QUANTITATIVE FINANCE
RESEARCH CENTRE

UNIVERSITY OF
TECHNOLOGY SYDNEY

QUANTITATIVE FINANCE
RESEARCH CENTRE

UTS

THINK.CHANGE.DO

QUANTITATIVE FINANCE RESEARCH CENTRE

Research Paper 198 June 2007

Pricing under the Real-World Probability Measure for Jump-Diffusion Term Structure Models

Nicola Bruti-Liberati, Christina Nikitopoulos-Sklibosios and Eckhard Platen

ISSN 1441-8010 www.qfrc.uts.edu.au

Pricing under the Real-World Probability Measure for Jump-Diffusion Term Structure Models

Nicola Bruti-Liberati, Christina Nikitopoulos-Sklibosios and Eckhard Platen[1]

June 8, 2007

Abstract. This paper considers interest rate term structure models in a market attracting both continuous and discrete types of uncertainty. The event driven noise is modelled by a Poisson random measure. Using as numeraire the growth optimal portfolio, interest rate derivatives are priced under the real-world probability measure. In particular, the real-world dynamics of the forward rates are derived and, for specific volatility structures, finite dimensional Markovian representations are obtained. Furthermore, allowing for a stochastic short rate, a class of tractable affine term structures is derived where an equivalent risk-neutral probability measure does not exist.

2000 *Mathematics Subject Classification:* primary 60H10; secondary 65C05.
JEL Classification: G10, G13.
Key words and phrases: jump diffusions, affine term structure, real-world pricing, growth optimal portfolio, benchmark approach, HJM.

1 Introduction

In financial markets, the observed trajectories of financial quantities exhibit occasionally jumps. For instance, on January 6, 2002, Argentina experienced an economic reform, from a fixed rate regime, where the Argentinean peso was pegged to the US dollar, to a floating exchange rate regime. Due to a number of political and economic factors, a sharp rise in the interest rate, combined with a sharp devaluation of the Argentinean currency, created an increased potential for default on the massive fiscal budget deficit. Many small to medium-size companies were brought close to bankruptcy. Argentina faced a financial crisis with devastating impact on the entire nation. The evolution of the exchange rate and the 30-day deposit rate, which we interpret as short rate, are shown in Figure 1.1. It is evident that this currency crisis has caused extreme jumps in the exchange rate and the interest rate.

[1]University of Technology Sydney, School of Finance & Economics and Department of Mathematical Sciences, PO Box 123, Broadway, NSW, 2007, Australia

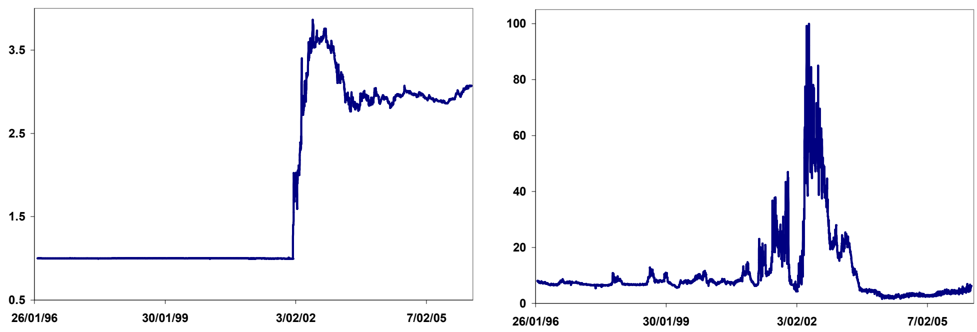

Figure 1.1: a) US dollar to Argentinean peso - exchange rate b)Argentinean peso 30-day deposit rate

Examples such as the Argentinean crisis in 2002, the Asian crisis in 1997 and the Russian crisis in 1998 provide compelling evidence for jumps in emerging markets of developing countries. Although less evident, this phenomenon is also present in financial markets of developed countries. Johannes (2004) highlights the important impact of jumps on pricing interest rate options and provides evidence of jumps in US Treasury bill rates. In addition, as several empirical studies have demonstrated, including Chan, Karolyi, Longstaff & Sanders (1992), Ait-Sahalia (1996) and Das (2002), jump-diffusion models provide a better fit to observed interest rates than pure diffusion models.

The arbitrage pricing theory, introduced by Ross (1976) and Harrison & Kreps (1979), has been the prevailing pricing approach over the past 30 years, with an extensive supporting literature and many practical applications. The fundamental assumption of this theory is the existence of an equivalent risk-neutral probability measure. In advanced modelling, for instance, when treating credit risk quantitatively, the choice of the appropriate equivalent risk-neutral probability measure is not a straightforward task. More importantly, realistic market models may not admit an equivalent risk-neutral probability measure. In particular, recent developments in financial theories generalize the classical risk-neutral approach, see Long (1990), Platen (2002) and Karatzas & Kardaras (2007). The benchmark approach in Platen & Heath (2006) proposes the consequent use of the *growth optimal portfolio* (GOP) as numeraire portfolio together with the real-world probability measure as pricing measure. This yields to, so called, real-world pricing which generalises the concept of risk-neutral pricing. It is applicable as long as a GOP exists and provides a framework that is more general than the classical risk-neutral theory which requires an equivalent risk-neutral probability measure. Numeraire portfolios for diffusion models have been studied by Long (1990), Bajeux-Besnainou & Portait (1997), Goll & Kallsen (2000) and Platen (2002). More recent research by Korn, Oertel & Schäl (2003), Platen (2004a) and Platen & Heath (2006) use numeraire portfolios for jump-diffusion models. In Miller & Platen (2005), diffusion type term structure models have been proposed and analysed under the benchmark approach.

The central finding of the current paper is the arbitrage-free characterisation of the dynamics of forward rates under the real-world probability measure for jump-diffusion models. A link between drift and diffusion coefficients has been derived similar to the one discovered in Heath, Jarrow & Morton (1992) for the risk-neutral case. The structure of the paper is as follows. Section 2 applies the benchmark approach to term structure models. Furthermore, under certain volatility specifications finite dimensional Markovian short rate dynamics are obtained. Section 3 describes a parsimonious model with closed form solutions for zero-coupon bonds and computationally tractable formulas for European interest rate derivatives.

2 Real-World Pricing and the Term Structure of Interest Rates

2.1 Modelling Traded Uncertainty

Given a filtered probability space $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$, with $\underline{\mathcal{A}} = (\mathcal{A}_t)_{t\in[0,\infty)}$, satisfying the usual conditions, the *continuous traded uncertainty* is modelled by an $\underline{\mathcal{A}}$-adapted, $m-$dimensional Wiener process $W = \{W_t = (W_t^1, \dots, W_t^m)^\top, t \in [0,T]\}$. The *event driven traded uncertainty* is modelled by a Poisson random measure p as follows. Given a mark space $(\mathcal{E}, \mathsf{B}(\mathcal{E}))$ with $\mathcal{E} \subseteq \mathbb{R}\backslash\{0\}$ we define on $\mathcal{E} \times [0,T]$ an $\underline{\mathcal{A}}$-adapted Poisson random measure $p(dv, dt)$ with intensity measure $\phi(dv)dt$. We assume finite total intensity $\lambda = \phi(\mathcal{E}) < \infty$. Thus, $p = \{p_t := p(\mathcal{E} \times [0,t]), t \in [0,T]\}$ is the stochastic process that counts the total number of jumps occurring in the time interval $[0,t]$. The Poisson random measure $p(dv, dt)$ generates a sequence of pairs $\{(\tau_i, \xi_i), i \in \{1, 2, \dots, p_T\}\}$. Here $\{\tau_i : \Omega \to \mathbb{R}_+, i \in \{1, 2, \dots, p_T\}\}$ is the sequence of jump times of the Poisson process p and $\{\xi_i : \Omega \to \mathcal{E}, i \in \{1, 2, \dots, p_T\}\}$ is the corresponding sequence of independent identically distributed marks ξ_i with probability density $\frac{\phi(dv)}{\lambda}$. One can interpret τ_i as the time of the ith event and the mark ξ_i as its magnitude. The corresponding jump martingale measure is given by $q(dv, dt) = p(dv, dt) - \phi(dv)dt$.

2.2 Growth Optimal Portfolio under Jump-Diffusions

The benchmark approach provides a consistent modelling framework under the real-world probability measure. Its central building block, the GOP, maximises the expected logarithmic utility from terminal wealth, see Kelly (1956), Long (1990) and Platen & Heath (2006). In a continuous time setting, the existence of a GOP implies no arbitrage in the sense of Platen & Heath (2006). Christensen & Platen (2005) establish a generalised GOP within a jump-diffusion setting with stochastic jump sizes.

Let us introduce the predictable vector process $\theta = \{\theta_t = (\theta_t^1, \theta_t^2, \ldots, \theta_t^m)^\top, t \in [0, T]\}$, for the market prices of Wiener process risk, and the predictable and bounded process $\psi(v) = \{\psi(v,t), t \in [0,T]\}$, for each mark $v \in \mathcal{E}$, as the corresponding density of the market price of jump risk. Moreover, assume that the density of the market price of jump risk is less than one and uniformly bounded as a function of (v,t) almost surely. The unique generalised GOP S_t, according to Christensen & Platen (2005), satisfies the SDE

$$\begin{aligned} dS_t = & S_{t-} \left[r_t dt + \sum_{i=1}^{m} \theta_t^i \left(\theta_t^i dt + dW_t^i \right) \right. \\ & \left. + \int_{\mathcal{E}} \frac{\psi(v,t)}{1-\psi(v,t)} \left(\psi(v,t)\phi(dv)dt + q(dv,dt) \right) \right], \end{aligned} \tag{2.1}$$

for all $t \in [0, \infty)$, with $S_0 = 1$. Here $r = \{r_t, t \in [0,T]\}$ denotes the predictable short rate process. Thus, the dynamics of the GOP is determined solely by the short rate r_t, the market prices of Wiener process risk θ_t and the density of the market price of jump risk $\psi(v,t)$. Equation (2.1) also implies that the diffusion volatilities θ_t^i, $i \in \{1, 2, \ldots, m\}$, of the GOP equal the market prices of Wiener process risk. The jump volatility $\frac{\psi(v,t)}{1-\psi(v,t)}$ of the GOP is expressed in terms of the density of the market price of jump risk. Moreover, the total *risk premium* of the GOP at time $t \in [0,T]$, is given by

$$p_t^S = \theta_t^\top \theta_t + \int_{\mathcal{E}} \frac{\psi(v,t)^2}{1-\psi(v,t)} \phi(dv). \tag{2.2}$$

According to a Diversification Theorem in Platen (2004b), the GOP can be approximated by any well-diversified world stock index (WSI) under rather general assumptions. For illustration, Figure 2.2 shows a WSI constructed by 104 world sector stock market accumulation indices in Le & Platen (2006), when denominated in Argentinean peso and US dollar for the period from 1996 until 2006. These can be interpreted as two different currency denominations of a proxy of the GOP.

