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Research Paper 179 **July 2006**

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ISSN 1441-8010 www.qfrc.uts.edu.au

On Weak Predictor-Corrector Schemes for Jump-Diffusion Processes in Finance

Nicola Bruti-Liberati[1] and **Eckhard Platen**[2]

Abstract. Event-driven uncertainties such as corporate defaults, operational failures or central bank announcements are important elements in the modelling of financial quantities. Therefore, stochastic differential equations (SDEs) of jump-diffusion type are often used in finance. We consider in this paper weak discrete time approximations of jump-diffusion SDEs which are appropriate for problems such as derivative pricing and the evaluation of risk measures. We present regular and jump-adapted predictor-corrector schemes with first and second order of weak convergence. The regular schemes are constructed on regular time discretizations that do not include jump times, while the jump-adapted schemes are based on time discretizations that include all jump times. A numerical analysis of the accuracy of these schemes when applied to the jump-diffusion Merton model is provided.

200 *Mathematics Subject Classification:* primary 60H10; secondary 65C05.
JEL Classification: G10, G13, C63.

Key words and phrases: weak approximations, Monte Carlo simulation, predictor-corrector schemes, jump diffusions.

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1 Introduction

Several empirical studies indicate that the dynamics of financial quantities exhibit jumps, see Jorion (1988), Bates (1996), Das (2002) and Johannes (2004). Announcements by central banks, for instance, create jumps in the evolution of interest rates. Moreover, events such as corporate defaults and operational failures have a strong impact on financial quantities. These events cannot be properly modelled by pure diffusion processes. Therefore, several financial models are specified in terms of jump diffusions via their corresponding stochastic differential equations (SDEs), see Merton (1976), Björk, Kabanov & Runggaldier (1997), Duffie, Pan & Singleton (2000), Kou (2002) and Glasserman & Kou (2003).

The class of jump-diffusion SDEs that admits explicit solutions is rather limited. Therefore, it is important to develop discrete time approximations. An important application of these methods arises in the pricing and hedging of interest rate derivatives under the LIBOR market model. Since the arbitrage-free dynamics of the LIBOR rates are specified by non-linear multi-dimensional SDEs, Monte Carlo simulation with discrete time approximations is the typical technique used for pricing and hedging. Recently, LIBOR market models with jumps have appeared in the literature, see Glasserman & Kou (2003) and Samuelides & Nahum (2001). Here efficient schemes for SDEs with jumps are needed.

Discrete time approximations of SDEs can be divided into the classes of strong and weak schemes. In the current paper we study weak schemes which provide an approximation of the probability measure and are suitable for problems such as derivative pricing, the evaluation of moments, risk measures and expected utilities. Strong schemes, instead, provide pathwise approximations which are appropriate for scenario simulation, filtering and hedge simulation, see Kloeden & Platen (1999).

A discrete time approximation Y^Δ converges weakly with order β to X at time T, if for each $g \in \mathcal{C}_P^{2(\beta+1)}(\mathbb{R}^d, \mathbb{R})$ there exists a positive constant C, independent of Δ, and a positive and finite number $\Delta_0 > 0$, such that

$$\varepsilon_w(\Delta) := |E(g(X_T)) - E(g(Y_T^\Delta))| \leq C\Delta^\beta, \tag{1.1}$$

for each $\Delta \in (0, \Delta_0)$. Here we denote by $\mathcal{C}_P^{2(\beta+1)}(\mathbb{R}^d, \mathbb{R})$ the space of $2(\beta+1)$ continuously differentiable functions which, together with their partial derivatives of order up to $2(\beta+1)$, have polynomial growth. This means that for any $g \in \mathcal{C}_P^{2(\beta+1)}(\mathbb{R}^d, \mathbb{R})$ there exist constants $K > 0$ and $r \in \{1, 2, \ldots\}$, depending on g, such that

$$|\partial_y^j g(y)| \leq K(1 + |y|^{2r}),$$

for all $y \in \mathbb{R}^d$ and any partial derivative $\partial_y^j g(y)$ of order $j \leq 2(\beta+1)$.

In the case of pure diffusion SDEs there is a substantial body of research on discrete time approximations, see Kloeden & Platen (1999). The literature on weak approximations of jump-diffusion SDEs, instead, is rather limited, see Mikulevicius & Platen (1988), Liu & Li (2000), Kubilius & Platen (2002), Glasserman & Merener (2003) and Higham & Kloeden (2005, 2006). In this paper we propose several new weak predictor-corrector schemes for jump-diffusion SDEs with first and second order of weak convergence.

