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Pricing of Defaultable Securities under Stochastic Interest

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Abstract. We reduce the problem of pricing *continuously monitored* defaultable securities (namely, barrier type options, corporate debts) under a stochastic interest rate framework to calculations of boundary crossing probabilities (BCP) for Brownian Motion (BM) with stochastic boundaries. For the case when the interest rate is governed by linear stochastic equation (Vasicek model) we suggest a numerical algorithm for calculation of BCP based on a piece-wise linear approximation for the stochastic boundaries. We also provide an estimation for a rate of convergence of the suggested approximation as a function of number of nodes and illustrate the results by numerical examples.

1. Introduction. Practitioners often acknowledge an existence of a common problem with pricing schemes of exotic (e.g. barrier, lookback) options for the contracts with long maturities. The prices of these instruments significantly depend on an interest rate term-structure. For the case of deterministic interest rates some results in this direction have been obtained by Roberts/Shortland (1997), Novikov/Frishling/Kordzakhia (1999,2003) (see also other references in these papers). For deterministic interest rates the pricing problem of barrier options is reduced to calculations of BCP for BM with deterministic boundaries. In Section 2 we present a modification of the algorithm from [9] and [10] to handle a general setup of stochastic interest rates. In Sections 3 and 4 we describe the numerical algorithm in details and provide an estimation for a rate of convergence of the suggested approximation as a function of number of nodes. In Section 5 the results are illustrated by numerical examples.

Further we use the standard notation S_t for a price of an underlying asset and r_t for a default-free short interest rate. We assume that S_t is a diffusion process of the form

$$S_t = S_0 e^{Y_t}$$

with the log-return process Y_t governed by the equation

$$dY_t = \mu(t)dt + \sigma dW_t,$$

where W_t is a standard BM with respect to a 'real-world' measure P and a filtration $\mathcal{F}_t$, σ is a constant volatility, $\mu(t)$ is a historical trend.

Let f_T be a payoff of an option at maturity T . In this paper we concentrate on two fundamental examples of payoff functions (there are many other examples that may be treated in a similar way).

Case 1. The payoff of *Up-and-Out European Barrier Call* option is

$$f_T = (S_T - K)^+ I\{\tau > T\},$$

where T is a maturity time, a default time $\tau = \inf\{t : S_t \geq G_+(t)\}$, $G_+(t)$ is a continuous *deterministic* barrier, and $I\{\tau > T\}$ is an indicator function.

Case 2. The payoff at maturity T of a *defaultable zero-coupon bond* S_t is

$$f_T = 1 - wI\{\tau \leq T\}, \quad 0 < w = \text{const} < 1,$$

where $\tau = \inf\{t : Y_t \leq b_t\}$, the default threshold b_t is defined as follows

$$b_t = b_0 + \lambda \int_0^t (Y_s - v - b_s) ds,$$

with some constants b_0 , v and $\lambda > 0$. An economic rationale behind the model is well justified in Collin-Dufresne/Goldstein (2001).

Put for convenience

$$b_t = Y_t - l_t.$$

Then

$$\tau = \inf\{t : l_t \leq 0\}, \quad dl_t = \mu(t)dt + \lambda(v - l_t)dt + \sigma dW_t. \quad (1)$$

Under the assumption $\lambda = 0$ this model has been studied by Longstaff - Schwartz (1997).

2. Pricing formulae. Under a free-arbitrage assumption a fair price of an option with a payoff f_T is

$$C_f = E^* \left(e^{-\int_0^T r_s ds} f_T \right), \quad (2)$$

where E^* is a symbol of expectation with respect to the equivalent risk-neutral (martingale) measure P^* such that the process

$$e^{-\int_0^t r_s ds} S_t$$

is $(P^*, \mathcal{F}_t)$ -martingale. It implies (see e.g. Liptser/Shiryaev (2003)) that

$$Y_t = \sigma W_t^* + \int_0^t r_s ds - \frac{\sigma^2}{2} t, \quad (3)$$

where W_t^* is a standard BM with respect P^* .

