alpha$, respectively, and the speed of the price adjustment of the market maker, μ. In this case, the system (2.10) has the unique positive equilibrium (F, F) and the the local stability of the fundamental steady state depends on the eigenvalue λ of the characteristic equation of the system at the fundamental steady state

$$\Delta(\lambda) := p(\lambda, \tau) + q(\lambda, \tau)e^{-\lambda\tau} = 0, \tag{3.2}$$

where

$$\begin{aligned} p(\lambda, \tau) &= \lambda^2 + (k + \gamma_f - \gamma_c)\lambda + k\gamma_f - k\gamma_c + \frac{k\gamma_c}{1 - e^{-k\tau}}, \\ q(\lambda, \tau) &= -\frac{k\gamma_c e^{-k\tau}}{1 - e^{-k\tau}}. \end{aligned} \tag{3.3}$$

It is easy to see that $\Delta(\lambda)$ is continuous for τ and analytic for λ. In addition, for $\tau = 0$,

$$\Delta(\lambda) = \lambda^2 + (k + \gamma_f)\lambda + k\gamma_f = 0$$

has two negative roots $\lambda_1 = -k < 0$ and $\lambda_2 = -\gamma_f < 0$. Hence, all roots of the characteristic equation (3.2) have negative real parts when τ is sufficient small. Based on this observation, we have the following result on the global stability of the fundamental steady state when the delay is sufficient small.

Theorem 3.1. *The fundamental steady state (F, F) of (2.10) is globally asymptotically stable when $\tau \to 0$.*

Proof. See Appendix A. □

In order to examine the local stability and bifurcation of the fundamental steady state, we now investigate the existence of the purely imaginary root $\lambda = i\omega(\omega > 0)$ of Eq. (3.2), which takes the form of a second-degree exponential polynomial in λ with $p(\lambda, \tau)$ and $q(\lambda, \tau)$ defined as in Eq. (3.3). By establishing a geometrical criterion, Beretta and Kuang (2002) provide conditions on the existence of purely imaginary roots of a characteristic equation with delay-dependent coefficients. It is easy to verify that, for the characteristic equation (3.2), these conditions are satisfied when $k + \gamma_f \neq \gamma_c$.

Now let $\lambda = i\omega(\omega > 0)$ be a root of Eq. (3.2). Substituting it into Eq. (3.2) and separating the real and imaginary parts yield that

$$\begin{aligned} \omega^2 - k\gamma_f - \frac{k\gamma_c e^{-k\tau}(1 - \cos\omega\tau)}{1 - e^{-k\tau}} &= 0, \\ \omega(k + \gamma_f - \gamma_c) + \frac{k\gamma_c e^{-k\tau}\sin\omega\tau}{1 - e^{-k\tau}} &= 0, \end{aligned} \tag{3.4}$$

which lead to

$$\begin{aligned} \sin\omega\tau &= \frac{-\omega(1 - e^{-k\tau})(k + \gamma_f - \gamma_c)}{k\gamma_c e^{-k\tau}}, \\ \cos\omega\tau &= 1 - \frac{(1 - e^{-k\tau})(\omega^2 - k\gamma_f)}{k\gamma_c e^{-k\tau}}. \end{aligned} \tag{3.5}$$

Following the definitions of $p(\lambda,\tau)$ and $q(\lambda,\tau)$ in (3.3), Eq. (3.5) can be written as

$$\sin\omega\tau = Im\left(\frac{p(i\omega,\tau)}{q(i\omega,\tau)}\right), \qquad \cos\omega\tau = -Re\left(\frac{p(i\omega,\tau)}{q(i\omega,\tau)}\right),$$

which yields

$$|p(i\omega,\tau)|^2 = |q(i\omega,\tau)|^2.$$

This equation can be written as

$$h(\omega^2,\tau) = 0, \tag{3.6}$$

where $h(W,\tau) := W^2 + a_1 W + a_2$ is a second degree polynomial with

$$a_1 = k^2 + \gamma_f^2 + \gamma_c^2 - 2\gamma_f\gamma_c - \frac{2k\gamma_c}{1-e^{-k\tau}}, \qquad a_2 = k^2\gamma_f^2 + \frac{2k^2\gamma_c\gamma_f e^{-k\tau}}{1-e^{-k\tau}}.$$

When $a_1 \geq 0$ or $a_1^2 - 4a_2 < 0$, because of $a_2 > 0$, equation $h(W,\tau) = 0$ has no positive real root. Therefore Eq. (3.2) has no purely imaginary root. When $a_1 < 0$ and $a_1^2 - 4a_2 \geq 0$, for

$$\tau \in (0,\ \widetilde{\tau}\ ] := I, \tag{3.7}$$

where

$$\widetilde{\tau} = ln\left[1 + \frac{2k\gamma_c}{(k+\gamma_f-\gamma_c)^2 + 2|k+\gamma_f-\gamma_c|\sqrt{k\gamma_f}}\right]^{1/k}, \tag{3.8}$$

equation $h(W,\tau) = 0$ has two positive real roots denoted by $W_+(\tau)$ and $W_-(\tau)$ satisfying $W_+(\tau) \geq W_-(\tau)$. Hence $h(\omega^2,\tau) = 0$ has two positive real roots denoted by $\omega_+(\tau)$ and $\omega_-(\tau)$ with $\omega_+(\tau) \geq \omega_-(\tau)$.

The upper bound $\widetilde{\tau}$ defined in (3.8) depends on the decay rate k and, most importantly, the balance between $k+\gamma_f$ and γ_c. In particular, when $k+\gamma_f-\gamma_c \to 0$, one can see from (3.8) that the upper bound $\widetilde{\tau} \to \infty$. For $\gamma_f = 0.3$ and $\gamma_c = 0.7$, Fig. 2 plots the upper bound $\widetilde{\tau}$ as a function of k. The function is convex for $k \in (0, \gamma_c - \gamma_f)$ with a positive minimum value, but convex and decreasing for $k > \gamma_c - \gamma_f$ satisfying $\widetilde{\tau} \to \infty$ as $k \to \gamma_c - \gamma_f$ and $\widetilde{\tau} \to 0$ as $k \to \infty$.

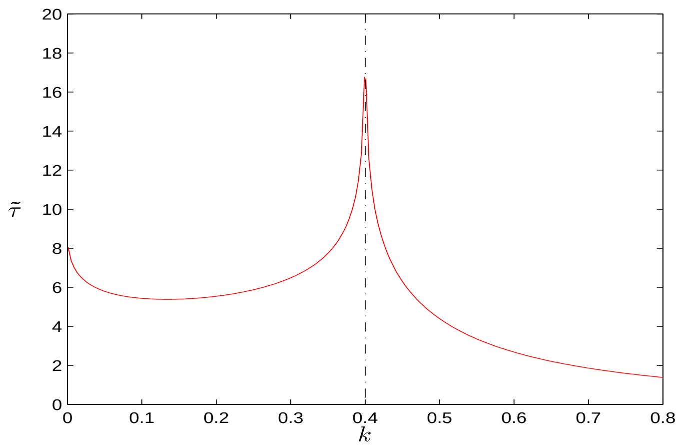


Figure 2: The upper bound $\widetilde{\tau}$ as a function of k. Here $\gamma_f = 0.3$ and $\gamma_c = 0.7$.