The Argentinean peso denomination shows that a realistic model for the GOP should involve jumps, whereas the US dollar denomination may be modelled without jumps. The Argentinean peso denominated market prices of the Wiener process risk are observable as the diffusion volatilities of the corresponding GOP denomination and the density of the market price of jump risk follows from its jump coefficient. More precisely, at a jump time τ_i we obtain from (2.1) that

$$\psi(\xi_i, \tau_i-) = 1 - \frac{S_{\tau_i-}}{S_{\tau_i}}. \tag{2.3}$$

Consequently, the market price of jump risk density can be observed in terms of the GOP values at jump times.

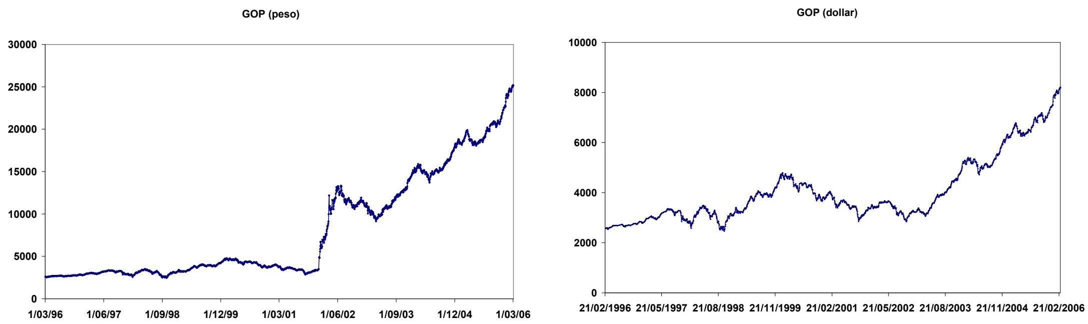

Figure 2.2: a) WSI denominated in Argentinean peso and b) WSI denominated in US dollar

2.3 Real-World Pricing

Any nonnegative portfolio when expressed in units of the GOP forms an $(\underline{\mathcal{A}}, P)$-supermartingale, see Platen & Heath (2006) and Christensen & Platen (2005). Among all supermartingales with same future value (payoff), the minimal supermartingale is obtained as the martingale that matches this future value (payoff). Thus by taking the GOP as numeraire and expressing GOP denominated derivative prices as martingales, the pricing of derivatives can be directly performed under the real-world probability measure. A price process which forms a martingale, when denominated in units of the GOP, is called fair. The concept of *real-world pricing* or fair pricing, which follows the lines of Long (1990) and has been derived in detail in Platen & Heath (2006), is now introduced.

Definition 2.1 *Assume for $T \in [0, \infty)$ that H_T is an $\mathcal{A}_T-$measurable nonnegative random contingent claim delivered at maturity T which satisfies*

$$E\left(\frac{H_T}{S_T}\right) < \infty \; a.s., \tag{2.4}$$

for all $t \in [0, T]$. Then the fair price process $U_{H_T} = \{U_{H_T}(t), t \in [0, T]\}$ of this contingent claim H_T is given by the real-world pricing formula

$$U_{H_T}(t) = S_t E\left(\frac{H_T}{S_T} | \mathcal{A}_t\right), \tag{2.5}$$

for all $t \in [0, T]$.

Thus real-world pricing requires the evaluation of real-world expectations of claims where the GOP serves as numeraire.

2.3.1 Risk-Neutral Pricing

When applying the standard risk-neutral approach, it is sometimes challenging to establish the existence of an equivalent risk-neutral probability measure which is often neglected in the literature. The existence of such a probability measure is not required under the above introduced concept of real-world pricing.

Let S_t^0 denote the locally riskless savings account at time t, which continuously accrues the short rate r_t, thus

$$S_t^0 = \exp\left\{\int_0^t r_s \, ds\right\}, \tag{2.6}$$

for $t \in [0, \infty)$. In a complete market, the Radon-Nikodym derivative process $\Lambda = \{\Lambda_t, t \in [0, \infty)\}$ for the candidate risk-neutral probability measure Q is given as

$$\Lambda_t = \frac{dQ}{dP}|_{\mathcal{A}_t} = \frac{S_t^0 S_0}{S_t} = \frac{\hat{S}_t^0}{\hat{S}_0^0}, \tag{2.7}$$

see Platen & Heath (2006). Using this notion, the real-world pricing formula (2.5) can be rewritten as

$$U_{H_T}(t) = E\left(\frac{\Lambda_T}{\Lambda_t}\frac{S_t^0}{S_T^0} H_T | \mathcal{A}_t\right), \tag{2.8}$$

for all $t \in [0, T]$. If the candidate Radon-Nikodym derivative process Λ is an $(\underline{\mathcal{A}}, P)$-martingale, then the risk-neutral probability measure Q exists and is equivalent to the real-world probability measure P. In this case, the well-known risk-neutral pricing formula is obtained from (2.8) by Girsanov's theorem and the Bayes formula. Thus, the concept of real-world pricing generalises that of risk-neutral pricing, as it does not impose the restrictive condition that Λ forms an $(\underline{\mathcal{A}}, P)$-martingale. In fact, as discussed in Platen & Heath (2006), since the stock index grows in the long-term significantly more than the savings account, realistic long-term market models are unlikely to admit an equivalent risk-neutral probability measure, see also Section 3. Still, the above more general modelling framework does not permit arbitrage in the sense that any market participant could generate strictly positive wealth as a going concern out of zero initial capital. See Platen & Heath (2006) for further details on an adequate definition of such a strong form of arbitrage and its absence in the proposed framework.

2.3.2 Zero-Coupon Bond

Consider for $T \in [0, \infty)$ the example of a zero-coupon bond $P(t, T)$ at time $t \in [0, T]$ which pays one unit of domestic currency at maturity T, i.e. $P(T, T) = 1$. Assuming $E\left((S_T)^{-1}\right) < \infty$, the real-world pricing formula (2.5) yields

$$P(t, T) = S_t E\left(\frac{1}{S_T} | \mathcal{A}_t\right), \tag{2.9}$$

for $t \in [0, T]$. Relationship (2.9) shows that the benchmarked zero-coupon bond

$$\hat{P}(t, T) = \frac{P(t, T)}{S_t}, \tag{2.10}$$

is an $(\underline{\mathcal{A}}, P)$-martingale. Therefore, it satisfies a driftless SDE of the form

$$d\hat{P}(t,T) = -\hat{P}(t-,T)\left(\sum_{i=1}^{m}\sigma^i(t,T)dW_t^i + \int_{\mathcal{E}}\beta(v,t,T)q(dv,dt)\right), \tag{2.11}$$

where $\sigma^i(\cdot,T)$, for each $i \in \{1,2,\ldots,m\}$, and $\beta(v,\cdot,T)$, for all $v \in \mathcal{E}$, are predictable stochastic processes, modelling the generalised benchmarked bond volatilities. Using the SDEs (2.1) and (2.11), and relationship (2.10), Itô's formula yields the SDE of the zero-coupon bond price in the form

$$\begin{aligned} dP(t,T) = P(t-,T)\Bigg[& r_t dt + \sum_{i=1}^{m}\tilde{\sigma}^i(t,T)\left(\theta_t^i dt + dW_t^i\right) \\ & + \int_{\mathcal{E}}\tilde{\beta}(v,t,T)\left(\psi(v,t)\phi(dv)dt + q(dv,dt)\right)\Bigg], \end{aligned} \tag{2.12}$$

for all $t \in [0,\infty)$, with generalised *bond volatilities*

$$\tilde{\sigma}^i(t,T) = \theta_t^i - \sigma^i(t,T), \tag{2.13}$$

and

$$\tilde{\beta}(v,t,T) = \frac{\psi(v,t) - \beta(v,t,T)}{1-\psi(v,t)}. \tag{2.14}$$

2.4 Real-World Forward Rate Dynamics

The well-known Heath, Jarrow & Morton (1992) (HJM) framework is a risk-neutral no-arbitrage framework for the study of the interest rate term structure. It accommodates consistency with the currently observed yield curve within an arbitrage-free environment. Under this framework, term structure models require only the information of the initial forward rate curve and the specification of the forward rate volatility. Extensions of the original diffusion based HJM framework to risk-neutral jump-diffusion models have been studied, for instance, by Shirakawa (1991), Björk, Kabanov & Runggaldier (1997) and Das (2000). The literature typically assumes the existence of an equivalent risk-neutral probability measure, which is rather restrictive. Next we will derive the forward rate dynamics for jump diffusions under the real-world probability measure, which avoids this restrictive and potentially unrealistic assumption. Moreover, we will recover the risk-neutral forward rate dynamics obtained in Björk, Kabanov & Runggaldier (1997) for the case when an equivalent risk-neutral probability measure exists.

By using (2.10) the instantaneous forward rate $f(t,T)$ at time $t \in [0,T]$, $T \in [0,\infty)$, can be defined as

$$f(t,T) = -\frac{\partial}{\partial T}\ln(P(t,T)) = -\frac{\partial}{\partial T}\ln(\hat{P}(t,T)). \tag{2.15}$$

As shown in Appendix A, this yields the following result, where $\sigma^i(t,T)$ and $\beta(v,t,T)$ are defined in (2.11) as the generalised benchmarked bond volatilities.

Proposition 2.2 *The real-world dynamics of the forward rate are given by*

$$df(t,T) = \mu_F(t,T)dt + \sum_{i=1}^{m} \sigma_F^i(t,T)dW_t^i + \int_{\mathcal{E}} \beta_F(v,t,T)p(dv,dt), \tag{2.16}$$

where

$$\begin{aligned} \mu_F(t,T) = &\sum_{i=1}^{m} \sigma_F^i(t,T) \left(\int_t^T \sigma_F^i(t,s)ds + \theta_t^i \right) \\ &- \int_{\mathcal{E}} \beta_F(v,t,T) \exp\left(- \int_t^T \beta_F(v,t,s)ds \right) (1-\psi(v,t))\phi(dv), \end{aligned} \tag{2.17}$$

and

$$\sigma_F^i(t,T) = \frac{\partial}{\partial T}\sigma^i(t,T), \tag{2.18}$$

$$\beta_F(v,t,T) = \frac{\frac{\partial}{\partial T}\beta(v,t,T)}{1-\beta(v,t,T)}. \tag{2.19}$$

The characterisation (2.17) of the drift $\mu_F(t,T)$ is similar to the HJM drift restriction for jump diffusions, as obtained in Björk, Kabanov & Runggaldier (1997), yet does not require the existence of an equivalent risk-neutral probability measure. Note that the jump intensity $\phi(dv)$ in (2.17) is the *real-world jump intensity*, which can be estimated from historical data by standard statistical methods.