In general, the fair price (2) can be approximated by Monte Carlo methods via discretization of a time parameter. However, such algorithms typically have a significant bias; this aspect is discussed in Section 5 for the case of *discretely monitored* options.

The price of a risk-free zero-coupon bond with maturity T is

$$P(0, T) = E^*(e^{-\int_0^T r_s ds})$$

Define the *forward measure* P^F as follows:

$$P^F(A) = E^*(I(A)e^{-\int_0^T r_s ds})/P(0, T), \quad A \in \mathcal{F}_T.$$

Then

$$C_f = E^* \left(e^{-\int_0^T r_s ds} f_T \right) = P(0, T) E^F(f_T), \quad (4)$$

where E^F is a symbol of expectation with respect to P^F .

For affine models (e.g. Vasicek, CIR models) the function $P(0, T)$ can be calculated analytically. Therefore, a pricing problem is reduced to evaluation of $E^F(f_T)$. Further we assume that the risk-free short rate r_t is a Ornstein-Uhlenbeck (or, Vasicek) process governed by a linear stochastic equation

$$dr_t = a_r(\bar{r} - r_t)dt + \sigma_r dW_t^{(r)}, \quad (5)$$

where a parameter $a_r > 0$, $W_t^{(r)}$ is another standard BM with respect to the risk-neutral measure P^* and given filtration $\mathcal{F}_t$ such that $E^*(W_t^{(r)} W_t^*) = 0$. A general case of correlated S_t and r_t we plan to discuss in a separate paper.

The solution of (5) has the following representation

$$r_t = a(t) + \sigma_r \xi_t, \quad a(t) = \bar{r} + (r_0 - \bar{r})e^{-a_r t},$$

where

$$d\xi_t = -a_r \xi_t dt + dW_t^{(r)}, \quad \xi_0 = 0$$

and, therefore,

$$\xi_t = e^{-a_r t} \int_0^t e^{a_r s} dW_s^{(r)}. \quad (6)$$

By direct calculations

$$P(0, T) = E^* (e^{-\int_0^T r_s ds}) = e^{-A(T) + D^2(T)/2}$$

with

$$\begin{aligned} A(T) &= \int_0^T a(s) ds = \bar{r}T + (r_0 - \bar{r})(1 - e^{-a_r T})/a_r, \\ D(T) &= \sigma_r^2 \text{Var}(\int_0^T \xi_s ds) = \frac{\sigma_r^2}{2a_r^3} (2Ta_r - e^{-2Ta_r} - 4a_r^2 + 4a_r^2 e^{-Ta_r} + 1). \end{aligned}$$

Also, by the Girsanov theorem one could show that under the forward measure P^F the BM $W_t^{(r)}$ has a drift term

$$q(t, T) = -\frac{\sigma_r}{a_r} \int_0^t (1 - e^{-a_r(T-u)}) du \quad (7)$$

(see e.g. Brigo/Mercurio (2006), p. 886).

Case 1. Using (3) we obtain

$$\tau = \inf\{t : S_t > G_+(t)\} = \inf\{t : \sigma W_t^* > \frac{\sigma^2}{2}t + \log(G_+(t)/S_0) - \int_0^t r_s ds\}.$$

The fair price C_f can be written as follows

$$C_f = E^*[e^{-\int_0^T r_s ds} (S_T - K) I\{\tau > T, S_T > K\}].$$

In view of (4) and (7) we have

$$\begin{aligned} &= S_0 [E^* (e^{\sigma W_T^* - \frac{\sigma^2}{2}T} I\{\tau > T, S_T > K\}) - KP(0, T)P^F\{\tau > T, S_T > K\}] = \\ &S_0 q_1 - P(0, T)Kq_2 \end{aligned} \quad (8)$$

with

$$q_1 = \tilde{P}\{\sigma W_t^* < \frac{\sigma^2}{2}t + \log(G_+(t)/S_0) - \int_0^t r_s ds, \sigma W_T^* > \frac{\sigma^2}{2}T + \int_0^T r_s ds + \log(K/S_0)\},$$

where the measure $\tilde{P}$ is defined as follows

$$\tilde{P}(A) = E^*(I\{A\}e^{\sigma W_T^* - \frac{\sigma^2}{2}T}), \quad A \in \mathcal{F}_T.$$

Note that, the BM W_t^* has a drift σt under the measure $\tilde{P}$.

