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On a solution of the optimal stopping problem for processes with independent increments

Alexander Novikov [1] *and Albert Shiryaev* [2]

Abstract

We discuss a solution of the optimal stopping problem for the case when a reward function is a power function of a process with independent stationary increments (random walks or Levy processes) on an infinite time interval. It is shown that an optimal stopping time is the first crossing time through a level defined as the largest root of the Appell function associated with the maximum of the underlying process.

1. Introduction. Let $X = (X_t)$ be a process with independent stationary increments with a discrete time parameter $t \in \mathbf{Z}^+ = \{0, 1, 2, \ldots\}$ or continuous time parameter $t \in \mathbf{R}^+ = [0, \infty)$, $X_0 = x \in \mathbf{R} = (-\infty, \infty)$. We suppose that X is defined on a probability space $(\Omega, \mathcal{F}, P)$ with a natural filtration $\mathcal{F}_t = \sigma\{X_s, s \le t\}$, $\mathcal{F}_0 = \{\varnothing, \Omega\}$.

The optimal stopping problem we study here consists in finding the "value" function

$$V(x) = \sup_{\tau \in \mathcal{M}} E(e^{-q\tau} g(X_\tau) I\{\tau < \infty\}), \qquad x \in \mathbf{R}, \quad q \ge 0,$$

where $g(x)$ is a measurable function, $\mathcal{M}$ is the class of all Markov times τ (with respect to $(\mathcal{F}_t)$) with values in $[0, \infty]$, $I\{A\}$ is the indicator function. We call τ^* as the *optimal* stopping time if

$$V(x) = E(e^{-q\tau^*} g(X_{\tau^*}) I\{\tau^* < \infty\}), \qquad x \in \mathbf{R}. \tag{1}$$

We discuss here only the case of power reward functions that is the case

$$g(x) = (x^+)^\nu, \qquad \nu > 0, \quad x^+ = \max(x, 0)$$

though the method developed below is quite general and can be used for finding explicit solutions for monotone functions $g(x)$.

The explicit solution of the problem under consideration for discrete time setting and the case $\nu = 1$ was found in [7] and [6]. We generalised their results for the case of integer $\nu = 1, 2, \ldots$ in [11] (we discussed the case $q = 0$ in [11]) using properties of the so-called Appell polynomials associated with the maximum of the process X_t. Kyprianou and Surya [8] have got an extension of our result to the continuous time setting with $q > 0$.

To solve the problem for *arbitrary* power $\nu > 0$ we had to study a generalisation of Appell polynomials which we call Appell functions. As in [11] and [8] we show the optimal stopping time has the threshold form that is

$$\tau^* = \tau_a = \inf\{t \ge 0 : X_t \ge a\}$$

where the optimal value of the parameter a is defined as a positive root of the Appell function associated with the maximum of the process X_t, see Theorem 1 in Section 4. Note that for the case $0 < \nu < 1$ this result can be derived also by methods of the

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paper Beibel [1] (we thank Prof. R. Lerche for this reference). Some necessary facts about Appell functions and the maximum of X_t are presented in Sections 2 and 3. In Section 5 we put two simple examples.

2. Appell functions. *Appell polynomials* (or, *Sheffer polynomials*, see e.g. [12]) generated by a random variable (r.v.) η such that $E|\eta|^n < \infty$ can be defined as follows:

$$Q_k(y;\eta) = (-1)^k \frac{d^k}{du^k}\Big(\frac{e^{-u\,y}}{Ee^{-u\eta}}\Big)\Big|_{u=0} \qquad k = 1,2,\ldots,n\,. \tag{2}$$

Based on this definition it is easy to derive the following properties of Appell polynomials which are valid under the assumption $E|\eta|^n < \infty$: for $k = 1,\ldots,n$

$$\frac{d}{dy}Q_k(y;\eta) = kQ_{k-1}(y;\eta), \tag{3}$$

$$E(Q_k(\eta+y;\eta)) = y^k. \tag{4}$$

Now we define continuous functions (we call them Appell functions) which have these two properties but with real parameter ν instead integer k. At first we find a function $Q_\nu(y;\eta)$ which satisfies both (3) and (4) with negative ν instead of integer k.

We assume further that η is a nonnegative random variable and

$$P(0 \le \eta < \varepsilon) > 0 \ \text{ for any } \ \varepsilon > 0. \tag{5}$$

Actually, Appell functions can be defined under more general assumptions but for purposes of this paper we shall need only this case. Condition (5) implies

$$Ee^{-u\eta} \ge P(0 \le \eta < \varepsilon)e^{-u\varepsilon} > 0, \qquad \varepsilon > 0,$$

and, hence, for any $\nu < 0$ and $y > 0$

$$\int_0^\infty u^{-\nu-1}\frac{e^{-u\,y}}{Ee^{-u\eta}}\,du < \infty. \tag{6}$$

For the proof of main results (see Section 4) we shall use the random variable $\eta = M_\theta = \sup_{0\le t<\theta}(X_t - X_0)$ which does satisfy (5) (see e.g. Lemma 2 in Section 3).