In Appendix A, we also derive the real-world dynamics of the forward rate in terms of generalised benchmarked bond volatilities, as

$$\begin{aligned} df(t,T) = &\sum_{i=1}^{m} \frac{\partial}{\partial T}\sigma^i(t,T) \left(\sigma^i(t,T)dt + dW_t^i \right) \\ &+ \int_{\mathcal{E}} \frac{\frac{\partial}{\partial T}\beta(v,t,T)}{1-\beta(v,t,T)} \left(\beta(v,t,T)\phi(dv)dt + q(dv,dt) \right), \end{aligned} \tag{2.20}$$

for $t \in [0,T]$. Now, using (2.18), (2.19), (2.13) and (2.14), we obtain

$$\int_t^T \sigma_F^i(t,s)ds = \sigma^i(t,T) - \theta_t^i = -\tilde{\sigma}^i(t,T), \tag{2.21}$$

and

$$\begin{aligned} 1 - \exp&\left(- \int_t^T \beta_F(v,t,s)ds \right) (1-\psi(v,t)) \\ &= \beta(v,t,T) = \psi(v,t) - (1-\psi(v,t))\tilde{\beta}(v,t,T). \end{aligned} \tag{2.22}$$

Then, the real-world dynamics of the forward rate (2.20) can also be expressed in terms of the bond volatilities, see (2.13) and (2.14), as

$$df(t,T) = \sum_{i=1}^{m} \frac{\partial}{\partial T}\tilde{\sigma}^i(t,T)\left(\tilde{\sigma}^i(t,T)dt - d\tilde{W}_t^i\right) \tag{2.23}$$
$$+ \int_{\mathcal{E}} \frac{\frac{\partial}{\partial T}\tilde{\beta}(v,t,T)}{1+\tilde{\beta}(v,t,T)}\left(\tilde{\beta}(v,t,T)(1-\psi(v,t))\phi(dv)dt + \tilde{q}(dv,dt)\right),$$

for $t \in [0,T]$, with $\tilde{W}^i = \{\tilde{W}_t^i = \int_0^t \theta_s^i ds + W_t^i, t \in [0,T]\}$, and $\tilde{q}(dv,dt) = \psi(v,t)\phi(dv)dt + q(dv,dt)$. If an equivalent risk-neutral probability measure exists, then under this measure the processes $\tilde{W}^i$ are standard Wiener processes and $\tilde{q}(dv,dt)$ is a jump martingale measure. The resulting risk-neutral forward rate dynamics (2.23) are the same as those obtained in Björk, Kabanov & Runggaldier (1997). However, under our approach an equivalent risk-neutral probability measure may not exist.

Pricing under an equivalent risk-neutral probability measure requires parameters to be estimated or calibrated under this measure. This leads to a challenging statistical problem; the estimation of parameters under an assumed measure that cannot observed directly. Consequently, such approach involves significant model risk concerning the specification of the unknown measure transformation. However, statistical parameter estimation using historical data can be directly performed under the real-world probability measure without extra complications. Note that although the volatility does not change under a risk-neutral measure, the intensity of the jump component does. Under the benchmark approach, this is not an issue, as we model the real-world dynamics of the term structure of interest rates and can use the statistical estimates of the historical jump intensities. In addition, using relation (2.3) the market price of jump risk can be directly observed and estimated. In markets where there are not enough liquid instruments to calibrate the model, parameter estimation using historical data and economic reasoning are necessary to fit the model. Here the benchmark approach has a major advantage since it provides some form of absolute pricing with the GOP as reference unit, whereas risk-neutral pricing is more a form of relative pricing, which uses other traded instruments as reference.

Let us finally consider the special case of finite dimensional jump-diffusion markets, as in Platen & Heath (2006). The ith type of event driven uncertainty is modelled by the counting process $p^i = \{p_t^i = p(A_i,[0,t]), t \in [0,\infty)\}$, for $i \in \{1,\dots,m_d\}$, where A_i is a one-dimensional interval of length l_i and $A_i \cap A_j = \emptyset$ for $i \neq j \in \{1,\dots,m_d\}$. Therefore, the jump process p^i has intensity $\lambda^i = \phi(l_i)$. We also introduce the corresponding jump martingale $q^i = \{q_t^i, t \in [0,\infty)\}$, with stochastic differential $dq_t^i = (dp_t^i - \lambda^i dt)$, for $i \in \{1,2,\dots,m_d\}$ and $t \in [0,\infty)$. Under these specifications the dynamics of the GOP is given by

$$dS_t = S_{t-}\left(r_t dt + \sum_{i=1}^{m}\theta_t^i\left(\theta_t^i dt + dW_t^i\right) + \sum_{i=1}^{m_d}\frac{\psi_t^i}{1-\psi_t^i}\left(\psi_t^i\lambda^i dt + dq_t^i\right)\right),$$

for all $t \in [0, \infty)$, where $\psi = \{\psi_t = (\psi_t^1, \ldots, \psi_t^{m_d})^\top, t \in [0,T]\}$ is the predictable and uniformly bounded vector process of the market prices of jump risk, with $\psi_t^i < 1$ for $i \in \{1, \ldots, m_d\}$. Moreover, the benchmarked bond price satisfies the driftless SDE

$$d\hat{P}(t,T) = -\hat{P}(t-,T)\left(\sum_{i=1}^{m} \sigma^i(t,T)dW_t^i + \sum_{i=1}^{m_d} \beta^i(t,T)dq_t^i\right),$$

where $\sigma^i(\cdot, T)$, for each $i \in \{1, 2, \ldots, m\}$, and $\beta^i(\cdot, T)$, for each $i \in \{1, 2, \ldots, m_d\}$, are predictable stochastic processes, modelling the generalised benchmarked bond volatilities. Thus the real-world forward rate dynamics (2.20), expressed in terms of the generalised benchmarked bond volatilities, reduce to

$$\begin{aligned} df(t,T) = &\sum_{i=1}^{m} \frac{\partial}{\partial T}\sigma^i(t,T)\left(\sigma^i(t,T)dt + dW_t^i\right) \\ &+ \sum_{i=1}^{m_d} \frac{\frac{\partial}{\partial T}\beta^i(t,T)}{1-\beta^i(t,T)}\left(\beta^i(t,T)\lambda^i dt + dq_t^i\right), \end{aligned} \tag{2.24}$$

for $t \in [0, T]$.

2.5 Finite Dimensional Markovian Term Structures

Though our extension of the HJM framework is very flexible, it possesses the drawback of generating models which are, in general, non-Markovian. It has been demonstrated in Chiarella & Nikitopoulos (2003) that, under the existence of a risk-neutral probability measure, specific forward rate volatility structures can produce finite dimensional Markovian jump-diffusion HJM models, which are computationally tractable. However, the assumption of a state dependent jump coefficient leads in this case to a non-Markovian term structure. We shall show next that in the above jump-diffusion model, by using the benchmark approach, certain forward rate volatility specifications also lead to finite dimensional Markovian dynamics for the interest rate term structure.

Assumption 2.3 *For $T \in (0, \infty)$ and $i \in \{1, \ldots, m\}$, the diffusion coefficients of the forward rate are of the form*

$$\sigma_F^i(t,T) = \bar{\sigma}_F^i(t, \mathrm{F}_t)\; e^{-\int_t^T k_\sigma^i(s)ds}, \tag{2.25}$$

and the jump coefficient is of the form

$$\beta_F(v,t,T) = \bar{\beta}_F(v,t)\, e^{-\int_t^T k_\beta(s)ds}, \tag{2.26}$$

where $k_\sigma^i(t)$, $k_\beta(t)$ are deterministic functions of time, integrable in [0,T], and $\bar{\sigma}_F^i = \{\bar{\sigma}_F^i(t, \mathrm{F}_t), t \in [0,T]\}$ and $\bar{\beta}_F = \{\bar{\beta}_F(v,t), (v,t) \in \mathcal{E} \times [0,T], \}$ characterise well-defined functions, with $\mathrm{F}_t = (r_t, f(t,T_1), \ldots, f(t,T_z))^\top$, for $z \in \mathbb{N}$ and $t < T_1 < \ldots < T_z$. In addition, the market price of jump risk $\psi = \{\psi(v,t), (v,t) \in \mathcal{E} \times [0,T]\}$ is a deterministic function.

The above volatility specification separates in the interest rate term structure dynamics, the time dependent component from the maturity dependent component, which is a key property that can provide a Markovian model. As we will see, Assumption 2.3 allows for a wide range of interesting and tractable term structure models. In Appendix B we prove the following result:

Proposition 2.4 *Under Assumption 2.3 the real-world dynamics of the short rate are given by*

$$dr_t = \Big[\xi_t + \mathcal{E}_\beta(t) + \sum_{i=1}^{m} \mathcal{E}^i_\sigma(t) - \sum_{i=2}^{m}(k^i_\sigma(t) - k^1_\sigma(t))\mathcal{D}^i_\sigma(t) - (k_\beta(t) - k^1_\sigma(t))\mathcal{D}_\beta(t)$$
$$+ \sum_{i=1}^{m} \bar{\sigma}^i_F(t, \mathrm{F}_t)\theta^i_t - k^1_\sigma(t) r_t\Big]dt + \sum_{i=1}^{m} \bar{\sigma}^i_F(t, \mathrm{F}_t)dW^i_t + \int_{\mathcal{E}} \bar{\beta}_F(v,t)p(dv,dt), \tag{2.27}$$

with the state variables $\mathcal{E}^i_\sigma(t)$, $\mathcal{D}^i_\sigma(t)$, $\mathcal{D}_\beta(t)$ defined as

$$\mathcal{E}^i_\sigma(t) = \int_0^t \left(\sigma^i_F(u,t)\right)^2 du, \tag{2.28}$$

$$\mathcal{D}^i_\sigma(t) = \int_0^t \sigma^i_F(u,t)\int_u^t \sigma^i_F(u,s)dsdu + \int_0^t \sigma^i_F(u,t)(dW^i_u + \theta^i_u du), \tag{2.29}$$

$$\mathcal{D}_\beta(t) = -\int_0^t \int_{\mathcal{E}} \beta_F(v,u,t)e^{-\int_u^t \beta_F(v,u,s)ds}(1-\psi(v,u))\phi(dv)du$$
$$+ \int_0^t \int_{\mathcal{E}} \beta_F(v,u,t)p(dv,du), \tag{2.30}$$

respectively, and the time varying coefficients $\mathcal{E}_\beta(t)$ and ξ_t determined by

$$\mathcal{E}_\beta(t) = \int_0^t \int_{\mathcal{E}} (\beta_F(v,u,t))^2 e^{-\int_u^t \beta_F(v,u,s)ds}(1-\psi(v,u))\phi(dv)du$$
$$- \int_{\mathcal{E}} \bar{\beta}_F(v,t)(1-\psi(v,t))\phi(dv), \tag{2.31}$$

$$\xi_t = \frac{\partial}{\partial T} f(0,T)|_{T=t} + k^1_\sigma(t) f(0,t), \tag{2.32}$$

respectively.

