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AGGREGATION OF HETEROGENEOUS BELIEFS AND ASSET PRICING THEORY: A MEAN-VARIANCE ANALYSIS

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ABSTRACT. Within the standard mean-variance framework, this paper provides a procedure to aggregate the heterogeneous beliefs in not only risk preferences and expected payoffs but also variances/covariances into a market consensus belief. Consequently, an asset equilibrium price under heterogeneous beliefs is derived. We show that the market aggregate behavior is in principle a weighted average of heterogeneous individual behaviors. The CAPM-like equilibrium price and return relationships under heterogeneous beliefs are obtained. The impact of diversity of heterogeneous beliefs on the market aggregate risk preference, asset volatility, equilibrium price and optimal demands of investors is examined. As a special case, our result provides a simple explanation for the empirical relation between cross-sectional volatility and expected returns.

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1. INTRODUCTION

The Sharpe-Lintner-Mossin (Sharpe (1964), Lintner (1965), Mossin (1966)) Capital Asset Pricing Model (CAPM) plays a central role in modern finance theory. It is founded on the paradigm of homogeneous beliefs and a rational representative agent. However, from the theoretical perspective this paradigm has been criticized on a number of grounds, in particular concerning its extreme assumptions about homogeneous beliefs and information of the economic environment and computational ability on the part of the rational representative economic agent. Within the standard mean-variance framework, this paper seeks to introduce heterogeneous beliefs in risk preferences, means and variances/covariances among agents, to analyze the aggregation properties of their heterogeneous beliefs, to examine the impact of the heterogeneity of beliefs on asset equilibrium price, and to establish a CAPM-like relationship under heterogeneous beliefs.

The impact of heterogeneous beliefs among agents on the market equilibrium price has been an important focus in the literature. It has been found that heterogeneous beliefs can affect aggregate markets returns. Models with agents who have heterogeneous beliefs have been previously studied (see, for example, Lintner (1969), Williams (1977), Huang and Litzenberger (1988), Abel (1989), Detemple and Murthy (1994), Zapatero (1998) and Basak (2000)). In much of these earlier work, the heterogeneous beliefs reflect either differences of opinion among the agents (see, for example, Lintner (1969), Miller (1977), Mayshar (1982), Varian (1985), Abel (1989, 2002), Cecchetti *et al.* (2000)) or differences in information upon which agents are trying to learn by using Bayesian updating rule (see, for example, Williams (1977), Detemple and Murthy (1994), Zapatero (1998)). Heterogeneity has been investigated in the context of either CAPM-like mean-variance models (see, for example Lintner (1969), Miller (1977), Williams (1977) and Mayshar (1982)) or an Arrow-Debreu contingent claims models (see, for example, Varian (1985), Abel (1989, 2002), Calvet *et al.* (2004) and Jouini and Napp (2006)).

In most of this literature, the impact of heterogeneous belief is studied for a portfolio of one risky asset and one risk-free asset (e.g. Abel (1989), Basak (2000), Zapatero (1998) and Johnson (2004)). In those papers that consider a portfolio of many risky assets and one risk-free asset, agents are assumed to be heterogeneous in the risk preferences and expected payoffs or returns of risky assets (e.g. Williams (1977), Varian (1985) and Jouini and Napp (2006)), but not in the variances and covariances. The only exception seems to have been the contribution of Lintner (1969) in which the heterogeneity in both means and variances/covariances is investigated in a mean-variance portfolio context. Variation of dispersion in the expected payoffs of risky assets among investors can be characterized by heterogeneous beliefs about the variance/covariance among investors. However, the impact of such heterogeneity has not been fully explored in the literature, including the contribution of Lintner (1969). Miller (1977) proposes a direct relationship between a stock's *risk* and its *divergence of opinion*. Variation in expectations among potential investors is characterized as the stock's divergence of opinion. He argue that "in practice, uncertainty, divergence of opinion about a security's return, and risk go together". Consequently, he proposed that "the riskiest stocks are also those about which there is the greatest divergence of opinion", thus, the market clearing price of a relatively high-risk stock will be greater than that for a relatively low-risk stock. The early empirical study by Bart and Masse (1981) supports Miller's proposition. Recently, Diether *et al.* (2002) provide empirical evidence that stocks with higher dispersion in analysts' earnings forecasts earn lower future returns than otherwise similar stocks, in particular for small stocks and stocks that have performed poorly over the past year. This is inconsistent with a view that dispersion in analysts' forecasts proxies for risk. Johnson (2004) offers a simple explanation for this phenomenon based on the interpretation of dispersion as a proxy for un-priced information risk arising when asset values are unobservable. Ang *et al.* (2006) examine the relation between cross-sectional volatility and expected returns and find that stocks with high sensitivities to innovations in aggregate volatility have low average returns. As suggested by the empirical study in Chan *et al.* (1999), while future variances and covariances are more easily predictable than expected future returns, the difficulties should not be understated. They argue that "While optimization (based on historical estimates of variances and covariances) leads to a reduction in volatility, the problem of forecasting covariance poses a challenge". Therefore, understanding the impact of heterogeneous beliefs in variances and covariances on equilibrium prices, volatility and cross-sectional expected returns is very important for a proper development of asset pricing theory. This paper is largely motivated by a re-reading of Lintner's early work and the recent empirical studies.

In this paper, we consider a portfolio of one risk-free asset and many risky assets and extend the mean-variance model to allow for heterogeneity in not only the means but also the variances/covariances across agents. The heterogeneous beliefs are considered as given. They reflect either differences of opinion among the agents or differences in information. By introducing the concept of a *consensus belief*, we first show that the consensus belief can be constructed as a weighted average of the heterogeneous beliefs and prove that the analysis of the heterogeneous beliefs model is equivalent to the analysis of a classical homogeneous model with the consensus belief. In particular, we show that the market aggregate expected payoffs of the risky assets can be measured by a weighted average of the heterogeneous expected payoffs of the risky assets across the agents, in which the weights are given by the heterogeneous covariance matrices adjusted by the risk aversion coefficients of the agents. We then examine various aggregation properties, including the impact of heterogeneity on the market equilibrium price, volatility, risk premium and agents' optimal demands in equilibrium. We show that the market equilibrium price is a weighted average of the equilibrium prices under the heterogeneous beliefs. We also establish an equilibrium relation between the market aggregate expected payoff of the risky assets and the market portfolio's expected payoff, leading to a CAPM-like relation under heterogeneous beliefs. An exact formula for the β coefficient under heterogeneous beliefs is derived. Consequently, the standard CAPM in return under homogeneous belief is extended to the one under heterogeneous beliefs. As a special case, our result provides a simple explanation for the empirical relation between cross-sectional volatility and expected returns.

An example of two risky assets and two heterogeneous beliefs is used to illustrate various impacts of heterogeneous beliefs on the equilibrium demands of heterogeneous agents, the equilibrium returns of the risky assets and the market portfolio. In particular we examine the impact of the heterogeneous beliefs on the β coefficient.

The paper is organized as follows. Heterogeneous beliefs are introduced and the standard mean-variance analysis is conducted in Section 2. In Section 3, we first introduce a consensus belief, and show how the consensus belief can be constructed from heterogeneous beliefs. We then derive the market equilibrium price of risky assets based on the consensus belief. Aggregation properties and the impact of diversified beliefs are examined in Section 4. In Section 5, we extend the traditional CAMP under homogeneous belief to the one under heterogeneous beliefs. An example of two agents and two beliefs is presented in Section 6 to illustrate the different impact of heterogeneity on the equilibrium optimal demands, returns of risky assets and market portfolio, and the corresponding β coefficients. Section 7 concludes.

2. Mean-Variance Analysis under Heterogeneous Beliefs

The static mean-variance model considered in this section is standard except that we allow the agents to have different risk preference, subjective means, variances and covariances. Consider a market with one risk-free asset and $K(\geq 1)$ risky assets. Let the current price of the risk-free asset be 1 and its payoff be $R_f = 1 + r_f$. Let $\tilde{\mathbf{x}} = (\tilde{x}_1, \cdots, \tilde{x}_K)^T$ be the payoff vector of the risky assets, where $\tilde{x}_k = \tilde{p}_k + \tilde{d}_k$ corresponds to the cum-prices.

Assume that there are I investors in the market indexed by $i = 1, 2, \cdots, I$. The heterogeneous (subjective) belief $\mathcal{B}_i = (\mathbb{E}_i(\tilde{\mathbf{x}}), \Omega_i)$ of investor i is defined with respect to the means, variances and covariances of the payoffs of the assets[1]

$$\mathbf{y}_i = \mathbb{E}_i(\tilde{\mathbf{x}}) = (y_{i,1}, y_{i,2}, \cdots, y_{i,K})^T, \qquad \Omega_i = (\sigma_{i,kl})_{K\times K},$$

where

$$y_{i,k} = E_i[\tilde{x}_k], \quad \sigma_{i,kl} = Cov_i(\tilde{x}_k, \tilde{x}_l) \tag{2.1}$$

for $i = 1, 2, \cdots I$ and $k, l = 1, 2, \cdots, K$.