At first, we define *the Appell function of order* ν for $\nu < 0$ using the following integral representation:

$$Q_\nu(y;\eta) = \int_0^\infty u^{-\nu-1}\frac{e^{-u\,y}}{Ee^{-u\eta}}\,\frac{du}{\Gamma(-\nu)} \qquad (\,y > 0\,,\ \nu < 0\,) \tag{7}$$

where $\Gamma(z)$ is the Gamma function. Accordingly to this representation the function $Q_\nu(y;\eta)$ is continuous with respect to both parameters ν and y. Note also that for all $y > 0$

$$\lim_{\nu\uparrow 0} Q_\nu(y;\eta) = 1. \tag{8}$$

To see this we write the integral in (7) as a sum of two integrals $\int_0^\varepsilon + \int_\varepsilon^\infty$ and then show that $\int_\varepsilon^\infty$ vanishes as $\nu \uparrow 0$ for any $\varepsilon > 0$. With help of the fundamental property of the Gamma function (that is $\Gamma(z+1) = z\Gamma(z)$, $\Gamma(1) = 1$) one can show that

$$\int_0^{\varepsilon} u^{-\nu-1} \frac{e^{-u\,y}}{Ee^{-u\eta}} \frac{du}{\Gamma(-\nu)} = 1 + o(1), \quad \nu \uparrow 0,$$

and that implies (8). So, to define a continuous function $Q_\nu(y;\eta)$ (as a function of parameter ν) we need to set

$$Q_0(y;\eta) = 1 \tag{9}$$

for all $y > 0$.

Definition (7) implies

$$\frac{d}{dy} Q_\nu(y;\eta) = -\int_0^{\infty} u^{-\nu} \frac{e^{-u\,y}}{Ee^{-u\eta}} \frac{du}{\Gamma(-\nu)} = \nu Q_{\nu-1}(y;\eta) \quad (\, y > 0\,,\ \nu < 0\,) \tag{10}$$

where we used the fundamental property of the Gamma function $\Gamma(z+1) = z\Gamma(z)$ and the last equation holds due to (7). Also,

$$E(Q_\nu(y+\eta;\eta)) = \int_0^{\infty} u^{-\nu-1} \frac{Ee^{-u\,(y+\eta)}}{Ee^{-u\eta}} \frac{du}{\Gamma(-\nu)} = y^\nu \quad (\, y > 0\,,\ \nu < 0\,). \tag{11}$$

So, we have got properties (3) and (4) with parameter $\nu < 0$ instead of k .

To define $Q_\nu(y;\eta)$ for real $\nu > 0$ we set as a definition the following relation:

$$Q_\nu(y;\eta) = Q_\nu(0;\eta) + \nu \int_0^y Q_{\nu-1}(z;\eta)\,dz, \qquad y > 0, \quad \nu > 0, \tag{12}$$

assuming that $Q_\nu(0;\eta)$ is a finite constant. In other words, we require the validity of (10) also for $\nu > 0$.

To implement this definition we define at first the function $Q_\nu(y;\eta)$ for $\nu \in (0,1)$ using the representation (7), then for $\nu \in (1,2)$ based on (12) and with $Q_\nu(y;\eta)$ just defined for $\nu \in (0,1)$ and so on. Doing so, we get the analog of property (3) for any real ν instead of integer k .

Note the defined function $Q_\nu(y;\eta)$ is continuous with respect to both parameters ν and $y > 0$ because the right-hand side of (12) is an integral of a continuous function.

To have an analog of property (4) we set

$$Q_\nu(0;\eta) = -\nu E\Big(\int_0^{\eta} Q_{\nu-1}(z;\eta)dz \Big) \quad \nu > 0, \tag{13}$$

assuming that the last expectation is finite. The finiteness of the last integral does hold under the condition

$$E(\eta^\nu) < \infty.$$

To see this, we may chose the constant $A > 0$ such that $P(0 \le \eta < A) > 0$ and apply the estimate

$$Ee^{-u\eta} \ge P(0 \le \eta < A) e^{-u\,A}.$$

Since

$$Q_{\nu-1}(z;\eta) = \int_0^{\infty} u^{-\nu} \frac{e^{-u\,z}}{Ee^{-u\eta}} \frac{du}{\Gamma(1-\nu)} \quad (\, y > 0\,,\ \nu < 1\,),$$

we get the following upper bound with A from above and any $y > 0$, $\nu \in (0,1)$:

$$
\begin{aligned}
\int_0^y Q_{\nu-1}(z;\eta)dz \;&\le\; I(y \le A) \int_0^A Q_{\nu-1}(z;\eta)\,dz \\
&+ I(y > A) \int_A^y \left(\int_0^\infty \frac{u^{-\nu} e^{-u(z-A)}}{\Gamma(1-\nu) P(0 \le \eta < A)}\, du \right) dz \\
&\le I(y \le A) A \max_{0 \le z \le A} Q_{\nu-1}(z;\eta) + I(y > A) \int_A^y \frac{(z-A)^{\nu-1}}{P(0 \le \eta < A)}\, dz \\
&\le C(A,\nu)(I(y \le A) + I(y > A)(y-A)^\nu).
\end{aligned}
$$

where $C(A,\nu)$ is a finite positive constant. By (13) the last estimate implies that for $\nu \in (0,1)$

$$
(14) \qquad |Q_\nu(0;\eta)| \le C(A,\nu)\big(1 + E(I(\eta > A)(\eta - A)^\nu)\big) < \infty, \qquad \nu \in (0,1),
$$

and also by (12)

$$
|Q_\nu(y;\eta)| \le |Q_\nu(0;\eta)| + C(A,\nu)(I(y \le A) + I(y > A)(y-A)^\nu), \quad \nu \in (0,1),
$$

where $C(A,\nu)$ is some finite positive constant. Now we can apply this estimate in (12) for the case $\nu \in (1,2)$ and similarly get

$$
|Q_\nu(0;\eta)| \;\le C(A,\nu)(|Q_\nu(0;\eta)|\, E(\eta) + 1 + E(I(\eta > A)(\eta - A)^\nu)) < \infty, \nu \in (1,2)
$$

with another finite positive constant $C(A,\nu)$. Continuing this procedure we get the estimate with the main term $E(I(\eta > A)(\eta - A)^\nu)$ like in (14) but for any $\nu > 0$.