For pure diffusion SDEs arising in applications to LIBOR market models, specific weak predictor-corrector schemes have been proposed and analyzed in Hunter, Jäckel & Joshi (2001) and Joshi & Stacey (2006). These authors show that for the numerical approximation of the non-linear dynamics of discrete forward rates, predictor-corrector schemes outperform the simpler Euler scheme and allow the use of a single time step within reasonable accuracy. The weak predictor-corrector schemes proposed in the current paper can be applied to pricing and hedging of complex interest rate derivatives under LIBOR market models with jumps.

The paper is organized as follows. Section 2 introduces the class of jump-diffusion SDEs under consideration. In Section 3 we propose several weak predictor-corrector schemes for SDEs with jumps. These are divided into regular predictor-corrector schemes and jump-adapted predictor-corrector schemes. Finally, we present in Section 4 a numerical study of these schemes applied to the jump-diffusion Merton model.

2 Model Dynamics

The continuous uncertainty is modeled with an $\underline{\mathcal{A}}$-adapted m-dimensional standard Wiener process denoted by $W = \{W_t = (W_t^1, \ldots, W_t^m)^\top, t \in [0,T]\}$, while the event-driven uncertainty is represented by an $\underline{\mathcal{A}}$-adapted r-dimensional compound Poisson process denoted by $J = \{J_t = (J_t^1, \ldots, J_t^r)^\top, t \in [0,T]\}$. Each component J_t^k, for $k \in \{1, 2, \ldots, r\}$, of the r-dimensional compound Poisson process $J = \{J_t = (J_t^1, \ldots, J_t^r)^\top, t \in [0,T]\}$ is defined by

$$J_t^k = \sum_{i=1}^{N_t^k} \xi_i^k,$$

where $N^1, \ldots, N^r$ are r independent standard Poisson processes with constant intensities $\lambda^1, \ldots, \lambda^r$, respectively. Let us note that each component of the compound Poisson process J^k generates a sequence of pairs $\{(\tau_i^k, \xi_i^k), i \in \{1, 2, \ldots, N_T^k\}\}$ of jump times and marks. We will denote with $F^k(\cdot)$ the distribution function of the marks ξ_i^k, for $i \in \{1, 2, \ldots, N_T^k\}$, generated by the kth Poisson process N^k.

We consider the dynamics of the underlying d-dimensional factors specified with the jump-diffusion SDE

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t + c(t, X_{t-})\, dJ_t, \tag{2.1}$$

for $t \in [0,T]$, with $X_0 \in \mathbb{R}^d$. Here $a(t,x)$ is a d-dimensional vector of real valued functions on $[0,T] \times \mathbb{R}^d$, while $b(t,x)$ and $c(t,x)$ are a $d \times m$-matrix of real valued functions on $[0,T] \times \mathbb{R}^d$ and a $d \times r$-matrix of real valued functions on $[0,T] \times \mathbb{R}^d$, respectively. Moreover, we denote by $Z_{t-} = \lim_{s \Uparrow t} Z_s$ the almost sure left-hand limit of $Z = \{Z_s, s \in [0,T]\}$ at time t. Let us note that in the following we adopt a superscript to denote vector components, which means, for instance, $a = (a^1, \ldots, a^d)^\top$. Moreover, we write b^i and c^i to denote the ith column of matrixes b and c, respectively.

We assume that the coefficient functions a, b and c satisfy the usual linear growth and Lipschitz conditions sufficient for the existence and uniqueness of a strong solution of (2.1), see Øksendal & Sulem (2005). Moreover, when we will indicate the orders of weak convergence of the approximations to be presented in Section 3 we will assume that smoothness and integrability conditions similar to those required in Kloeden & Platen (1999) for pure diffusion SDEs are satisfied. The specific conditions along with a proof of the convergence theorem will be given in forthcoming work.

If we choose multiplicative coefficients in the one-dimensional case with one Wiener and one Poisson process, $d = m = r = 1$, then we obtain the SDE

$$dX_t = X_{t-} \left(\mu dt + \sigma dW_t + dJ_t \right), \tag{2.2}$$

which describes the *jump-diffusion Merton model*, see Merton (1976). For this linear SDE we have the explicit solution

$$X_t = X_0 \, e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \prod_{i=1}^{N_t} (1 + \xi_i), \tag{2.3}$$

which we will use in Section 4 for a numerical study. In Merton (1976) $(1+\xi_i) = e^{\zeta_i}$ is the ith outcome of a log-normal random variable with $\zeta_i \sim \mathcal{N}(\varrho, \varsigma)$. If instead $(1+\xi_i)$ is drawn from a log-Laplace random variable we recover the *Kou model*, see Kou (2002). Moreover, a simple degenerate case arises when $(1+\xi_i)$ is a positive constant.

