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Nicola Bruti-Liberati and Eckhard Platen

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Approximation of Jump Diffusions in Finance and Economics

Nicola Bruti-Liberati[1] and **Eckhard Platen**[2]

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Abstract. In finance and economics the key dynamics are often specified via stochastic differential equations (SDEs) of jump-diffusion type. The class of jump-diffusion SDEs that admits explicit solutions is rather limited. Consequently, discrete time approximations are required. In this paper we give a survey of strong and weak numerical schemes for SDEs with jumps. Strong schemes provide pathwise approximations and therefore can be employed in scenario analysis, filtering or hedge simulation. Weak schemes are appropriate for problems such as derivative pricing or the evaluation of risk measures and expected utilities. Here only an approximation of the probability distribution of the jump-diffusion process is needed. As a framework for applications of these methods in finance and economics we use the benchmark approach. Strong approximation methods are illustrated by scenario simulations. Numerical results on the pricing of options on an index are presented using weak approximation methods.

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Key words and phrases: jump-diffusion processes, discrete time approximation, simulation, strong convergence, weak convergence, benchmark approach, growth optimal portfolio.

[1]University of Technology Sydney, School of Finance & Economics, PO Box 123, Broadway, NSW, 2007, Australia

[2]University of Technology Sydney, School of Finance & Economics and Department of Mathematical Sciences, PO Box 123, Broadway, NSW, 2007, Australia

1 Introduction

The dynamics of financial and economic quantities are often described by stochastic differential equations (SDEs). In order to capture the dynamics observed it is important to model also the impact of event-driven uncertainty. Events such as corporate defaults, operational failures, market crashes or governmental macroeconomic announcements cannot be properly modelled by purely continuous processes. Therefore, SDEs of jump-diffusion type receive much attention in financial and economic modelling, see Merton (1976) or Cont & Tankov (2004). Since only a small class of jump-diffusion SDEs admits explicit solutions, one needs, in general, time discrete approximations.

The aim of the current paper is to provide an introductory survey to the numerical solution of jump-diffusion SDEs. To illustrate applications of time discrete approximations in finance we also give a brief introduction to the benchmark approach, which provides a general modelling framework for derivative pricing and portfolio optimization, see Platen & Heath (2006).

Discrete time approximations of SDEs can be divided into two classes: strong approximations and weak approximations, see Kloeden & Platen (1999). We say that a discrete time approximation Y^{Δ}, corresponding to a time discretization $(t)_{\Delta}$, where Δ is the time step size, converges strongly with order γ at time T to the solution X of a given SDE, if there exists a positive constant C, independent of Δ, and a finite number, $\Delta_0 > 0$, such that

$$\varepsilon_s(\Delta) := \sqrt{E(|X_T - Y_T^{\Delta}|^2)} \leq C\Delta^{\gamma}, \tag{1.1}$$

for each maximum time step size $\Delta \in (0, \Delta_0)$. As one can notice from the definition of the strong error (1.1), strong schemes provide pathwise approximations. Therefore, these methods are suitable for problems such as filtering, scenario analysis and hedge simulation.

We say that a discrete time approximation Y^{Δ} converges weakly with order β to X at time T, if for each $g \in \mathcal{C}_P^{2(\beta+1)}(\mathbb{R}^d, \mathbb{R})$ there exists a positive constant C, independent of Δ, and a finite number, $\Delta_0 > 0$, such that

$$\varepsilon_w(\Delta) := |E(g(X_T)) - E(g(Y_T^{\Delta}))| \leq C\Delta^{\beta}, \tag{1.2}$$

for each $\Delta \in (0, \Delta_0)$. Here $\mathcal{C}_P^{2(\beta+1)}(\mathbb{R}^d, \mathbb{R})$ denotes the space of $2(\beta+1)$ continuously differentiable functions which, together with their partial derivatives of order up to $2(\beta+1)$, have polynomial growth. This means that for $g \in \mathcal{C}_P^{2(\beta+1)}(\mathbb{R}^d, \mathbb{R})$ there exist constants $K > 0$ and $r \in \{1, 2, \ldots\}$, depending on g, such that

$$|\partial_y^j g(y)| \leq K(1 + |y|^{2r}), \tag{1.3}$$

for all $y \in \mathbb{R}^d$ and any partial derivative $\partial_y^j g(y)$ of order $j \leq 2(\beta+1)$. Weak schemes provide approximations of the probability measure and are appropri-

ate for problems such as derivative pricing and the evaluation of moments, risk measures and expected utilities.