Let $z_{i,o}$ and $\bar{z}_{i,o}$ be the absolute amount and the endowment of investor i in the risk-free asset, respectively, and

$$\mathbf{z}_i = (z_{i,1}, z_{i,2}, \cdots, z_{i,K})^T \qquad \text{and} \qquad \bar{\mathbf{z}}_i = (\bar{z}_{i,1}, \bar{z}_{i,2}, \cdots, \bar{z}_{i,K})^T$$

be the risky portfolio and the endowment, respectively, of investor i in absolute amount of the risky assets. Then the end-of-period wealth of the portfolio for investor i is

$$\tilde{W}_i = R_f z_{io} + \tilde{\mathbf{x}}^T \mathbf{z}_i.$$

Then, under the belief $\mathcal{B}_i$, the expected value and variance of portfolio wealth $\tilde{W}_i$ are given, respectively, by

$$\mathbb{E}_i(\tilde{W}_i) = R_f z_{i,o} + \mathbf{y}_i^T \mathbf{z}_i, \qquad \sigma_i^2(\tilde{W}_i) = \mathbf{z}_i^T \Omega_i \mathbf{z}_i. \tag{2.2}$$

We now make the following standard assumptions under the mean-variance framework.

(H1) *Assume the expected utility of the wealth generated from the portfolio* $(z_{i,o}, \mathbf{z}_i)$ *of investor* i *has the form* $V_i(\mathbb{E}_i(\tilde{W}_i), \sigma_i^2(\tilde{W}_i))$*, where* $V_i(x, y)$ *is continuously differentiable and satisfies* $V_{i1}(x, y) = \partial V_i(x, y)/\partial x > 0$ *and* $V_{i2}(x, y) = \partial V_i(x, y)/\partial y < 0$*.*

(H2) *Assume* $-2V_{i2}(x, y)/V_{i1}(x, y)$ *to be a constant* θ_i *for all* (x, y)*, i.e.*

$$\theta_i = \frac{-2V_{i2}(x, y)}{V_{i1}(x, y)} = const.$$

Assumption (H1) is in particular consistent with the constant absolute risk aversion (CARA) utility function $U_i(w) = -e^{-A_i w}$ with normally distributed w. Here $A_i > 0$ corresponds to the CARA coefficient. In this case, investor-i's optimal investment portfolio is obtained by maximizing the certainty-equivalent of his/her future wealth, $C_i(\tilde{W}_i) = \mathbb{E}_i(\tilde{W}_i) - \frac{A_i}{2} Var_i(\tilde{W}_i)$, and therefore $V_i(x, y) = x - \frac{A_i}{2} y$. Under assumption (H2), $\theta_i = A_i$, which is the absolute risk aversion of investor i. Based on this, we refer θ_i as the risk aversion measure of investor i.

Under (H1), the optimal portfolio of investor-i of risky assets $\mathbf{z}_i^*$ and risk-free asset z_{io}^* is determined by

$$\max_{z_{io}, \mathbf{z}_i} V_i(\mathbb{E}_i(\tilde{W}_i), \sigma_i^2(\tilde{W}_i))$$

[1] The heterogeneity considered in this paper is quite general. It may be due to the heterogeneous probability beliefs in an Arrow-Debreu economy, or heterogeneous information, or differences of opinion among agents.

subject to the budget constraint

$$z_{i,o} + \mathbf{p}_o^T \mathbf{z}_i = \bar{z}_{i,o} + \mathbf{p}_o^T \bar{\mathbf{z}}_i, \tag{2.3}$$

where $\mathbf{p}_o = (p_{1o}, p_{2o}, \cdots, p_{Ko})^T$ is the vector of market equilibrium prices of the risky assets, which is to be determined. We can then obtain the following Lemma 2.1 for the optimal demand of investor i in equilibrium.

Lemma 2.1. *Under assumptions (H1) and (H2), the optimal risky portfolio $\mathbf{z}_i^*$ of investor i at the market equilibrium is given by*

$$\mathbf{z}_i^* = \theta_i^{-1} \Omega_i^{-1} [\mathbf{y}_i - R_f \mathbf{p}_o]. \tag{2.4}$$

Proof. Let λ_i be the Lagrange multiplier and set

$$L(z_{i,o}, \mathbf{z}_i, \lambda_i) := V_i(\mathbb{E}_i(\tilde{W}_i), \sigma_i^2(\tilde{W}_i)) + \lambda_i[(\bar{z}_{i,o} + \mathbf{p}_o^T \bar{\mathbf{z}}_i) - (z_{i,o} + \mathbf{p}_o^T \mathbf{z}_i)].$$

Then the optimal portfolio of agent i is determined by the first order conditions

$$V_{i1} \frac{\partial \mathbb{E}_i(\tilde{W}_i)}{\partial z_{i,o}} = \lambda_i, \tag{2.5}$$

$$V_{i1} \frac{\partial \mathbb{E}_i(\tilde{W}_i)}{\partial z_{i,k}} + V_{i2} \frac{\partial \sigma^2(\tilde{W}_i)}{\partial z_{i,k}} = \lambda_i p_{ko}, \qquad k = 1, 2, \cdots, K. \tag{2.6}$$

From equation (2.2) we have

$$\frac{\partial \mathbb{E}_i(\tilde{W}_i)}{\partial z_{i,o}} = R_f, \quad \frac{\partial \mathbb{E}_i(\tilde{W}_i)}{\partial z_{i,k}} = y_{i,k}, \qquad \frac{\partial \sigma^2(\tilde{W}_i)}{\partial z_{i,k}} = 2 \sum_{l=1}^{K} \sigma_{i,kl} z_{i,l}$$

for $k = 1, 2, \cdots, K$. Then (2.5) and (2.6) become

$$V_{i1} R_f = \lambda_i, \tag{2.7}$$

$$V_{i1} y_{i,k} + 2V_{i2} \sum_{l=1}^{K} \sigma_{i,kl} z_{i,l} = \lambda_i p_{ko}, \qquad k = 1, 2, \cdots, K. \tag{2.8}$$

Substituting (2.7) into (2.8) leads to

$$V_{i1}[y_{i,k} - R_f p_{ko}] + 2V_{i2} \sum_{l=1}^{K} \sigma_{i,kl} z_{i,l} = 0, \qquad k = 1, 2, \cdots, K, \tag{2.9}$$

which in matrix notation can be written as

$$V_{i1}[\mathbf{y}_i - R_f \mathbf{p}_o] + 2V_{i2} \Omega_i \mathbf{z}_i = 0.$$

This, together with assumption (H2), leads to the optimal portfolio (2.4) of investor i at the market equilibrium. □

Lemma 2.1 shows that the optimal demand of investor-i is determined by his/her risk aversion θ_i and his/her belief about the expected payoffs and variance/covariance matrix of the risky assets' payoffs. We will see that, in the market equilibrium, the optimal demand depends on the dispersion of expected payoffs of investor-i from the market.

3. Consensus Belief and Equilibrium Asset Prices

In this section, we first define a consensus belief. By construction, we show the existence and uniqueness of the consensus belief. The market equilibrium prices of risky assets are then derived by using the consensus belief.

A market **equilibrium** is a vector of asset prices $\mathbf{p}_o$ determined by the individual demands (2.4) together with the market aggregation condition

$$\sum_{i=1}^{I} \mathbf{z}_i^* = \sum_{i=1}^{I} \bar{\mathbf{z}}_i = \mathbf{z}_m, \tag{3.1}$$

which defines a **market portfolio**. To characterize the market equilibrium, we introduce the following definition of **consensus belief**.

Definition 3.1. *A belief $\mathcal{B}_a = (\mathbb{E}_a(\tilde{\mathbf{x}}), \Omega_a)$, defined by the expected payoff of the risky assets $\mathbb{E}_a(\tilde{\mathbf{x}})$ and the variance and covariance matrix of the risky asset payoffs Ω_a, is called a* **consensus belief** *if and only if the equilibrium price under the heterogeneous beliefs is also the equilibrium price under the homogeneous belief $\mathcal{B}_a$.*

We now show how such a consensus belief can be uniquely constructed and how the market equilibrium price can be characterized by the consensus belief.

Proposition 3.2. *Under assumptions (H1) and (H2), let*

$$\Theta = \left[\frac{1}{I}\sum_{i=1}^{I}(1/\theta_i)\right]^{-1}.$$

Then

(i) *the consensus belief $\mathcal{B}_a$ is given by*

$$\Omega_a = \Theta^{-1}\left(\frac{1}{I}\sum_{i=1}^{I}\theta_i^{-1}\Omega_i^{-1}\right)^{-1}, \tag{3.2}$$

$$\mathbf{y}_a = \mathbb{E}_a(\tilde{\mathbf{x}}) = \Theta\Omega_a\left(\frac{1}{I}\sum_{i=1}^{I}\theta_i^{-1}\Omega_i^{-1}\mathbb{E}_i(\tilde{\mathbf{x}})\right); \tag{3.3}$$

(ii) *the market equilibrium price $\mathbf{p}_o$ is determined by*

$$\mathbf{p}_o = \frac{1}{R_f}\left[\mathbb{E}_a(\tilde{\mathbf{x}}) - \frac{1}{I}\Theta\Omega_a\mathbf{z}_m\right]; \tag{3.4}$$

(iii) *the equilibrium optimal portfolio of agent i is given by*

$$\mathbf{z}_i^* = \theta_i^{-1}\Omega_i^{-1}\left[(\mathbf{y}_i - \mathbf{y}_a) + \frac{1}{I}\Theta\Omega_a\mathbf{z}_m\right]. \tag{3.5}$$

Proof. It follows from the individuals demand (2.4) and the market clearing condition (3.1) that

$$\mathbf{z}_m = \sum_{i=1}^{I}\bar{\mathbf{z}}_i = \sum_{i=1}^{I}\mathbf{z}_i^* = \sum_{i=1}^{I}\theta_i^{-1}\Omega_i^{-1}[\mathbf{y}_i - R_f\mathbf{p}_o]. \tag{3.6}$$

Under the definitions (3.2) and (3.3), equation (3.6) can be rewritten as

$$\begin{aligned}\mathbf{z}_m &= \sum_{i=1}^{I} \theta_i^{-1}\Omega_i^{-1}\mathbf{y}_i - IR_f\Theta^{-1}\Omega_a^{-1}\mathbf{p}_o \\ &= I\Theta^{-1}\Omega_a^{-1}\left[\Theta\Omega_a \frac{1}{I}\sum_{i=1}^{I}\theta_i^{-1}\Omega_i^{-1}\mathbf{y}_i - R_f\mathbf{p}_o\right] \\ &= I\Theta^{-1}\Omega_a^{-1}[\mathbb{E}_a(\tilde{\mathbf{x}}) - R_f\mathbf{p}_o]. \end{aligned} \tag{3.7}$$

This leads to the market equilibrium price (3.4). Inserting (3.4) into the optimal demand function of investor-i in (2.4) we obtain the equilibrium demand (3.5) of investor-i for the risky assets. The uniqueness of the consensus belief follows from the uniqueness of the equilibrium price and the construction. □

Proposition 3.2 shows not only the existence of the unique consensus belief but also how it can be constructed from heterogeneous beliefs. The equilibrium asset pricing formula is the standard one under the consensus belief. As one of the main results of this paper, the implications of Proposition 3.2 are explored in the following section.