For $\tau \in I$, let $\theta_\pm(\tau) \in [0, 2\pi)$ be defined by

$$\sin\theta_\pm(\tau) = \frac{-\omega_\pm(1-e^{-k\tau})(k+\gamma_f-\gamma_c)}{k\gamma_c e^{-k\tau}},$$
$$\cos\theta_\pm(\tau) = 1 - \frac{(1-e^{-k\tau})(\omega_\pm^2 - k\gamma_f)}{k\gamma_c e^{-k\tau}}.$$

Define two sequences of functions on I by

$$S_n^+(\tau) = \tau - \frac{\theta_+(\tau)+2n\pi}{\omega_+(\tau)}, \qquad S_n^-(\tau) = \tau - \frac{\theta_-(\tau)+2n\pi}{\omega_-(\tau)},$$

where $n \in N_0 = \{0, 1, 2, \cdots\}$. One can verify that $i\omega^*(\omega^* = \omega(\tau^*) > 0)$ is a purely imaginary root of Eq. (3.2) if and only if τ^* is a root of the function S_n^+ or S_n^- for some $n \in N_0$. Based on the above analysis, we have the following result on the existence of the Hopf bifurcation characterized by the existence of the purely imaginary root of the characteristic equation.

Theorem 3.2. *For the system* (2.10),

(i) if either $a_1 \geq 0$ or $a_1^2 - 4a_2 < 0$, then Eq. (3.2) *has no purely imaginary root;*

(ii) if $a_1 < 0$ and $a_1^2 - 4a_2 \geq 0$, then $i\omega^*(\omega^* = \omega(\tau^*) > 0)$ is a purely imaginary root of Eq. (3.2) *if and only if τ^* is a root of the function S_n^+ or S_n^- for some $n \in N_0$.*

Following Beratta and Kuang (2002), we have the following properties of the functions $S_n^\pm(\tau)$.

Theorem 3.3. *Assume that the function $S_n^+(\tau)$ or $S_n^-(\tau)$ has a positive root $\tau^* \in I$, for some $n \in N_0$, then a pair of simple purely imaginary roots $\pm i\omega(\tau^*)$ of Eq.* (3.2) *exists at $\tau = \tau^*$. In addition,*

(i) if $S_n^+(\tau^*) = 0$, then this pair of simple conjugate pure imaginary roots crosses the imaginary axis from left to right if $\delta_+(\tau^*) > 0$ and from right to left if $\delta_+(\tau^*) < 0$, where

$$\delta_+(\tau^*) := sign\left\{\frac{dRe(\lambda)}{d\tau}\bigg|_{\lambda=i\omega(\tau^*)}\right\} = sign\left\{\frac{dS_n^+(\tau)}{d\tau}\bigg|_{\tau=\tau^*}\right\}.$$

(ii) if $S_n^-(\tau^*) = 0$, then this pair of simple conjugate purely imaginary roots crosses the imaginary axis from left to right if $\delta_-(\tau^*) > 0$ and from right to left if $\delta_-(\tau^*) < 0$, where

$$\delta_-(\tau^*) := sign\left\{\frac{dRe(\lambda)}{d\tau}\bigg|_{\lambda=i\omega(\tau^*)}\right\} = -sign\left\{\frac{dS_n^-(\tau)}{d\tau}\bigg|_{\tau=\tau^*}\right\}.$$

(iii) $S_n^+(\tau) > S_{n+1}^+(\tau)$ and $S_n^-(\tau) > S_{n+1}^-(\tau)$ for all $n \in N_0$.

(iv) if $S_0^+(\tau) > S_0^-(\tau)$ on I, then $S_n^+(\tau) > S_n^-(\tau)$ on I for all $n \in N_0$.

The properties of $S_n^{\pm}(\tau)$ as functions of τ are illustrated in Fig. 3 (a). Let $S_n(\tau)$ be either $S_n^+(\tau)$ or $S_n^-(\tau)$. It clearly indicates that $\{S_n(\tau)\}$ is a decreasing series of n. Therefore, if $S_0(\tau)$ have no zero on I, then function $S_n(\tau)$ has no zero on I for all $n \geq 1$. If for some $n \in N_0$, the functions $S_n^{\pm}(\tau)$ become zero for some time delay lag, say at $\{\tau_{n_j}^{\pm}\} \in I$, then there exists at least one $\tau_{n_j}^{\pm}$ satisfying $\frac{dS_n^{\pm}(\tau_{n_j}^{\pm})}{d\tau} \neq 0$. Define

$$\{\tau_j |\ \tau_j < \tau_{j+1},\ j = 0, 1, 2, \cdots, j_0\} = \cup_{n \in N_0} \{\tau_{n_j}^{\pm}\} := J_+, \tag{3.9}$$

where $j_0 = \#\{\tau_{n_j}^{\pm}\}$. If j_0 is finite, then $\tau_{j_0+1} = \infty$. It is easy to see that $S_0^+(\tau_0) = 0$ and $\frac{dS_0^+(\tau_0)}{d\tau} > 0$.

Applying Proposition 2.1 and the Hopf bifurcation theorem for functional differential equations (see Hale, 1997, Chapter 11, Theorem 1.1), we obtain the following result on the existence of a Hopf bifurcation.

Theorem 3.4. *Assume $k \neq \gamma_c - \gamma_f$. Then for system* (2.10)*,*

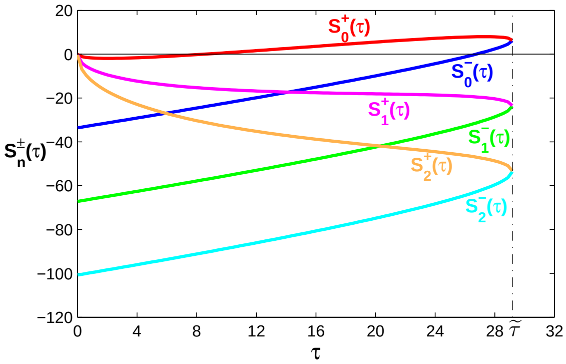
(i) if functions $S_0^+(\tau)$ and $S_0^-(\tau)$ have no positive zero on I defined in Eq. (3.7)*, then the fundamental steady state is asymptotically stable for all $\tau > 0$;*

(ii) if function $S_n^+(\tau)$ or $S_n^-(\tau)$ has positive zeros on I for some $n \in N_0$, then the fundamental steady state is asymptotically stable for $0 < \tau < \tau_0$, and becomes unstable for τ staying in a right neighborhood of τ_0. In addition, system (2.10) *undergoes a Hopf bifurcation when $\tau = \tau_j$ for $j = 0, 1, 2, \cdots$.*

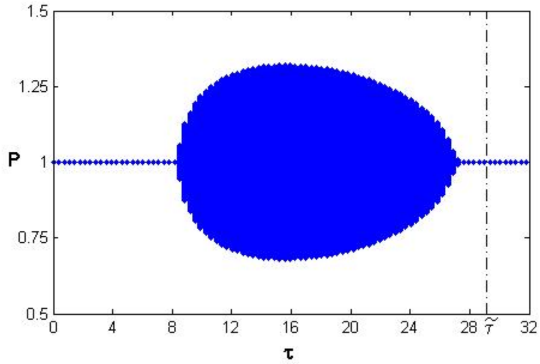
3.2 Stability Switching

To illustrate the stability switching, we consider two cases $k+\gamma_f > \gamma_c$ and $k+\gamma_f < \gamma_c$. In the first case, we choose $k = 0.05, \gamma_f = 0.7$ and $\gamma_c = 0.7$ and plot the function $S_n^{\pm}(\tau)$ in Fig. 3(a). In Fig. 3(a), there are two Hopf bifurcation values for τ, say $\tau_0 < \tau_1$. The first one occurs when $S_0^+(\tau)$ crosses 0 at $\tau = \tau_0 = 8.5612$ and the second one occurs when $S_0^-(\tau)$ crosses 0 at $\tau = \tau_1 = 26.7457$. We also plot the corresponding bifurcation diagram of the market price with respect to τ indicated in Theorem 3.4 in Fig. 3(b) showing that the fundamental steady state is stable for $\tau \in [0, \tau_0) \cup (\tau_1, \infty)$ and Hopf bifurcations occur at $\tau = \tau_o$ and $\tau = \tau_1$. The stability of the fundamental steady state switches at τ_0 and τ_1. In Figs. 3(c)-(e), we show the phase plots of the $(P(t), u(t))$ with the same initial value for three values of $\tau = 5(< \tau_0)$, $\tau = 16(\in (\tau_0, \tau_1))$ and $\tau = 32(> \tau_1)$. They show that the fundamental steady state is stable for both $\tau = 5$ and $\tau = 32$ and a stable cycle appears for $\tau = 16$, verifying the stability switching indicated by the bifurcation plot in Fig. 3(b).