The second probability in (8) has a similar form:

$$q_2 = P^F\{\sigma W_t^* < \frac{\sigma^2}{2}t + \log(G_+(t)/S_0) - \int_0^t r_s ds, \sigma W_T^* > \frac{\sigma^2}{2}T + \int_0^T r_s ds + \log(K/S_0)\},$$

(recall that under P^F the BM W_t^r has the drift $q(t, T)$ defined in (7)).

Case 2. We have

$$C_f = E^* e^{-\int_0^T r_s ds} (1 - w + wI\{\tau > T\}) = P(0, T)[1 - w + P^F\{l_t \geq 0, t \leq T\}],$$

where due to (3) and (1) we have

$$dl_t = r_t dt + \lambda(v - l_t)dt + \sigma dW_t^*, \quad l_0 = -b_0.$$

With a simple algebra one can show that the problem of computation of $P^F\{l_t \geq 0, t \leq T\}$ can be reduced to finding of probabilities

$$P\{\eta_t < g(t) - e^{-\lambda t} \int_0^t e^{\lambda s} \xi_s ds, t \leq T\},$$

where $g(t)$ is a deterministic function, ξ_s is defined in (6) and η_t is a Ornstein-Uhlenbeck process defined by the equation

$$d\eta_t = -\lambda\eta_t dt + \sigma dW_t, \eta_0 = 0$$

and therefore

$$\eta_t = \sigma e^{-\lambda t} \int_0^t e^{\lambda s} dW_s.$$

Applying the change of time

$$u = \int_0^t e^{2\lambda s} ds$$

we obtain

$$\int_0^{t(u)} e^{\lambda s} dW_s = \tilde{W}_u, t(u) = \frac{\log(2\lambda u + 1)}{2\lambda},$$

where $\tilde{W}_u$ is another standard BM. Hence, we can reduce the pricing problem to computation of the probability

$$P\{\sigma W_u < \sqrt{2\lambda u + 1} g(t(u)) - \int_0^{t(u)} e^{\lambda s} \xi_s ds, u \leq \int_0^T e^{2\lambda s} ds\}.$$

3. Approximations to BCP. For **Cases 1** and **2** discussed above we have reduced the option pricing problem to finding BCP of the form

$$P(b, h) := P\{W_T > b; W_t < h(t), t \leq T\},$$

where b is a random variable, $h(t)$ is a stochastic boundary such that

$$h(t) = g(t) - zR_t, R_t = \int_0^t \xi_s ds,$$

$g(t)$ is smooth deterministic function, $z = \text{const}$, ξ_t has the form (6) with a standard BM W_t .

Further we study **Case 1** in somewhat more details. Note that for this case $\lambda = 0$ and for probabilities $q_i = P(b_i, h_i), i = 1, 2$ defined in (8) we have

$$\sigma b_i = \sigma_r \int_0^T \xi_s ds + H_i,$$

$$\sigma h_i(t) = g_i(t) - \sigma_r R_t^{(i)},$$

where

$$g_1(t) = -\frac{\sigma^2}{2}t + \log(G_+(t)/S_0) - \int_0^t a(s)ds,$$

$$H_1 = -\frac{\sigma^2}{2}T + \int_0^T a(s)ds + \log(K/S_0), R_t = \int_0^t \xi_s ds$$