In the sequel we give an overview of the still rather limited literature on approximations of jump-diffusion SDEs driven by Wiener processes and Poisson random measures. The early paper by Platen (1982a) describes a convergence theorem for strong schemes of any given strong order $\gamma \in \{0.5, 1, 1.5, \ldots\}$ and introduces jump-adapted approximations. The work in Maghsoodi & Harris (1987) analyzes the so-called in-probability approximations. In Mikulevicius & Platen (1988) a theorem for the weak convergence of jump-adapted weak Taylor schemes of any weak order $\beta \in \{1, 2, \ldots\}$ is derived. The papers by Li (1995) and Liu & Li (2000a) analyze the almost sure convergence of jump-diffusion approximations. Maghsoodi (1996, 1998) presents an analysis of some discrete time approximations up to strong order $\gamma = 1.5$. The Euler scheme for the approximation of SDEs driven by rather general semimartingales is studied in Kohatsu-Higa & Protter (1994), Protter & Talay (1997), Jacod & Protter (1998), Jacod (2004) and Jacod, Kurtz, Méléard & Protter (2005). The paper Liu & Li (2000b) analyzes weak Taylor schemes of any weak order $\beta \in \{1, 2, \ldots\}$ which are based on time discretizations that do not include jump times. Here a weak convergence theorem is given and the leading coefficients of the global error are derived for the Euler method and the order 2 weak Taylor scheme. Extrapolation methods are also presented. In Kubilius & Platen (2002) the weak convergence of the jump-adapted Euler scheme in the case of Hölder continuous coefficients is treated. The paper by Glasserman & Merener (2003) considers the weak convergence of the jump-adapted Euler scheme under weak assumptions on the jump coefficient. Gardoṅ (2004) presents a convergence theorem for strong schemes of any given order $\gamma \in \{0.5, 1, 1.5, \ldots\}$, similar to that presented in Platen (1982a). However, the results are limited to SDEs driven by Wiener processes and homogeneous Poisson processes and jump-adapted approximations are not considered. Higham & Kloeden (2005, 2006) propose a class of implicit schemes for SDEs that are also driven by Wiener processes and homogeneous Poisson processes. These papers also analyze numerical stability properties. In Bruti-Liberati & Platen (2005a) convergence theorems for strong approximations of jump-diffusion SDEs of any strong order $\gamma \in \{0.5, 1, 1.5, \ldots\}$, covering also derivative free, implicit and jump-adapted schemes, are proposed. Finally, Bruti-Liberati & Platen (2005b) present convergence theorems for weak approximations of any weak order $\beta \in \{1, 2, \ldots\}$, including derivative free, implicit, predictor-corrector and jump-adapted schemes.

The current paper is organized as follows. In Section 2 we present the class of jump-diffusion SDEs under consideration. Section 3 presents strong approximations which are divided into strong schemes and jump-adapted strong schemes. In Section 4 we present weak approximations, separated into weak schemes and jump-adapted weak schemes. In Section 5 we first give a brief introduction to the benchmark approach and then present numerical results on scenario simulation and Monte Carlo simulation.

2 Jump-Diffusion Stochastic Differential Equations

The securities and other financial and economic qunatities are driven by a Markovian factor process. Let us consider a filtered probability space $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$ satisfying the usual conditions. We consider a d-dimensional factor process $X = \{X_t, t \in [0,T]\}$ whose dynamics are described by the following jump-diffusion SDE

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t + c(t-, X_{t-})\, dJ_t, \tag{2.1}$$

for $t \in [0,T]$, with $X_0 \in \mathbb{R}^d$. Here $W = \{W_t = (W_t^1, \ldots, W_t^m)^\top, t \in [0,T]\}$ denotes an $\underline{\mathcal{A}}$-adapted m-dimensional standard Wiener process and $J = \{J_t = (J_t^1, \ldots, J_t^r)^\top, t \in [0,T]\}$ an $\underline{\mathcal{A}}$-adapted r-dimensional compound Poisson process. Each component J_t^j, for $j \in \{1, 2, \ldots, r\}$, of the r-dimensional compound Poisson process $J = \{J_t = (J_t^1, \ldots, J_t^r)^\top, t \in [0,T]\}$ is defined by

$$J_t^j = \sum_{i=1}^{N_t^j} \xi_i^j, \tag{2.2}$$

where $N^1, \ldots, N^r$ denote r independent Poisson processes with constant intensities $\lambda^1, \ldots, \lambda^r$, respectively. Let us note that each component of the compound Poisson process J^j generates a sequence of pairs $\{(\tau_i^j, \xi_i^j), i \in \{1, 2, \ldots, N^j(T)\}\}$. Here $\{\tau_i^j : \Omega \to \mathbb{R}_+, i \in \{1, 2, \ldots, N^j(T)\}\}$ is a sequence of increasing nonnegative random variables representing the jump times of the jth Poisson process N^j and $\{\xi_i^j : \Omega \to \mathbb{R}, i \in \{1, 2, \ldots, N^j(T)\}\}$ is a sequence of independent identically distributed (i.i.d.) random variables representing the corresponding jump sizes, drawn from a probability density $f^j(x)$.