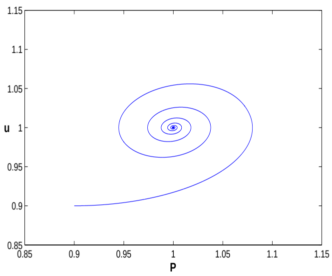
In the second case, we choose $k = 0.06, \gamma_f = 1$ and $\gamma_c = 1.1$ so that $k + \gamma_f < \gamma_c$. We plot the function $S_n^{\pm}(\tau)$ in Fig. 4(a). In Fig. 4(a), there are three Hopf bifurcation values for τ, say $\tau_0 < \tau_1 < \tau_2$. The first one occurs when $S_0^+(\tau)$ crosses 0 at $\tau = \tau_0 = 4.1764$, the second one occurs when $S_1^-(\tau)$ crosses 0 at $\tau = \tau_1 = 27.705$, and the third one occurs



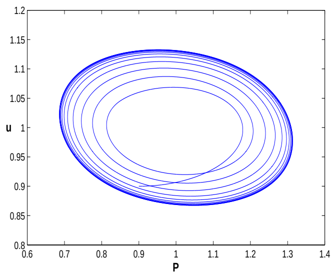
(a) Functions $S_n^{\pm}$



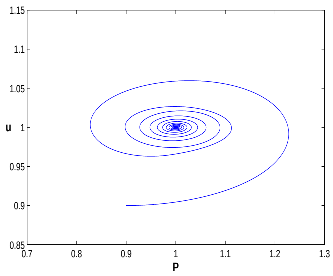
(b) Price bifurcation



(c) Phase plot for $\tau = 5$



(d) Phase plot for $\tau = 16$

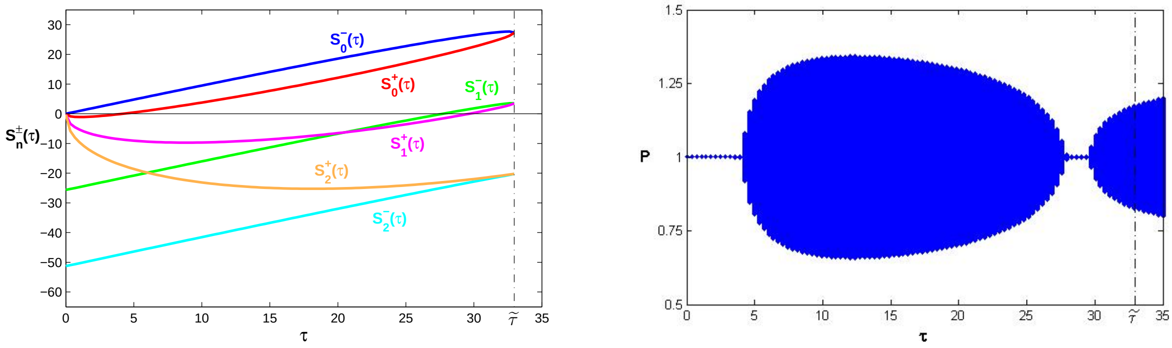


(e) Phase plot for $\tau = 32$

Figure 3: (a) The plots of $S_n^{\pm}$ as functions of τ; (b) the corresponding bifurcation diagrams of the market price with respect to τ, and phase plots for (c) $\tau = 5$, (d) $\tau = 16$, and (e) $\tau = 32$. Here $k = 0.05$, $\gamma_f = 0.7$, $\gamma_c = 0.7$, $\mu = 1$, $\alpha = 0.5$ and $F = 1$.

when $S_1^+(\tau)$ crosses 0 at $\tau = \tau_2 = 29.6524$. We also plot the corresponding bifurcation diagram of the market price with respect to τ indicated in Theorem 3.4 in Fig. 4 (b) showing that the fundamental steady state is stable for $\tau \in [0, \tau_0) \cup (\tau_1, \tau_2)$ and Hopf bifurcations occur at $\tau = \tau_i\,(i = 0, 1, 2)$. The stability of the fundamental steady state switches from unstable to stable at τ_1 and from stable to unstable at τ_2.

In both cases, we have demonstrated that an increases in the time delay can result in the stability switching of the fundamental price either from stable to unstable or from unstable to stable and such switching can happen many times in general. It is commonly believed in the discrete-time HAM literature that the more delayed prices used for the trend followers to form the price trend, the more smoothing the price trend becomes, the more sensible the demand function of the trend followers react to price changes, the less stable the market price becomes. Hence the trend followers plays a destabilizing role in general. For the continuous-time HAM in this paper, the destabilizing role of the time delay for the trend followers is expected, however the stabilizing role and stability switching as the time delay increases are rather surprising and less intuitive. From the numerical example in Figs 3(c) and (e), one interesting feature we have found is that the speed of the convergence of the market price to the fundamental price when $\tau = 32$ is



(a) Functions $S_n^\pm$ (b) Price bifurcation

Figure 4: (a) The plot of $S_n^\pm$ as functions of τ; (b) The corresponding bifurcation diagrams of the market price with respect to τ with $k = 0.06$, $\gamma_f = 1$ and $\gamma_c = 1.1$.

much slower than that when $\tau = 5$.

3.3 Stability of the Hopf Bifurcation

By using the normal form theory and the center manifold argument presented by Hassard, Kazarinoff and Wan (1981), we can establish an explicit formula in determining the direction and stability of periodic solutions bifurcating from the fundamental steady state at a Hopf bifurcation value, say $\tau = \tau^*$. In fact, let $\omega^* = \omega(\tau^*)$. Then the normal form of system (2.10) has the following form (see Appendix B for the details)

$$\frac{dz(t)}{dt} = i\omega^*\tau^* z(t) + g(z(t), \bar{z}(t)), \tag{3.10}$$

where

$$g(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots,$$

$$g_{20} = 0,\ g_{11} = 0,\ g_{02} = 0,\ g_{21} = -\frac{(\gamma_f^2 + \omega^{*2})a_3}{\mu^2(1-\alpha^2)a_4} \tag{3.11}$$

and

$$\begin{aligned}
a_3 &= (1-e^{-k\tau^*})^2(\gamma_f\omega^{*2} + \gamma_c\omega^{*2} + k^2\gamma_f + k\gamma_f^2 - k\gamma_c\gamma_f + k\omega^{*2}) \\
&\quad + k\gamma_c\tau^* e^{-k\tau^*}(1-e^{-k\tau^*})(k\gamma_f\cos\omega^*\tau^* - \omega^{*2}\cos\omega^*\tau^* - k\omega^*\sin\omega^*\tau^* - \omega^*\gamma_f\sin\omega^*\tau^*) \\
&\quad + i(1-e^{-k\tau^*})^2(2\omega^{*3} + k^2\omega^* - k\gamma_c\omega^* - \gamma_c\gamma_f\omega^* + \gamma_f^2\omega^*) \\
&\quad + ik\gamma_c\tau^* e^{-k\tau^*}(1-e^{-k\tau^*})(k\omega^*\cos\omega^*\tau^* + \omega^*\gamma_f\cos\omega^*\tau^* + k\gamma_f\sin\omega^*\tau^* - \omega^{*2}\sin\omega^*\tau^*), \\
a_4 &= (1-e^{-k\tau^*})^2[(k+\gamma_f-\gamma_c)^2 + 4\omega^{*2}] + k^2\gamma_c^2\tau^{*2}e^{-2k\tau^*} \\
&\quad + 2k\gamma_c\tau^* e^{-k\tau^*}(1-e^{-k\tau^*})[(k+\gamma_f-\gamma_c)\cos\omega^*\tau^* - 2\omega^*\sin\omega^*\tau^*].
\end{aligned}$$