Now we claim that under the condition $E(\eta^\nu) < \infty$ we have the analog of the property (4) for any ν :

$$
(15) \qquad E(Q_\nu(\eta + y;\eta)) = y^\nu, \quad y > 0.
$$

Indeed, we have shown above that it is true for $\nu < 0$ (see (11)). For $\nu = 0$ it is true by definition (9). For $\nu \in (0,1)$ and $y > 0$ we have by definitions (12) and (13) that

$$
E(Q_\nu(\eta + y;\eta)) = \nu E \int_\eta^{\eta+y} Q_{\nu-1}(z;\eta)\, dz = \nu \int_0^y E Q_{\nu-1}(z+\eta;\eta)\, dz,
$$

where $EQ_{\nu-1}(z+\eta;\eta) = \; z^{\nu-1}$ due to (11). So we have shown the validity of (15) for $\nu \in (0,1)$. Applying this consideration recursively (of course, always assuming the existence of integrals) we get the validity of (15) for all real $\nu > 0$.

The case of integer $\nu = 0,1,2,\ldots$ now can viewed as a limiting case of functions $Q_\nu(y;\eta)$ but, of course, the original definition (2) is easier to use.

Below we shall use the following property of the Appell functions.

Lemma 1. *Let* (5) *hold and let* $E(\eta^\nu) < \infty$. *Then for any* $\nu > 0$ *there exists* a_ν *such that*

$$
(16) \qquad Q_\nu(y;\eta) \le 0 \;\; \text{for} \;\; 0 < y < a_\nu, \qquad Q_\nu(a_\nu;\eta) = 0
$$

and $Q_\nu(y;\eta)$ *is an increasing function for* $y \ge a_\nu$.

Proof. For integer $\nu = 1,2,\ldots$ the statement of this lemma was proved in [11, Lemma 5].

For the case $\nu \in (0,1)$ we note at first that due to the assumption $\eta \geq 0$ we have the estimate

$$Q_{\nu-1}(y;\eta) \geq \int_0^\infty u^{-\nu-1} e^{-u\,y} \frac{du}{\Gamma(-\nu)} = y^{\nu-1} \qquad (\nu < 1)$$

and so by (13) $Q_\nu(0;\eta) < 0$. Also, by (12) we have that $Q_\nu(y;\eta)$ is a nondecreasing function (of the variable y) such that

$$Q_\nu(y;\eta) \geq Q_\nu(0;\eta) + y^\nu.$$

So, it grows to infinity and, hence, Lemma 1 does hold for $\nu \in (0,1)$.

Next, consider the case $\nu \in (1,2)$ and $y > 0$. Then due the fact just proved

$$Q_{\nu-1}(y;\eta) < 0 \quad \text{for} \quad y \in (0, a_{\nu-1}).$$

So, on the interval $(0, a_{\nu-1})$ the function $Q_\nu(y;\eta)$ is negative and decreasing. Obviously, it reaches its minimum at point $y = a_{\nu-1}$. For $y \geq a_{\nu-1}$ the function $Q_\nu(y;\eta)$ is ultimately increasing to infinity due to the estimate

$$Q_\nu(y;\eta) \geq Q_\nu(0;\eta) + Q_{\nu-1}(0;\eta)y + y^\nu \quad \text{for} \quad y > 0.$$

Hence, there exists a root $a_\nu > a_{\nu-1} > 0$.

Using (12) recursively and the consideration presented above we see that the statement of Lemma 1 holds for all $\nu > 0$.

3. Some facts about the distribution of maximum

Writing $t \in \mathbf{Z}^+$ or $t \in \mathbf{R}^+$ we will indicate that the discrete time or continuous time cases are under consideration correspondingly. We formulate here all results in a form which is valid for the both cases $t \in \mathbf{Z}^+$ and $t \in \mathbf{R}^+$ but proofs of corresponding results we will have to discuss separately.

We assume always below that (X_t) is a process with independent homogeneous increments, $X_0 = x$. Let a random variable θ be independent of X_t such that

$$P(\theta > t) = e^{-tq}, \qquad q > 0,$$

where $t \in \mathbf{Z}^+$ or $t \in \mathbf{R}^+$. Set

$$M_{\theta,q} = \sup_{0 \leq t < \theta} (X_t - x),$$

and by definition set for the case $q = 0$

$$M_{\theta,0} = M_\infty \stackrel{\text{def}}{=} \sup_{0 \leq t < \infty} (X_t - x).$$

Further we always assume that for the case $q = 0$

$$E(X_1^+) < \infty \quad n\text{and} \quad E(X_1 - x) < 0. \tag{17}$$

Lemma 2. *If* $q \geq 0$ *then*

$$P\{M_{\theta,q} < \varepsilon\} > 0 \quad \textit{for any} \quad \varepsilon > 0. \tag{18}$$

Proof.