4. Aggregation Properties and Impact of Heterogeneity in Beliefs

In this paper, heterogeneity is characterized by the diversity in risk aversion coefficients, expected payoffs and variance/covariance matrices of the payoffs of the risky assets. Understanding the impact of such diversity under market aggregation is important for a proper understanding of asset pricing theory. We examine the impact of heterogeneity from several different perspectives.

4.1. **The aggregation effect of diversity in risk aversion coefficients.** If we treat θ_i as the absolute risk aversion coefficient of investor-i, then the coefficient Θ corresponds to the *harmonic mean* of the absolute risk aversion of all the investors. The aggregate property of the risk aversion coefficient can be examined from two different perspectives.

First, given the fact that $f(x) = 1/x, x > 0$ is a decreasing and convex function, we have[2]

$$\Theta \le \frac{1}{I}\sum_{i=1}^{I}\theta_i. \tag{4.1}$$

This implies that the aggregate risk aversion coefficient Θ is smaller than the average of the risk aversion coefficients among investors.

Secondly, the aggregation property of the risk aversions can be characterized via a *mean-preserving spread* in the distribution of the risk aversion coefficients θ_i. The mean-preserving spread is a standard technique developed in Rothschild-Stiglitz (1970) to measure the stochastic dominance among risky assets. We extend this technique to examine the effect of the diversity of the risk aversions.

To illustrate, assume $I = 2$ and let the risk aversion coefficients be $\{\theta_1, \theta_2\}$, with $\theta_1 < \theta_2$. That is investor-2 is more risk averse than investor-1. Define $\overline{\theta} := (\theta_1 + \theta_2)/2$ as the mean (or average) risk aversion. The aggregate risk aversion in this case can be written as

$$\Theta = 2\frac{\theta_1\theta_2}{\theta_1+\theta_2} = \frac{\theta_1\theta_2}{\overline{\theta}}$$

[2]In fact, for any continuous convex function $f(x)$, $f(\sum_{i=1}^n \alpha_i x_i) \le \sum_{i=1}^n \alpha_i f(x_i)$ holds for $\alpha_i > 0$ satisfying $\sum_{i=1}^n \alpha_i = 1$. The equality holds if and only if all x_i are the same.

Assume now that the risk aversion coefficients change into the following $\{\theta_1', \theta_2'\} = \{\theta_1 - \varepsilon, \theta_2 + \varepsilon\}$ with $\theta_1 \geq \varepsilon > 0$, this represents a mean-preserving spread in the risk aversion coefficients and $\epsilon > 0$ measure the dispersion of heterogeneous belief in the risk aversions around the mean. The mean risk aversion is again $\overline{\theta}$, but the aggregate risk aversion becomes

$$\Theta' = \frac{(\theta_1 - \varepsilon)(\theta_2 + \varepsilon)}{\overline{\theta}}$$

Given that

$$(\theta_1 - \varepsilon)(\theta_2 + \varepsilon) = \theta_1\theta_2 - \varepsilon(\theta_2 - \theta_1) - \varepsilon^2 < \theta_1\theta_2,$$

it turns out that $\Theta' < \Theta$. This implies that diversity of a mean-preserving spread in risk-aversion coefficients can reduce the risk aversion coefficient under aggregation. In particular, if ϵ is very close to θ_1, the aggregate risk aversion Θ' is very close to 0, and hence the market is close to a risk-neutral market.

The above analysis indicates that aggregation of diversified risk preferences among heterogeneous agents makes the market become less risk averse.

4.2. **The aggregation effect of diversity in variances and covariances.** It follows from (3.2) that the inverse of the aggregate covariance matrix is a risk-adjusted weighted average (with weights $\Theta/(I\theta_i)$) of the inverse of the covariance matrices of the heterogeneous investors.

To investigate the aggregation property of the variance and covariance, we first compare the variances of any portfolio under both the aggregate covariance matrix and average of the heterogeneous covariance matrices. More precisely, we use both the aggregated covariance matrix Ω_a and the weighted average covariance matrix $\bar{\Omega} = (\Theta/I)\sum_{i=1}^{n}\theta_i^{-1}\Omega_i$ and calculate the respective variances of any given portfolio $\mathbf{z}$, namely

$$\sigma_a^2(\mathbf{z}) = \mathbf{z}^T\Omega_a\mathbf{z}, \qquad \overline{\sigma^2}(\mathbf{z}) = \mathbf{z}^T\bar{\Omega}\mathbf{z}.$$

It would be interesting to know if and under what conditions $\sigma_a^2(\mathbf{z}) \leq \overline{\sigma^2}(\mathbf{z})$.

Numerical simulations show that this is not true in general, however, this result is true when the payoffs of different assets are uncorrelated. As a matter of a fact, in this case, it follows from (3.2) that the aggregate variance of asset j is given by

$$(\sigma_{a,j}^2)^{-1} = \frac{\Theta}{I}\sum_{i=1}^{I}\theta_i^{-1}(\sigma_{i,j}^2)^{-1} \tag{4.2}$$

and therefore $\sigma_{a,j}^2$ is a (weighted) harmonic mean of the variance beliefs. Equation (4.2) implies that

$$\sigma_{a,j}^2 \leq \frac{1}{I}\sum_{i=1}^{I}\frac{\Theta}{\theta_i}\sigma_{i,j}^2 = \bar{\sigma}_j^2. \tag{4.3}$$

Hence, when asset payoffs are uncorrelated, the variance of any portfolio under the aggregate variance is smaller than that under the weighted average variance.

Similarly to the discussion in Section 4.1, it is interesting to examine the effect of diversity in variance/covariance beliefs. The effect is not clear in general. We only consider the case when asset payoffs are uncorrelated. In this case, when the beliefs about the risk aversion coefficients are homogeneous (i.e. $\theta_i = \theta$ for all i), $\sigma_{a,j}^2$ is a harmonic mean of the variance beliefs. Applying the same argument as in Section 4.1, we can conclude that a mean-preserving spread in variance beliefs can reduce the asset risk under aggregation. However, this result is also true under certain conditions when the beliefs about the risk aversion coefficients are heterogeneous. This is illustrated by the following example.

Example. Let $I = 2$ and the risk aversion coefficients be θ_1, θ_2. Assume the payoffs of the risky assets are uncorrelated and the variance beliefs are $\sigma^2_{1,j}$, $\sigma^2_{2,j}$, with $\sigma^2_{1,j} < \sigma^2_{2,j}$, for asset j. Define

$$\overline{\sigma}^2_j = \frac{\theta_1\sigma^2_{1,j} + \theta_2\sigma^2_{2,j}}{\theta_1 + \theta_2}$$

as the weighted average variance. In this particular case $(\sigma^2_{a,j})^{-1}$ can be rewritten as

$$(\sigma^2_{a,j})^{-1} = \frac{\Theta}{2}\left(\frac{1}{\theta_1\sigma^2_{1,j}} + \frac{1}{\theta_2\sigma^2_{2,j}}\right) = \frac{\overline{\sigma}^2_j}{\sigma^2_{1,j}\sigma^2_{2,j}},$$

that is,

$$\sigma^2_{a,j} = \frac{\sigma^2_{1,j}\sigma^2_{2,j}}{\overline{\sigma}^2_j}$$

Assume that the variance beliefs change to

$$\left\{\sigma^{2\prime}_{1,j}, \sigma^{2\prime}_{2,j}\right\} = \left\{\sigma^2_{1,j} - \varepsilon, \sigma^2_{2,j} + \delta\right\}, \qquad \sigma^{2\prime}_{1,j} > \varepsilon > 0, \qquad \delta = \varepsilon\theta_1/\theta_2,$$

this is a mean-preserving spread in variance beliefs. The weighted average variance is again $\overline{\sigma}^2_j$, but the aggregate variance becomes

$$\sigma^{2\prime}_{a,j} = \frac{(\sigma^2_{1,j} - \varepsilon)(\sigma^2_{2,j} + \delta)}{\overline{\sigma}^2_j}.$$

In this case we obtain that $\sigma^{2\prime}_a < \sigma^2_a$ iff

$$(\sigma^2_{1,j} - \varepsilon)(\sigma^2_{2,j} + \varepsilon\theta_1/\theta_2) < \sigma^2_{1,j}\sigma^2_{2,j},$$

which is equivalent to

$$\theta_2 > \frac{\sigma^2_{1,j} - \epsilon}{\sigma^2_{2,j}}\theta_1. \tag{4.4}$$

Condition (4.4) implies that, on the one hand, a mean-preserving spread in variance beliefs reduces the aggregate market risk of the risky asset when an investor (here investor-2) who believes the asset is more risky (measured by higher $\sigma_{2,j}$) is more risk averse (in the sense of (4.4)). On the other hand, a mean-preserving spread in variance beliefs increases the aggregate market risk of the risky asset when an investor (here investor-2) who believes the asset is more risky is less risk averse.