ξ_t has the form (6) with a standard BM $W_t^{(r)}$;

$$g_2(t) = \frac{\sigma^2}{2}t + \log(G_+(t)/S_0) - \int_0^t a(s)ds,$$

$$H_2 = \frac{\sigma^2}{2}T + \int_0^T a(s)ds + \log(K/S_0)\}.$$

We suggest to use approximating probabilities $P(\hat{b}, \hat{h})$ with piece-wise linear stochastic boundaries $\hat{h}(t)$ instead of $h(t) = g(t) - zR_t$ such that for some partition $\{t_i\}, 0 = t_0 < t_1 < \dots < t_m = T$,

$$\hat{h}(t_j) = h(t_j) = g(t_j) - zR_{t_j}, j = 0, \dots, m. \quad (9)$$

Further we assume that $z = 1$.

Note that the vector $\mathbf{R} = (R_{t_1}, \dots, R_{t_m})$ and the process W_t^* are independent due to the assumption on independency of S_t and r_t .

The approximating probabilities $P(\hat{b}, \hat{h})$ can be calculated using the following formula

$$P(\hat{b}, \hat{h}) = E[I\{W_T > \hat{b}\} \prod_{j=0}^{m-1} (1 - e^{-\frac{2(\hat{h}(t_j) - W_{t_j})^+ (\hat{h}(t_{j+1}) - W_{t_{j+1}})^+}{t_{j+1} - t_j}})]. \quad (10)$$

This formula was derived by Wang/Pötzelberger (1997) for the special case when $b = -\infty$ and boundaries $\hat{h}(t)$ are deterministic piece-wise linear functions. In a more general context, including double barriers, although restricted to deterministic boundaries, a similar result was derived in [9]. To prove formula (10) for stochastic boundaries one could follow the proof in [9] or, alternatively, Proposition 1 formulated below may be helpful.

Let us consider

$$\eta_j(u) := W_{t_j+u} - W_{t_j} - u \frac{W_{t_{j+1}} - W_{t_j}}{\Delta t_j}, \quad 0 \leq u \leq \Delta t_j = t_{j+1} - t_j, \quad j = 0, \dots, m-1$$

and

$$\mathbf{W} = \{W_{t_1}, \dots, W_{t_m}\}.$$

Proposition 1. *For any u and v*

$$\text{Cov}(\eta_i(u), \eta_j(v) | \mathbf{W}) = 0, i \neq j,$$

$$\text{Cov}(\eta_i(u), \eta_i(v) | \mathbf{W}) = \min(u, v) - uv / \Delta t_i.$$

The proof of this result is based on standard properties of Gaussian random variables.

As a simple consequence of Proposition 1 one can see that the random processes $\{\eta_j(u), j = 0, \dots, m-1\}$ are jointly independent Brownian bridges conditioned to $\mathbf{W}$. Recall another well-known fact that for a linear (nonrandom) function $h(t)$

$$P\{W_t < h(t), t_i \leq t \leq t_{i+1} | (W_{t_i}, W_{t_{i+1}})\} = 1 - e^{-\frac{2(h(t_i) - W_{t_i})^+ (h(t_{i+1}) - W_{t_{i+1}})^+}{t_{i+1} - t_i}}.$$

Now using conditioning with respect to a vector $(\mathbf{W}, \mathbf{R})$ with $\mathbf{R} = (R_{t_1}, \dots, R_{t_m})$ one can obtain (10) for the general case under discussion.

Note that to compute (10) we need to simulate $2m$ -Gaussian r.v.'s, W_{t_j} and R_{t_j} , $j = 1, \dots, m$ or, alternatively, we could use $2m$ numerical integrations with respect to space variables.

4. Accuracy of the approximation. Our estimation of accuracy of the suggested approximation will be based on the following result proved in Borovkov/Novikov (2005).

Notation. $Lip(\kappa)$ is the class of Lipschitz functions $h(t)$ on $[0, T]$:

$$|h(t+h) - h(t)| \leq \kappa h, 0 = t < t+h = T,$$

where κ is a finite nonrandom constant.