Other important examples of jump-diffusion dynamics of the form (2.1) arise in LIBOR market models. Samuelides & Nahum (2001), for instance, consider a LIBOR market model with jumps for pricing short-term interest rate derivatives. Given a set of equidistant tenor dates $T_1, \ldots, T_{d+1}$, with $T_{i+1} - T_i = \delta$ for $i \in \{1, \ldots, d\}$, the components of the vector $X_t = (X_t^1, \ldots, X_t^d)^\top$ represent discrete forward rates at time t maturing at tenor dates $T_1, \ldots, T_d$, respectively. Moreover, they consider one driving Wiener process, $m = 1$, and two driving Poisson processes, $r = 2$. The diffusion coefficient is specified as $b(t,x) = \sigma x$, with σ a d-dimensional vector of positive numbers, and the jump coefficient $c(t,x) = \beta x$, where β is a $d \times 2$-matrix with $\beta^{i,1} > 0$ and $\beta^{i,2} < 0$, for $i \in \{1, \ldots, d\}$. In this way the first jump process generates upward jumps, while the second jump process creates downward jumps. Moreover, the marks are set to $\xi_i = 1$ so that the two driving jump processes are standard Poisson processes. A no-arbitrage restriction on the evolution of forward rates under the T_{d+1}-forward measure, see Björk, Kabanov & Runggaldier (1997) and Glasserman & Kou (2003), imposes a particular form on the non-linear drift coefficient $a(t, x)$ whose ith component is given by

$$\begin{aligned} a^i(t,x) &= -\Big\{ \sum_{j=i+1}^{d} \frac{\delta x^j}{1+\delta x^j}\sigma^j + \lambda^1 \prod_{j=i+1}^{d} \big(1+\beta^{j,1}\frac{\delta x^j}{1+\delta x^j}\big) \\ &\quad +\lambda^2 \prod_{j=i+1}^{d} \big(1+\beta^{j,2}\frac{\delta x^j}{1+\delta x^j}\big)\Big\}. \end{aligned} \tag{2.4}$$

A complex non-linear drift coefficient, as that in (2.4), is a typical feature of LIBOR market models. Therefore, it makes the application of numerical techniques essential in the pricing of complex interest rate derivatives.

To recover some empirical features observed in the market, it is sometimes important to consider a jump behavior more general than that driving the SDE (2.1). By considering jump-diffusion SDEs driven by a Poisson random measure it is possible to introduce, for instance, state-dependent intensities. The numerical schemes to be presented can be naturally extended to the case with Poisson random measures, see Bruti-Liberati & Platen (2006).

3 Weak Predictor-Corrector Schemes

In this section we present several discrete time weak approximations of the jump-diffusion SDE (2.1). First we consider regular schemes based on regular time discretizations which do not include jump times of the Poisson processes. Then we present jump-adapted schemes constructed on time discretizations which include all jump times.

3.1 Regular Weak Predictor-Corrector Schemes

We consider an equidistant time discretization $0 = t_0 < t_1 < \ldots < t_{\bar{n}} = T$, with $t_n = n\Delta$ and step size $\Delta = \frac{T}{\bar{n}}$, for $n \in \{0, 1, \ldots, \bar{n}\}$ and $\bar{n} \in \{1, 2, \ldots\}$. We denote a corresponding discrete time approximation of the solution X of the SDE (2.1) by $Y^\Delta = \{Y_n^\Delta, n \in \{0, 1, \ldots, \bar{n}\}\}$.

Before introducing advanced predictor-corrector schemes, we present the *Euler scheme* which is given by

$$Y_{n+1} = Y_n + a\Delta + \sum_{j=1}^{m} b^j \Delta W_n^j + \sum_{k=1}^{r} c^k \hat{\xi}_n^k \Delta p_n^k, \tag{3.1}$$

for $n \in \{0, 1, \ldots, \bar{n} - 1\}$, with initial value $Y_0 = X_0$. For ease of notation we omit here and in the following the dependence on time and state variables in the coefficients of the scheme, this means we simply write a for $a(t_n, Y_n)$, etc.