We can compute the following quantities:

$$\begin{aligned}
c_1(0) &= \frac{i}{2\omega^*\tau^*}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2} = \frac{g_{21}}{2},\\
\mu_2 &= -\frac{Re(c_1(0))}{Re(\lambda'(\tau^*))},\\
\beta_2 &= 2Re(c_1(0)),\\
T_2 &= -\frac{Im(c_1(0)) + \mu_2 Im(\lambda'(\tau^*))}{\omega^*\tau^*},
\end{aligned} \tag{3.12}$$

which determine the properties of bifurcating periodic solutions at the critical value τ^*. In particular, μ_2 determines the direction of the Hopf bifurcation. The coefficient β_2 determines the stability of bifurcating periodic solutions. The parameter T_2 determines the period of the bifurcating periodic solution. In summary, we obtain the following result on the direction and stability of the Hopf bifurcation.

Theorem 3.5. *The Hopf bifurcation of the fundamental steady state of the system* (2.10) *at $\tau = \tau^*$ results in periodic solutions in a right (left) neighborhood of τ^* when $\mu_2 > 0(< 0)$. In addition, the bifurcating periodic solutions are stable (unstable) if $Re(c_1(0)) < 0\ (> 0)$.*

Remark 3.6. From the previous discussion, we have $Re(\lambda'(\tau_0)) > 0$. It then follows from Theorem 3.5 that, at $\tau = \tau_0$, the bifurcating periodic solutions exist and are stable (unstable) for $\tau > \tau_0$ ($\tau < \tau_0$) when $Re(c_1(0)) < 0(> 0)$.

For the two cases considered above, we can use Theorem 3.5 to verify the numerical results in Figs 3 and 4. In the first case, $Re(c_1(0)) = -0.1306(-0.0689) < 0$ at $\tau = \tau_0(\tau_1)$. From Theorems 3.4 and 3.5, stable periodic solutions occur as τ crosses τ_0 increasingly and τ_1 decreasingly. This is partially verified by the stable limit cycle in Fig. 3(d) for $\tau = 16 \in (\tau_0, \tau_1)$. In the second case, $Re(c_1(0)) = -0.5436, \ -0.3197$ and -0.4616 at $\tau = \tau_0, \tau_1$ and τ_2, respectively. From Theorems 3.4 and 3.5, stable periodic solutions occur as τ crosses τ_0 and τ_2 increasingly and τ_1 decreasingly.

3.4 Global Existence of Periodic Solutions

The analysis on the stability of the fundamental price and Hopf bifurcation in the above provides a local analysis of the fundamental steady state of the system near the bifurcation values. Therefore the theoretical results on the existence and stability of the periodic solution induced from the Hopf bifurcation characterize the price dynamics of the system near the steady state and near the bifurcation values of delay τ. Numerically, we have shown in Figs 3 and 4 that such periodic solution bifurcated from the Hopf bifurcation continues to exist for all $\tau \in (\tau_0, \tau_1)$ in Fig. 3 and $\tau \in (\tau_0, \tau_1)$ and $\tau \in (\tau_2, \infty)$ in Fig. 4. It would be interesting, but very challenge, to provide a theoretical support on the global continuation of periodic solutions of the system (2.10) bifurcating from the bifurcating values τ_i. In the following, we present two results on such global continuation of periodic solutions.

Theorem 3.7. *Assume that $\gamma_f > 4\gamma_c$ and $\tau_0 > \frac{1}{k} ln \frac{15+\sqrt{33}}{12}$. Then (i) the system* (2.10) *has at least one nonconstant periodic solution bifurcating from τ_0 for all $\tau \in (\tau_0, \tau_1)$; (ii) for any fixed j $(j = 1, 2, \cdots, j_0)$, the system* (2.10) *has at least one nonconstant periodic solution bifurcating from $\tau = \tau_j$ for either all $\tau \in (\tau_{j-1}, \tau_j)$ or all $\tau \in (\tau_j, \tau_{j+1})$.*

Proof. See Appendix C. □

Theorem 3.7 provides sufficient conditions, in terms of the behavior parameters and time delay, on the global existence of the nonconstant periodic solution when the fundamental steady state becomes unstable. It provides a theoretical support on the observation obtained in the two numerical examples in Figs 3 and 4. Essentially, it implies that the periodic solution bifurcated from the Hopf bifurcation value τ_0 continues to exist for (τ_0, τ_1). For $j \geq 1$, the periodic solution bifurcated from the Hopf bifurcation value τ_j continues to exist for either all $\tau \in (\tau_{j-1}, \tau_j)$ or all $\tau \in (\tau_j, \tau_{j+1})$, as in Figs 3 and 4. Note that the upper limit of the interval can be infinite as in Fig. 4. The periodic solution may not be unique in general. The global existence result in Theorem 3.7 is quite strong, which is at a price of the very restrictive sufficient conditions. The sufficient condition in next result is rather less restrictive.

Theorem 3.8. *If $k > \gamma_c - \gamma_f$ or $\frac{4\gamma_c}{(e^{\frac{2\gamma_c}{\gamma_f}} + e^{-\frac{2\gamma_c}{\gamma_f}})^2} > k + \gamma_f$, then for any fixed j $(j = 1, 2, 3, \cdots, j_0)$, the system* (2.10) *has at least one nonconstant periodic solution bifurcating from $\tau = \tau_j$ for either all $\tau \in (\tau_{j-1}, \tau_j)$ or all $\tau \in (\tau_j, \tau_{j+1})$.*

Proof. See Appendix C. □

Comparing Theorem 3.7, the condition of Theorem 3.8 implies that the existence of periodic solutions for all $\tau \in (\tau_0, \tau_1)$ is not necessarily bifurcated from τ_0. Note that the parameter region indicated by the conditions in Theorem 3.8 is rather smaller than the region indicated by the numerical simulations.

4 Conclusion

This paper develops a continuous-time heterogeneous agent model when the price trend of the trend followers is formed by a geometrically weighted and continuously distributed lagged prices. The model provides a unified treatment to the discrete-time HAMs where the price trend follows weighted moving average rules. However, the correspondence between the behavior of high dimensional discrete-time models and infinite dimensional continuous-time models with delays such as (2.10) may be severely limited. In particular, the stabilizing effect of an increase in time delay is apparently little known for the current discrete-time HAM literature. It is clear from the present work and the

HAM literature (see for example, Chiarella, He and Hommes, 2006) that, when agents use lagged information such as price to form the expectation, an increase in the time lag is potentially a destabilizing factor. However, the analysis presented in this paper shows that, under certain circumstance, a further increase in the time delay for an unstable system can stabilize the system. Furthermore, we have shown analytically and demonstrated numerically that the stability of the fundamental price can switch many times as the time delay increases. In addition, we provide some sufficient conditions on the existence and stability of periodic solution resulted from the Hopf bifurcation. The results obtained in this paper provide some newly interesting insight into the stabilizing role of the trend followers and the generating mechanism on market instability. The paper also demonstrates the advantage of continuous-time models over the discrete models and we hope the continuous-time framework established in this paper will provide an alternative approach to the current discrete-time financial market modeling with bounded rational and heterogeneous agents.