Proposition 2 (Borovkov-Novikov (2005)). *Let stochastic boundaries $h(t) \in Lip(\kappa)$. Then for any $\varepsilon > 0$*

$$P(b, h + \varepsilon) - P(b, h) \leq (5\kappa/2 + 1/\sqrt{T})\varepsilon.$$

The proof of this results is given in [3], Lemma 1. The latter was formulated for the case of nonstochastic boundaries but one can easy to check that the result does hold for stochastic boundaries $h(t) \in Lip(\kappa)$ as well.

Further we assume: $T = 1$, ξ_t has the form (6) with a standard BM $W_t^{(r)}$,

$$h(t) = g(t) - \int_0^t \xi_s ds,$$

$g(t)$ is a twice continuously differentiable deterministic function, b is a constant.

Let $\hat{g}(t)$ be a piece-wise linear function such that $\hat{g}(t_j) = g(t_j)$ and R_t be a piece-wise linear process, the stochastic boundary $\hat{h}$ is defined in (9).

Theorem 1. *Let $t_j = \frac{j}{m}$. Then*

$$|P(b, \hat{h}) - P(b, h)| = O\left(\frac{\log(m)}{m^{3/2}}\right).$$

For the proof we will need the following

Proposition 3. *Let $x_m = C\sqrt{\log(m)}$ with large enough constant C . Then*

$$P\left\{\max_{t \leq T} \left| \int_0^t \xi_s ds - R_t \right| > \frac{x_m}{m^{3/2}}\right\} = o\left(\frac{1}{m^{3/2}}\right).$$

Proof of Theorem 1. Since $g(t)$ is a twice continuously differentiable function, for the linear piece-wise function $\hat{g}_t$ such that $\hat{g}(t_j) = g(t_j)$ we have ¹

$$|\hat{g}(t) - g(t)| \leq \frac{C}{m^2}.$$

By virtue of Proposition 3 proved below with $x_m = C\sqrt{\log(m)}$ we have

$$P\left(b, \hat{h} - \frac{x_m}{m^{3/2}} - \frac{C}{m^2}\right) - o\left(\frac{1}{m^{3/2}}\right) \leq P(b, g) \leq P\left(b, \hat{h} + \frac{x_m}{m^{3/2}} + \frac{C}{m^2}\right) + o\left(\frac{1}{m^{3/2}}\right).$$

Now applying Proposition 2, for any $\kappa > 0$ we obtain

$$|P(b, g) - P(b, \hat{h})| \leq (5\kappa/2 + 1)\left(\frac{2x_m}{m^{3/2}} + \frac{C}{m^2}\right) + P\left\{\max_{s \leq 1} |\xi_s| > \kappa\right\} + o\left(\frac{1}{m^{3/2}}\right). \quad (11)$$

Furthermore, using properties of Gaussian distribution one can show that

$$P\left(\max_{s \leq 1} |\xi_s| > \kappa\right) \leq Ce^{-\lambda\kappa^2}/\kappa$$

with some $\lambda > 0$. Choosing $\kappa = \kappa(m) = \sqrt{\frac{2\log(m)}{\lambda}}$ we obtain

$$P\left\{\max_{s \leq 1} |\xi_s| > \kappa(m)\right\} = O\left(\frac{1}{m^{3/2}}\right).$$

Combining this estimate with (11) we complete the proof of Theorem 1.