Moreover, in (2.1) $a(t,x)$ is a d-dimensional vector of real valued functions on $[0,T] \times \mathbb{R}^d$, while $b(t,x)$ and $c(t,x)$ are a $d \times m$-matrix of real valued functions on $[0,T] \times \mathbb{R}^d$ and a $d \times r$-matrix of real valued functions on $[0,T] \times \mathbb{R}^d$, respectively. Here and in the sequel we adopt the notation a^i to denote the ith component of any vector a. Similarly, $b^{i,j}$ denotes the element in the ith row and jth column of a given matrix b. Finally, we denote the almost sure left-hand limit of $X = \{X_t, t \in [0,T]\}$ by $X_{t-} = \lim_{s \Uparrow t} X_s$.

For ease of presentation, in (2.1) we have modelled the jump processes as compound Poisson processes with fixed intensities. For a detailed presentation of jump-diffusion models we refer to Runggaldier (2003) and Øksendal & Sulem (2005). In a more general framework, which allows the modelling of events with state-dependent intensities, one can describe the driving jump processes by a Poisson random measure. We refer to Bruti-Liberati & Platen (2005a) and Bruti-Liberati & Platen (2005b) for numerical approximations of SDEs driven by Wiener processes and Poisson random measures.

It is common to assume standard Lipschitz and linear growth conditions on the

coefficients a, b and c, which ensure the existence and uniqueness of a strong solution of the SDE (2.1), see Ikeda & Watanabe (1989). Moreover, to simplify our presentation, whenever we present a numerical approximation we assume sufficient smoothness, integrability and growth conditions on the coefficients a, b and c, so that the corresponding strong or weak convergence theorems, presented in Bruti-Liberati & Platen (2005a) and Bruti-Liberati & Platen (2005b), are satisfied for the case at hand.

3 Strong Schemes

For simplicity, we consider in the current and the next sections the autonomous one-dimensional jump-diffusion SDE

$$dX_t = a(X_t)dt + b(X_t)dW_t + c(X_{t-})\, dJ_t, \tag{3.1}$$

for $t \in [0, T]$, with $X_0 \in \mathbb{R}$, where $W = \{W_t, t \in [0, T]\}$ is an $\underline{\mathcal{A}}$-adapted one-dimensional Wiener process. We assume that $J = \{J_t, t \in [0, T]\}$ is an $\underline{\mathcal{A}}$-adapted compound Poisson process, defined by

$$J_t = \sum_{i=1}^{N_t} \xi_i, \tag{3.2}$$

where $N = \{N_t, t \in [0, T]\}$ is an $\underline{\mathcal{A}}$-adapted standard Poisson process with intensity $\lambda > 0$ and ξ_i i.i.d. distributed according to a given probability density function $f(\cdot)$. The SDE (3.1) may be written in integral form as

$$X_t = X_0 + \int_0^t a(X_s)ds + \int_0^t b(X_s)dW_s + \sum_{i=1}^{N_t} c(X_{\tau_i-})\xi_i, \tag{3.3}$$

where $\{(\tau_i, \xi_i), i \in \{1, 2 \ldots, N(t)\}\}$ is the double sequence of jump times and marks generated by the compound Poisson process J. The numerical approximations to be presented in the current and next sections can be extended to the non-autonomous multi-dimensional SDE (2.1) and, in general, to multi-dimensional SDEs driven by Wiener processes and Poisson random measures, as described in Bruti-Liberati & Platen (2005a) and Bruti-Liberati & Platen (2005b).

In this section we present numerical schemes suitable for strong approximations. We emphasize that it is important, for theoretical and also practical reasons, to distinguish between strong and weak approximations and to choose an appropriate scheme based on the nature of the problem under consideration. The strong schemes to be presented in this section provide pathwise approximations suitable for problems such as filtering, scenario analysis and hedge simulation.

3.1 Strong Taylor Schemes

Let us construct an equidistant time discretization $\{0 = t_0, t_1, \ldots, t_{\bar{n}} = T\}$, with $t_n = n\Delta$, and step size $\Delta = \frac{T}{\bar{n}}$. We now consider discrete time approximations $Y^\Delta = \{Y_n^\Delta, n \in \{0, 1, \ldots, \bar{n}\}\}$ of the solution X of the autonomous SDE (3.3).