By assuming that investors are risk averse, we can use the above example to explain the empirical relation between cross-sectional volatility and expected returns reported by Diether et al. (2002) and Ang *et al.* (2006). They found empirical evidence that stocks with higher dispersion in analysts' earnings forecasts earn lower future returns than otherwise similar stocks. Assume that both investors in the above example have homogeneous beliefs about are expected payoffs of risky assets j and j' but heterogeneous about risk aversion coefficients and variances of the assets. We also assume the variance beliefs for asset j' is a mean-preserving spread of variance beliefs for asset j. If investor-2 is more risk averse than investor-1 (in the sense of condition (4.4)), then it follows from the example that the aggregate variance of asset j' is less than that of asset j. Thus, from the equilibrium price equation (3.4), the equilibrium price for asset j' is higher than the equilibrium price for asset j. This in turn implies that asset j' has lower expected return than asset j. In other word, stocks with higher dispersion in expected payoffs have higher market clearing prices and earn lower future expected returns than otherwise similar stocks. This result is consistent with Miller's proposition that divergence of opinion and risk "go together". It is also interesting to see that this kind of argument cannot hold when investors have homogeneous beliefs.

4.3. **The aggregation effect of diversity in expected payoffs.** Given that $(\Theta\Omega_a)^{-1} = (1/I)\sum_{i=1}^{I}\theta_i^{-1}\Omega_i^{-1}$, equation (3.3) indicates that the aggregate expected payoff of risky assets under the consensus belief $\mathcal{B}_a$ is a weighted average of the heterogeneous expected payoffs of the risky assets. On the one hand, if investors agree on the expected payoff $\mathbb{E}_i(\tilde{\mathbf{x}}) = \mathbb{E}_o(\tilde{\mathbf{x}})$, then it follows from (3.3) that $\mathbb{E}_a(\tilde{\mathbf{x}}) = \mathbb{E}_o(\tilde{\mathbf{x}})$, although they may disagree on their risk preferences, variances and covariances. On the other hand, if investors agree on the variance and covariance, then

$$\mathbb{E}_a(\tilde{\mathbf{x}}) = \frac{1}{I}\sum_{i=1}^{I}\frac{\Theta}{\theta_i}\mathbb{E}_i(\tilde{\mathbf{x}}), \tag{4.5}$$

which reflects a weighted average opinion of the market on the expected payoff of risky assets. In this case, the expected market payoff is dominated by investors who are less (more) risk averse and believe in a higher (lower) expected payoff, as we would expect in bull (bear) market, although such dominance may be asymmetric for bull and bear markets. Otherwise, the aggregate expected payoff may be unchanged even if investors have divergent opinions on their expected payoffs, as long as they are *balanced*.

Based on the above discussion, one can see that the aggregate payoff $\mathbb{E}_a(\tilde{\mathbf{x}})$ is affected by the covariance beliefs only when investors disagree on both the expected payoffs and covariances. The impact of a mean-preserving spread in either risk aversion coefficients or variance matrices on the expected aggregate payoffs is less clear in general and we leave an analysis of this issue to future research.

4.4. **The impact of heterogeneity on the market equilibrium price.** The market equilibrium price (3.4) in Proposition 3.2 (ii) is exactly the same as the traditional equilibrium price for a representative agent holding the consensus belief $\mathcal{B}_a$. If we define $\mathbf{p}_{i,o}$ as the equilibrium price vector of the risky assets for investor i as if he/she were the only investor in the market, then we would have

$$\mathbf{p}_{i,o} = \frac{1}{R_f}[\mathbb{E}_i(\tilde{\mathbf{x}}) - \theta_i\Omega_i\bar{z}_i].$$

Equation (3.4) can then be rewritten as

$$\mathbf{p}_o = \Theta\Omega_a\left[\frac{1}{I}\sum_{i=1}^{I}\theta_i^{-1}\Omega_i^{-1}\mathbf{p}_{i,o}\right]. \tag{4.6}$$

Therefore, the aggregate market equilibrium price is a weighted average of each agent's equilibrium prices under his/her belief if he/she were the only agent in the market. Consistent with Miller's argument, the market price may reflect the expectations of only the most optimistic minority, as long as this minority can absorb the entire supply of stock.

Equation (3.4) indicates that the market equilibrium price depends on the aggregate expected payoff $\mathbb{E}_a(\tilde{\mathbf{x}})$ and the equity risk premium $\Theta\Omega_a\mathbf{z}_m/I$. The equity risk premium is proportional to both the aggregate risk aversion coefficient Θ and the covariance between the risky assets and the average market portfolio $\Omega_a\mathbf{z}_m/I$. The diversity of heterogeneous beliefs in variances and covariance will affect the equity risk premium. In particular, a mean-preserving spread in variance beliefs when asset payoffs are uncorrelated will reduces the aggregate variances of stocks, leading to a lower equity risk premium and therefore a higher market price. When both the risk aversion coefficients and the market portfolio are bounded (as is often the case), the equity risk premium becomes smaller when the number of investors increases. In the limiting case, the equity risk premium tends to zero as $I \to \infty$, and hence

$$\mathbf{p}_o \approx \frac{1}{R_f}\mathbb{E}_a(\tilde{\mathbf{x}}) = \frac{1}{R_f}\mathbb{E}_a(\tilde{\mathbf{p}} + \tilde{\mathbf{d}}). \tag{4.7}$$

This is the traditional *risk-neutral discount equity value formula* under the expected aggregate payoff of heterogeneous beliefs, which we see may be a reasonable approximation in a market with heterogeneous beliefs if the number of different beliefs is sufficient large.

4.5. **The impact of heterogeneity on the optimal demands and trading volume.** Proposition 3.2 (iii) indicates that the equilibrium demand of an individual investor has two components. The first term $\theta_i^{-1}\Omega_i^{-1}[\mathbb{E}_i(\tilde{\mathbf{x}}) - \mathbb{E}_a(\tilde{\mathbf{x}})]$ corresponds to the standard demand. It reflects the dispersion of the investor's expected payoff from the aggregate expected payoff. The second term $(\Theta/\theta_i)\Omega_i^{-1}\Omega_a\mathbf{z}_m/I$ reflects the dispersion of the investor's belief on variance and covariance from the aggregate variance and covariance. When an investor's expected payoff is the same as the aggregate expected payoff, that is, $\mathbb{E}_i(\tilde{\mathbf{x}}) = \mathbb{E}_a(\tilde{\mathbf{x}})$, the investor's demand is simply determined by the second component. When investors are homogeneous in the risk aversion coefficient $\theta_i = \theta_o$ and the covariance matrix $\Omega_i = \Omega_o$, the second component reduces to $\mathbf{z}_m/I$, which is the average share of the market portfolio. In this case, the equilibrium demand of investor i is reduced to

$$\mathbf{z}_i^* = \theta_o^{-1}\Omega_o^{-1}[\mathbb{E}_i(\tilde{\mathbf{x}}) - \mathbb{E}_a(\tilde{\mathbf{x}})] + \mathbf{z}_m/I, \tag{4.8}$$

and the market equilibrium price is reduced to

$$\mathbf{p}_o = \frac{1}{R_f}[\mathbb{E}_a(\tilde{\mathbf{x}}) - \theta_o\Omega_o\mathbf{z}_m/I], \qquad \text{where} \quad \mathbb{E}_a(\tilde{\mathbf{x}}) = \frac{1}{I}\sum_{i=1}^{I}\mathbb{E}_i(\tilde{\mathbf{x}}). \tag{4.9}$$

From (4.8) and (4.9), one can see that a mean-preserving spread in the distribution of the expected payoffs among investors will not change the equilibrium price, but will spread optimal demands among investors around the average market portfolio, this in turn will increase the trading volume in the market, assuming a uniform initial endowment among investors. This implies that a high trading volume due to diversified beliefs about asset expected payoffs may not necessarily lead to high volatility of asset prices. If the expected payoff dispersion of investors from the average expected payoff does not change, investors demands will not change. However a high average of the expected payoffs will lead to a high market equilibrium asset price. This suggests that a higher (or lower) market price due to a higher (or lower) averaged expected payoff may not necessarily lead to high trading volume.

5. The CAPM-like Relationship under Heterogeneous Beliefs

We now explore the impact of heterogeneity on the CAPM relationship, which constitutes the second main set of results of this paper. For the market portfolio $\mathbf{z}_m$, its value in the market equilibrium is given by $W_{m,o} = \mathbf{z}_m^T\mathbf{p}_o$ and its future payoff is given by $\tilde{W}_m = \tilde{\mathbf{x}}^T\mathbf{z}_m$. Hence, under the consensus belief $\mathcal{B}_a$,

$$W_m = \mathbb{E}_a(\tilde{W}_m) = \mathbb{E}_a(\tilde{\mathbf{x}})^T\mathbf{z}_m, \qquad \sigma_m^2 = Var(\tilde{W}_m) = \mathbf{z}_m^T\Omega_a\mathbf{z}_m. \tag{5.1}$$

Based on Proposition 3.2 and the above observation, we obtain the following *CAPM-like price relation under heterogeneous beliefs*. We shall call this relationship the *Heterogeneous CAPM* (**HCAPM**) in price.