We shall show next that the stochastic quantities $\mathcal{E}^i_\sigma(t)$, $\mathcal{D}^i_\sigma(t)$ and $\mathcal{D}_\beta(t)$ form state variables of linearly mean reverting SDEs.

Proposition 2.5 *Under Assumption 2.3, the stochastic quantities $\mathcal{E}^i_\sigma(t)$, $\mathcal{D}^i_\sigma(t)$ and $\mathcal{D}^i_\beta(t)$ satisfy the SDEs,*

$$d\mathcal{E}^i_\sigma(t) = [(\bar{\sigma}^i_F(t, \mathrm{F}_t))^2 - 2\kappa^i_\sigma(t)\mathcal{E}^i_\sigma(t)]dt,$$

$$d\mathcal{D}^i_\sigma(t) = [\mathcal{E}^i_\sigma(t) - \kappa^i_\sigma(t)\mathcal{D}^i_\sigma(t)]dt + \bar{\sigma}^i_F(t, \mathrm{F}_t)(dW^i_t + \theta^i_t dt),$$

and

$$d\mathcal{D}_\beta(t) = [\mathcal{E}_\beta(t) - \kappa_\beta(t)\mathcal{D}_\beta(t)]\,dt + \int_{\mathcal{E}} \bar{\beta}_F(v,t)p(dv,dt).$$

Proof. Taking the differentials of the stochastic quantities (2.28), (2.29) and (2.30), the desired results are obtained. $\square$

A closed Markovian system will be obtained if, in addition, the stochastic market prices of Wiener process risk θ_t^i satisfy an SDE with coefficients depending on the state variables of this system. Thus, the short rate dynamics (2.27) can be described by a finite dimensional Markovian system with state vector (r_t, $f(t,T_1)$, $\ldots, f(t,T_z)$, θ_t^1, $\mathcal{E}_\sigma^1(t)$, $\mathcal{D}_\sigma^1(t)$, $\ldots$, θ_t^m, $\mathcal{E}_\sigma^m(t)$, $\mathcal{D}_\sigma^m(t)$, $\mathcal{D}_\beta(t))^\top$, for $z, m \in \mathbb{N}$. Note that the state variables $f(t,T_k), k \in (1,2,\ldots,z)$, satisfy the SDE (2.16) under the specifications of Assumption 2.3. Alternatively, an additional set of state variables maybe assumed in the Markovian structure to model the market prices of Wiener process risk, as these are the only state variables driving the dynamics of the discounted GOP.

In Assumption 2.3 we restrict the density of the market price of jump risk $\psi(v,t)$ to a deterministic function. For a general stochastic market price of jump risk, an approximate Markovianisation can be achieved, in the spirit of Chiarella & Nikitopoulos (2003).

For a $\bar{\sigma}_F^i$ function depending only on the short rate, we obtain a jump-diffusion extension of a short rate model, similar to the two-state Markovian representation studied by Ritchken & Sankarasubramanian (1995). A feasible parameterisation is $\bar{\sigma}_F^i(t,r_t) = cr_t^\gamma$, with c constant and $\gamma \geq 0$, see Ritchken & Sankarasubramanian (1995). Then, we obtain an extended Cox, Ingersoll & Ross (1985) model for $\gamma = 0.5$, while for $\gamma = 0$ we obtain an extended Vasicek model, see Vasicek (1977). The above proposed class of models incorporates jumps and describes the short rate evolution using a multi-dimensional Markovian representation. Such Markov short rate models can be treated by using numerical methods, see Chiarella & Nikitopoulos (2003). However, this class includes also rather tractable general multi-factor affine term structure models, see Filipović (2005), as will be discussed later.

3 Alternative Term Structure Models

In this section we present a class of parsimonious two-factor term structure models which recover the typical shapes of forward rates and lead to computationally tractable formulas for pricing zero-coupon bonds and options on zero-coupon bonds.

3.1 Forward Rate

The zero-coupon bond $P(t,T)$, evaluated in (2.9), can be expressed in terms of the discounted GOP $\bar{S}_t = \frac{S_t}{S_t^0}$ and the savings account (2.6) as

$$P(t,T) = E\left(\frac{S_t}{S_T}\,|\mathcal{A}_t\right) = E\left(\frac{\bar{S}_t}{\bar{S}_T}\exp\left\{-\int_t^T r_s ds\right\}|\mathcal{A}_t\right), \tag{3.33}$$

for $t \in [0,T]$, $T \in [0,\infty)$. For simplicity let us assume independence between the short rate and the discounted GOP. Then (3.33) yields

$$P(t,T) = E\left(\frac{\bar{S}_t}{\bar{S}_T}\,|\mathcal{A}_t\right) E\left(\exp\left\{-\int_t^T r_s ds\right\}|\mathcal{A}_t\right). \tag{3.34}$$

Note that the above independence is not a necessary condition for the approach. However, it yields tractable explicit pricing formulas for the particular models that we will consider. Alternatively, numerical methods could be employed in the more general case.

Using the definition (2.15), we express at time t the forward rate

$$f(t,T) = n(t,T) + \varrho(t,T), \tag{3.35}$$

as the sum of the *market price of risk contribution*

$$n(t,T) = -\frac{\partial}{\partial T}\ln\left[E\left(\frac{\bar{S}_t}{\bar{S}_T}\,|\mathcal{A}_t\right)\right], \tag{3.36}$$

and the *short rate contribution*

$$\varrho(t,T) = -\frac{\partial}{\partial T}\ln\left[E\left(e^{-\int_t^T r_s ds}\,|\mathcal{A}_t\right)\right]. \tag{3.37}$$

In the following sections we propose specific models for the two factors, the discounted GOP and the short rate, which will produce closed form expressions for forward rates.

3.2 Minimal Market Model

For simplicity, assume that the GOP follows continuous dynamics as is typical for denominations in major currencies including the US dollar. In this case, due to the continuity of the GOP, the market price of jump risk is zero. Thus, the GOP follows the dynamics (2.1) with $\psi(v,t) = 0$. From equation (2.6) the discounted GOP dynamics are given by

$$d\bar{S}_t = \bar{S}_{t-}\sum_{i=1}^m \theta_t^i \left(\theta_t^i dt + dW_t^i\right), \tag{3.38}$$

which solely depend on the total market price of risk $|\theta_t| = \sqrt{\sum_{i=1}^{m}(\theta_t^i)^2}$. The *net market trend* is defined by

$$\alpha_t = \bar{S}_t|\theta_t|^2, \tag{3.39}$$

and thus the total market price of risk can be expressed as

$$|\theta_t| = \sqrt{\frac{\alpha_t}{\bar{S}_t}}. \tag{3.40}$$

By using (3.39) and (3.40) in (3.38), Platen & Heath (2006) show that the discounted GOP is a time transformed squared Bessel process. The stylised minimal market model, proposed in Platen (2001) and Platen (2002), assumes that the net market trend α_t, $t \in [0,T]$, has the form

$$\alpha_t = \alpha_0 \exp\{\eta t\}, \tag{3.41}$$

where $\eta > 0$ is the constant *net growth rate* and $\alpha_0 > 0$ is an initial parameter. The discounted GOP $\bar{S}_t$ is then a time transformed squared Bessel process of dimension four with deterministic transformed time. By applying the explicit transition density of $\bar{S}_t$, the market price of risk contribution to the bond price, see (3.34), is obtained by the formula

$$M_t(T,\bar{S}_t) = E\left(\frac{\bar{S}_t}{\bar{S}_T}\,|\mathcal{A}_t\right) = 1 - \exp\left\{-\frac{2R(t,T)\bar{S}_t}{\alpha_t}\right\}, \tag{3.42}$$

with

$$R(t,T) = \frac{\eta}{\exp\{\eta(T-t)\}-1},$$

see Platen (2002). The market price of risk contribution function to the forward rate (3.36) follows then as

$$n(t,T) = \frac{2R(t,T)(\eta + R(t,T))\bar{S}_t}{\alpha_t[\exp\{\frac{2R(t,T)\bar{S}_t}{\alpha_t}\}-1]}. \tag{3.43}$$

The only stochastic parameter determining the market price of risk contribution (3.43) is the discounted GOP value $\bar{S}_t$ at time t. In Figure 3.3 the market price of risk contribution is displayed as a function of maturity and net growth rate $\eta \in [0.01, 0.25]$. The initial market price of risk is set to the value $\theta_0 = \sqrt{\frac{\alpha_0}{\bar{S}_0}} = 0.2$. The market price of risk contribution appears to be practically zero for short maturities and approaches asymptotically the value of the net growth rate η for extremely long maturities. This offers an interesting economic interpretation of the forward rate dynamics; at the short end only some contribution from the short rate is observed, where at the long end the net growth rate η of the equity market is added to the short rate contribution.

We note that under the above model, the Radon-Nikodym derivative process $\Lambda = \{\Lambda_t = \frac{S_t^0 S_0}{S_t}, t \in [0,\infty)\}$ for the candidate risk-neutral measure, see (2.7), is the inverse of a time transformed squared Bessel process of dimension four, which is a strict $(\underline{\mathcal{A}}, P)$-local martingale, see Revuz & Yor (1999). Since it is not a martingale, this demonstrates that an equivalent risk-neutral probability measure does not exist for the above model. For derivatives, real-world pricing can be applied using the real-world probability when evaluating the expectation in (2.5) with the GOP as numeraire.