The simplest scheme is the well-known *Euler scheme*, given by

$$Y_{n+1} = Y_n + a\Delta + b\Delta W_n + c\Delta J_n, \tag{3.4}$$

for $n \in \{0, 1, \ldots, \bar{n} - 1\}$, with initial value $Y_0 = X_0$. Note that we use the abbreviations $a = a(Y_n)$, $b = b(Y_n)$ and $c = c(Y_n)$. Also in the sequel, when no misunderstanding is possible, for any coefficient function $g(\cdot)$, along with its derivatives, we will write $g = g(Y_n)$.

In (3.4), $\Delta W_n = W_{t_{n+1}} - W_{t_n} \sim \mathcal{N}(0, \Delta)$ is the nth increment of the Wiener process W and $\Delta J_n = J_{t_{n+1}} - J_{t_n}$ is the nth increment of the compound Poisson process J, which can be expressed as

$$\Delta J_n = \sum_{i=N_{t_n}+1}^{N_{t_{n+1}}} \xi_i.$$

Here N is the underlying Poisson process with intensity λ and for $i \in \{1, \ldots, N(T)\}$ the mark ξ_i is the outcome of a random variable with probability density function $f(\cdot)$. The Euler scheme (3.4) achieves a strong order of convergence $\gamma = 0.5$, in general.

To obtain more accuracy it is important to construct numerical schemes with a higher order of convergence. By including more terms from the Wagner-Platen expansion, which is the extension of the Taylor expansion to the stochastic setting, see Platen (1982b), we obtain the *order 1.0 strong Taylor scheme*, given by

$$\begin{aligned} Y_{n+1} \;=\; & Y_n + a\Delta + b\Delta W_n + c\Delta J_n \\ & + \frac{b\,b'}{2}\{(\Delta W_n)^2 - \Delta\} + \Big\{c\big(Y_n + c\big) - c\Big\} \sum_{j=N(t_n)+1}^{N(t_{n+1})} \sum_{i=N(t_n)+1}^{j} \xi_j\,\xi_i \\ & + b\,c' \sum_{i=N(t_n)+1}^{N(t_{n+1})} \xi_i\,\{W(\tau_i) - W(t_n)\} \\ & + \Big\{b\big(Y_n + c\big) - b\Big\} \sum_{i=N(t_n)+1}^{N(t_{n+1})} \xi_i\,\{W(t_{n+1}) - W(\tau_i)\}, \end{aligned} \tag{3.5}$$

where

$$b'(x) := \frac{d\,b(x)}{d\,x} \quad \text{and} \quad c'(x) := \frac{d\,c(x)}{d\,x}. \tag{3.6}$$

The scheme (3.5) achieves a strong order of convergence $\gamma = 1.0$.

By comparing the order 1.0 strong Taylor scheme (3.5) with the Euler scheme (3.4) one notices that (3.5) is more complex. First, it requires the computation of derivatives of the diffusion and the jump coefficient. Furthermore, one needs to sample the Wiener process W at the jump times τ_i, for $i \in \{1, \ldots, N_T\}$. Therefore, the computational effort of the order 1.0 strong Taylor scheme depends heavily on the intensity λ of the Poisson process. For this reason it is of particular importance that one carefully studies the structure of the SDE under consideration before choosing a numerical scheme. Indeed, if so-called commutativity conditions are satisfied, then the order 1.0 strong Taylor scheme has only a complexity comparable to that of the Euler scheme, and its computational effort can become independent of the intensity λ, see Bruti-Liberati & Platen (2005a). We emphasize that commutativity conditions are practically very important for multi-dimensional SDEs.

The computation of the derivatives of the SDE coefficient functions can be avoided by so-called derivative free schemes. If we replace the derivatives in the scheme (3.5) with corresponding difference ratios, then we obtain the *order 1.0 strong derivative free scheme*

$$\begin{aligned} Y_{n+1} &= Y_n + a\Delta + b\Delta W_n + c\Delta J_n \\ &+ \frac{b(\overline{Y}_n) - b}{2\sqrt{\Delta}} \{(\Delta W_n)^2 - \Delta\} + \Big\{c\big(Y_n + c\big) - c\Big\} \sum_{j=N(t_n)+1}^{N(t_{n+1})} \sum_{i=N(t_n)+1}^{j} \xi_j \, \xi_i \\ &+ \frac{c(\overline{Y}_n) - c}{\sqrt{\Delta}} \sum_{i=N(t_n)+1}^{N(t_{n+1})} \xi_i \, \{W(\tau_i) - W(t_n)\} \\ &+ \Big\{b\big(Y_n + c\big) - b\Big\} \sum_{i=N(t_n)+1}^{N(t_{n+1})} \xi_i \, \{W(t_{n+1}) - W(\tau_i)\}, \end{aligned} \tag{3.7}$$

with the supporting value

$$\overline{Y}_n = Y_n + b\sqrt{\Delta}. \tag{3.8}$$

This scheme achieves strong order of convergence $\gamma = 1.0$, see Bruti-Liberati & Platen (2005a).