In (3.1) we denote by $\Delta W_n^j = W_{t_{n+1}}^j - W_{t_n}^j \sim \mathcal{N}(0, \Delta)$ the nth increment of the jth Wiener process W^j and by $\Delta p_n^k = N_{t_{n+1}}^k - N_{t_n}^k \sim Poiss(\lambda^k \Delta)$ the nth increment of the kth Poisson process N^k with intensity λ^k. Moreover, $\hat{\xi}_n^k$ is the nth independent outcome of a random variable with given probability distribution function $F^k(\cdot)$. The Euler scheme achieves, in general, weak order of convergence $\beta = 1$.

It is possible to replace the Gaussian and Poisson random variables ΔW_n^j and Δp_n^k with simpler multi-point distributed random variables that satisfy certain moment-matching conditions, see Bruti-Liberati & Platen (2006). For instance, if we use in (3.1) the two-point distributed random variables $\Delta \widehat{W}_n^j$ and $\Delta \hat{p}_n^k$, where

$$P(\Delta \widehat{W}_n^j = \pm\sqrt{\Delta}) = \frac{1}{2} \tag{3.2}$$

for $j \in \{1, \ldots, m\}$, and

$$P\Big(\Delta \hat{p}_n^k = \frac{1}{2}(1 + 2\lambda^k \Delta \pm \sqrt{1 + 4\lambda^k \Delta})\Big) = \frac{1 + 4\lambda^k \Delta \mp \sqrt{1 + 4\lambda^k \Delta}}{2\,(1 + 4\lambda^k \Delta)} \tag{3.3}$$

for $k \in \{1, \ldots, r\}$, then we obtain the *simplified Euler scheme* which still achieves weak order of convergence $\beta = 1$. Let us note that this scheme can be implemented in a highly efficient manner by resorting to random bit generators and hardware accelerators, as shown for pure diffusion SDEs in Bruti-Liberati & Platen (2004) and Bruti-Liberati, Platen, Martini & Piccardi (2005).

As indicated in Hofmann & Platen (1996) for pure diffusion SDEs and in Higham & Kloeden (2005, 2006) for jump-diffusion SDEs, explicit schemes have narrower regions of numerical stability than corresponding implicit schemes. For this reason implicit schemes for diffusion and jump-diffusion SDEs have been proposed. Despite their better numerical stability properties, implicit schemes carry, in general, an additional computational burden since they usually require the solution of an algebraic equation at each time step. Therefore, in choosing between an explicit and an implicit scheme one faces a trade-off between computational efficiency and numerical stability.

Predictor-corrector schemes are designed to retain the numerical stability properties of similar implicit schemes, while avoiding the additional computational effort required for solving an algebraic equation in each time step. This is achieved with the following procedure implemented at each time step: at first an explicit scheme is generated, the so-called predictor, and afterwards a de facto implicit scheme is used as corrector. The corrector is made explicit by using the predicted value $\bar{Y}_{n+1}$, instead of Y_{n+1}. The orders of weak convergence of the predictor-corrector schemes to be presented can be obtained by applying the Wagner-Platen expansion for jump-diffusion SDEs, see Platen (1982). We refer to Bruti-Liberati & Platen (2006) for the weak convergence of explicit and implicit approximations for SDEs with jumps.

The *weak order one predictor-corrector scheme* has corrector

$$Y_{n+1} = Y_n + \frac{1}{2}\left\{a(t_{n+1}, \bar{Y}_{n+1}) + a\right\}\Delta + \sum_{j=1}^{m} b^j \Delta W_n^j + \sum_{k=1}^{r} c^k\, \hat{\xi}_n^k\, \Delta p_n^k, \tag{3.4}$$

and predictor

$$\bar{Y}_{n+1} = Y_n + a\Delta + \sum_{j=1}^{m} b^j \Delta W_n^j + \sum_{k=1}^{r} c^k\, \hat{\xi}_n^k \Delta p_n^k. \tag{3.5}$$

The predictor-corrector scheme (3.4)–(3.5) achieves first order of weak convergence. Also in this case we can use the two-point distributed random variables (3.2) and (3.3) without affecting the order of weak convergence of the scheme. Let us note that the difference $Z_{n+1} := \bar{Y}_{n+1} - Y_{n+1}$ between the predicted and the corrected value provides an indication of the local error. This can be used to implement more advanced schemes with step size control based on Z_{n+1}.