Proposition 5.1. *In equilibrium the market aggregate expected payoff of the risky assets are related to the expected payoff of the market portfolio* $\mathbf{z}_m$ *by the CAPM-like price relation*

$$\mathbb{E}_a(\tilde{\mathbf{x}}) - R_f\mathbf{p}_o = \frac{1}{\sigma_m^2}\Omega_a\mathbf{z}_m[\mathbb{E}_a(\tilde{W}_m) - R_fW_{m,o}], \tag{5.2}$$

or equivalently,

$$\mathbb{E}_a(\tilde{x}_k) - R_fp_{k,o} = \frac{\sigma(\tilde{W}_m, \tilde{x}_k)}{\sigma_m^2}[\mathbb{E}_a(\tilde{W}_m) - R_fW_{m,o}], \qquad k = 1, 2, \cdots, K, \tag{5.3}$$

where $\Omega_a = (\sigma_{kj})_{K\times K}$ *and* $\sigma(\tilde{W}_m, \tilde{x}_k) = \sum_{j=1}^K z_{m,j}\sigma_{kj}$ *for* $k = 1, \cdots, K$ *corresponds to the covariance of the market aggregate payoffs of the risky asset* k *and the aggregate market portfolio payoff* $\tilde{W}_m$.

Proof. From (3.7) and (5.1),

$$0 < \sigma_m^2 = I\Theta^{-1}[\mathbb{E}_a(\tilde{W}_m) - R_fW_{m,o}]$$

and hence

$$\mathbb{E}_a(\tilde{W}_m) - R_fW_{m,o} = \Theta\sigma_m^2/I. \tag{5.4}$$

On the other hand, from (3.4),

$$\mathbb{E}_a(\tilde{\mathbf{x}}) - R_f\mathbf{p}_o = \Theta\Omega_a\mathbf{z}_m/I.$$

This last equation, together with (5.4), lead to the CAPM-like price relation (5.2) under heterogeneous beliefs in a vector form. □

The HCAPM price relation (5.2) can be converted to the standard CAPM-like return relation. Define the returns

$$\tilde{r}_j = \frac{\tilde{x}_j}{p_{j,o}} - 1, \qquad \tilde{r}_m = \frac{\tilde{W}_m}{W_{m,o}} - 1.$$

and set

$$\mathbb{E}_a(\tilde{r}_j) = \frac{\mathbb{E}_a(\tilde{x}_j)}{p_{j,o}} - 1, \qquad \mathbb{E}_a(\tilde{r}_m) = \frac{\mathbb{E}_a(\tilde{W}_m)}{W_{m,o}} - 1.$$

With these notations, we can obtain from (5.2) the following HCAPM relation between returns of risky assets and the market portfolio.

Corollary 5.2. *In equilibrium, the HCAPM price relation (5.2) can be expressed in terms of returns as*

$$\mathbb{E}_a[\tilde{\mathbf{r}}] - r_f\mathbf{1} = \boldsymbol{\beta}[\mathbb{E}_a(\tilde{r}_m) - r_f], \tag{5.5}$$

where

$$\boldsymbol{\beta} = (\beta_1, \beta_2, \cdots, \beta_K)^T, \qquad \beta_k = \frac{cov_a(\tilde{r}_m, \tilde{r}_k)}{\sigma_a^2(\tilde{r}_m)}, \qquad k = 1, \cdots, K,$$

and the mean and variance/covariance of returns under the consensus belief $\mathcal{B}_a$ *are defined similarly.*

Proof. Based on the previous notations, we divide thoroughly $p_{k,o}$ on both sides of (5.3), then

$$[\mathbb{E}_a(\tilde{r}_k)+1]-[r_f+1] = \frac{W_{mo}\sigma(\tilde{W}_m, \tilde{x}_k)}{p_{ko}\sigma_m^2}[(\mathbb{E}_a(\tilde{r}_m)+1)-(r_f+1)], \qquad k = 1, 2, \cdots, K.$$

That is,

$$\mathbb{E}_a\tilde{(}r_k) - r_f = \beta_k[\mathbb{E}_a(\tilde{r}_m) - r_f], \qquad k = 1, 2, \cdots, K,$$

where

$$\beta_k = \frac{W_{m,o}}{p_{k,o}} \frac{\sigma(\tilde{W}_m, \tilde{x}_k)}{\sigma_m^2} = \frac{cov_a(\tilde{x}_k/p_{k,o}, \tilde{W}_m/W_{m,o})}{Var_a(\tilde{W}_m/W_{m,o})} = \frac{cov_a(\tilde{r}_m, \tilde{r}_k)}{\sigma_a^2(\tilde{r}_m)}.$$

□

The equilibrium relation (5.5) is the standard CAPM except that the mean and variance/covariance are calculated based on the consensus belief $\mathcal{B}_a$.

6. The Case of Two Risky Assets and Two Beliefs

In this section, we illustrate the different impact of heterogeneity on the equilibrium optimal demands (of heterogeneous agents), returns of risky assets and market portfolio and the corresponding beta's of risky assets, by considering a simple market with two risky assets and one risk-free asset. We assume that there are two agents who may have different beliefs.

To facilitate our analysis, we recall the connection between asset payoffs and asset returns. For asset j ($j = 1, 2$), the rate of return $\tilde{r}_j$ and the payoff $\tilde{x}_j$ are related by $\tilde{x}_j = x_{jo}(1 + \tilde{r}_j)$, where $x_{jo} > 0$ is a constant. For $i, j = 1, 2$, set

$$\mu_{i,j} = \mathbb{E}_i(\tilde{r}_j), \quad \bar{\sigma}_{i,j}^2 = Var_i(\tilde{r}_j), \quad \bar{\sigma}_{i,12} = Cov_i(\tilde{r}_1, \tilde{r}_2), \quad \bar{\rho}_i = \frac{\bar{\sigma}_{i,12}}{\bar{\sigma}_{i,1}\bar{\sigma}_{i,2}}.$$

Then

$$y_{i,j} = \mathbb{E}_i(\tilde{x}_j) = x_{j,o}(1 + \mu_{i,j}), \quad \sigma_{i,j} = x_{j,o}\bar{\sigma}_{i,j},$$

and

$$\sigma_{i,12} = x_{1o}x_{2o}\bar{\sigma}_{i,12}, \quad \rho_i = \frac{\sigma_{i,12}}{\sigma_{i,1}\sigma_{i,2}} = \bar{\rho}_i.$$

Hence the expected payoffs and variance/covariance matrix of the two risky asset payoffs are, respectively,

$$y_i = \begin{pmatrix} y_{i1} \\ y_{i2} \end{pmatrix}, \qquad \Omega_i = \begin{pmatrix} \sigma_{i1}^2 & \rho_i\sigma_{i1}\sigma_{i2} \\ \rho_i\sigma_{i1}\sigma_{i2} & \sigma_{i2}^2 \end{pmatrix}, \qquad i = 1, 2.$$

As a benchmark, we consider the corresponding homogeneous case where

$$\mu_1 = \mu_{i,1}, \quad \mu_2 = \mu_{i,2}, \quad \bar{\sigma}_1 = \sigma_{i,1}, \quad \bar{\sigma}_2 = \sigma_{i,2}, \quad \bar{\rho} = \rho_i \tag{6.1}$$

for $i = 1, 2$. Let $r_j = \mathbb{E}_a(\tilde{r}_j), r_m = \mathbb{E}_a(\tilde{r}_m)$ be the equilibrium return of asset $j(j = 1, 2)$ and market portfolio respectively, and r_f be the risk-free rate.

To explore the different impact of agent heterogeneity on the equilibrium portfolio of agents and market equilibrium returns of the risky assets, the market portfolio and the betas, we consider the following six cases.

6.1. **Case 1.**

We first consider the homogenous case and examine the impact of changing the absolute risk aversion coefficient, the mean, variance, and the correlation coefficient on the optimal demands (of investors), the equilibrium returns and the corresponding betas of the risky assets. The following Proposition 6.1 on the homogeneous agent case is helpful in understanding the impact of different aspects of heterogeneity in the subsequent cases.

Proposition 6.1. *For a market with two risky assets and one risk-free asset, if agents are homogeneous (with respect to the risk aversion coefficient θ, the expected payoffs $\mathbf{y} = (y_1, y_2)$, and the variance and covariance structure ρ, σ_1, σ_2, then*

(i) *the equilibrium demand of investor i is an equal share of the market portfolio $\mathbf{z}_i^* = \mathbf{z_m}/2$;*

(ii) *in terms of the expected payoff of the 1st risky asset* (y_1)*, we have*

$$\frac{\partial r_2}{\partial y_1} = 0, \qquad \frac{\partial \beta_2}{\partial y_1} > 0$$

and

$$\frac{\partial r_1}{\partial y_1} < (>,=)0, \quad \frac{\partial \beta_1}{\partial y_1} < (>,=)0 \qquad \textit{iff} \qquad \sigma_1 + \rho\sigma_2 > (<,=)0;$$

(iii) *in terms of the volatility of the 1st risky asset* (σ_1)*, we have*

$$\frac{\partial r_1}{\partial \sigma_1} > 0 \quad \textit{iff} \quad \rho > \frac{-2\sigma_1}{\sigma_2}; \qquad \frac{\partial r_2}{\partial \sigma_1} > 0 \quad \textit{iff} \quad \rho > 0$$

and

$$\frac{\partial \beta_1}{\partial \sigma_1} > 0 \quad \textit{iff} \quad \rho > \frac{-2\sigma_1}{\sigma_2} \quad \textit{and} \quad \frac{r_1}{\sigma_1} > \frac{r_m}{\sigma_m}\frac{2(\sigma_1+\rho\sigma_2)^2}{\sigma_m(2\sigma_1+\rho\sigma_2)},$$

$$\frac{\partial \beta_2}{\partial \sigma_1} < 0 \quad \textit{iff} \quad \rho > -\frac{\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2+\sigma_1\sigma_2};$$