Proof of Proposition 3. Due to the definition of R_t , we have

$$\begin{aligned} \max_{t \leq 1} \left| \int_0^t \xi_s ds - R_t \right| &= \max_j \left| \int_{t_{j-1}}^{t_j} \xi_s ds - m \int_{t_{j-1}}^{t_j} \xi_u du \right| \leq \\ &\leq \frac{2}{m} \max_j \left(\max_{t_{j-1} \leq s \leq t_j} |\xi_s - \xi_{t_{j-1}}| \right). \end{aligned}$$

¹ C is a generic constant

Applying Girsanov transform, for any $x > 0$ we have

$$\begin{aligned} & P\{\max_j(\max_{t_{j-1} \leq s \leq t_j} |\xi_s - \xi_{t_{j-1}}|) > x\} = \\ & E[I\{\max_j(\max_{t_{j-1} \leq s \leq t_j} |W_s - W_{t_{j-1}}|) > x\} e^{-a_r \int_0^1 W_s dW_s - \frac{a_r^2}{2} \int_0^1 W_s^2 ds}] \\ & \leq e^{a_r/2} P\{\max_j(\max_{t_{j-1} \leq s \leq t_j} |W_s - W_{t_{j-1}}|) > x\}, (as \int_0^1 W_s dW_s = (W_1^2 - 1)/2). \end{aligned}$$

The random variables $\max_{t_{j-1} \leq s \leq t_j} |W_s - W_{t_{j-1}}|$, $j = 1, \dots, m$, are iid and by scaling and homogeneity properties we have

$$\max_{t_{j-1} \leq s \leq t_j} |W_s - W_{t_{j-1}}| \stackrel{d}{=} \max_{0 \leq s \leq 1} |W_s| / \sqrt{m},$$

Note that

$$\max_{s \leq 1} |W_s| \leq \max_{s \leq 1} (W_s) + \max_{s \leq 1} (-W_s)$$

and

$$\max_{0 \leq s \leq 1} (-W_s) \stackrel{d}{=} \max_{0 \leq s \leq 1} (W_s) \stackrel{d}{=} |W_1|.$$

It implies that for any $x > 0$

$$P\{\max_j(\max_{t_{j-1} \leq s \leq t_j} |W_s - W_{t_{j-1}}|) > x\} \leq 2P\{\max_j \gamma_j > x/2\}$$

where iid r.v.'s $\gamma_j \stackrel{d}{=} |W_1|$. and, hence,

$$\begin{aligned} & P\{\max_{t \leq 1} |\int_0^t \xi_s ds - R_t| > \frac{x}{m^{3/2}}\} \leq 2e^{a_r/2} P\{\max_j(\gamma_j) > x/2\} = \\ & = 2e^{a_r/2} (1 - (2\Phi(x/2) - 1)^m) \leq 2e^{a_r/2} (1 - (1 - \frac{2e^{-\frac{x^2}{8}}}{x\sqrt{2\pi}})^m), \end{aligned}$$

here we use the well-known inequality $1 - \Phi(x) \leq \frac{e^{-\frac{x^2}{2}}}{x\sqrt{2\pi}}$ for standard normal distribution $\Phi(x)$.

For $x = x_m \rightarrow \infty$

$$(1 - \frac{4e^{-\frac{x_m^2}{8}}}{x_m\sqrt{2\pi}})^m = e^{-\frac{4me^{-\frac{x_m^2}{8}}}{x_m\sqrt{2\pi}}(1+o(1))}$$

Choose now $x_m = C\sqrt{\log(m)}$ with large enough C . Then $\frac{4me^{-\frac{x_m^2}{8}}}{x_m\sqrt{2\pi}} = o(\frac{1}{m^{3/2}})$ and so

$$1 - e^{-\frac{2e^{-\frac{x_m^2}{8}}}{x_m\sqrt{2\pi}}m(1+o(1))} = o(\frac{1}{m^{3/2}}).$$

That completes the proof of Proposition 3.

5. Numerical examples. Here we consider model (5) with parameters

$$T = 1, a_r = 1, \bar{r} = 0.1, r_0 = 0.15.$$

For the asset price S_t we consider the same set of parameters as in [12], namely,

$$S_0 = 10, \sigma = 0.1, K = 11, G_+(t) = 12.$$

In terms of our notation used in Section 2

$$r_t = a(t) + \sigma_r \xi_t, E(r_t) = a(t) = 0.1 + 0.05e^{-t}, \int_0^t a(s) ds = 0.1t + 0.05(1 - e^{-t}),$$

$$d\xi_t = -\xi_t dt + \sigma_r dW_t^{(r)}, \xi_0 = 0,$$

where $W_t^{(r)}$ is a standard BM.