In order to make the model parsimonious and to focus on the delay effect, we consider a very simple financial market with heterogeneous agents in this paper. The demand functions of the heterogeneous agents are assumed based on agents' behavior rather than on utility maximization in the standard financial economics theory. Justification and variation of the behavior demand functions using utility maximization are of interest. It is also of interest to extend the analysis to a stochastic model in which the fundamental price is driven by a stochastic process and the behavior of noise traders is taking into account. In addition, similar to the discrete-time models, the market fractions of the fundamentalists and trend followers can endogenously change when agents are allowed to switch among different types of beliefs or strategies based on certain fitness or performance measures. This paper provides a first step on the applications of delay differential equations to finance. We leave these issues for future research.

Appendix A. Proof of Theorem 3.1

Let $P_0(t) = P(t) - F, u_0(t) = u(t) - F$. Then system (2.10) becomes

$$\begin{aligned} \frac{dP_0(t)}{dt} &= -\gamma_f P_0(t) + \mu(1-\alpha)\tanh(\beta_c(P_0(t) - u_0(t))), \\ \frac{du_0(t)}{dt} &= \frac{k}{1-e^{-k\tau}}[P_0(t) - e^{-k\tau}P_0(t-\tau) - (1-e^{-k\tau})u_0(t)]. \end{aligned} \tag{A.1}$$

When $\tau \to 0$, the second equation in the system (A.1) becomes

$$\frac{d(P_0(t) - u_0(t))}{dt} = -k[P_0(t) - u_0(t)].$$

Since $k > 0$, we have

$$\lim_{t\to\infty}(P_0(t) - u_0(t)) = 0.$$

Applying variation of constants to the first equation in the system (A.1), we have

$$P_0(t) = e^{-\gamma_f t}P(0) + \mu(1-\alpha)e^{-\gamma_f t}\int_0^t e^{\gamma_f s}\tanh(\beta_c(P_0(s) - u_0(s)))ds. \tag{A.2}$$

Let $G(t) = \tanh\Big(\beta_c(P_0(t) - u_0(t))\Big)$. Then $\lim_{t\to\infty} G(t) = 0$. If $\int_0^t e^{\gamma_f s} G(s)ds$ is bounded, then $P_0(t) \to 0$ when $t \to \infty$. If $\int_0^t e^{\gamma_f s} G(s)ds$ is unbounded, then we have

$$\lim_{t\to\infty} \frac{\int_0^t e^{\gamma_f s} G(s)ds}{e^{\gamma_f t}} = \lim_{t\to\infty} \frac{G(t)}{\gamma_f} = 0,$$

leading to $\lim\limits_{t\to\infty} P_0(t) = 0$. Therefore, the equilibrium (F, F) of the system (2.10) is globally asymptotically stable when $\tau \to 0$.

Appendix B. Proof of Theorem 3.5

Let $\tau = \tau^* + \upsilon$, then $\upsilon = 0$ is a Hopf bifurcation value of Eq.(2.10). Let $P_0(t) = P(t) - F, u_0(t) = u(t) - F$ and, for convenience, denote $(P_0(t), u_0(t))$ by $(P(t), u(t))$. We re-scale the time by $t \mapsto (t/\tau)$ to normalize the delay so that system (2.10) can be written as

$$\begin{pmatrix} \dot{P}(t) \\ \dot{u}(t) \end{pmatrix} = \tau B(\tau) \begin{pmatrix} P(t) \\ u(t) \end{pmatrix} + \tau C(\tau) \begin{pmatrix} P(t-1) \\ u(t-1) \end{pmatrix} + \tau f(P, u), \tag{B.1}$$

where

$$B(\tau) = \begin{pmatrix} \gamma_c - \gamma_f & -\gamma_c \\ \frac{k}{1-e^{-k\tau}} & -k \end{pmatrix}, \quad C(\tau) = \begin{pmatrix} 0 & 0 \\ \frac{-ke^{-k\tau}}{1-e^{-k\tau}} & 0 \end{pmatrix},$$

and

$$f(P, u) = \begin{pmatrix} -\frac{\gamma_c\beta_c^2}{3} P^3(t) + \gamma_c\beta_c^2 P^2(t)u(t) - \gamma_c\beta_c^2 P(t)u^2(t) + \frac{\gamma_c\beta_c^2}{3} u^3(t) + \cdots \\ 0 \end{pmatrix},$$

The linearization of Eq. (B.1) around the origin is given by

$$\dot{\widetilde{u}}(t) = \tau B(\tau)\widetilde{u}(t) + \tau C(\tau)\widetilde{u}(t-1),$$

where $\widetilde{u}(t) = (P(t), u(t))^T$.

For $\phi = (\phi_1,\ \phi_2)^T \in C\ ([-1, 0],\ R^2)$, define

$$L_\upsilon(\phi) = (\tau^* + \upsilon)(B(\tau^* + \upsilon)\phi(0) + C(\tau^* + \upsilon)\phi(-1)).$$

By the Riesz Representation Theorem, there exists a 2×2 matrix, $\eta(\vartheta, \upsilon)(-1 \le \vartheta \le 0)$, whose elements are of bounded variation functions such that

$$L_\upsilon(\phi) = \int_{-1}^{0} [d\eta(\vartheta, \upsilon)]\ \phi(\vartheta), \qquad for\ \ \phi \in C([-1, 0],\ R^2). \tag{B.2}$$

In fact, we can choose

$$\eta(\vartheta, \upsilon) = (\tau^* + \upsilon)(B(\tau^* + \upsilon)\delta(\vartheta) - C(\tau^* + \upsilon)\delta(\vartheta + 1)),$$

where $\delta(\theta)$ is the indicated function; that is

$$\eta(\vartheta, \upsilon) = \begin{cases} (\tau^* + \upsilon)B(\tau^* + \upsilon), & \vartheta = 0, \\ 0, & \vartheta \in (-1, 0), \\ -(\tau^* + \upsilon)C(\tau^* + \upsilon), & \vartheta = -1. \end{cases}$$

Then Eq. (B.2) is satisfied.

For $\phi \in C^1([-1,0],\ \mathbb{R}^2)$, define an operator $A(\upsilon)$ as

$$A(\upsilon)\phi(\vartheta) = \begin{cases} \dfrac{d\phi(\vartheta)}{d\vartheta}\ , & \vartheta \in [-1,0), \\ \int_{-1}^{0}[d\eta(\xi,\upsilon)]\phi(\xi)\ , & \vartheta = 0. \end{cases} \tag{B.3}$$

For $\phi = (\phi_1,\ \phi_2)^T \in C\ ([-1,0],\mathbb{R}^2)$, let

$$h(\upsilon,\phi) = (\tau^* + \upsilon)\begin{pmatrix} h_1 \\ 0 \end{pmatrix}, \tag{B.4}$$

where

$$h_1 = -\frac{\gamma_c\beta_c^2}{3}\phi_1^3(0) + \gamma_c\beta_c^2\phi_1^2(0)\phi_2(0) - \gamma_c\beta_c^2\phi_1(0)\phi_2^2(0) + \frac{\gamma_c\beta_c^2}{3}\phi_2^3(0) + \cdots .$$

If we further define an operator $R(\upsilon)$ as

$$R(\upsilon)\phi(\vartheta) = \begin{cases} 0, & \vartheta \in [-1,0), \\ h(\upsilon,\phi), & \vartheta = 0, \end{cases} \tag{B.5}$$

then the system (2.10) is equivalent to the following operator equation

$$\dot{\widetilde{u}}_t = A(\upsilon)\widetilde{u}_t + R(\upsilon)\widetilde{u}_t, \tag{B.6}$$

where $\widetilde{u}_t = \widetilde{u}(t+\vartheta)$ for $\vartheta \in [-1,0]$.