(iv) *in terms of the risk aversion coefficient* (θ)*, we have*

$$\frac{\partial r_1}{\partial \theta} > 0 \quad \textit{iff} \quad \sigma_1 > -\rho\sigma_2; \qquad \frac{\partial r_2}{\partial \theta} > 0 \quad \textit{iff} \quad \sigma_2 > -\rho\sigma_1$$

and

$$\frac{\partial \beta_1}{\partial \theta} < 0 \quad \textit{iff} \quad r_1 W_{m,o}^2 < r_m; \qquad \frac{\partial \beta_2}{\partial \theta} < 0 \quad \textit{iff} \quad r_2 W_{m,o}^2 < r_m;$$

(v) *in terms of the correlation coefficient* (ρ)*, we have*

$$\frac{\partial r_1}{\partial \rho} > 0, \qquad \frac{\partial r_2}{\partial \rho} > 0$$

and

$$\frac{\partial \beta_1}{\partial \rho} > 0 \qquad \textit{iff} \qquad \frac{r_1}{\sigma_1} > 2\frac{r_m}{\sigma_m}\frac{\sigma_1+\rho\sigma_2}{\sigma_m W_{m,o}^2},$$

$$\frac{\partial \beta_2}{\partial \rho} > 0 \qquad \textit{iff} \qquad \frac{r_2}{\sigma_2} > 2\frac{r_m}{\sigma_m}\frac{\sigma_2+\rho\sigma_1}{\sigma_m W_{m,o}^2},$$

Proof. See Appendix A. □

When agents are homogeneous, the result in Proposition 6.1(i) is very intuitive. Changing mean and variance/covariance does not change the equilibrium demands for risky assets. This simply illustrates the no-trad theorem in the homogeneous and representative agent literature. Propositions 6.1(ii)-(v) indicate that changes in expected payoff, variance, correlation coefficient and the risk aversion coefficient have different impacts on equilibrium returns and beta coefficients of the risky assets. To illustrate these impacts, we choose

$$\mu_1 = 0.1, \mu_2 = 0.12, \bar{\sigma}_1 = 0.12, \bar{\sigma}_2 = 0.15, \bar{\rho} = 0.5, r_f = 0.05, \theta = 1, x_{j,o} = 10. \tag{6.2}$$

Hence

$$y_1 = 11, y_2 = 11.2, \sigma_1 = 1.2, \sigma_2 = 1.5, \rho = 0.5. \tag{6.3}$$

Assume that the initial endowment of the two risky assets are 0.5 and 0.5, respectively for both investors. This leads to a market portfolio of one share for both risky assets. These are the parameters we will use in our following discussion on various cases unless stated otherwise. The discussions are illustrated by using various figures, in which the blue (or dark) surface corresponds to asset-1 and the green (or grey) surface corresponds to asset-2.

Proposition 6.1 is illustrated in Figure 6.1. Proposition 6.1 (ii) shows that, an increase of the expected payoff of asset-1 doesn't change the equilibrium return of asset-2, but increases the beta coefficient for asset-2, and correspondingly the expected return of the market portfolio decreases. Also, an increase of expected payoff of asset-1 decreases (increases) the equilibrium return and the beta coefficient of asset-1 when $\sigma_1 + \rho\sigma_2 > 0 (< 0)$. A similar argument can be used for the case when the expected payoff of asset-2 changes. This result is illustrated in Figure 6.1 panels (A3) and (B3)[3].

The impact of changing variance on the equilibrium returns and betas is more complicated. Assume that both asset payoffs are positively correlated. From Proposition 6.1 (iii), the equilibrium returns for both assets increase as the volatility of asset-1 increases. Also, the beta coefficient increases for asset-1 but decreases for asset-2. This is illustrated in Figure 6.1 panels (A2) and (B2). For fixed σ_2, a high volatility in σ_1 is associated with high return for asset-1 while the return for asset-2 is almost unchanged. Figure 6.1 (B2) demonstrates that changing volatility (and hence the covariance) has a significant impact on beta coefficients of the risky assets.

Figure 6.1 panels (A1) and (B1) illustrate the equilibrium returns (r_1, r_2) and β coefficients of the risky assets for changing absolute risk aversion (CARA) coefficient θ and correlation coefficient ρ. It is found that $r_1 < r_2$ and $\beta_1 < 1 < \beta_2$. With respect to the risk aversion coefficient, one can see from Figure 6.1 panels (A1) and (B1) that, as investors become more risk averse, returns of the risky assets increase significantly and the beta coefficient of the first risky assets decreases while the beta coefficient of the second risky assets increases. Also, it follows from Proposition 6.1 (iv) that, for the given $\sigma_i (i = 1, 2)$, if $\rho < -\sigma_1/\sigma_2 (= -0.8)$, the return of the first asset will decrease as agents become more risk averse. Hence one of the asset returns may decrease when two risky assets are highly negatively correlated.

From Proposition 6.1 (v), one can see that an increase in correlation of asset payoffs improves the returns of the risky assets and the market portfolio. This is clearly indicated in Figure 6.1 panel (A1). More interestingly, Figure 6.1 panels (A1) and (B1) indicate that the correlation coefficient ρ plays a less significant role in determining the equilibrium return but a more significant role in determining the β of the assets. On the other hand, the risk aversion coefficient has a more significant impact on the equilibrium return but a less significant impact on the β of the assets.

Based on the above analysis, one can see that the equilibrium returns of the risky assets are strongly influenced by the change of the CARA coefficient, followed by the standard deviation, the correlation coefficient, and the expected payoff of the assets. As far as the beta coefficients are concerned, they are mostly influenced by changes of the correlation coefficient, followed by the standard deviation, the CARA coefficient and the expected payoff of the assets. Overall, both the returns and beta coefficients are strongly influenced by changes in the standard deviation and weakly influenced by changes in the expected payoff of the assets. This observation underlines the significant impact of heterogeneity in the variance/covariance to be discussed below.

We now consider various aspects of heterogeneity among the two agents and examine the impacts of these heterogeneities on the equilibrium demands in the optimal portfolio of investors, the equilibrium returns of risky assets and the market portfolio, and the corresponding β coefficients for the risky assets.

6.2. **Case 2.** First, we assume that agents are homogeneous except for having heterogeneous beliefs about the correlation coefficients of the risky assets ρ_1 and ρ_2. Figure 6.2 panels (a2), (b2) and (c2) illustrate the impact on the equilibrium demands for

[3]In all the figures, the expected return $\mu_{i,j}$, rather than the expected payoff $y_{i,j}$, is used for convenience. Since $dy_{i,j}/d\mu_{i,j} > 0$, this replacement does not change the results.

the risky assets (z_{11}, z_{12}) for investor-1, the equilibrium returns of risky assets (r_1, r_2), and the corresponding beta coefficients $(\beta_i, i = 1, 2)$, respectively. Unlike Case 1, the optimal demand for risky asset-j of agent 1 satisfies

$$\frac{\partial z_{1i}}{\partial \rho_1} < 0, \quad i = 1, 2.$$

Intuitively, because of $r_1 < r_2$, the optimal demand of investor-1 for asset 1 (asset 2) is lower (higher) when the asset returns are highly correlated. It is also found that

$$r_1 < r_m < r_2, \qquad \beta_1 < 1 < \beta_2$$

and

$$\frac{\partial r_i}{\partial \rho_i} > 0, \qquad \frac{\partial \beta_1}{\partial \rho_i} > 0, \qquad \frac{\partial \beta_2}{\partial \rho_i} < 0 \qquad (i = 1, 2).$$

The impact of heterogeneous risk aversion coefficients is illustrated in Figure 6.2 panels (a1), (b1) and (c1). We observe very similar features to the homogeneous case except that the optimal demands of the investors change dramatically.

6.3. **Case 3.** We now consider the case in which two agents are heterogeneous in their expected payoffs of the risky assets but homogeneous in their variance/covariance beliefs. For fixed expected payoff for agent-2, the impact of the heterogeneous expected payoffs of agent-1 is illustrated in Figure panels 6.3 (a3), (b3) and (c3). The optimal demand of agent-1 changes as his/her expected payoffs change. Intuitively, agent-1 optimally holds less (more) share of the asset with lower (higher) expected return. For agent-1, given an expected return of asset-2, as his/her expected return of asset-1 increases, the equilibrium return of asset-1 decreases slightly while the equilibrium return of asset-2 does not change. Correspondingly, $\beta_1 < 1 < \beta_2$. We observe that changing heterogeneous expected returns has a significant impact on the optimal demands of investors, but has an insignificant effect on the equilibrium returns and beta coefficients.

6.4. **Case 4.** We now add one more dimension to the discussion in Case 3 by assuming that agents can have different beliefs on the correlation coefficients of the two risky asset returns, for example, $(\rho_1, \rho_2) = (0, 0)$ and $(-0.5, 0.5)$. It is found that there is no significant difference from what we have observed in Case 3, except lower or negative correlation among two assets reduces the overall returns of the risky assets.

Based on the above two cases, we have found that, with respect to the equilibrium returns and the betas of risky assets, heterogeneous beliefs in mean and correlation structure do not generate much difference from the benchmark homogeneous case. However, such heterogeneity leads to significant changes in agents' optimal portfolio positions, which may contribute to high trading volumes in the market.