A more general *family of weak order one predictor-corrector schemes* is given by the corrector

$$\begin{aligned} Y_{n+1} &= Y_n + \left\{\theta\, \bar{a}(t_{n+1}, \bar{Y}_{n+1}) + (1-\theta)\, \bar{a}\right\}\Delta \\ &\quad + \sum_{j=1}^{m} \left\{\eta\, b^j(t_{n+1}, \bar{Y}_{n+1}) + (1-\eta)\, b^j\right\}\Delta W_n^j + \sum_{k=1}^{r} c^k\, \hat{\xi}_n^k\, \Delta p_n^k, \end{aligned} \tag{3.6}$$

for $\theta, \eta \in [0,1]$, where

$$\bar{a} = a - \eta \sum_{j=1}^{m} \sum_{i=1}^{d} b^{i,j} \frac{\partial b^j}{\partial x^i}, \tag{3.7}$$

and the predictor (3.5). Here one can tune the degree of implicitness in the drift coefficient and in the diffusion coefficient by changing the parameters $\theta, \eta \in [0,1]$, respectively. Note that when the degree of implicitness η is different from zero, it is important to use bounded random variables as $\Delta \widehat{W}_n^j$ and $\Delta \hat{p}_n^k$ in an implicit scheme. These prevent the effect of possible divisions by zero in the algorithm, see Kloeden & Platen (1999). For a predictor-corrector method this can be computationally advantageous, but it is no longer required. One can still use the Gaussian and Poisson random variables, ΔW_n^j and Δp_n^k, in the above scheme (3.6).

By using the Wagner-Platen expansion for jump-diffusion SDEs, it is possible to derive higher order regular weak predictor-corrector schemes. However, these schemes are quite complex as they involve the generation of multiple stochastic integrals with respect to time, Wiener processes and Poisson processes.

3.2 Jump-Adapted Weak Predictor-Corrector Schemes

As introduced in Platen (1982), let us consider a *jump-adapted time discretization* $0 = t_0 < t_1 < \ldots < t_M = T$ constructed as follows. First, as in Section 3.1, we choose an equidistant time discretization $0 = \bar{t}_0 < \bar{t}_1 < \ldots < \bar{t}_{\bar{n}} = T$, with $\bar{t}_n = n\Delta$, for $n \in \{1, \ldots, \bar{n}\}$, and step size $\Delta = \frac{T}{\bar{n}}$. Then we simulate all jump times τ_i^k, for $i \in \{1, 2, \ldots, N_T^k\}$ and $k \in \{1, \ldots, r\}$, generated by the r Poisson processes, and superimpose these on the equidistant time discretization. The resulting jump-adapted time discretization includes all jump times τ_i^k of the r Poisson processes and all equidistant time points $\bar{t}_1, \ldots, \bar{t}_n$. Its maximum step size is then guaranteed to be not greater than $\Delta = \frac{T}{\bar{n}}$. Note that the number $M+1$ of points in the jump-adapted time discretization is random and, thus, changes in each simulation. It equals the total number of jumps τ_i^k of the r Poisson processes plus $\bar{n} + 1$. Therefore, the average number of grid points and, thus, of operations of jump-adapted schemes is for large intensity almost proportional to the total intensity $\bar{\lambda} = \sum_{k=1}^r \lambda^k$, which is defined as the sum of the intensities of the r Poisson processes.

From now on for convenience we use the notation $Y_{t_n} = Y_n$ and denote by $Y_{t_{n+1}-} = \lim_{s \Uparrow t_{n+1}} Y_s$ the almost sure left-hand limit of Y at time t_{n+1}.

Within a jump-adapted time discretization, by construction jumps arise only at discretization times and we can separate the diffusion part of the dynamics from the jump part. Therefore, the *jump-adapted Euler scheme* is given by

$$Y_{t_{n+1}-} = Y_{t_n} + a\Delta_{t_n} + \sum_{j=1}^{m} b^j \Delta W_{t_n}^i \tag{3.8}$$

and

$$Y_{t_{n+1}} = Y_{t_{n+1}-} + \sum_{k=1}^{r} c^k(t_{n+1}-, Y_{t_{n+1}-})\Delta J_{t_{n+1}}^k, \tag{3.9}$$