Besides their order of convergence, an important property of numerical schemes is their numerical stability. Especially when solving stiff SDEs with very different time scales, it is important to use numerical schemes with wide stability regions. As in the analysis of ordinary differential equations, implicit schemes generally exhibit wider regions of numerical stability than their explicit counterparts for SDEs with jumps. For instance, when considering an SDE with multiplicative noise as a test equation, it has been shown that explicit schemes have narrower regions of numerical stability than the corresponding implicit schemes, see Hof-

mann & Platen (1996) for diffusions and Higham & Kloeden (2005, 2006) for jump diffusions.

By introducing implicitness in the drift of the Euler scheme (3.4) we obtain the *drift-implicit Euler scheme*

$$Y_{n+1} = Y_n + \{\zeta a(Y_{n+1}) + (1-\zeta)a\}\Delta + b\Delta W_n + c\Delta J_n, \tag{3.9}$$

where the parameter $\zeta \in [0,1]$ characterizes the degree of implicitness. The drift-implicit Euler scheme (3.9) achieves strong order of convergence $\gamma = 0.5$. For an analysis of the stability properties of this scheme we refer to Higham & Kloeden (2006). It should be noted that in order to achieve better stability properties one has to pay a price in terms of computational efficiency, as the scheme (3.9) generally requires the solution of an additional algebraic equation at each time step.

Similarly, by introducing implicitness in the drift of the order 1.0 strong Taylor scheme (3.5) we obtain the *order 1.0 drift-implicit strong Taylor scheme*, given by

$$\begin{aligned} Y_{n+1} \;=\; & Y_n + \{\zeta a(Y_{n+1}) + (1-\zeta)a\}\Delta + b\Delta W_n + c\Delta J_n \\ & + \frac{b\,b'}{2}\{(\Delta W_n)^2 - \Delta\} + \Big\{c\big(Y_n + c\big) - c\Big\} \sum_{j=N(t_n)+1}^{N(t_{n+1})} \sum_{i=N(t_n)+1}^{j} \xi_j\,\xi_i \\ & + b\,c' \sum_{i=N(t_n)+1}^{N(t_{n+1})} \xi_i\,\{W(\tau_i) - W(t_n)\} \\ & + \Big\{b\big(Y_n + c\big) - b\Big\} \sum_{i=N(t_n)+1}^{N(t_{n+1})} \xi_i\,\{W(t_{n+1}) - W(\tau_i)\}, \end{aligned} \tag{3.10}$$

which achieves strong order of convergence $\gamma = 1.0$. As in (3.9), by changing the parameter $\zeta \in [0,1]$ one can vary the degree of implicitness.

By including additional terms from the Wagner-Platen expansion in a scheme, see Platen (1982a, 1982b), it is, in principle, possible to construct numerical approximations with higher strong orders of convergence. However, these schemes become difficult to implement, as the additional terms contain complex multiple stochastic integrals involving time, Wiener process and compound Poisson process. Approximations which lead to much simpler higher order schemes are presented in the next section.

3.2 Strong Jump-Adapted Schemes

Now we present the so-called jump-adapted schemes, originally introduced in Platen (1982a), which are based on time discretizations that include all jump

times. Note that the waiting time between two consecutive jump times of a Poisson process with intensity λ is exponentially distributed with parameter λ. We consider a *jump-adapted time discretization* $0 = t_0 < t_1 < \ldots < t_M = T$, which is constructed by a superposition of the jump times $\{\tau_1, \ldots, \tau_{N(T)}\}$ generated by the Poisson process N and an equidistant time discretization with step size $\Delta = \frac{T}{\bar{n}}$, as in Section 3.1. Therefore, simply by construction, the jump-adapted time discretization includes all jump times of the Poisson process. The maximum step size of this discretization is Δ. Note that the number of time steps in the jump-adapted time discretization is random, as it equals $\bar{n}$ plus the number of jump times $N(T)$ of the Poisson process.

By including all jump times in the jump-adapted time discretization, we know that the solution X of (3.1) follows a diffusion process between discretization points. It can jump only at a discretization time. Therefore, it is possible to derive simple schemes, similar to those for diffusion SDEs, see Kloeden & Platen (1999). Let us note that in this section and in Section 4.2 we use a different notation by setting $Y_{t_n} = Y_n$ and we define

$$Y_{t_{n+1-}} = \lim_{s \Uparrow t_{n+1}} Y_s,$$

as the almost sure left-hand limit at time t_{n+1}.