5.1. Deterministic interest rate. In [12] the authors have set $\sigma_r = 0$ and then using some probabilistic arguments obtained the following bounds for a fair price C_f of *Up-and-In European barrier call* option

$$0.516758 \leq C_f \leq 0.517968. \quad (12)$$

Denote by $C_f^d(m)$ the fair prices of the discretely monitored options with m monitoring dates and let $\hat{C}_f^d(m)$ be a Monte Carlo estimation for $C_f^d(m)$.

With the number of paths $N = 10^7$ we obtained the following results

$$\begin{aligned} \hat{C}_f^d(1000) &= 0.5125, \\ \hat{C}_f^d(100) &= 0.5036, \\ \hat{C}_f^d(50) &= 0.4985. \end{aligned} \quad (13)$$

The significant difference of 0.8%, even for $m = 1000$, compared to bounds of the continuously monitored barrier price from (12) can be explained by a slow rate of convergence of discretely monitored prices to continuous monitored ones. The magnitude of convergence rate is known to be of order $O(\frac{1}{\sqrt{m}})$, (see e.g. Nagaev (1970), Borovkov (1982)).

Denote by $\hat{C}_f(m)$ a Monte Carlo estimation for the fair prices of the *continuously monitored* options obtained through the numerical procedure described in Section 3; here m is a number of node points for a piece-wise linear boundary. With the number of paths $N = 10^7$ we obtained the following results

$$\begin{aligned} \hat{C}_f(1000) &= 0.5167, \\ \hat{C}_f(100) &= 0.5167, \\ \hat{C}_f(50) &= 0.5168. \end{aligned} \quad (14)$$

The contrast difference with the results for discretely monitored options is due to a better rate of convergence which, accordingly to Theorem 1, is at least of order $O(\frac{\log(m)}{m^{3/2}})$. As a matter of fact, for smooth deterministic boundaries the rate of convergence is even faster, namely, it is of order $O(\frac{1}{m^2})$ as shown in [11], [3].

In practice, the price $C_f^d(m)$ as a function of m is also of interest. Based on the known rate of convergence $C_f^d(m)$ to the limit as $m \rightarrow \infty$, we suggest to use for any m a simple approximation $\tilde{C}_f^d(m)$ defined by the following formula:

$$\tilde{C}_f^d(m) = \hat{C}_f(50) + \frac{A}{\sqrt{m}}, \quad (15)$$

where A is a constant; further we define the constant A from the equation $\hat{C}_f^d(50) = \hat{C}_f(50) + \frac{A}{\sqrt{50}}$.

The following table contains the simulated prices $\hat{C}_f^d(m)$ and the relative error of approximation (15) in the range $m \in [100, 250]$.

Table 1. Prices of discretely monitored option, $\sigma_r = 0$, $\hat{C}_f(50) = 0.51680$

m	Price $\hat{C}_f^d(m)$	Relative Error(%)
100	0.5036	0.06
150	0.5058	0.09
200	0.5073	0.08
250	0.5083	0.07

5.2. Stochastic interest rate. For comparison, we included results of simulation of prices $\hat{C}_f^d(m)$ and the relative error of approximation (15) for $\sigma_r = 0.2$. Note that the relative errors of approximating formula (15) in the range $[10, 250]$ is consistently less than 0.1%.

Table 2. Prices of discretely monitored option, $\sigma_r = 0.2, \hat{C}_f(50) = 0.62524$

m	Price $\hat{C}_f^d(m)$	Relative Error(%)
100	0.6148	0.05
150	0.6172	-0.04
200	0.6183	-0.03
250	0.6191	-0.03

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