for $n \in \{0 \ldots, M-1\}$, where $\Delta_{t_n} = t_{n+1} - t_n$ and $\Delta W_{t_n}^j = W_{t_{n+1}}^j - W_{t_n}^j \sim \mathcal{N}(0, \Delta_{t_n})$. Here $\Delta J_{t_{n+1}}^k$ equals $\xi_{N_{t_{n+1}}^k}^k$ if t_{n+1} is a jump time of the kth Poisson process or zero otherwise. The solution X follows a diffusion process between discretization points and is approximated by (3.8). If we encounter a jump time as discretization time, then the jump impact is simulated by (3.9). The jump-adapted Euler scheme has first order of weak convergence. By replacing the Gaussian random variable $\Delta W_{t_n}^j$ in (3.8) with the two-point random variable

$$P(\Delta \widehat{W}_{t_n}^j = \pm\sqrt{\Delta_{t_n}}) = \frac{1}{2}, \tag{3.10}$$

for $j \in \{1, \ldots, m\}$, we obtain the *jump-adapted simplified Euler scheme* which still achieves first order of weak convergence.

The *jump-adapted weak order one predictor-corrector scheme* is given by the corrector

$$Y_{t_{n+1}-} = Y_{t_n} + \frac{1}{2}\left\{a(t_{n+1}-, \bar{Y}_{t_{n+1}-}) + a\right\}\Delta + \sum_{j=1}^{m} b^j \Delta W_{t_n}^j, \tag{3.11}$$

the predictor

$$\bar{Y}_{t_{n+1}-} = Y_{t_n} + a\Delta_{t_n} + \sum_{j=1}^{m} b^j \Delta W_{t_n}^j, \tag{3.12}$$

and (3.9). This scheme achieves the same first order of weak convergence of the jump-adapted Euler scheme. Thanks to the quasi-implicitness in the drift it has, in general, better numerical stability properties. Also in this case it is possible to replace the Gaussian random variables in (3.11) and (3.12) with the two-point random variables (3.10).

A more general *family of jump-adapted weak order one predictor-corrector schemes* is given by the corrector

$$\begin{aligned} Y_{t_{n+1}-} &= Y_{t_n} + \left\{\theta\bar{a}(t_{n+1}-, \bar{Y}_{t_{n+1}-}) + (1-\theta)\bar{a}\right\}\Delta \\ &\quad + \sum_{j=1}^{m}\left\{\eta\, b^j(t_{n+1}-, \bar{Y}_{t_{n+1}-}) + (1-\eta)\, b^j\right\}\Delta W_n^j \end{aligned} \tag{3.13}$$

for $\theta, \eta \in [0,1]$. Here $\bar{a}$ is defined as in (3.7) and the predictor as in (3.12) again together with relation (3.9). This scheme achieves in general first order of weak convergence. Also in this case one can use the two-point random variables (3.10).

Within the class of jump-adapted schemes we can derive higher order weak predictor-corrector schemes which do not involve multiple stochastic integrals with respect to the Poisson processes. By using a second order weak implicit scheme as corrector and a second order weak explicit scheme as predictor, we obtain the *jump-adapted weak order two predictor-corrector scheme*. It is given by the corrector

$$Y_{t_{n+1}-} = Y_{t_n} + \frac{1}{2}\Big\{a(t_{n+1}-, \bar{Y}_{t_{n+1}-}) + a\Big\}\Delta_{t_n} + \Psi_{t_n}, \tag{3.14}$$

with

$$\Psi_{t_n} = \sum_{j=1}^{m}\Big\{b^j + \frac{1}{2}L^0 b^j \Delta_{t_n}\Big\}\Delta W_{t_n}^j + \frac{1}{2}\sum_{j_1,j_2=1}^{m} L^{j_1} b^{j_2}\Big(\Delta W_{t_n}^{j_1}\Delta W_{t_n}^{j_2} + V_{t_n}^{j_1,j_2}\Big), \tag{3.15}$$

the predictor

$$\bar{Y}_{t_{n+1}-} = Y_{t_n} + a\Delta_{t_n} + \Psi_{t_n} + \frac{1}{2}L^0 a\,(\Delta_{t_n})^2 + \frac{1}{2}\sum_{j=1}^{m} L^j a \Delta W_{t_n}^j \Delta_{t_n} \tag{3.16}$$

and relation (3.9). The differential operator L^0 is defined by

$$L^0 := \frac{\partial}{\partial t} + \sum_{i=1}^{d} a^i(t,x)\frac{\partial}{\partial x^i} + \frac{1}{2}\sum_{i,k=1}^{d}\sum_{j=1}^{m} b^{i,j}(t,x)b^{k,j}(t,x)\frac{\partial^2}{\partial x^i \partial x^j} \tag{3.17}$$