6.5. **Case 5.** In this case we assume that agents have heterogeneous beliefs about the variance of asset returns but have homogeneous beliefs about the expected returns. Fig. 6.3 panels (a4), (b4) and (c4) illustrate the impact of such heterogeneity. One can see that

$$\frac{\partial z_{11}}{\partial \bar{\sigma}_{11}} < 0, \quad \frac{\partial z_{12}}{\partial \bar{\sigma}_{11}} > 0, \quad \frac{\partial r_1}{\partial \bar{\sigma}_{11}} > 0, \quad \frac{\partial \beta_1}{\partial \bar{\sigma}_{11}} > 0, \quad \frac{\partial \beta_2}{\partial \bar{\sigma}_{11}} < 0.$$

For fixed $\bar{\sigma}_{12}$, there exists a $\bar{\sigma}_{11}^* = \bar{\sigma}_{11}^*(\bar{\sigma}_{12})$ (in fact $\bar{\sigma}_{11}^* \approx 0.114$ and 0.214 for $\bar{\sigma}_{12} = 0.1$ and 0.15, respectively) such that $\frac{\partial \bar{\sigma}_{11}^*}{\partial \bar{\sigma}_{12}} > 0$,

$$r_1 = r_2 = r_m, \quad \beta_1 = \beta_2 = 1 \quad \text{for } \bar{\sigma}_{11} = \bar{\sigma}_{11}^*$$

and

$$r_1 < r_m < r_2, \quad \beta_1 < 1 < \beta_2 \quad \text{for } \bar{\sigma}_{11} < \bar{\sigma}_{11}^*,$$

$$r_1 > r_m > r_2, \quad \beta_1 > 1 > \beta_2 \quad \text{for } \bar{\sigma}_{11} > \bar{\sigma}_{11}^*$$

Similar features are also found for various combinations of (ρ_1, ρ_2), such as $(\rho_1, \rho_2) = (0, 0), (-0.5, -0.5)$, except that the levels of returns increase as $\rho = \rho_1 = \rho_2$ increases. This feature is also found in the homogeneous case.

6.6. **Case 6.** We now assume that agents have heterogeneous beliefs about both expected returns and variance/covariance. Calculations (not reported here) show that there is no significant difference for the equilibrium returns and betas compared to Case 5.

Based on the discussion in Cases 5 and 6, we can see that heterogeneity in variance/covariance has a significant impact on agents' equilibrium demands of the risky assets, equilibrium returns and beta coefficients of the risky assets, in particular, for volatility $\bar{\sigma}_{11}$ near the critical value $\bar{\sigma}_{11}^*$. For example, for fixed $\bar{\sigma}_{12} = 0.1, \bar{\sigma}_{21} = 0.12, \bar{\sigma}_{22} = 0.15$ and $\rho_1 = \rho_2 = 0.5$, the following table shows the impact of different subjective volatilities of agent-1 on asset-1. A 2% difference of agent-1's subjective volatility $\bar{\sigma}_{11}$ on the first risky asset generates an excess return of 1.7% for the first risky asset, 0.2% for the second asset and 1% for the market portfolio. It also generates a significant change for both beta coefficients. The first asset changes from the least risky (with $\beta_1 = 0.945$) to the most risky (with $\beta = 1.018$) while the changes are other way around for the second asset. This simple example suggests that a higher risk premium of a risky asset may be due to the heterogeneous beliefs about variance and covariance among the agents.

$\bar{\sigma}_{11}$	r_1	r_2	r_m	β_1	β_2
0.10	0.144	0.155	0.149	0.945	1.055
0.12	0.161	0.157	0.159	1.018	0.982

TABLE 6.1. Impact of heterogeneity of σ_{11} for fixed $\sigma_{12} = 0.10$.

7. CONCLUSION

This paper provides an aggregation procedure for the construction of a market consensus belief from the heterogeneous beliefs of different investors. This allows us to characterize the market equilibrium in the traditional mean-variance model under the consensus belief. Various impacts of heterogeneity are discussed. In particular, the impact of diversity of heterogeneous beliefs is examined. In principle, we show that the market aggregation behavior is a weighted average of heterogeneous individual behavior, a very intuitive result. These weights are proportional to the individual risk tolerance and covariance matrix. For example, the market equilibrium price reflects a weighted average of the individuals equilibrium prices under their beliefs. We have established an equilibrium relation between the market aggregate expected payoff of the risky assets and the market portfolio's expected payoff, which leads to the CAPM-like relationship under heterogeneous beliefs. Our result also provides a simple explanation for the empirical relation between cross-sectional volatility and expected returns.

This paper provides a simple framework for dealing with heterogeneous beliefs and aggregation. The intuition and results obtained in this paper can be extended to a dynamic setting and this may help us to understand various types of market behaviors, such as, long swings of the market price away from the fundamental price, market booms and crashes, herding, volatility clustering, long-range dependence, the risk premium puzzle and the relation between cross-sectional volatility and expected returns to name the most significant. This task is left for future research.

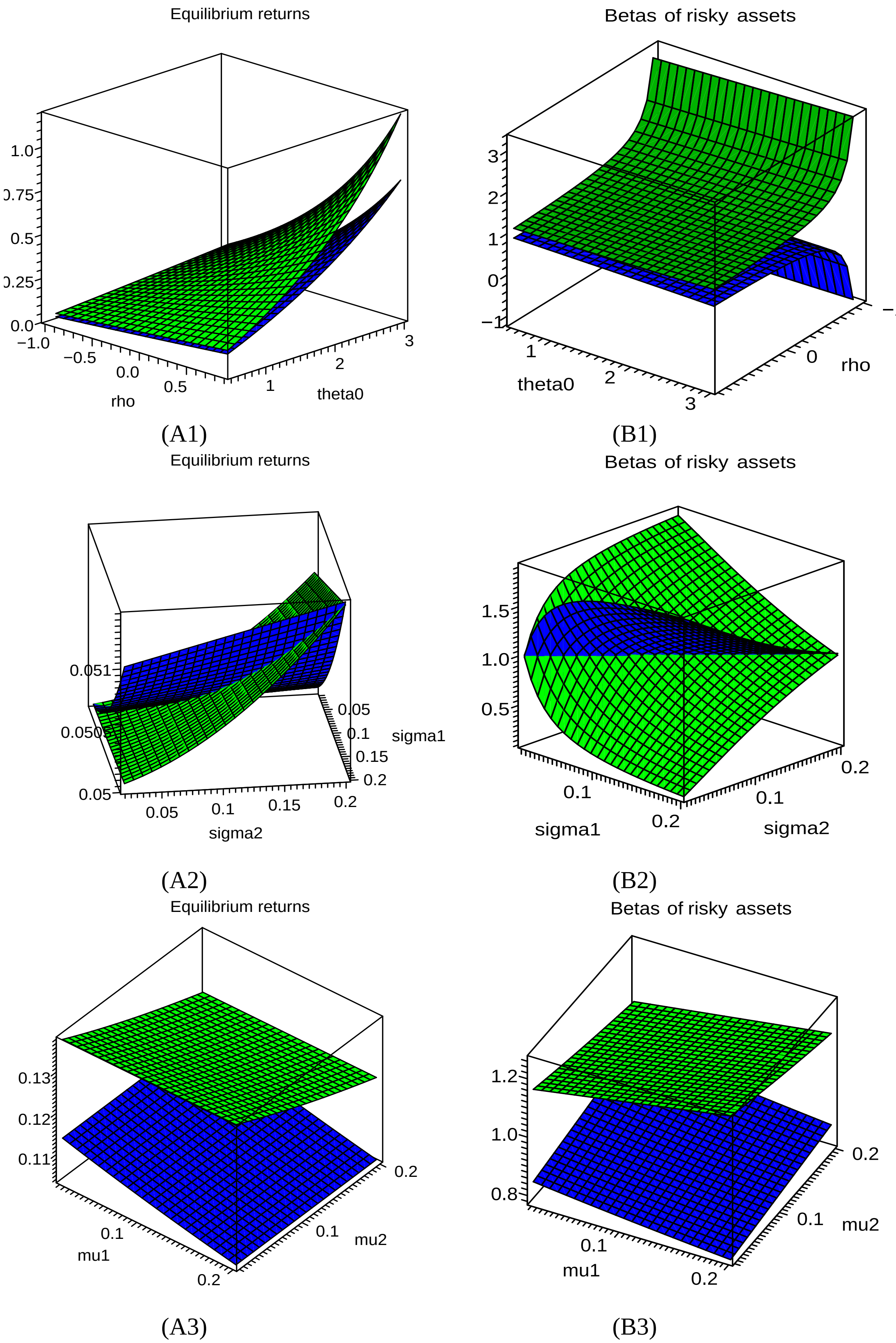

FIGURE 6.1. Effect of homogeneous risk aversion, correlation coefficient, expected return and variance.