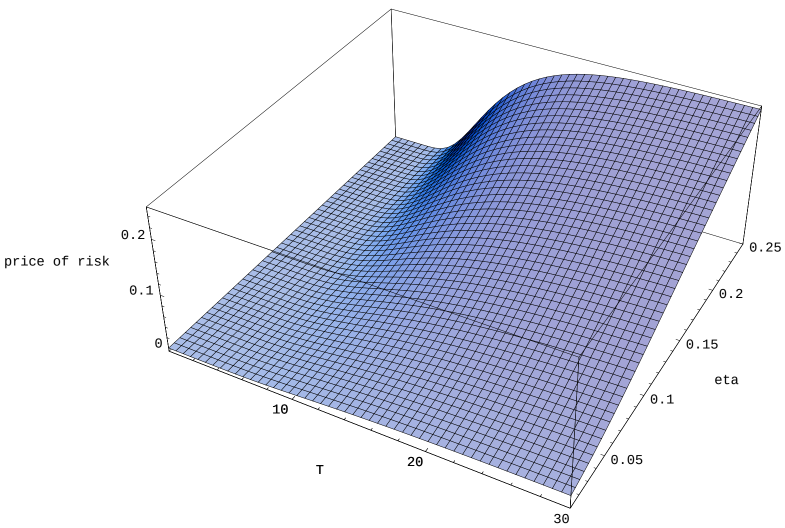

Figure 3.3: Market price of risk contribution

3.3 Affine Term Structure

Affine term structure models include the models proposed by Vasicek (1977), Cox, Ingersoll & Ross (1985), Hull & White (1990) and their affine extensions to the jump-diffusion case.

Definition 3.1 *For $T \in (0, \infty)$, a short rate process $r = \{r_t, t \in [0, T]\}$ generates an affine term structure if*

$$E\left(e^{-\int_t^T r_s ds} |\mathcal{A}_t\right) = e^{A(t,T)-B(t,T)r_t}, \tag{3.44}$$

for $t \in [0, T]$, where $A(t, T)$ and $B(t, T)$ are deterministic functions of time.

Note that the expectation in equation (3.44) is taken under the real-world probability measure and does not represent the bond price, see (3.34). Filipović (2005) provides under mild conditions a full characterization of multi-dimensional jump-diffusion Markovian factor processes that lead to affine term structures. For simplicity, we restrict our analysis to the case of the one-dimensional short rate process given by

$$dr_t = (\alpha_1(t) + \alpha_2(t)r_t)dt + \sqrt{\beta_1(t) + \beta_2(t)r_t}dW_t + \int_{\mathcal{E}} c\big(v,t\big)p(dv,dt). \quad (3.45)$$

Then, as shown in Appendix C, this short rate generates an affine term structure, where the functions $A(t,T)$ and $B(t,T)$ solve a two-dimensional system of ordinary differential equations. Consequently, the forward rate (3.35) can be expressed as

$$f(t,T) = n(t,T) - \frac{\partial A(t,T)}{\partial T} + \frac{\partial B(t,T)}{\partial T} r_t, \quad (3.46)$$

for $t \in [0,T]$. A trivial case arises for a deterministic short rate which yields the forward rate $f(t,T) = n(t,T) + r_T$, see (3.37) and (3.35). Thus, the forward rate dynamics are expressed as

$$df(t,T) = dn(t,T) + \left(-\frac{\partial^2 A(t,T)}{\partial t \partial T} + \frac{\partial^2 B(t,T)}{\partial t \partial T} r_t\right) dt + \frac{\partial B(t,T)}{\partial T} dr_t, \quad (3.47)$$

with the short rate dynamics described by (3.45). Two novelties are presented with the result (3.47). First, these are the dynamics under the real-world probability measure. Second, the forward rate evolution depends also on the market price of risk contribution $n(t,T)$, which is absent in any risk-neutral setting. Appendix D demonstrates that the forward rate dynamics (3.47) have indeed the jump-diffusion HJM form revealed in (2.20).

Two examples of affine term structures, which provide closed form solutions for the functions $A(t,T)$ and $B(t,T)$ appearing in (3.46), are presented in the next section.

3.3.1 Vasicek Model with Jumps

The short rate dynamics of the jump augmented Vasicek model are specified by the following SDE

$$dr_t = \kappa\big(\bar{r} - r_t\big)dt + \sigma dW_t + \int_{\mathcal{E}} v\, p(dv,dt), \quad (3.48)$$

where κ, $\bar{r}$, and σ are positive constants. The probability density $\frac{\phi(dv)}{\lambda}$ of the marks $v \in \mathcal{E} = \mathbb{R}$ is given by

$$f(v) = w h_1 e^{-h_1 v}\mathbf{1}_{\{v \geq 0\}} + (1-w)h_2 e^{h_2 v}\mathbf{1}_{\{v<0\}}, \quad (3.49)$$

where $h_1, h_2 > 0$, $w \in [0,1]$ and $\mathbf{1}_{\{.\}}$ is the indicator function defined by

$$\mathbf{1}_{\{v \in A\}} = \begin{cases} 1, & \text{for} \quad v \in A, \\ 0, & \text{for} \quad v \notin A, \end{cases}$$

for a given set $A \subseteq \mathcal{E}$. Note that at a jump time, according to (3.49), with probability w the sign of the jump is positive and its amplitude is exponentially distributed with mean $1/h_1$, while with probability $1-w$ the sign of the jump is negative and its amplitude is exponentially distributed with mean $1/h_2$. Under this model the short rate contribution function (3.37) can be obtained in closed form, since the expectation (3.44) is evaluated using

$$B(t,T) = \frac{1-e^{-\kappa(T-t)}}{\kappa}, \tag{3.50}$$

and

$$\begin{aligned} A(t,T) &= (\bar{r} - \frac{\sigma^2}{2\kappa^2})\Big(B(t,T) - (T-t)\Big) - \frac{\sigma^2}{4\kappa}B(t,T)^2 \\ &+ \lambda\Big(h_1 \frac{\ln\big(e^{\kappa(T-t)}(1+\kappa h_1) - 1\big) - \ln(\kappa h_1)}{(1+\kappa h_1)} \\ &+ h_2 \frac{\ln\big(1 - e^{\kappa(T-t)}(1-\kappa h_2)\big) - \ln(\kappa h_2)}{(1-\kappa h_2)} - (T-t)\Big). \end{aligned} \tag{3.51}$$

This particular choice for the distribution of the marks, and the corresponding closed form solution for the bond, generalizes that of Das & Foresi (1996), who consider the mark density (3.49) with $h_1 = h_2$. As noticed by Das & Foresi (1996), large negative jumps can lead to infinite bond prices. To prevent this to happen, the speed of mean reversion κ should be higher than the mean of the negative jumps, that is $\kappa h_2 > 1$. By considering $h_1 \neq h_2$, we still obtain a closed form solution for the function $A(t,T)$ without imposing any constraint on the mean of the positive jumps.

In Figure 3.4 we plot the forward rate (3.46) as function of maturity T and initial short rate $r_0 \in [0.04, 0.08]$. The net growth rate is set to $\eta = 0.05$ and the initial market price of risk is $\theta_0 = 0.2$. We have also used $\kappa = 2$, $\bar{r} = 0.05$, $\sigma = 0.1$, $w = 0.8$, $h_1 = 5$, $h_2 = 10$ and $\lambda = 0.2$.

Figure 3.4 shows that the short end of the forward rate curve is driven by the short rate dynamics, while the long end is determined by the market price of risk. A drawback of the Vasicek model is that its interest rates can become negative. A remedy for the diffusion case has been proposed in Miller & Platen (2005), where the short rate is truncated at zero and numerical integration has to be used to price a bond.

3.3.2 CIR Model with Jumps

Let us now restrict our attention to nonnegative affine short rate models, as defined in Definition 3.1, but with time-homogeneous functions $A(t,T) = A(T-t)$

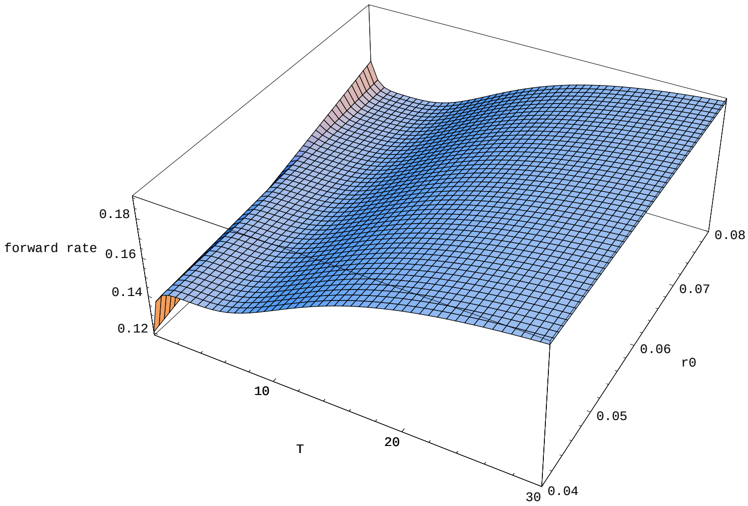

Figure 3.4: Forward rate curves for Vasicek model with jumps.

and $B(t,T) = B(T-t)$. Filipović (2001) provides a full characterization of these time-homogeneous models, which are of the CIR type, see Cox, Ingersoll & Ross (1985), with jump component. The case of exponentially distributed jumps and time independent coefficients leads to closed form solutions for the functions $A(t,T)$ and $B(t,T)$ in (3.44), as we will show next. We refer to Duffie & Garleanu (2001) for applications of this model to credit risk.

The short rate dynamics can be specified by the following SDE

$$dr_t = \kappa\big(\bar{r} - r_t\big)dt + \sigma\sqrt{r_t}\,dW_t + \int_{\mathcal{E}} v\,p(dv,dt), \tag{3.52}$$

where $\kappa, \bar{r}, \sigma$ are positive constants (with $2\kappa\bar{r} > \sigma^2$, to ensure strictly positive solutions to (3.52)) and the mark $v \in \mathcal{E}$ is exponentially distributed with mean $1/h$. We obtain

$$B(t,T) = \frac{L_1(T-t)}{L_2(T-t)}, \tag{3.53}$$

and

$$\begin{aligned} A(t,T) &= \frac{2\kappa\bar{r}}{\sigma^2}\ln\Big(\frac{L_3(T-t)}{L_2(T-t)}\Big) \\ &\quad + \frac{\lambda h}{1+\kappa h - 0.5\sigma^2 h^2}\ln\Big(e^{-\frac{t}{h}}\frac{L_1(T-t)+hL_2(T-t)}{hL_3(T-t)}\Big), \end{aligned} \tag{3.54}$$

where

$$\begin{aligned} L_1(t) &= 2(e^{\varpi_1 t}-1), \\ L_2(t) &= \varpi_1(e^{\varpi_1 t}+1) + \kappa(e^{\varpi_1 t}-1), \\ L_3(t) &= 2\varpi_1 e^{(\varpi_1+\kappa)t/2}, \end{aligned} \tag{3.55}$$

with $\varpi_1 = \sqrt{\kappa^2 + 2\sigma^2}$, see Filipović (2001). If one plots the forward rate (3.46), then similar forward rate curves as those shown in Figure 3.4 are obtained.