1) For the case $t \in \mathbf{Z}^+$ and $q = 0$ this result is a consequence of the observation that

$$P\{M_\infty < \varepsilon\} \geq P\{M_\infty = 0\}$$

and the fact that under imposed conditions

$$P\{M_\infty = 0\} > 0 \tag{19}$$

(see e.g. [3, pp. 91–92]).

2) For the case $t \in \mathbf{Z}^+$ and $q > 0$ note that if $\theta = 1$ then $M_{\theta,q} = 0$ and, hence,

$$P\{M_{\theta,q} < \varepsilon\} \geq P\{\theta = 1\} = e^{-q} > 0.$$

So (18) holds as well.

3) Consider now the case $t \in \mathbf{R}^+$ and $q = 0$. Let (R_t) be a compound Poisson process generated by jumps of (X_t) which are greater than 1. Set

$$Q_t = X_t - x - R_t.$$

Due to this definition the process (Q_t) does not contain jumps exceeding 1. Note that (R_t) and (Q_t) are independent processes with stationary increments, $E(R_1) \geq 0$.

To prove Lemma 2 we note that for any m

$$M_\infty = \sup_{t \geq 0}(Q_t + mt + R_t - mt) \leq \sup_{t \geq 0}(Q_t + mt) + \sup_{t \geq 0}(R_t - mt)$$

and so due to independency of (R_t) and (Q_t) we have for any $\varepsilon > 0$

$$P\{M_\infty < \varepsilon\} \geq P\Big\{\sup_{t \geq 0}(Q_t + mt) < \varepsilon/2\Big\} P\Big\{\sup_{t \geq 0}(R_t - mt) < \varepsilon/2\Big\}. \tag{20}$$

We shall estimate the both last probabilities separately under a proper choice of m.

To estimate $P\{\sup_{t\geq 0}(Q_t + mt) < \varepsilon/2\}$ note that we may choose the constant $m > E(R_1) \geq 0$ such that

$$E(Q_1) + m = E(X_1 - x) + m - E(R_1) < 0$$

(see assumption (17)). Now we show that with the such choice of m for any $\varepsilon > 0$

$$P\Big(\sup_{t \geq 0}(Q_t + mt) < \varepsilon\Big) > 0. \tag{21}$$

To see this, consider the exponential martingale

$$Z_t(u) = \exp\{u(Q_t + mt) - t\varphi(u)\}$$

with

$$\varphi(u) = \log E e^{u(Q_1 + m)}.$$

Then

$$\varphi(0) = 0, \quad \varphi'(0) = E(Q_1 + m) < 0.$$

and, as well known, $\varphi(u)$ is a continuous convex function.

Suppose the function $\varphi(u)$ has a root $u^* > 0$. (It is certainly true when $\varphi(u) \to \infty$, e.g. when Q_t contains a diffusion component or a component with positive jumps.) Then the process $Z_t(u^*) = \exp\{u^* Q_t\}$ is an exponential martingale. Applying the optional stopping theorem for the stopping time

$$\tau_\varepsilon = \inf\{t : Q_t + mt \geq \varepsilon\}$$

and the fact that $E(\exp\{u^*(Q_t + mt)\}) = 1$ we get the inequality

$$EI\{\tau_\varepsilon < \infty\} \exp\{u^*(Q_{\tau_\varepsilon} + m\tau_\varepsilon)\} \leq 1. \tag{22}$$

Since $Q_{\tau_\varepsilon} + m\tau_\varepsilon \geq \varepsilon$ on the set $\{\tau_\varepsilon < \infty\} = \{\sup_{t\geq 0}(Q_t + mt) \geq \varepsilon\}$ it implies that $P\{\tau_\varepsilon < \infty\} \leq e^{-u^*\varepsilon}$ and so

$$P\Big\{\sup_{t\geq 0}(Q_t + mt) < \varepsilon\Big\} \geq 1 - e^{-u^*\varepsilon} > 0. \tag{23}$$

Hence, (21) does hold under the assumption that the function $\varphi(u)$ has a root $u^* > 0$.

Consider now the alternative case when $\varphi(u) \leq 0$ for all $u > 0$. Then we may choose $u = 1$ and similar to (22) we get

$$EI\{\tau_\varepsilon < \infty\} \exp\{Q_{\tau_\varepsilon} + m\tau_\varepsilon - \tau_\varepsilon\varphi(1)\} \leq 1. \tag{24}$$

Since $\varphi(1) \leq 0$ and $Q_{\tau_\varepsilon} + m\tau_\varepsilon \geq \varepsilon$ on the set $\{\tau_\varepsilon < \infty\}$, this inequality implies (23) with $u^* = 1$ and so (21) does hold for all possible cases.