The *jump-adapted Euler scheme* is then given by

$$Y_{t_{n+1-}} = Y_{t_n} + a\Delta_{t_n} + b\Delta W_{t_n} \tag{3.11}$$

and

$$Y_{t_{n+1}} = Y_{t_{n+1-}} + c(Y_{t_{n+1-}})\{J(t_{n+1}) - J(t_{n+1-})\}, \tag{3.12}$$

where $\Delta_{t_n} = t_{n+1} - t_n$ and $\Delta W_{t_n} = W_{t_{n+1}} - W_{t_n} \sim \mathcal{N}(0, \Delta_{t_n})$. With (3.11) we approximate the diffusion between discretization points, while (3.12) adds the jumps. Indeed, if t_{n+1} is a jump time, then $J(t_{n+1}) - J(t_{n+1-}) = \xi_{N(t_{n+1})}$ and

$$Y_{t_{n+1}} = Y_{t_{n+1-}} + c(Y_{t_{n+1-}})\xi_{N(t_{n+1})}, \tag{3.13}$$

while if t_{n+1} is not a jump time then $Y_{t_{n+1}} = Y_{t_{n+1-}}$. The jump-adapted Euler scheme (3.11)–(3.12) achieves strong order of convergence $\gamma = 0.5$.

By approximating the diffusion part with a Milstein scheme, we obtain the *jump-adapted order 1.0 strong scheme*

$$Y_{t_{n+1-}} = Y_{t_n} + a\Delta_{t_n} + b\Delta W_{t_n} + \frac{b\,b'}{2}\{(\Delta W_{t_n})^2 - \Delta_{t_n}\} \tag{3.14}$$

and

$$Y_{t_{n+1}} = Y_{t_{n+1-}} + c(Y_{t_{n+1-}})\{J(t_{n+1}) - J(t_{n+1-})\}, \tag{3.15}$$

which achieves strong order of convergence $\gamma = 1.0$.

By comparing the jump-adapted order 1.0 strong scheme (3.14)–(3.15) with the order 1.0 strong Taylor scheme (3.5), one notices that the jump-adapted scheme

is much easier to implement. By using an order 1.5 scheme for approximating the diffusion part, see Kloeden & Platen (1999), we obtain the *jump-adapted order 1.5 strong scheme*

$$\begin{aligned} Y_{t_{n+1-}} &= Y_{t_n} + a\Delta_{t_n} + b\Delta W_{t_n} + \frac{b\,b'}{2}\{(\Delta W_{t_n})^2 - \Delta_{t_n}\} \\ &\quad + a'\,b\Delta Z_{t_n} + \frac{1}{2}\left(a\,a' + \frac{1}{2}b^2\,a''\right)(\Delta_{t_n})^2 \\ &\quad + \left(a\,b' + \frac{1}{2}b^2\,b''\right)(\Delta W_{t_n}\Delta_{t_n} - \Delta Z_{t_n}) \\ &\quad + \frac{1}{2}b\left(b\,b'' + (b')^2\right)\left\{\frac{1}{3}(\Delta W_{t_n})^2 - \Delta_{t_n}\right\}\Delta W_{t_n}, \end{aligned} \tag{3.16}$$

and

$$Y_{t_{n+1}} = Y_{t_{n+1-}} + c(Y_{t_{n+1-}})\{J(t_{n+1}) - J(t_{n+1-})\}, \tag{3.17}$$

where

$$\Delta Z_{t_n} = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_2} dW_{s_1}\, ds_2. \tag{3.18}$$

One can show that ΔZ_{t_n} has a Gaussian distribution with mean $E(\Delta Z_{t_n}) = 0$, variance $E((\Delta Z_{t_n})^2) = \frac{1}{3}(\Delta_{t_n})^3$ and covariance $E(\Delta Z_{t_n}\,\Delta W_{t_n}) = \frac{1}{2}(\Delta_{t_n})^2$. Therefore, with two independent standard Gaussian random variables U_1 and U_2, we can simulate the correlated random variables ΔZ_{t_n} and ΔW_{t_n} in each time step, by setting:

$$\Delta W_{t_n} = U_1\sqrt{\Delta_{t_n}} \quad \text{and} \quad \Delta Z_{t_n} = \frac{1}{2}(\Delta_{t_n})^{\frac{3}{2}}\left(U_1 + \frac{1}{\sqrt{3}}U_2\right). \tag{3.19}$$

The scheme (3.16)–(3.17) achieves strong order of convergence $\gamma = 1.5$.