and the operator L^j by

$$L^j := \sum_{i=1}^{d} b^{i,j}(t,x)\frac{\partial}{\partial x^i}, \tag{3.18}$$

for $j \in \{1,\dots,m\}$. The random variables $V_{t_n}^{j_1,j_2}$ are two-point distributed with

$$P(V_{t_n}^{j_1,j_2} = \pm\sqrt{\Delta_{t_n}}) = \frac{1}{2}, \tag{3.19}$$

for $j_2 \in \{1,\dots,j_1-1\}$, where

$$V_{t_n}^{j_1,j_1} = -\Delta_{t_n} \tag{3.20}$$

and

$$V_{t_n}^{j_1,j_2} = -V_{t_n}^{j_2,j_1} \tag{3.21}$$

for $j_2 \in \{j_1+1,\dots,m\}$ and $j_1 \in \{1,\dots,m\}$. The Gaussian random variable $\Delta W_{t_n}^k$ can be replaced by the three-point random variable $\Delta\widetilde{W}_{t_n}^k$ defined by

$$P(\Delta\widetilde{W}_{t_n}^k = \pm\sqrt{3\Delta_{t_n}}) = \frac{1}{6}, \qquad P(\Delta\widetilde{W}_{t_n}^k = 0) = \frac{2}{3}, \tag{3.22}$$

for $k \in \{1,\dots,m\}$.

Let us finally remark that, although jump-adapted schemes are easier to derive and implement than corresponding regular schemes, their computational complexity is for large intensities almost proportional to the sum of the intensities of the driving Poisson processes. Therefore, while jump-adapted schemes should be in general preferred, in the approximation of SDEs driven by high intensity Poisson processes regular schemes are normally more efficient.

4 Numerical Results

For illustration in this section we present some numerical results obtained by applying some of the schemes described in Section 3 to the evaluation of a payoff function g of the solution X of (2.1) at a terminal time T. We discretize in time the dynamics of the solution X of the linear SDE (2.2) with one of the schemes in Section 3 and perform a Monte Carlo simulation to estimate $E\big(g(X_T)\big)$. Note that this numerical approximation generates a systematic error, resulting from the time discretization of X, and also a statistical error caused by the finite sample size used in the Monte Carlo simulation. In this paper we address the problem of reducing the systematic error. Therefore, in our experiments the number of simulation paths is chosen so large that the statistical error becomes negligible when compared to the systematic error. For variance reduction techniques, which reduce the statistical error, we refer to Kloeden & Platen (1999) and Glasserman (2004).

The theorems which establish the weak order of convergence of the discrete time approximations presented in Section 3 require a certain degree of smoothness for the payoff function g. Recall that we assumed in Section 1 $g \in \mathcal{C}_P^{2(\beta+1)}(\mathbb{R}^d, \mathbb{R})$, see also Bruti-Liberati & Platen (2006). The same conditions are usually required for weak convergence in the case of pure diffusion SDEs. For results with non-smooth payoffs limited to the Euler scheme for pure diffusion SDEs we refer to Bally & Talay (1996).

As particular example we consider the evaluation of the expectation of the non-smooth payoff of a call option $g(x) = (x - K)^+$, where K is the strike price, that means we evaluate $E\big((X_T - K)^+\big)$. Since we have modeled the dynamics of the security with the jump-diffusion Merton model, by using (2.3) we obtain a closed form solution given by

$$E((X_T - K)^+) = e^{\left(\mu+\lambda\left(E(\xi)-1\right)\right)T} \sum_{n=0}^{\infty} \frac{e^{-\lambda' T}(\lambda' T)^n}{n!} f_n, \tag{4.1}$$

where $\lambda' = \lambda\big(E(\xi) - 1\big)$. Here

$$f_n = X_0 \mathcal{N}(d_{1,n}) - e^{-\mu_n T} K \mathcal{N}(d_{2,n}), \tag{4.2}$$

denotes the Black-Scholes price of a European call option with the parameters specified as

$$d_{1,n} = \frac{\ln(\frac{X_0}{K}) + (\mu_n + \frac{\sigma_n^2}{2})\,T}{\sigma_n \sqrt{T}}, \tag{4.3}$$

$d_{2,n} = d_{1,n} - \sigma_n \sqrt{T}$, $\mu_n = \mu - \lambda(E(\xi)) + \frac{n \ln\{E(\xi)-1\}}{T}$ and $\sigma_n^2 = \sigma^2 + \frac{n\varsigma}{T}$. We denote by $\mathcal{N}(\cdot)$ the probability distribution of a standard Gaussian random variable. Therefore, the weak error $\varepsilon_w(\Delta)$, defined in (1.1), can be explicitly computed.