For $\psi \in C^1([0,1],\ (R^2)^*)$, define

$$A^*\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & s \in (0,1], \\ \int_{-1}^{0}\psi(-\xi)d\eta(\xi,0), & s = 0, \end{cases}$$

and a bilinear form

$$\langle\psi(s)\ ,\ \phi(\vartheta)\rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\vartheta}\bar{\psi}(\xi-\vartheta)d\eta(\vartheta)\phi(\xi)d\xi,$$

where $\eta(\vartheta) = \eta(\vartheta,0)$. Then $A(0)$ and A^* are adjoint operators. From the previous discussion, we know that $\pm i\omega^*\tau^*$ are eigenvalues of $A(0)$ and therefore they are also eigenvalues of A^*.

It can be verified that the vector $q(\vartheta) = (q_1,q_2)^T e^{i\omega^*\tau^*\vartheta}$ $(\vartheta \in [-1,0])$ and $q^*(s) = \dfrac{1}{\bar{d}}(q_1^*,q_2^*)e^{i\omega^*\tau^* s}$ $(s \in [0,1])$ are the eigenvectors of $A(0)$ and A^* corresponding to the eigenvalue $i\omega^*\tau^*$ and $-i\omega^*\tau^*$, respectively, where

$$(q_1,q_2) = \left(1, \frac{\gamma_c - \gamma_f - i\omega^*}{\gamma_c}\right),\quad (q_1^*,q_2^*) = \left(1, \frac{\gamma_c}{i\omega^* - k}\right).$$

Let

$$\begin{aligned} d &= (\bar{q_1^*}q_1 + \bar{q_2^*}q_2) - \int_{-1}^{0}\int_{\xi=0}^{\vartheta}\bar{q}^*(\xi-\vartheta)d\eta(\vartheta)q(\xi)d\xi \\ &= \bar{q_1^*}(q_1 + \tau^*e^{-i\omega^*\tau^*}\sum_{j=1}^{2}c_{1j}q_j) + \bar{q_2^*}(q_2 + \tau^*e^{-i\omega^*\tau^*}\sum_{j=1}^{2}c_{2j}q_j) \\ &= 1 + \tfrac{(i\omega^*-k)(\gamma_c-\gamma_f-i\omega^*)}{\omega^{*2}+k^2} + \tfrac{-(i\omega^*-k)k\gamma_c\tau^*e^{-i\omega^*\tau^*-k\tau^*}}{(\omega^{*2}+k^2)(1-e^{-k\tau^*})}, \end{aligned}$$

where c_{ij} $(i = 1,2)$ represents the element of row i and column j in matrix $C(\tau)$, then $\langle q^*(s),q(\vartheta)\rangle = 1,\ \langle q^*(s),\bar{q}(\vartheta)\rangle = 0$.

Following the algorithm in Hassard, Kazarinoff and Wan (1981) and using a computation process similar to that in Wei and Li (2005), we can obtain the normal form (3.10) and the coefficients.

Appendix C. Proofs of Theorems 3.7 and 3.8

Throughout this appendix, we follow closely the notations used in Wei and Li (2005) and define

$$
\begin{aligned}
&X = C([-\tau, 0], \mathbb{R}^2),\\
&\Sigma = \mathrm{Cl}\{(x,\tau,l): \ (x,\tau,l) \in X \times \mathbb{R}_+ \times \mathbb{R}_+, \ \text{x is a } l\text{-periodic solution of Eq. (2.10) }\},\\
&\mathcal{S} = \left\{(\hat{x},\tau,l): \ \hat{x} = (\hat{P},\hat{u}), \ \alpha\beta_f(\hat{P}-\bar{F}) = (1-\alpha)\frac{e^{\beta_c(\hat{P}-\hat{u})} - e^{-\beta_c(\hat{P}-\hat{u})}}{e^{\beta_c(\hat{P}-\hat{u})} + e^{-\beta_c(\hat{P}-\hat{u})}}, \ \hat{P} = \hat{u}\right\},\\
&\triangle_{(x^*, \ \tau, \ l)}(\lambda) = p(\lambda,\tau) + q(\lambda,\tau)e^{-\lambda\tau},
\end{aligned}
$$

and let $C(x^*, \tau_j, 2\pi/\omega)$ denote the connected component of $(x^*, \tau_j, 2\pi/\omega)$ in Σ, where ω and τ_j are defined in Eqs (3.6) and (3.9), respectively. To obtain the main result, we first introduce three lemmas.

Lemma C1. *All periodic solutions of the system* (2.10) *are uniformly bounded.*

Proof. Let $(P(t), u(t))$ be a nonconstant periodic solution of the system (2.10) and $\overline{P} = P(t_1) = \max\{P(t)\}$, $\underline{P} = P(t_2) = \min\{P(t)\}$ be the maximum and minimum of $P(t)$, respectively. Then $P'(t_1) = P'(t_2) = 0$. It follows from the first of Eq. (2.10) that

$$
\overline{P} = F + \frac{1-\alpha}{\alpha\beta_f}\tanh(\beta_c(\overline{P} - u(t_1))) \quad \text{and} \quad \underline{P} = F + \frac{1-\alpha}{\alpha\beta_f}\tanh(\beta_c(\underline{P} - u(t_2))).
$$

It then follows from $|\tanh(x)| < 1$ for any x that $F - \frac{1-\alpha}{\alpha\beta_f} < P(t) < F + \frac{1-\alpha}{\alpha\beta_f}$.

Similarly, let $\overline{U} = u(t_3) = \max\{u(t)\}$ and $\underline{U} = u(t_4) = \min\{u(t)\}$ be the maximum and minimum of $u(t)$, respectively. Then $u'(t_3) = u'(t_4) = 0$, and by the second of Eq. (2.10) we have

$$
P(t_3) - e^{-k\tau}P(t_3 - \tau) = (1 - e^{k\tau})\overline{U} \quad \text{and} \quad P(t_4) - e^{-k\tau}P(t_4 - \tau) = (1 - e^{k\tau})\underline{U}.
$$

Then the uniform boundedness of $u(t)$ follows from the uniform boundednedd of $P(t)$. The proof is complete. □

Lemma C2. *The system* (2.10) *has no nontrivial τ-periodic solution when either $\gamma_f + k > \gamma_c$ or $\frac{4\gamma_c}{\left(e^{\frac{2\gamma_c}{\gamma_f}} + e^{-\frac{2\gamma_c}{\gamma_f}}\right)^2} > k + \gamma_f$.*

Proof. For a contradiction, suppose that the system (2.10) has a τ-periodic solution. Then the following system of ODE has a τ-periodic solution.

$$
\begin{aligned}
\frac{dP}{dt} =& \mu\left[\alpha\beta_f(F - P(t)) + (1-\alpha)\tanh(\beta_c(P(t) - u(t)))\right],\\
\frac{du}{dt} =& k[P(t) - u(t)].
\end{aligned}
\tag{C.1}
$$

Let $P_0(t) = P(t) - F, u_0(t) = u(t) - F$ and still denote $P_0(t)$ and $u_0(t)$ by $P(t)$ and $u(t)$, respectively. Then Eq. (C.1) becomes

$$
\begin{aligned}
\frac{dP}{dt} =& -\gamma_f P + \mu(1-\alpha)\tanh(\beta_c(P - u)),\\
\frac{du}{dt} =& k(P - u).
\end{aligned}
\tag{C.2}
$$

Applying variation of constants, we have

$$P(t) = e^{-\gamma_f t}[P(0) + \mu(1-\alpha)\int_0^t e^{\gamma_f s}\tanh(\beta_c P(s) - \beta_c u(s))ds],$$