By replacing the schemes in the diffusion parts with derivative free or implicit schemes for diffusion SDEs, see Kloeden & Platen (1999), we can construct the corresponding derivative free and implicit jump-adapted schemes with desired order of strong convergence, see Bruti-Liberati & Platen (2005a).

4 Weak Schemes

By looking at the definition (1.2) for the weak error of a numerical scheme, one notices that only an approximation of the probability distribution of the solution X has to be sought. We now present weak schemes which provide approximations for the probability measure of the original solution X of the SDE. As we will see in the current section, when developing weak schemes one has more freedom in the generation of the necessary random variables. This leads to the design of so-called simplified weak Taylor schemes, which rely on simple random variables that match

appropriate moments of the involved multiple stochastic integrals. This contrasts with the strong schemes presented in Section 3, for which moment-matching properties are clearly not sufficient, since we are seeking pathwise approximations.

Weak schemes are appropriate for problems such as derivative pricing or the evaluation of risk measures and expected utilities.

4.1 Weak Taylor Schemes

In this section we consider an equidistant time discretization with time step size Δ, as in Section 3.1 and not a jump-adapted time discretization. The simplest weak scheme one can use is the Euler scheme (3.4) presented in Section 3.1 as a strong scheme. The Euler scheme achieves a weak order of convergence $\beta = 1$, which is different from its strong order. Moreover, as already indicated, it is possible to develop weak schemes which rely on very simple random variables. The *simplified Euler scheme* is given by

$$Y_{n+1} = Y_n + a\Delta + b\Delta\widehat{W}_n + c\,\widehat{\xi}_n\Delta\widehat{p}_n. \tag{4.1}$$

Here $\widehat{\xi}_n$ is a random variable drawn from the probability density $f(\cdot)$. If the random variables $\Delta\widehat{W}_n$ and $\Delta\widehat{p}_n$ match the first three moments of ΔW_n and $\Delta p_n = N(t_{n+1}) - N(t_n) \sim \text{Poiss}(\lambda\Delta)$, respectively, then the simplified Euler scheme (4.1) also achieves weak order of convergence $\beta = 1$. For instance, we can choose the following two-point distributed random variables, see Kloeden & Platen (1999) and Bruti-Liberati & Platen (2005b),

$$P(\Delta\widehat{W}_n = \pm\sqrt{\Delta}) = \frac{1}{2}, \tag{4.2}$$

and

$$P\Big(\Delta\widehat{p}_n = \frac{1}{2}(1 + 2\lambda\Delta \pm \sqrt{1 + 4\lambda\Delta})\Big) = \frac{1 + 4\lambda\Delta \mp \sqrt{1 + 4\lambda\Delta}}{2\,(1 + 4\lambda\Delta)}, \tag{4.3}$$

which match the first three moments of ΔW_n and Δp_n, respectively.

The two-point distributed random variables (4.2) and (4.3) can be efficiently generated using random bit generators and hardware accelerators, leading to highly efficient schemes, see Bruti-Liberati & Platen (2004) and Bruti-Liberati, Platen, Martini & Piccardi (2005). Let us finally note that by choosing simple random variables that adequately match the moments of the multiple stochastic integrals present in weak schemes, one can construct simplified versions of any such weak scheme presented in the current paper, see Bruti-Liberati & Platen (2005b).

By using further terms from the Wagner-Platen expansion, the weak convergence theorem in Bruti-Liberati & Platen (2005b) constructs the *order 2 weak Taylor*

scheme

$$
\begin{aligned}
Y_{n+1} \;=\;& Y_n + a\Delta + b\Delta W_n + c\Delta J_n \\
&+\frac{b\,b'}{2}\{(\Delta W_n)^2 - \Delta\} + \Big\{c\big(Y_n + c\big) - c\Big\} \sum_{j=N(t_n)+1}^{N(t_{n+1})} \sum_{i=N(t_n)+1}^{j} \xi_j\,\xi_i \\
&+b\,c' \sum_{i=N(t_n)+1}^{N(t_{n+1})} \xi_i\,\{W(\tau_i) - W(t_n)\} \\
&+\Big\{b\big(Y_n + c\big) - b\Big\} \sum_{i=N(t_n)+1}^{N(t_{n+1})} \xi_i\,\{W(t_{n+1}) - W(\tau_i)\} \\
&+\big(a\,a' + \frac{a''\,b^2}{2}\big)\frac{\Delta^2}{2} + a'\,b\Delta Z_n + \big(a\,b' + \frac{b''\,b^2}{2}\big)\{\Delta W_n\Delta - \Delta Z_n\} \\
&+\big(a\,c' + \frac{c''\,b^2}{2}\big) \sum_{i=N(t_n)+1}^{N(t_{n+1})} \xi_i\,\{\tau_i - t_n\} \\
&+\Big\{a\big(Y_n + c\big) - a\Big\} \sum_{i=N(t_n)+1}^{N(t_{n+1})} \xi_i\,\{t_{n+1} - \tau_i\}, \qquad (4.4)
\end{aligned}
$$

which achieves, in general, weak order of convergence $\beta = 2$. We refer to Bruti-Liberati & Platen (2005b) for further weak schemes based on deterministic time discretizations which do not include jump times.