In Figure 4.1 we report a log-log plot with the logarithm $\log_2(\varepsilon_w(\Delta))$ of the weak error versus the logarithm $\log_2(\Delta)$ of the time step size. The parameters of the linear SDE (2.2) are set as $\mu = 0.05$, $\sigma = 0.2$, and $\beta = 0.2$, with initial value $X_0 = 1$. The driving jump process is a standard Poisson process with intensity $\lambda = 0.2$. Therefore, the marks ξ are constant and equal one. Moreover, we chose the final time $T = 0.5$ and the strike price $K = 1.25$. We consider the Euler, the weak order one predictor-corrector, the jump-adapted Euler, the jump-adapted weak order one predictor-corrector and the jump-adapted weak order two predictor-corrector schemes. These are labelled "Eul", "PredCorr", "JAEuler", "JAPredCorr" and "JA2PredCorr", respectively in Figure 4.1. Note that in the log-log plot the

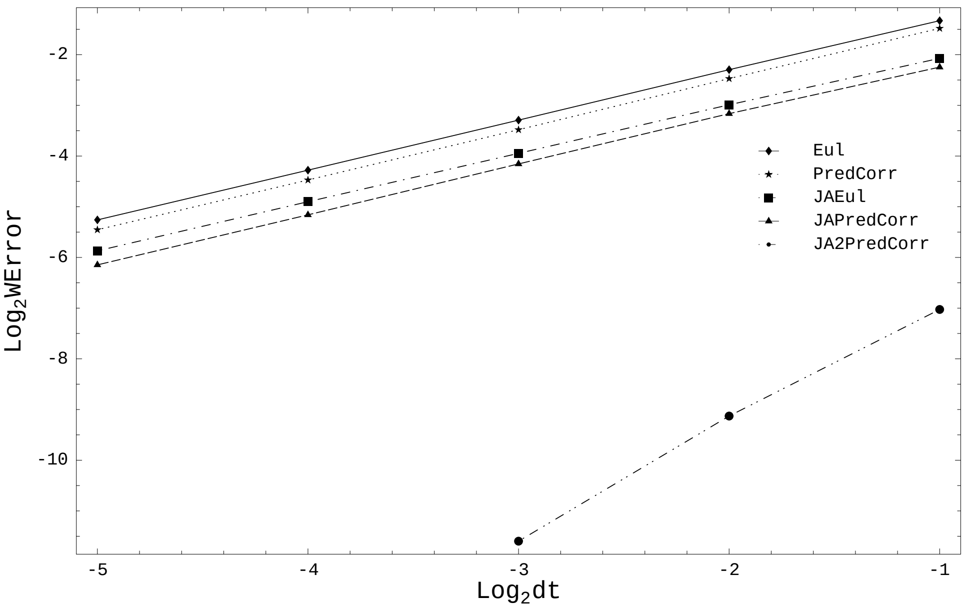

Figure 4.1: Weak error for Euler, predictor-corrector, jump-adapted Euler, jump-adapted predictor-corrector and jump-adapted weak order two predictor-corrector schemes.

achieved orders of convergence are given by the slopes of the observable lines. Figure 4.1 indicates that the Euler, the predictor-corrector, the jump-adapted Euler and the jump-adapted predictor-corrector schemes achieve an order of weak convergence of about $\beta = 1$. By comparing the two Euler schemes to the corresponding predictor-corrector schemes, one notices that predictor-corrector schemes are more accurate. This effect is due to the implicitness in the drift of predictor-corrector schemes and is expected to be more pronounced for more complex nonlinear SDEs, which has been reported also in Hunter, Jäckel & Joshi (2001) for diffusion SDEs arising in BGM models. In particular, for stiff SDEs with widely varying time scales predictor-corrector schemes should be preferred to the traditional Euler scheme. When comparing the first order regular schemes,"Eul" and "PredCorr" in the figure, to the first order jump-adapted schemes, "JAEuler" and "JAPredCorr", we note that jump-adapted schemes are more accurate. This is due to the simulation of the jump impact at the correct jump times. Finally, the jump-adapted weak order two predictor-corrector scheme is the most accurate and seems to achieve second order of weak convergence.

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