Next, we show that for any constant $m > E(R_1) \geq 0$ and any $\varepsilon > 0$

$$P\Big\{\sup_{t\geq 0}(R_t - mt) < \varepsilon\Big\} > 0. \tag{25}$$

This estimate is, actually, a consequence of the fact (19) and the estimate (23) proved above. Recall that (R_t) is a compound Poisson process generated by jumps of (X_t) which are greater than 1 and so it has the representation

$$R_t = \sum_{k=1}^{N_t} \xi_k,$$

where (N_t) is a Poisson process with the rate $\lambda \geq 0$, ξ_k are independent identically distributed (iid) random variables, $\xi_k > 1$, $\{\xi_k\}$ and N_t are independent. Assume further $\lambda > 0$ (otherwise $P\{\sup_{t\geq 0}(R_t - mt) < \varepsilon\} = 1$), choose b such that

$$E(R_1) = \lambda E(\xi_1) < \lambda b < m$$

and note

$$P\Big\{\sup_{t\geq 0}(R_t - mt) < \varepsilon\Big\} \geq P\Big\{\sup_{t\geq 0}(R_t - bN_t) = 0,\ \sup_{t\geq 0}(bN_t - mt) < \varepsilon\Big\}.$$

Here $R_t - bN_t = \sum_{k=1}^{N_t}(\xi_k - b)$ and so

$$\sup_{t\geq 0}(R_t - bN_t) = \sup_{k\geq 1} S_k^+, \tag{26}$$

where (S_k) is a random walk with negative drift as $E(S_1) = E(\xi_k - b) < 0$. This implies

$$P\Big\{\sup_{t\geq 0}(R_t - bN_t) = 0\Big\} = P\Big\{\sup_{k\geq 1} S_k^+ = 0\Big\} > 0$$

(see the step 1).

Since the set $\{\sup_{k\geq 1} S_k^+ = 0\}$ and the process (N_t) are independent we get

$$P\Big\{\sup_{t\geq 0}(R_t - mt) < \varepsilon\Big\} \geq P\Big\{\sup_{k\geq 1} S_k^+ = 0\Big\}\, P\Big\{\sup_{t\geq 0}(N_t - tm/b) < \varepsilon/b\Big\}.$$

Now we need just to note that the inequality

$$P\Big\{\sup_{t\geq 0}(N_t - tm/b) < \varepsilon\Big\} > 0$$

is a particular case of (23) because (N_t) is a Poisson process with unit jumps and $E(N_1) = \lambda < m/b$.

To complete the proof of Lemma 2 we need just to note that under the choice of m indicated above we have shown that the lower bound in (20) $\varepsilon > 0$.

4) For the case $t \in \mathbf{R}^+$ and $q > 0$ note that due to independency of (R_t) and (Q_t) we have for any $\varepsilon > 0$

$$P\{M_{\theta,q} < \varepsilon\} \geq P\Big\{\sup_{0\leq t<\theta} R_t < \varepsilon/2\Big\} P\Big\{\sup_{0\leq t<\theta} Q_t < \varepsilon/2\Big\}.$$

Due to independency of θ and $\sup_{0\leq s\leq t} R_s$ we have

$$P\Big\{\sup_{0\leq s<\theta} R_s < \varepsilon/2\Big\} \geq P\Big\{\sup_{0\leq s\leq 1} R_s < \varepsilon/2\Big\}\, P\{\theta < 1\},$$

where $P\{\theta < 1\} = 1 - e^{-q} > 0$ and so, obviously,

$$P\Big\{\sup_{0\leq s\leq 1} R_s < \varepsilon/2\Big\} \geq P\Big\{\sup_{0\leq s\leq 1} R_s = 0\Big\} > 0$$

for any $\varepsilon > 0$.

To estimate from below $P\{\sup_{0\leq t<\theta} Q_t < \varepsilon\}$ we can use the consideration from the previous step 3) in the part related to the process Q_t . At first note

$$P\Big\{\sup_{0\leq s<\theta} Q_s < \varepsilon\Big\} \geq P\Big\{\sup_{0\leq s\leq t} Q_s < \varepsilon\Big\} P\{\theta < t\}.$$

Consider the exponential martingale $Z_t(u) = \exp\{uQ_t - t\varphi(u)\}$, $u > 0$. Applying the optional stopping theorem for the stopping time $\tau_\varepsilon = \inf\{t : Q_t > \varepsilon\}$ we get

$$E\big(I\{\tau_\varepsilon < t\} e^{uQ_{\tau_\varepsilon} - \tau_\varepsilon \varphi(u)}\big) \leq 1.$$

Then due to the estimate $Q_{\tau_\varepsilon} \geq \varepsilon$ we get

$$P\{\tau_\varepsilon < t\} \leq e^{-u\varepsilon + t\varphi(u)}$$

and so

$$P\Big\{\sup_{0\leq s\leq t}(Q_s) < \varepsilon\Big\} = 1 - P\{\tau_\varepsilon < t\} \geq 1 - e^{-u\varepsilon + t\varphi(u)}.$$

Fixing $\varepsilon > 0$ we can find small t such that $u\varepsilon > t\varphi(u)$ and this implies the required fact that for any $\varepsilon > 0$

$$P\{\sup_{0\le s<\theta}(Q_s) < \varepsilon\} > 0.$$

The proof of Lemma 2 is completed.

Lemma 3. *Let* $\nu > 0$,

$$q = 0, \quad E(X_1) < 0, \quad E((X_1^+)^{\nu+1}) < \infty$$

or

$$q > 0, \quad E((X_1^+)^{\nu}) < \infty.$$

Then

$$E(M_{\theta,q}^{\nu}) < \infty. \tag{27}$$

Proof. 1) For the case $t \in \mathbf{Z}^+$ and $q = 0$ this result is well know, see e.g. [3, pp. 91–92].