3.3.3 Bond Options

The pricing of bond options under the real-world probability measure can be treated by numerical approximation of the conditional real-world expectations. For instance, the value of a European call option $c(t,T,T^*)$, with maturity T and strike K, on a zero-coupon bond with maturity $T^* > T$ is given by

$$\begin{aligned} c(t,T,T^*) &= E\left(\frac{S_t}{S_T}(P(T,T^*)-K)^+|\mathcal{A}_t\right) \\ &= E\left(\frac{\bar{S}_t}{\bar{S}_T}\exp\left\{-\int_t^T r_s ds\right\}(P(T,T^*)-K)^+|\mathcal{A}_t\right), \end{aligned} \tag{3.56}$$

for all $t \in [0,T]$. Due to the presence of the discounted GOP, the stochastic short rate and the fact that an equivalent risk-neutral measure does not always exist this is not a straightforward approximation.

An alternative way is proposed next, using a conditional expectation over the path of the discounted GOP up to a fixed point in time. This methodology allows us to employ directly explicit formulas known from standard derivative pricing. As in Miller & Platen (2005), let us introduce the filtration $\mathcal{A}_t^{\bar{S}}$ generated by the $\mathcal{A}_t$ and the paths of the discounted GOP until time T^*, which is

$$\mathcal{A}_t^{\bar{S}} = \sigma\{\bar{S}_u \in [0,T^*]\} \cup \mathcal{A}_t.$$

Since $\mathcal{A}_t \subseteq \mathcal{A}_t^{\bar{S}}$, we obtain

$$c(t,T,T^*) = E\big(\frac{\bar{S}_t}{\bar{S}_T} M_T(T^*,\bar{S}_T)\Psi_{T,T^*}(t,r_t,\bar{S}_T)|\mathcal{A}_t\big), \tag{3.57}$$

where $M_T(T^*,\bar{S}_T) = 1 - \exp\left\{-\frac{2R(T,T^*)\bar{S}_T}{\alpha_T}\right\}$ is the market price of risk contribution to the bond price, see (3.42), $K(\bar{S}_T) = K/M_T(T^*,\bar{S}_T)$ and

$$\Psi_{T,T^*}(t,r_t,\bar{S}_T) = E\Big(\exp\left\{-\int_t^T r_s ds\right\}\big(e^{A(t,T)-B(t,T)r_t} - K(\bar{S}_T)\big)^+|\mathcal{A}_t^{\bar{S}}\Big). \tag{3.58}$$

The discounted GOP at time T, $\bar{S}_T$, is $\mathcal{A}_t^{\bar{S}}$-measurable, therefore the inner conditional expectation in (3.57) can be computed with the same formulas used to price a bond option with strike $K(\bar{S}_T)$ in a risk-neutral framework. For the CIR model with jumps, for instance, the inner conditional expectation in (3.57) can be evaluated, as in Filipović (2001), by

$$\begin{aligned} \Psi_{T,T^*}(t,r_t,\bar{S}_T) &= e^{A(t,T^*)-B(t,T^*)r_t}\, u_{T,T^*}(t,r_t,[0,r^*]) \\ &\quad - K(\bar{S}_T)e^{A(t,T)-B(t,T)r_t}\, u_{T,T}(t,r_t,[0,r^*]). \end{aligned} \tag{3.59}$$

Here $r^* = \big(A(T,T^*) - \ln(K(\bar{S}_T))\big)^+/B(T,T^*)$ and $u_{T,T^*}(t,x,[0,r^*])$ is a given distribution function. This distribution function can be characterized by its Laplace transform

$$\int_0^\infty e^{-sy} u_{T,T^*}(t,x,dy) = e^{A(t,T^*)-A(T,T^*)-\varsigma_1(T,s+B(T,T^*))} e^{-\big(\varsigma_2(T,s+B(T,T^*))-B(t,T^*)\big)x}.$$

Here

$$\begin{aligned}
\varsigma_1(t,s) =& \frac{L_4(t)s + L_1(t)}{L_5(t)s + L_2(t)},\\
\varsigma_2(t,s) =& -2\frac{\kappa\bar{r}}{\sigma^2}\ln\big(\frac{L_3(t)}{L_5(t)s + L_4(t)}\big)\\
& - \frac{c}{1+\kappa h + 0.5\sigma^2 h^2}\ln\Big(e^{-\frac{t}{h}}\frac{\big(L_4(t)+hL_5(t)\big)s + L_1(t) + hL_2(t)}{L_3(t)\big(s+h\big)}\Big),
\end{aligned}$$

with $L_4(t) = \varpi_1(e^{\varpi_1 t}+1) - \kappa(e^{\varpi_1 t}-1)$, $L_5(t) = \sigma^2(e^{\varpi_1 t}-1)$ and $L_1(t)$, $L_2(t)$, $L_3(t)$ defined in (3.55). Via Laplace inversion and for some parameter values, we can identify the distribution function $u_{T,T^*}(t,x,[0,r^*])$ in closed form. For instance, as shown in Filipović (2001), if $h < (\kappa+\varpi_1)/\sigma^2$ and

$$\frac{c}{1+\kappa h + 0.5\sigma^2 h^2} = n \in \mathbb{N},$$

then $u_{T,T^*}(t,x,[0,r^*])$ is a finite sum of convolutions of gamma distributions with a non-central chi-square distribution. The computation of the bond option under the real-world probability measure can be then performed numerically by using the transition density of the discounted GOP, see Platen & Heath (2006), when evaluating the expectation in (3.57). The price of the corresponding European put option can be obtained by using the fair put-call parity relationship, see Platen & Heath (2006), namely

$$p(t,T,T^*) = c(t,T,T^*) + KP(t,T) - P(t,T^*), \tag{3.60}$$

$t \in [0,T]$. Furthermore, the value of a cap can be obtained by expressing the cap as a portfolio of put options on zero-coupon bonds. It is also easy to add a deterministic shift to the above model, as proposed in Brigo & Mercurio (2005), which will allow automatic calibration to the initial yield curve.

Another interesting insight obtained from the proposed model is that instead of a change of probability measure, which has been traditionally employed for derivative pricing, a simple change of variables may provide significant computational advantages. As the existence of an equivalent probability measure is not required, a transformation of the benchmarked variables, guided by risk-neutral modelling principles, could be advantageous for discovering computationally tractable pricing formulas.

Acknowledgement

The authors would like to thank Carl Chiarella and Damir Filipović for their valuable comments and interesting discussions.

A Appendix: Proof of Proposition 2.2

By using the real-world dynamics (2.11) of the benchmarked bond price and the Itô formula, we obtain

$$
\begin{aligned}
d\ln(\hat{P}(t,T)) = &-\sum_{i=1}^{m}\left(\frac{1}{2}(\sigma^i(t,T))^2 dt + \sigma^i(t,T)dW_t^i\right) \\
&+ \int_{\mathcal{E}} \left(\beta(v,t,T) + \ln(1-\beta(v,t,T))\right)\phi(dv)dt + \int_{\mathcal{E}} \ln\left(1-\beta(v,t,T)\right) q(dv,dt).
\end{aligned}
\tag{A.61}
$$

Further, using definition (2.15) and (A.61) we obtain the forward rate dynamics under the real-world probability measure as

$$
\begin{aligned}
df(t,T) = &\sum_{i=1}^{m} \frac{\partial}{\partial T}\sigma^i(t,T)\left(\sigma^i(t,T)dt + dW_t^i\right) \\
&+ \int_{\mathcal{E}} \frac{\frac{\partial}{\partial T}\beta(v,t,T)}{1-\beta(v,t,T)}\left(\beta(v,t,T)\phi(dv)dt + q(dv,dt)\right),
\end{aligned}
\tag{A.62}
$$

for $t \in [0,T]$, which yields (2.20). Next we express the dynamics of the forward rates (A.62) in terms of the diffusion coefficients (2.18) and the jump coefficient (2.19), where we have

$$
\int_t^T \sigma_F^i(t,s)ds = \sigma^i(t,T) - \sigma^i(t,t), \tag{A.63}
$$

$$
\int_t^T \beta_F(v,t,s)ds = -\ln\left(\frac{1-\beta(v,t,T)}{1-\beta(v,t,t)}\right). \tag{A.64}
$$

Obviously, the bond volatilities (2.13) and (2.14) must satisfy the conditions

$$
\begin{aligned}
0 &= \tilde{\sigma}^i(t,t) = \theta_t^i - \sigma^i(t,t), \\
0 &= \tilde{\beta}(v,t,t) = \frac{\psi(v,t) - \beta(v,t,t)}{1-\psi(v,t)},
\end{aligned}
$$

for all $t \in [0,\infty)$. Therefore, by (A.63) and (A.64), the benchmarked bond volatilities are linked to the forward rate volatilities by the relations

$$
\sigma^i(t,T) = \int_t^T \sigma_F^i(t,s)ds + \theta_t^i, \tag{A.65}
$$

$$
\beta(v,t,T) = 1 - \exp\left(-\int_t^T \beta_F(v,t,s)ds\right)(1-\psi(v,t)), \tag{A.66}
$$

which yields (2.16). $\square$

B Appendix: Proof of Proposition 2.4

Using the dynamics of the forward rate (2.16), and setting $t = T$, we can use the identity $r_t = f(t,t)$ to obtain the short rate dynamics

$$r_t = f(0,t) + \int_0^t \mu_F(u,t)du + \sum_{i=1}^m \int_0^t \sigma_F^i(u,t)dW_u^i + \int_0^t \int_{\mathcal{E}} \beta_F(v,u,t)p(dv,du), \tag{B.67}$$

with μ_F as in (2.17). Thus

$$\begin{aligned} dr_t &= \frac{\partial}{\partial T} f(t,T)|_{T=t}dt + df(t,T)|_{T=t} \\ &= \left[\frac{\partial}{\partial T} f(0,T)|_{T=t} + \left(\int_0^t \frac{\partial}{\partial T}\mu_F(u,T)|_{T=t}du + \sum_{i=1}^m \int_0^t \frac{\partial}{\partial T}\sigma_F^i(u,T)|_{T=t}dW_u^i \right.\right. \\ &\qquad \left.\left. + \int_0^t \int_{\mathcal{E}} \frac{\partial}{\partial T}\beta_F(v,u,T)|_{T=t}p(dv,du)\right)\right] dt \\ &\quad + \mu_F(t,t)dt + \sum_{i=1}^m \sigma_F^i(t,t)dW_t^i + \int_{\mathcal{E}} \beta_F(v,t,t)p(dv,dt). \end{aligned} \tag{B.68}$$