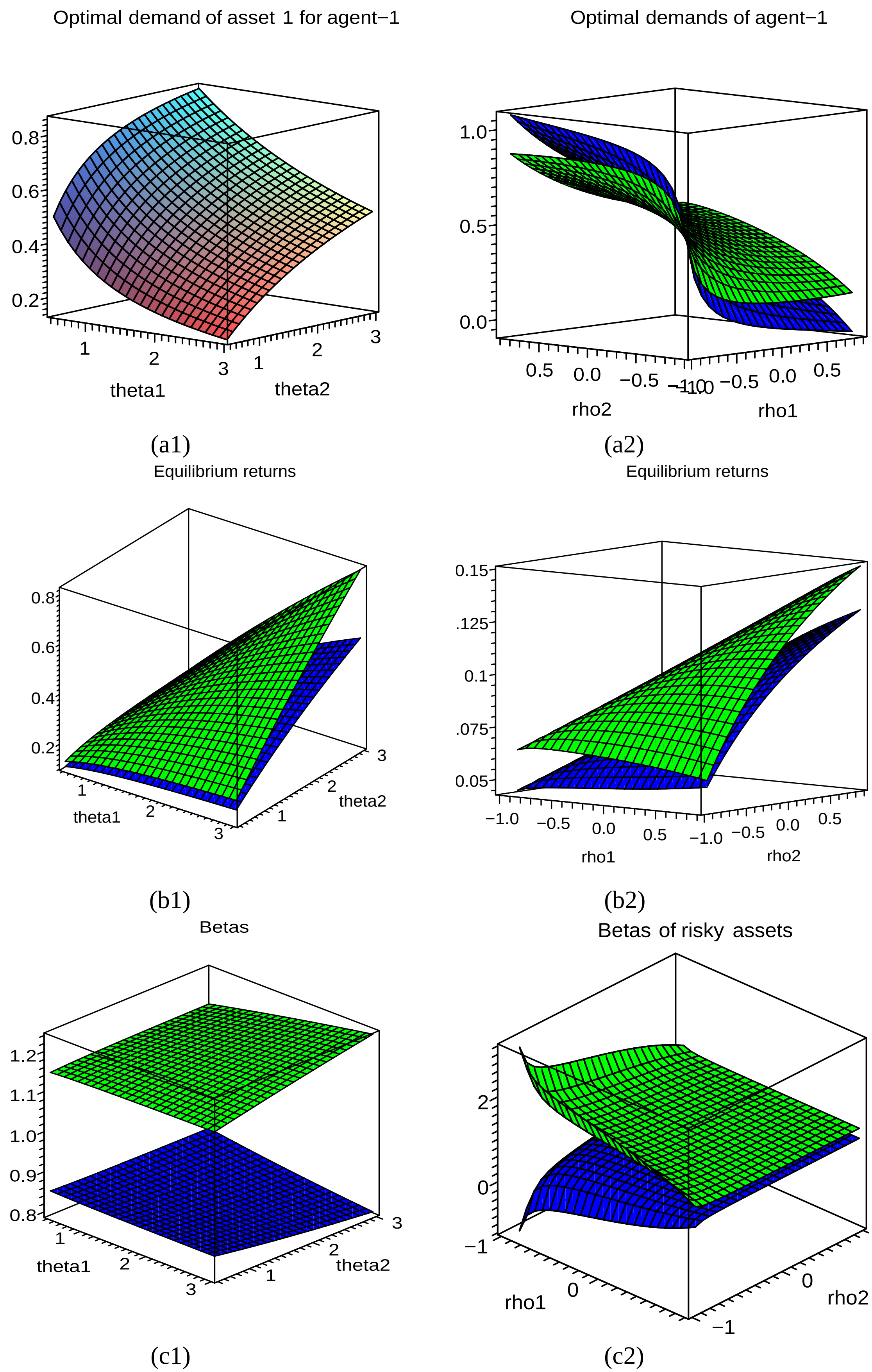

FIGURE 6.2. Effect of heterogeneous risk aversion (a1, b1, c1) and correlation coefficient (a2, b2, c2).

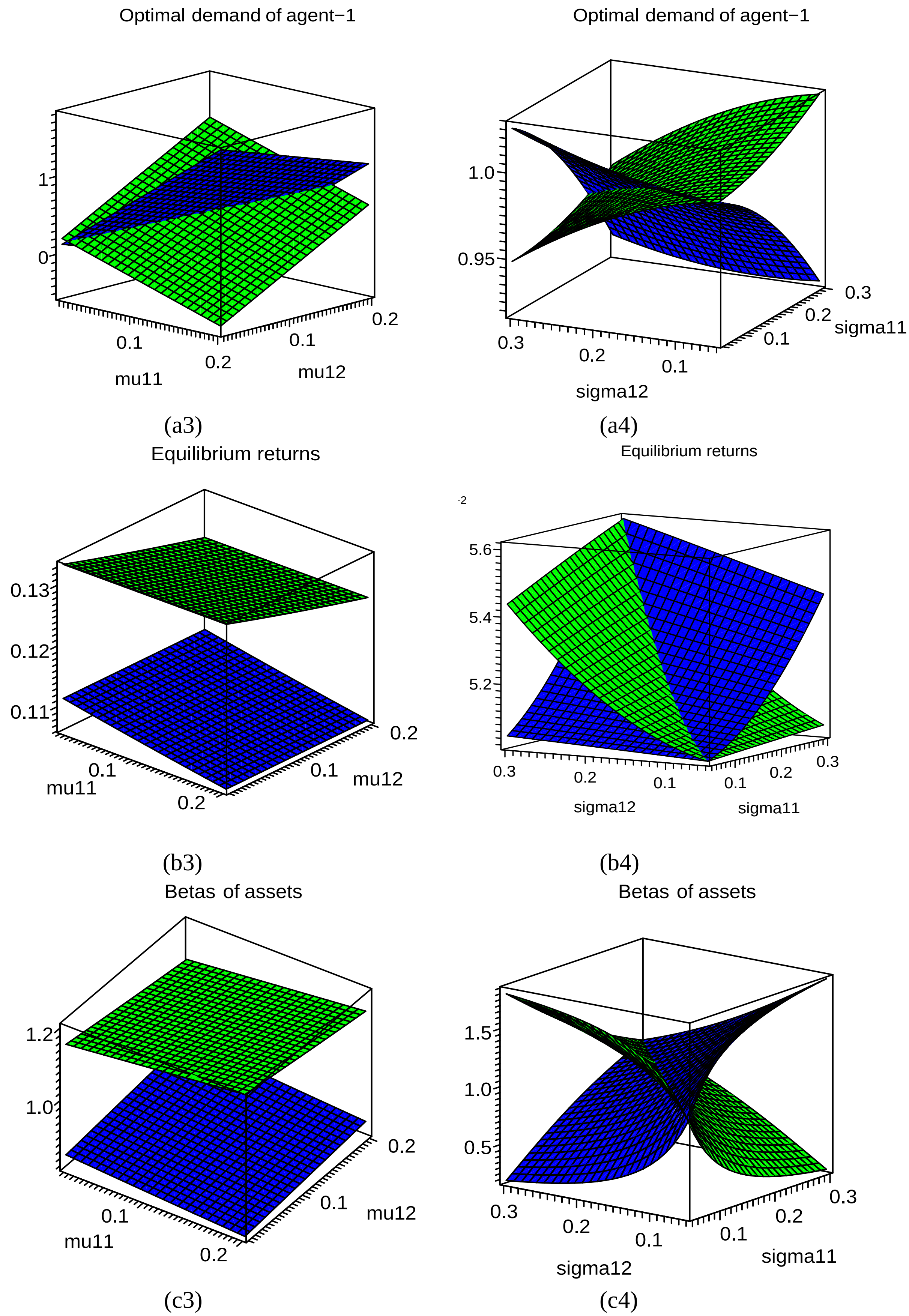

FIGURE 6.3. Effect of heterogeneous expected return (a3, b3, c3) and standard deviation (a4, b4, c4).

APPENDIX A. PROOF OF PROPOSITION 6.1

In the homogeneous case, we have $\theta_i = \theta, \sigma_{ij} = \sigma_j, y_{ij} = y_j, \rho_j = \rho$ for $i, j = 1, 2$. It follows that $\Theta = \theta$, $W_m = \mathbf{a}_m^T \mathbf{p}_o = p_{1o} + p_{2o}$, $\sigma_m^2 = \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2$ and $\mathbf{z}_i = \mathbf{z}_m/2$. Also, from Proposition 3.2,

$$p_{1o} = \frac{y_1}{R_f} - \Theta\sigma_1(\sigma_1 + \rho\sigma_2), \qquad p_{2o} = \frac{y_2}{R_f} - \Theta\sigma_2(\sigma_2 + \rho\sigma_1) \tag{A.1}$$

and

$$\beta_1 = \frac{\sigma_1(\sigma_1 + \rho\sigma_2)}{p_1^*(p_1^* + p_2^*)\sigma_m^2}, \qquad \beta_2 = \frac{\sigma_2(\sigma_2 + \rho\sigma_2)}{p_2^*(p_1^* + p_2^*)\sigma_m^2}. \tag{A.2}$$

Note that $r_i = y_i/p_{io} - 1 (i = 1, 2)$. In the following, we illustrate just the proof of (ii) since the rest of Proposition 6.1 follows similarly and hence is omitted.

It follows from (A.1) that

$$\frac{\partial r_1}{\partial y_1} = \frac{-\Theta\sigma_1(\sigma_1 + \rho\sigma_2)}{(p_1^*)^2}, \qquad \frac{\partial r_2}{\partial y_1} = 0.$$

By using $r_i = R_f + \beta_i(r_m - R)$, we have

$$\frac{\partial r_1}{\partial y_1} = \frac{\partial \beta_1}{\partial y_1}[r_m - R_f] + \beta_1 \frac{\partial r_m}{\partial y_1}.$$

Note that $r_m = \frac{y_1 + y_2}{p_{1o} + p_{2o}}$. Then using (A.2), we obtain

$$[\mu_m - R_f]\frac{\partial \beta_1}{\partial y_1} = \frac{\Theta\sigma_1(\sigma_1 + \rho\sigma_2)}{p_1^*}\left[\frac{1}{(p_1^* + p_2^*)^3} - \frac{1}{(p_1^* + p_2^*)}\right].$$

This implies that $\frac{\partial \beta_1}{\partial y_1} > 0$ if and only if $\sigma_1 + \rho\sigma_2 > 0$.

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