2) Consider here the case $t \in \mathbf{Z}^+$ and $q > 0$. At first note that

$$M_{\theta,q} \le \sum_{k=1}^{\theta}(\Delta X_k)^+,$$

where ΔX_k are iid r.v. For the case $0 < \nu \le 1$ by Hölder inequality and Wald's identity

$$E(M_\theta^\nu) \le E\biggl(\sum_{k=1}^{\theta}((\Delta X_k)^+)^\nu\biggr) = E(\theta)E((\Delta X_k)^+)^\nu) < \infty.$$

For the case $1 < \nu \le 2$ we note that

$$Z_t = \sum_{k=1}^{t}[(\Delta X_k)^+ - E(\Delta X_k)^+] \tag{28}$$

is a martingale and so we can use well-known martingale inequalities (see e.g. [5]) which lead to the estimate

$$E(M_\theta^\nu) \le C_\nu(E(|Z_\theta|^\nu) + C_\nu E(\theta^\nu)(E((\Delta X_k)^+)^\nu))$$

and

$$E(|Z_\theta|^\nu) \le C_\nu E(\theta^\nu)E((\Delta X_k)^+)^\nu) < \infty$$

with some finite constants C_ν .

For $\nu \ge 2$ the process (Z_t) from (28) is a square integrable martingale and with help of the same martingale inequalities from [5] we get

$$E(M_\theta^\nu) \le C_\nu(E(|Z_\theta|^2)^{\nu/2} + C_\nu E(\theta^\nu)E((\Delta X_k)^+)^\nu)$$

and

$$E(|Z_\theta|^2) \le C_\nu E(\theta^\nu) E((\Delta X_k)^+)^\nu) < \infty$$

with some finite constants C_ν.

So, (27) is proved for the case $t \in \mathbf{Z}^+$ and $q > 0$.

3) Consider now the case $t \in \mathbf{R}^+$ and $q = 0$. We may use considerations which are similar to the proof of Lemma 2. With the same choice of constants m and b as in the proof of Lemma 2 (step 3) we have for any $x > 0$

$$P\{M_\infty > x\} \le P\Big\{ \sup_{t\ge 0}(R_t - bN_t) > \frac{x}{3}\Big\} + P\Big\{ \sup_{t\ge 0}(Q_t + mt) > \frac{x}{3}\Big\} + P\Big\{ \sup_{t\ge 0}(N_t - tm/b) > \frac{x}{3b}\Big\}.$$

Integrating both sides of this inequality with respect to the measure

$$I\{x > 0\}\nu x^{\nu-1}\, dx$$

we get

$$E(M_\infty^\nu) \le 3^\nu E\Big(\sup_{t\ge 0}(R_t - bN_t)^\nu\Big) + 3^\nu E\Big(\sup_{t\ge 0}(Q_t + mt)^\nu\Big) + (3b)^\nu E\Big(\sup_{t\ge 0}(N - tm/b)^\nu\Big). \tag{29}$$

Note that due to the relation (26)

$$E\Big(\sup_{t\ge 0}(R_t - bN_t)^\nu\Big) = E\Big(\sup_{k\ge 1}(S_k^+)^\nu\Big)$$

and so this term in (29) is finite because we have assumed that $E(X_1^+)^{\nu+1} < \infty$ (see step 1) above).

Now we show that for any $\nu > 0$

$$E\Big(\sup_{t\ge 0}(Q_t + mt)^\nu\Big) < \infty \tag{30}$$

and

$$E(\sup_{t\ge 0}(N - tm/b)^\nu) < \infty. \tag{31}$$

To prove these facts we can use again the exponential martingale $Z_t(u^*) = \exp\{u^*(Q_t + mt)\}$ and apply the optional stopping theorem with the stopping time

$$\tau_x = \inf\{t : Q_t + mt > x\}.$$

With some standard considerations involving uniform integrability we get the identity

$$E\big(I\{\tau_x < \infty\} \exp\{u^*(Q_{\tau_x} + m\tau_x)\}\big) = 1.$$

Since $Q_{\tau_\varepsilon} \ge x$ it implies that $P\{\tau_x < \infty\} \le e^{-u^* x}$ and so

$$P\Big\{\sup_{t\ge 0}(Q_t + mt) > x\Big\} = P\{\tau_x < \infty\} \le e^{-u^* x}$$

and so (30) does hold. To complete the step 3) we need just note that the inequality (31) is a particular case of (30).

4) The result of this part of Lemma 3 was proved in [8]. We would like to mention here that, actually, it is a simple consequence of general martingale inequalities proved originally in [10].

Lemma 4.