Using the volatility specifications of Assumption 2.3, we obtain

$$\frac{\partial\sigma_F^i(u,T)}{\partial T}|_{T=t} = -k_\sigma^i(t)\sigma_F^i(u,t) \tag{B.69}$$

and

$$\frac{\partial\beta_F(v,u,T)}{\partial T}|_{T=t} = -k_\beta(t)\beta_F(v,u,t). \tag{B.70}$$

By introducing

$$V_\sigma^i(u,T) = \sigma_F^i(u,T)\int_u^T \sigma_F^i(u,s)ds, \tag{B.71}$$

and

$$V_\beta(v,u,T) = \beta_F(v,u,T)\exp\left(-\int_u^T \beta_F(v,u,s)ds\right), \tag{B.72}$$

we have by (2.17) that

$$\mu_F(u,T) = \sum_{i=1}^m V_\sigma^i(u,T) + \sum_{i=1}^m \sigma_F^i(u,T)\theta_u^i - \int_{\mathcal{E}} V_\beta(v,u,T)(1-\psi(v,u))\phi(dv). \tag{B.73}$$

Also note that by (2.25), for $i \in \{1,2,\ldots,m\}$,

$$\frac{\partial V_\sigma^i(u,T)}{\partial T}|_{T=t} = -k_\sigma^i(t)V_\sigma^i(u,t) + (\sigma_F^i(u,t))^2, \tag{B.74}$$

and by (2.26)

$$\frac{\partial V_\beta(v,u,T)}{\partial T}|_{T=t} = -k_\beta(t)V_\beta(v,u,t) - (\beta_F(v,u,t))^2 e^{-\int_u^t \beta_F(v,u,s)ds}. \tag{B.75}$$

Using the above results, the dynamics of the spot rate (B.68) are expanded to

$$\begin{aligned} dr_t = &\left\{ \frac{\partial}{\partial T} f(0,T)|_{T=t} + \int_0^t \sum_{i=1}^m \left[-k_\sigma^i(t)V_\sigma^i(u,t) + (\sigma_F^i(u,t))^2 - k_\sigma^i(t)\sigma_F^i(u,t)\theta_u^i \right] du \right. \\ &+ \int_0^t \int_{\mathcal{E}} \left[\left\{ k_\beta(t)V_\beta(v,u,t) + (\beta_F(v,u,t))^2 e^{-\int_u^t \beta_F(v,u,s)ds} \right\} (1-\psi(v,u))\phi(dv) \right] du \\ &\left. - \sum_{i=1}^m k_\sigma^i(t) \int_0^t \sigma_F^i(u,t)dW_u^i - k_\beta(t)\int_0^t \int_{\mathcal{E}} \beta_F(v,u,t)p(dv,du) \right\} dt \\ &+ \left(\sum_{i=1}^m \sigma_F^i(t,t)\theta_t^i - \int_{\mathcal{E}} \beta_F(v,t,t)(1-\psi(v,t))\phi(dv) \right) dt \\ &+ \sum_{i=1}^m \sigma_F^i(t,t)dW_t^i + \int_{\mathcal{E}} \beta_F(v,t,t)p(dv,dt). \end{aligned} \tag{B.76}$$

By using the definitions (2.28), (2.29), (2.30) and (2.31), (B.76) reduces to

$$\begin{aligned} dr_t = &\left\{ \frac{\partial}{\partial T} f(0,T)|_{T=t} + \mathcal{E}_\beta(t) + \sum_{i=1}^m \mathcal{E}_\sigma^i(t) - \sum_{i=1}^m k_\sigma^i(t)\mathcal{D}_\sigma^i(t) - k_\beta(t)\mathcal{D}_\beta(t) \right. \\ &\left. + \sum_{i=1}^m \bar{\sigma}_F^i(t,\mathrm{F}_t)\theta_t^i \right\} dt + \sum_{i=1}^m \bar{\sigma}_F^i(t,\mathrm{F}_t)dW_t^i + \int_{\mathcal{E}} \bar{\beta}_F(v,t)p(dv,dt). \end{aligned} \tag{B.77}$$

Further from definitions (2.29) and (2.30), (B.67) yields

$$\mathcal{D}_\sigma^1(t) = r_t - f(0,t) - \sum_{i=2}^m \mathcal{D}_\sigma^i(t) - \mathcal{D}_\beta(t), \tag{B.78}$$

which reduces (B.77) to (2.27). The above volatility parameterisation also produces a variety of well-known short rate term structure models. □

C Appendix: Affine Term Structure

Let the short rate r_t follow the one-dimensional affine dynamics in (3.45). Consider the functions $A(t,T)$ and $B(t,T)$ which solve the system of ordinary differential equations

$$\frac{\partial B(t,T)}{\partial t} + \alpha_2(t)B(t,T) - \frac{1}{2}\beta_2(t)B^2(t,T) = -1, \tag{C.79}$$

$$B(T,T) = 0,$$

$$\frac{\partial A(t,T)}{\partial t} - \alpha_1(t)B(t,T) + \frac{1}{2}\beta_1(t)B^2(t,T) - \int_{\mathcal{E}} (1 - e^{-B(t,T)c(v,t)})\phi(dv) = 0, \tag{C.80}$$

$$A(T,T) = 0.$$

Then by the Feynman-Kac theorem it follows that the functional

$$u(t,x) := E\left(e^{-\int_t^T r_s ds}\middle| r_t = x\right), \tag{C.81}$$

satisfies the partial-integro differential equation

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + \mathcal{L}u(t,x) - u(t,x)x = 0, \\ u(T,x) = 1, \end{cases} \tag{C.82}$$

for $(t,x) \in [0,T] \times \mathbb{R}$, with

$$\begin{aligned} \mathcal{L}u(t,x) &= (\alpha_1(t) + \alpha_2(t)x)\frac{\partial u(t,x)}{\partial x} + \frac{\beta_1(t) + \beta_2(t)x}{2}\frac{\partial^2 u(t,x)}{\partial x^2} \\ &\quad + \int_{\mathcal{E}} \Big(u(t, x + c(t,x,v)) - u(t,x)\Big)\phi(dv). \end{aligned}$$

It is easy to show that the functional (3.44) satisfies (C.82), and thus generates an affine term structure, where the expectation in (3.44) is taken under the real-world probability measure. Such a result was presented in Björk, Kabanov & Runggaldier (1997) under a risk-neutral probability measure.

D Appendix: Forward Rate Equation

The assumed independence between the discounted GOP and the short rate implies that only W_t^1 is driving the $\bar{S}_t$ dynamics (3.38), and W_t^2 is driving the r_t dynamics (3.45). Thus (using also (3.40))

$$d\bar{S}_t = \bar{S}_t\theta_t^1\left(\theta_t^1 dt + dW_t^1\right) = \alpha_t dt + \sqrt{\bar{S}_t\alpha_t}dW_t^1, \tag{D.83}$$

$$dr_t = (\alpha_1(t) + \alpha_2(t)r_t)dt + \sqrt{\beta_1(t) + \beta_2(t)r_t}dW_t^2 + \int_{\mathcal{E}} c\big(v,t\big)p(dv,dt). \tag{D.84}$$

Note that $\theta_t^2 = 0$ and $\psi(v,t) = 0$. Taking the forward rate dynamics (3.47) and substituting the short rate dynamics (D.84) we obtain

$$\begin{aligned} df(t,T) = dn(t,T) &+ \left(-\frac{\partial^2 A(t,T)}{\partial t \partial T} + \frac{\partial^2 B(t,T)}{\partial t \partial T}r_t + \frac{\partial B(t,T)}{\partial T}(\alpha_1(t) + \alpha_2(t)r_t)\right)dt \\ &+ \frac{\partial B(t,T)}{\partial T}\sqrt{\beta_1(t) + \beta_2(t)r_t}dW_t^2 + \int_{\mathcal{E}} \frac{\partial B(t,T)}{\partial T}c\big(v,t\big)p(dv,dt). \end{aligned} \tag{D.85}$$

By the Itô formula and (D.83), the $n(t,T)$ dynamics are expressed as

$$dn(t,T) = \left(\frac{\partial n(t,T)}{\partial t} + \frac{1}{2}\bar{S}_t\alpha_t\frac{\partial^2 n(t,T)}{\partial \bar{S}_t^2} + \frac{\partial n(t,T)}{\partial \bar{S}_t}\alpha_t\right)dt + \frac{\partial n(t,T)}{\partial \bar{S}_t}\sqrt{\bar{S}_t\alpha_t}dW_t^1. \tag{D.86}$$

Therefore, (D.85) becomes

$$
\begin{aligned}
df(t,T) = \Bigg(& \frac{\partial n(t,T)}{\partial t} + \frac{\bar{S}_t\alpha_t}{2}\frac{\partial^2 n(t,T)}{\partial \bar{S}_t^2} + \frac{\partial n(t,T)}{\partial \bar{S}_t}\alpha_t \\
& - \frac{\partial^2 A(t,T)}{\partial t \partial T} + \frac{\partial^2 B(t,T)}{\partial t\partial T} r_t + \frac{\partial B(t,T)}{\partial T}\big(\alpha_1(t) + \alpha_2(t) r_t\big)\Bigg)\, dt \qquad \text{(D.87)} \\
+ \frac{\partial n(t,T)}{\partial \bar{S}_t} & \sqrt{\bar{S}_t\alpha_t}dW_t^1 + \frac{\partial B(t,T)}{\partial T}\sqrt{\beta_1(t) + \beta_2(t) r_t}dW_t^2 + \int_{\mathcal{E}} \frac{\partial B(t,T)}{\partial T} c\big(v,t\big) p(dv,dt).
\end{aligned}
$$

Then by (2.16), (D.83) and (D.84), the forward rate volatilities are expressed as

$$
\sigma_F^1(t,T) = \frac{\partial n(t,T)}{\partial \bar{S}_t}\sqrt{\bar{S}_t\alpha_t}, \qquad \text{(D.88)}
$$

$$
\sigma_F^2(t,T) = \frac{\partial B(t,T)}{\partial T}\sqrt{\beta_1(t) + \beta_2(t) r_t}, \qquad \text{(D.89)}
$$

$$
\beta_F(v,t,T) = \frac{\partial B(t,T)}{\partial T} c\big(v,t\big), \qquad \text{(D.90)}
$$

where the drift should be expressed as shown in (2.17). For this purpose we define

$$
H(t,T,\bar{S}_t) := \ln\left[E\left(\frac{\bar{S}_t}{\bar{S}_T}|\mathcal{A}_t\right)\right] = \ln\left[1 - \exp\left\{-\frac{2R(t,T)\bar{S}_t}{\alpha_t}\right\}\right], \qquad \text{(D.91)}
$$

which yields

$$
n(t,T) = -\frac{\partial H(t,T,\bar{S}_t)}{\partial T}. \qquad \text{(D.92)}
$$