Hence

$$\begin{aligned} \mid P(t) \mid \leq & \mid P(0) \mid e^{-\gamma_f t} + \mu(1-\alpha)e^{-\gamma_f t}\int_0^t e^{\gamma_f s} \mid \tanh(\beta_c P(s) - \beta_c u(s)) \mid ds \\ < & \mid P(0) \mid e^{-\gamma_f t} + \frac{1-\alpha}{\alpha\beta_f}(1-e^{-\gamma_f t}) \to \frac{1-\alpha}{\alpha\beta_f}, \text{ when } t \to +\infty. \end{aligned}$$

We can also obtain that

$$\begin{aligned} \mid u(t) \mid \leq & \mid u(0) \mid e^{-kt} + e^{-kt}\int_0^t e^{ks} k \mid P \mid ds \\ \leq & \mid u(0) \mid e^{-kt} + \frac{1-\alpha}{\alpha\beta_f}(1-e^{-kt}) \to \frac{1-\alpha}{\alpha\beta_f}, \text{ when } t \to +\infty. \end{aligned}$$

Thus $(P(t), u(t))$ is ultimately uniformly bounded. Consider the vector field:

$$\begin{aligned} \frac{dP}{dt} =& -\gamma_f P + \mu(1-\alpha)\tanh(\beta_c(P-u)) := f^{(1)}(P,u), \\ \frac{du}{dt} =& k(P-u) := f^{(2)}(P,u). \end{aligned}$$

We have

$$\frac{\partial f^{(1)}}{\partial P} + \frac{\partial f^{(2)}}{\partial u} = -k + \frac{4\gamma_c}{[e^{\beta_c(P-u)} + e^{-\beta_c(P-u)}]^2} - \gamma_f.$$

On the simply connected region $D^* = \{(P,u) :\mid P \mid\leq \frac{1-\alpha}{\alpha\beta_f}, \ \ \mid u \mid\leq \frac{1-\alpha}{\alpha\beta_f}\}$, we have

$$\frac{4\gamma_c}{(e^{\frac{2\gamma_c}{\gamma_f}} + e^{-\frac{2\gamma_c}{\gamma_f}})^2} - k - \gamma_f \leq \frac{\partial f^{(1)}}{\partial P} + \frac{\partial f^{(2)}}{\partial u} \leq \gamma_c - \gamma_f - k.$$

Thus, $\frac{\partial f^{(1)}}{\partial P} + \frac{\partial f^{(2)}}{\partial u} < 0$ when $k > \gamma_c - \gamma_f$, and $\frac{\partial f^{(1)}}{\partial P} + \frac{\partial f^{(2)}}{\partial u} > 0$ when $\frac{4\gamma_c}{(e^{\frac{2\gamma_c}{\gamma_f}} + e^{-\frac{2\gamma_c}{\gamma_f}})^2} > k + \gamma_f$. According to the classical Bendixson's negative criterion, Eq. (C.2) has no periodic solutions in region D^* when

$$\gamma_f + k > \gamma_c \qquad \text{or} \qquad \frac{4\gamma_c}{(e^{\frac{2\gamma_c}{\gamma_f}} + e^{-\frac{2\gamma_c}{\gamma_f}})^2} > k + \gamma_f. \tag{C.3}$$

Under (C.3), the unique equilibrium $(0,0)$ of Eq. (C.2) in region D^* is globally asymptotically stable, leading to that the system (C.1) has no nonconstant periodic solutions. This completes the proof. □

Lemma C3. *Assume that $\gamma_f > 4\gamma_c$ and $\tau > \frac{1}{k}ln\frac{15+\sqrt{33}}{12}$, then the system* (2.10) *has no periodic solution of period 2τ.*

Proof. Let $P_0(t) = P(t) - F, u_0(t) = u(t) - F$ and, for convenience, denote $(P_0(t), u_0(t))$ by $(P(t), u(t))$. Let $x(t) = (P(t), u(t))$ be a 2τ-periodic solution. Set

$$y_k(t) = x(t - (k-1)\tau), \quad k = 1, 2.$$

Then $y(t) = (y_1(t), y_2(t))$ is a periodic solution to the system of ODE

$$\begin{cases} \dot{P}_1 = \mu\Big[-\alpha\beta_f P_1 + (1-\alpha)\tanh(\beta_c(P_1 - u_1))\Big], \\ \dot{u}_1 = \dfrac{k}{1-e^{-k\tau}}\Big[P_1 - e^{-k\tau}P_2 - (1-e^{-k\tau})u_1\Big], \\ \dot{P}_2 = \mu\Big[-\alpha\beta_f P_2 + (1-\alpha)\tanh(\beta_c(P_2 - u_2))\Big], \\ \dot{u}_2 = \dfrac{k}{1-e^{-k\tau}}\Big[P_2 - e^{-k\tau}P_1 - (1-e^{-k\tau})u_2\Big]. \end{cases} \tag{C.4}$$

From Lemma C1, the periodic orbit of the system (C.4) belongs to the region:

$$G = \left\{ y \in \mathbb{R}^4 \middle| \begin{pmatrix} -\frac{1-\alpha}{\alpha\beta_f} \\ -\frac{1-\alpha}{\alpha\beta_f} \end{pmatrix} < y_k < \begin{pmatrix} \frac{1-\alpha}{\alpha\beta_f} \\ \frac{1-\alpha}{\alpha\beta_f} \end{pmatrix}, \quad k = 1, 2. \right\}. \tag{C.5}$$

If we want to prove there is no 2τ-periodic solution, it suffices to prove that there is no nonconstant periodic solution of the system (C.4) in the region G. To do this, we apply the general Bendixson's criterion in higher dimensions developed by **?**. It is easy to compute the Jacobian matrix $J(y)$ of the system (C.4), for $y \in \mathbb{R}^4$:

$$J(y) = \begin{pmatrix} f_3(P_1,u_1) & f_4(P_1,u_1) & 0 & 0 \\ \frac{k}{1-e^{-k\tau}} & -k & \frac{-ke^{-k\tau}}{1-e^{-k\tau}} & 0 \\ 0 & 0 & f_3(P_2,u_2) & f_4(P_2,u_2) \\ \frac{-ke^{-k\tau}}{1-e^{-k\tau}} & 0 & \frac{k}{1-e^{-k\tau}} & -k \end{pmatrix},$$

where

$$f_3(P_i,u_i) = -\gamma_f + \frac{4\gamma_c}{[e^{\beta_c(P_i-u_i)} + e^{-\beta_c(P_i-u_i)}]^2},$$

and

$$f_4(P_i,u_i) = \frac{-4\gamma_c}{[e^{\beta_c(P_i-u_i)} + e^{-\beta_c(P_i-u_i)}]^2}, \; i = 1, 2.$$

Then the second additive compound matrix $J^{[2]}(y)$ of $J(y)$ is a $\binom{4}{2} \times \binom{4}{2}$ matrix defined by

$$J^{[2]}(y) =$$

$$\begin{pmatrix} f_3(P_1,u_1) - k & \frac{-ke^{-k\tau}}{1-e^{-k\tau}} & 0 & 0 & 0 & 0 \\ 0 & \sum_i f_3(P_i,u_i) & f_4(P_2,u_2) & f_4(P_1,u_1) & 0 & 0 \\ 0 & \frac{k}{1-e^{-k\tau}} & f_3(P_1,u_1) - k & 0 & f_4(P_1,u_1) & 0 \\ 0 & \frac{k}{1-e^{-k\tau}} & 0 & f_3(P_2,u_2) - k & f_4(P_2,u_2) & 0 \\ \frac{ke^{-k\tau}}{1-e^{-k\tau}} & 0 & \frac{k}{1-e^{-k\tau}} & \frac{k}{1-e^{-k\tau}} & -2k & \frac{-ke^{-k\tau}}{1-e^{-k\tau}} \\ 0 & \frac{ke^{-k\tau}}{1-e^{-k\tau}} & 0 & 0 & 0 & f_3(P_2,u_2) - k \end{pmatrix}.$$