(a) *Let* $\tau_a = \inf\{t \geq 0 : X_t \geq a\}$, $a \geq x$. *Then for all* $u \leq 0$

$$E(I\{\tau_a < \infty\}\, e^{uX_{\tau_a}} e^{-q\tau_a}) = \frac{E(I\{M_\theta + x \geq a\}\, e^{u(M_\theta + x)})}{E(e^{uM_\theta})} \tag{32}$$

(b) *Let the conditions of Lemma 3 hold. Then for all* $a \geq x$ *and* ν

$$E(I\{\tau_a < \infty\} X_{\tau_a}^{\nu} e^{-q\tau_a}) = E(I\{M_\theta + x \geq a\}\, Q_\nu(M_\theta + x; M_\theta)). \tag{33}$$

Proof. **(a)** This result for the case $t \in \mathbf{Z}^+$ and $q = 0$ was proved in [11]. For the case $t \in \mathbf{Z}^+$ with $q > 0$ the proof needs just some minor modifications by taking into account, in particular, the memoryless property of geometric distribution. We need to note that on the set $\{M_\theta + x \geq a\} = \{\tau_a < \theta\}$

$$\widehat{M_\theta} := M_\theta - (X_{\tau_a} - x) \stackrel{law}{=} M_\theta$$

and $\widehat{M_\theta}$ is independent of X_{τ_a} on the event $\{\tau_a < \theta\}$. Here we show details of the proof for the case $t \in \mathbf{Z}^+$ and $q > 0$ only.

We have $P(\theta = k) = e^{-kq}(e^q - 1)$ for $k = 1, 2, \ldots$ and so

$$E(e^{uM_\theta}) = \sum_{k=1}^{\infty} E(I\{\theta = k\} e^{uM_{k-1}}) = \sum_{k=1}^{\infty} e^{-kq}(e^q - 1) E(e^{uM_{k-1}}).$$

Note also that $X_{\tau_a} = x + M_{\tau_a}$ and so

$$E(I\{M_\theta + x \geq a\}\, e^{u(M_\theta + x)}) = E(I\{\tau_a < \theta\}\, e^{u(M_\theta + x - X_{\tau_a})} e^{uX_{\tau_a}})$$

$$= \sum_{k=1}^{\infty} E(I\{\tau_a = k\} I\{k < \theta\}\, e^{u(M_\theta - M_k)} e^{uX_k})$$

$$= E(\sum_{k=1}^{\infty}\sum_{n=k}^{\infty} I\{\tau_a = k\} I\{k < n\} I\{\theta = n\}\, e^{u(M_{n-1} - M_k)} e^{uX_k})$$

$$= E(\sum_{k=1}^{\infty} I\{\tau_a = k\} e^{uX_k} I\{k < n\} \sum_{n=k+1}^{\infty} E(I\{\theta = n\}\, e^{u\hat{M}_{n-1-k}} | \mathcal{F}_k)) =$$

(setting $n = k + i$ and taking into account that both θ and $\hat{M}_{i-1}$ are independent and also independent of $\mathcal{F}_k$)

$$= E[\sum_{k=1}^{\infty} I\{\tau_a = k\} e^{uX_k} \sum_{i=1}^{\infty} P\{\theta = i + k\}\, E e^{u\hat{M}_{i-1}}]$$

$$= \sum_{k=1}^{\infty} E(I\{\tau_a = k\} e^{uX_k} e^{-kq} \sum_{i=1}^{\infty} e^{-iq}(e^q - 1) E e^{u\hat{M}_i})$$

$$= \sum_{k=1}^{\infty} E(I\{\tau_a = k\}\, e^{uX_k} e^{-kq} E e^{uM_\theta}) = E(e^{uX_{\tau_a}} I\{\tau_a < \infty\}\, e^{uX_{\tau_a}} e^{-q\tau_a}) E(e^{uM_\theta}). \blacksquare$$

For the case $t \in R^+$ considerations are similar (see [8] for more details).

(b). Relation (33) for $\nu < 0$ is obtained by integrating of both sides of (32) with respect to the measure

$$I\{u > 0\} u^{-\nu-1} du / \Gamma(-\nu), \qquad \nu < 0.$$

Then use relation (12) for $\nu \in (0,1)$ and so on.

Lemma 5. *Let $t \in \mathbf{Z}^+$, $q \geq 0$ and let $f(x)$ and $g(x)$ be nonnegative functions such that for all x*

$$f(x) \geq g(x) \tag{34}$$

and

$$f(x) \geq e^{-q} E f(X_1). \tag{35}$$

Then for all x

$$f(x) \geq \sup_{\tau \in M} E\, I\{\tau < \infty\}\, e^{-q\tau} g(X_\tau). \tag{36}$$

Proof. Condition (35) implies the fact that the process $e^{-qt}(X_t)$ is a nonnegative supermartingale and, hence, by the supermartingale property we have for any stopping time τ

$$f(x) \geq E\, I\{\tau < \infty\}\, e^{-q\tau} f(X_\tau).$$

Now one can see that inequality (36) is a consequence of condition (34).

Remark. Lemma 5 is just a slight generalisation of Lemma 7 from [11] (see also [2]).

4. Main result

Theorem 1. *Let $g(x) = (x^+)^\nu$, $\nu > 0$, the conditions of Lemma 3 hold and let a_ν be the positive root of the equation*

$$Q_\nu(y; M_\theta) = 0. \tag{37}$$

Then the stopping time τ_{a_ν} is optimal and

$$\nu(x) = E(e^{-q\tau_a} X_{\tau_{a_\nu}} I\{\tau_{a_\nu} < \infty\}) = E\,(Q_\nu(M_\theta + x; M_\theta)\, I\{M_\theta + x \geq a_\nu\}).$$

Proof. For integer $\nu = 1, 2, \ldots$ the proof was given for the case $t \in \mathbf{Z}^+$ and $q = 0$ in [11] and for the case $t \in \mathbf{R}^+$ and $q \geq 0$ in [8]. (Note that the condition of Theorem 2 in [8] for the case $q = 0$ should be changed as we formulated in Lemma 3 above.) The proof for real $\nu > 0$, actually, coincides with the lines of the proof in the mentioned papers. By this reason we just outline it here, omitting obvious details.