Therefore (note that $H(t,t,\bar{S}_t) = 0$)

$$
\int_t^T \sigma_F^1(t,s)ds = \int_t^T \frac{\partial n(t,s)}{\partial \bar{S}_t} ds \sqrt{\bar{S}_t\alpha_t} = -\sqrt{\bar{S}_t\alpha_t}\frac{\partial H(t,T,\bar{S}_t)}{\partial \bar{S}_t}. \qquad \text{(D.93)}
$$

Using (3.40) we obtain

$$
\sigma_F^1(t,T)\left(\int_t^T \sigma_F^1(t,s)ds + \theta_t^1\right) = -\bar{S}_t\alpha_t\frac{\partial H(t,T,\bar{S}_t)}{\partial \bar{S}_t}\frac{\partial n(t,T)}{\partial \bar{S}_t} + \alpha_t\frac{\partial n(t,T)}{\partial \bar{S}_t}, \qquad \text{(D.94)}
$$

where, by differentiation of $n(t,T)$, it can be shown that

$$
-\bar{S}_t\alpha_t\frac{\partial H(t,T,\bar{S}_t)}{\partial \bar{S}_t}\frac{\partial n(t,T)}{\partial \bar{S}_t} = \frac{\partial n(t,T)}{\partial t} + \frac{\bar{S}_t\alpha_t}{2}\frac{\partial^2 n(t,T)}{\partial \bar{S}_t^2}. \qquad \text{(D.95)}
$$

Thus the drift, resulting from the $n(t,T)$ dynamics in (D.87), is of the type as the one in (2.17). In addition, from (D.89) and (D.90)

$$
\int_t^T \sigma_F^2(t,x)dx = B(t,T)\sqrt{\beta_1(t) + \beta_2(t) r_t}, \qquad \text{(D.96)}
$$

$$
\int_t^T \beta_F(v,t,x)dx = B(t,T)c\big(v,t\big). \qquad \text{(D.97)}
$$

By using (D.96) and (D.89)

$$\sigma_F^2(t,T)\int_t^T \sigma_F^2(t,x)dx = \frac{\partial B(t,T)}{\partial T} B(t,T)[\beta_1(t)+\beta_2(t)r_t]. \tag{D.98}$$

By differentiating (C.79) and (C.80) we obtain

$$\frac{\partial^2 B(t,T)}{\partial T \partial t} + \alpha_2(t)\frac{\partial B(t,T)}{\partial T} = \beta_2(t)B(t,T)\frac{\partial B(t,T)}{\partial T}, \tag{D.99}$$

$$\begin{aligned} -\frac{\partial^2 A(t,T)}{\partial T \partial t} + \alpha_1(t)\frac{\partial B(t,T)}{\partial T} &= \beta_1(t)B(t,T)\frac{\partial B(t,T)}{\partial T} \\ &\quad - \int_{\mathcal{E}} c(v,t)\frac{\partial B(t,T)}{\partial T} e^{-B(t,T)c(v,t)}\phi(dv). \end{aligned} \tag{D.100}$$

Thus from (D.99), (D.100) and (D.98) we obtain

$$\begin{aligned} &-\frac{\partial^2 A(t,T)}{\partial t \partial T} + \frac{\partial^2 B(t,T)}{\partial t \partial T} r_t + \frac{\partial B(t,T)}{\partial T}(\alpha_1(t)+\alpha_2(t)r_t) \\ &= \frac{\partial B(t,T)}{\partial T} B(t,T)[\beta_1(t)+\beta_2(t)r_t] - \int_{\mathcal{E}} c(v,t)\frac{\partial B(t,T)}{\partial T} e^{-B(t,T)c(v,t)}\phi(dv) \\ &= \sigma_F^2(t,T)\int_t^T \sigma_F^2(t,x)dx - \int_{\mathcal{E}} c(v,t)\frac{\partial B(t,T)}{\partial T} e^{-B(t,T)c(v,t)}\phi(dv). \end{aligned} \tag{D.101}$$

Using (D.88), (D.89), (D.90), (D.94), (D.95) and (D.101) in (D.87) and considering that $\theta_t^2 = 0$ and $\psi(v,t) = 0$, we derive

$$df(t,T) = \mu_F(t,T)dt + \sigma_F^1(t,T)dW_t^1 + \sigma_F^2(t,T)dW_t^2 + \int_{\mathcal{E}} \beta_F(v,t,T)p(dv,dt), \tag{D.102}$$

where

$$\begin{aligned} \mu_F(t,T) =& \sigma_F^1(t,T)\left(\int_t^T \sigma_F^1(t,s)ds + \theta_t^1\right) + \sigma_F^2(t,T)\left(\int_t^T \sigma_F^2(t,s)ds\right) \\ & - \int_{\mathcal{E}} \beta_F(v,t,T)\exp\left(-\int_t^T \beta_F(v,t,s)ds\right)\phi(dv). \end{aligned} \tag{D.103}$$

This shows that the forward rate dynamics (D.87) reveal the jump-diffusion HJM SDE (2.16).

References

Ait-Sahalia, Y. (1996). Testing continuous-time models for the spot interest rate. *Rev. Financial Studies* **9**, 385–426.

Bajeux-Besnainou, I. & R. Portait (1997). The numeraire portfolio: A new perspective on financial theory. *The European Journal of Finance* **3**, 291–309.

Björk, T., Y. Kabanov, & W. Runggaldier (1997). Bond market structure in the presence of marked point processes. *Math. Finance* **7**, 211–239.

Brigo, D. & F. Mercurio (2005). *Interest Rate Models - Theory and Practice* (2nd ed.). Springer Finance.

Chan, K. C., G. A. Karolyi, F. A. Longstaff, & A. B. Sanders (1992). An empirical comparison of alternative models of the short-term interest rate. *J. Finance* **47**, 1209–1227.

Chiarella, C. & C. S. Nikitopoulos (2003). A class of jump-diffusion bond pricing models within the HJM framework with state dependent volatilities. *Asia-Pacific Financial Markets* **10**(2-3), 87–127.

Christensen, M. M. & E. Platen (2005). A general benchmark model for stochastic jump sizes. *Stochastic Analysis and Applications* **23**(5), 1017–1044.

Cox, J. C., J. E. Ingersoll, & S. A. Ross (1985). A theory of the term structure of interest rates. *Econometrica* **53**, 385–407.

Das, S. R. (2000). *Interest Rate Modelling with Jump-Diffusion Processes.* Chapter 6 in Advanced Fixed-Income Valuation Tools by N. Jegadeesh and B. Tuckman, John Wiley & Sons.

Das, S. R. (2002). The surprise element: Jumps in the interest rates. *Journal of Econometrics* **106**, 27–65.

Das, S. R. & S. Foresi (1996). Exact solutions for bond and option prices with systematic jump risk. *Review of Derivatives Research* **1**, 7–24.

Duffie, D. & N. Garleanu (2001). Risk and valuation of collateralized debt obligations. *Financial Analysts Journal* **57**(1), 41–59.

Filipović, D. (2001). A general characterization of one factor affine term structure models. *Finance and Stochastics* **5**, 389–412.

Filipović, D. (2005). Time-inhomogeneous affine processes. *Stochastic Processes and their Applications* **11**, 639–659.

Goll, T. & J. Kallsen (2000). Optimal portfolios for logarithmic utility. *Stochastic Processes and their Applications* **89**, 31–48.

Harrison, J. M. & D. M. Kreps (1979). Martingale and arbitrage in multiperiod securities markets. *J. Economic Theory* **20**, 381–408.

Heath, D., R. Jarrow, & A. Morton (1992). Bond pricing and the term structure of interest rates: A new methodology for contingent claim valuation. *Econometrica* **60**(1), 77–105.

Hull, J. & A. White (1990). Pricing interest rate derivative securities. *Rev. Financial Studies* **3**, 573–592.

Johannes, M. (2004). The statistical and economic role of jumps in continuous-time interest rate models. *Journal of Finance* **59**(1), 227–260.

Karatzas, I. & C. Kardaras (2007). The numéraire portfolio in semimartingale financial models. *Finance and Stochastics*. To appear.

Kelly, J. R. (1956). A new interpretation of information rate. *Bell Syst. Techn. J.* **35**, 917–926.

Korn, R., F. Oertel, & M. Schäl (2003). The numeraire portfolio in financial markets modeled by a multi-dimensional jump diffusion process. *Decisions in Economics and Finance* **26**(2), 153–166.

Le, T. & E. Platen (2006). Approximating the growth optimal portfolio with a diversified world stock index. *Journal of Risk Finance* **7**(5), 559–574.

Long, J. B. (1990). The numeraire portfolio. *J. Financial Economics* **26**, 29–69.

Miller, S. & E. Platen (2005). A two-factor model for low interest rate regimes. *Asia-Pacific Financial Markets* **11**(1), 107–133.

Platen, E. (2001). A minimal financial market model. In *Trends in Mathematics*, pp. 293–301. Birkhäuser.

Platen, E. (2002). Arbitrage in continuous complete markets. *Adv. in Appl. Probab.* **34**(3), 540–558.

Platen, E. (2004a). A class of complete benchmark models with intensity based jumps. *J. Appl. Probab.* **41**, 19–34.

Platen, E. (2004b). Modeling the volatility and expected value of a diversified world index. *Int. J. Theor. Appl. Finance* **7**(4), 511–529.

Platen, E. & D. Heath (2006). *A Benchmark Approach to Quantitative Finance.* Springer Finance. Springer.

Revuz, D. & M. Yor (1999). *Continuous Martingales and Brownian Motion* (3rd ed.). Springer.

Ritchken, D. & L. Sankarasubramanian (1995). Volatility structures of forward rates and the dynamics of the term structure. *Math. Finance* **5**(1), 55–72.

Ross, S. A. (1976). The arbitrage theory of capital asset pricing. *J. Economic Theory* **13**, 341–360.

Shirakawa, H. (1991). Interest rate option pricing with Poisson-Gaussian forward rate curve processes. *Math. Finance* **1**, 77–94.

Vasicek, O. (1977). An equilibrium characterisation of the term structure. *Journal of Financial Economics* **5**, 177–188.