Choose a vector form in $\mathbb{R}^6$ as

$$|(x_1,x_2,x_3,x_4,x_5,x_6)| = \max\left\{ (3+\sqrt{33})|x_1|, 12|x_2|, 4|x_3|, 4|x_4|, \frac{4}{3}|x_5|, (3+\sqrt{33})|x_6| \right\}.$$

With respect to this norm, we can obtain that the Lozinskiĭ measure $\mu(J^{[2]}(y))$ of the matrix $J^{[2]}(y)$ is given by (see Coppel, 1965)

$$\begin{aligned}\mu(J^{[2]}(y)) = \max\Big\{ & -\gamma_f + \frac{4\gamma_c}{[e^{\beta_c(P_1-u_1)}+e^{-\beta_c(P_1-u_1)}]^2} + \frac{k}{1-e^{-k\tau}}\Big(-1+\frac{15+\sqrt{33}}{12}e^{-k\tau}\Big),\\ & -2\gamma_f + \frac{16\gamma_c}{[e^{\beta_c(P_1-u_1)}+e^{-\beta_c(P_1-u_1)}]^2} + \frac{16\gamma_c}{[e^{\beta_c(P_2-u_2)}+e^{-\beta_c(P_2-u_2)}]^2},\\ & -\gamma_f + \frac{16\gamma_c}{[e^{\beta_c(P_1-u_1)}+e^{-\beta_c(P_1-u_1)}]^2} + \frac{k}{1-e^{-k\tau}}\Big(-\frac{2}{3}+e^{-k\tau}\Big),\\ & \frac{k}{1-e^{-k\tau}}\Big(-\frac{4}{3}+\frac{15+\sqrt{33}}{9}e^{-k\tau}\Big)\Big\}.\end{aligned} \tag{C.6}$$

By Corollary 3.5 of Li and Muldowney (1994), the system (C.4) has no periodic orbits in G if $\mu(J^{[2]}(y)) < 0$ for all $y \in G$. By (C.6), we have $\mu(J^{[2]}(y)) < 0$ if

$$\gamma_f > 4\gamma_c \quad \text{and} \quad \tau > \frac{1}{k} ln \frac{15+\sqrt{33}}{12}. \tag{C.7}$$

□

Remark C4. Note that condition (C.7) implies condition (C.3).

Proof of Theorem 3.7: By Lemmas C1 and C3, there exist $\varepsilon > 0$, $\delta > 0$ and a smooth curve $\lambda : (\tau_j - \delta, \tau_j + \delta) \to C$ such that

$$\triangle(\lambda(\tau)) = 0, \quad |\lambda(\tau) - i\omega| < \varepsilon,$$

for all $\tau \in [\tau_j - \delta, \tau_j + \delta]$, and

$$\lambda(\tau_j) = i\omega, \quad \frac{dRe(\lambda(\tau))}{d\tau}\Big|_{\tau=\tau_j} \neq 0.$$

Denote $l_j = 2\pi/\omega$, and let

$$\Omega_\epsilon = \{(y, l) : 0 < y < \varepsilon, |l - l_j| < \varepsilon\}.$$

Obviously, if $|\tau - \tau_j| \le \delta$ and $(y, l) \in \partial\Omega_\varepsilon$ such that $\triangle_{(x^*,\tau,l)}(y + 2\pi i/l) = 0$, and then $\tau = \tau_j$, $y = 0$, $l = l_j$. Set

$$H^{\pm}(x^*, \tau_j, 2\pi/\omega)(y, l) = \triangle_{(x^*,\tau_j\pm\delta,l)}(y + 2\pi i/l).$$

We obtain the crossing number

$$\begin{aligned}\gamma_1(x^*, \tau_j, 2\pi/\omega_j) &= \deg_B(H^-(x^*, \tau_j, 2\pi/\omega_j), \Omega_\varepsilon) - \deg_B(H^+(x^*, \tau_j, 2\pi/\omega_j), \Omega_\varepsilon)\\ &= \begin{cases} -1, & S_n^{+\prime}(\tau_j) > 0 \text{ or } S_n^{-\prime}(\tau_j) < 0,\\ 1, & S_n^{+\prime}(\tau_j) < 0 \text{ or } S_n^{-\prime}(\tau_j) > 0.\end{cases}\end{aligned}$$

By Theorem 3.2 of Wu (1998), we conclude that the connected component $C(x^*, \tau_j, 2\pi/\omega_j)$ through $(x^*, \tau_j, 2\pi/\omega_j)$ in Σ is nonempty. Meanwhile, we have by Theorem 3.3 of Wu (1998) that either

(i) $C(x^*, \tau_j, 2\pi/\omega_j)$ is unbounded or

(ii) $C(x^*, \tau_j, 2\pi/\omega_j)$ is bounded, $C(x^*, \tau_j, 2\pi/\omega_j) \bigcap \mathcal{S}$ is finite and $\sum_{(\hat{y},\tau,l)\in C(0,\tau_j,2\pi/(\tau_j\omega_j))} \gamma_1(\hat{y}, \tau, l) = 0$.

By the definition of τ_j, we know that,

$$\pi < \tau_0\omega_0 < 2\pi, \quad 3\pi < \tau_j\omega_j < 2(j+1)\pi, \quad \tau_j \in J_+,$$

where J_+ is defined by (3.9). This implies that

$$\tau_0 < \frac{2\pi}{\omega_0} < 2\tau_0, \quad \frac{\tau_j}{j+1} < \frac{2\pi}{\omega_j} < \frac{2\tau_j}{3}.$$

Therefore, we have that $\tau < l < 2\tau$ if $(x, \tau, l) \in C(x^*, \tau_0, 2\pi/\omega_0)$, and $\frac{\tau}{j+1} < l < \frac{2\tau}{3}$ if $(x, \tau, l) \in C(x^*, \tau_j, 2\pi/\omega_j)$ for $j \geq 1$. Lemma C1 implies that the projection of $C(x^*, \tau_j, 2\pi/\omega_j)$ onto the x-space is bounded. Similar to Lemma C2, one can prove that the system (2.10) with $\tau = 0$ has no nonconstant periodic solutions. This fact and Lemmas C2 and C3 show that the projection of $C(x^*, \tau_j, 2\pi/\omega_j)$ onto the l-space is bounded under the condition (C.3) and (C.7).

Consequently, if $C(x^*, \tau_j, 2\pi/\omega_j)$ is unbounded, then the projection of $C(x^*, \tau_j, 2\pi/\omega_j)$ onto the τ-space is unbounded. If $C(x^*, \tau_j, 2\pi/\omega_j)$ is bounded, that is (ii) of Theorem 3.3 of Wu (1998) is satisfied, then for any fixed j ($j = 0, 1, 2, \cdots$), the system (2.10) has at least one nonconstant periodic solution for either all $\tau \in (\tau_{j-1}, \tau_j)$ or all $\tau \in (\tau_j, \tau_{j+1})$, so is our theorem. This completes the proof.

Proof of Theorem 3.8: From the proof of Theorem 3.7, we know that the first global Hopf branch contains periodic solutions of period between τ and 2τ. These are the slowly-oscillating periodic solutions. For $j \geq 1$, the τ_j branches contain fast-oscillating periodic solutions since the periods are less than $\frac{2\tau_j}{3}$. In addition, for $j \geq 1$, $\frac{\tau_j}{j+1} < \frac{2\pi}{\omega} < \frac{2\tau_j}{3}$ is satisfied automatically. The bounds on the period l for $(x, \tau, l) \in C(x^*, \tau_j, 2\pi/\omega)$ hold without resulting to Lemma C3. Thus, the global extension of the τ_j-branches for $j \geq 1$ can be proved without the restrictions (C.7) in Lemma C3.

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