At first we show that the function $E(X^\nu_{\tau_a} I\{\tau_a < \infty\})$ achieves its maximum at the point $a = a_\nu$ where a_ν is the positive root of the equation (37) and so by Lemma 4

$$\widehat{\nu}(x) = E(\, X^\nu_{\tau_{a_\nu}} I\{\tau_{a_\nu} < \infty\}) = E(Q_\nu(M + x; M)\, I\{M + x \geq a_\nu\}).$$

This fact is a direct consequence of Lemmas 1, 3, 4(b) and (15).

Next, we note that, obviously,

$$\hat{V}(x) \leq V(x).$$

At the final step we show that

$$\hat{V}(x) \geq \nu(x) \tag{38}$$

and conclude that the optimal stopping time is $\tau = \tau_{a_\nu}$.

The proof of (38) for the case $t \in \mathbf{Z}^+$ and $q = 0$ follows the lines of the proof from the paper [11] given there for integer ν; for the case $t \in \mathbf{R}^+$ and $q \geq 0$ it follows the lines of the paper [8].

Here we present some details of the proof for (38) only for the case $t \in \mathbf{Z}^+$ and $q > 0$. The idea of our proof is similar to that one used in [6] and [11] and it is based on Lemma 5 and the following fact known as *Lindley recursion*:

$$\hat{M}_\theta = (\gamma \hat{M}_\theta + \xi)^+ \quad \text{(by law)}$$

where $\hat{M}_\theta$, γ, and ξ are independent r.v.'s,

$$\xi = X_1 - x, \quad P(\gamma = 1) = e^{-q} = 1 - P(\gamma = 0).$$

Using this equation we check that the function

$$f(x) = \nu(x) = E(Q_\nu(M_\theta + x; M_\theta)\, I\{M_\theta + x \geq a_\nu\})$$

satisfies conditions (35) and (34) with $g(x) = (x^+)^\nu$ and so by Lemma 5 it implies the required inequality (38).

Condition (35) holds because the function $f(x) = \widehat{\nu}(x)$ is nonnegative increasing function as x increases and, therefore,

$$\begin{aligned}
f(x) &= E(I\{(\gamma M_\theta + \xi)^+ + x \geq a_\nu\}\, Q_\nu((\gamma M_\theta + \xi)^+ + x)) \\
&= e^{-q} E(I\{(M_\theta + \xi)^+ + x \geq a_\nu\}\, Q_\nu((M_\theta + \xi)^+ + x)) \\
&\geq e^{-q} E(I\{x \geq a^*,\, M + \xi < 0\}\, Q_\nu(M + \xi + x)) \\
&\quad + e^{-q} E(I\{M + \xi + x \geq a^*,\, M + \xi \geq 0\}\, Q_\nu(M + \xi + x)) \\
&= e^{-q} E(f(x + \xi)).
\end{aligned}$$

Condition (34) holds because for any $x > 0$ $f(x) = E(Q_\nu(M_\theta + x; M_\theta)^+$ and so by Jensen's inequality and Lemma 4

$$f(x) \geq (E(Q_\nu(M_\theta + x; M_\theta))^+ = (x^+)^\nu) = g(x).$$

5. Examples.

(1) *Discrete time case.* Let ξ_1^+ have the density $pe^{-\lambda x}$, $x > 0$; $p > 0$, $0 < \lambda < \infty$, $E(\xi_1) < 0$. Then using martingale considerations like in the proof of Lemma 3 we can show that

$$P(M_\infty > a) = \frac{\lambda e^{-u_0 a}}{\lambda + u_0}, \quad P(M_\infty = 0) = \frac{u_0}{\lambda + u_0},$$

where u_0 is a positive root of the equation

$$E(e^{u\xi_1}) = 1.$$

By direct calculations one can get that for any ν and $y > 0$

$$Q_\nu(y;\eta) = (2/3)^\nu \exp(3y/2)(-3\nu\Gamma[\nu, y/2] + \Gamma[1+\nu, 3y/2]).$$

where $\Gamma[z, y]$ is the incomplete Gamma function.

(2) *Brownian Motion case.* Let $L_t = W_t - mt$, $g(x) = (x^+)^\nu$, $\nu > 0$.
If $m > 0$ then $M_\infty = \sup_{t\geq 0}(W_t - mt) \sim \mathrm{Exp}(2m)$ and the Appell function for any ν and $y > 0$ is

$$Q_\nu(y; M) = y^{\nu-1}\Big(y - \frac{\nu}{2m}\Big).$$

So, the optimal threshold is

$$a_\nu = \frac{\nu}{2m}.$$

If $q > 0$, $m = 0$ then $M_\theta = \sup_{s\leq\theta}(W_s) = |W_\theta|$ (by distribution), where $\theta \sim \mathrm{Exp}(q)$. We get

$$a_1 = E(|W_\theta|) = \sqrt{\frac{1}{2q}},$$

$$a_2 = E(|W_\theta|) + \sqrt{\mathrm{var}(W_\theta)} = \sqrt{\frac{1}{2q}}(1+\sqrt{2}),$$

and so on.

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