As noted for the case of strong schemes, higher order schemes based on non jump-adapted grids are quite complex. Although it is possible to develop simplified higher order weak schemes using simple random variables satisfying sufficient moment-matching conditions, these still remain complicated when compared to higher order jump-adapted weak schemes, as we will see below.

4.2 Weak Jump-Adapted Schemes

We consider now jump-adapted weak schemes constructed on a jump-adapted time discretization as defined in Section 3.2. Let us note that, when performing a Monte Carlo simulation with a jump-adapted weak scheme, one can easily compute the jump times for each sample path in order to obtain the jump-adapted time grid.

The simplest scheme is again the jump-adapted Euler scheme (3.11)–(3.12) introduced in Section 3.2, which achieves weak order $\beta = 1$. By replacing the Gaussian random variable ΔW_n with the two-point distributed random variable

$\Delta\widehat{W}_{t_n}$, where

$$P(\Delta\widehat{W}_{t_n} = \pm\sqrt{\Delta_{t_n}}) = \frac{1}{2}, \tag{4.5}$$

we obtain the *jump-adapted simplified Euler scheme*

$$Y_{t_{n+1-}} = Y_{t_n} + a\Delta_{t_n} + b\Delta\widehat{W}_{t_n} \tag{4.6}$$

and

$$Y_{t_{n+1}} = Y_{t_{n+1-}} + c(Y_{t_{n+1-}})\{J(t_{n+1}) - J(t_{n+1-})\}. \tag{4.7}$$

The order of weak convergence of the scheme (4.6)–(4.7) is $\beta = 1$.

By using an order 2 weak scheme for the diffusion part of SDE, we obtain the *jump-adapted order 2 weak scheme* given by

$$\begin{aligned} Y_{t_{n+1-}} &= Y_{t_n} + a\Delta_{t_n} + b\Delta W_{t_n} + \frac{b\,b'}{2}\{(\Delta W_{t_n})^2 - \Delta_{t_n}\} \\ &\quad + \frac{1}{2}\left(aa' + \frac{1}{2}a''b^2\right)\Delta_{t_n}^2 + \frac{1}{2}\left(a'b + ab' + \frac{1}{2}b''b^2\right)\Delta W_{t_n}\Delta_{t_n} \end{aligned} \tag{4.8}$$

and

$$Y_{t_{n+1}} = Y_{t_{n+1-}} + c(Y_{t_{n+1-}})\{J(t_{n+1}) - J(t_{n+1-})\}, \tag{4.9}$$

which achieves weak order of convergence $\beta = 2$. If we replace the Gaussian random variable ΔW_{t_n} in the scheme (4.8)–(4.9) by the three-point distributed random variable $\Delta\widetilde{W}_{t_n}$, where

$$P(\Delta\widetilde{W}_{t_n} = \pm\sqrt{3\Delta_{t_n}}) = \frac{1}{6}, \qquad P(\Delta\widetilde{W}_{t_n} = 0) = \frac{2}{3}, \tag{4.10}$$

then we obtain the *jump-adapted second order simplified method*, which still achieves weak order of convergence $\beta = 2$.

One can also construct a *jump-adapted order 3 weak scheme* given by

$$\begin{aligned} Y_{n+1} &= Y_n + a\Delta + b\Delta W_{t_n} + \frac{1}{2}L^1 b\left\{(\Delta W_{t_n})^2 - \Delta_{t_n}\right\} \\ &\quad + L^1 a\Delta Z_{t_n} + \frac{1}{2}L^0 a\Delta_{t_n}^2 + L^0 b\left\{\Delta W_{t_n}\,\Delta_{t_n} - \Delta Z_{t_n}\right\} \\ &\quad + \frac{1}{6}\left(L^0L^0b + L^0L^1a + L^1L^0a\right)\left\{\Delta W_{t_n}\,\Delta_{t_n}^2\right\} \\ &\quad + \frac{1}{6}\left(L^1L^1a + L^1L^0b + L^0L^1b\right)\left\{(\Delta W_{t_n})^2 - \Delta_{t_n}\right\}\,\Delta_{t_n} \\ &\quad + \frac{1}{6}L^0L^0a\,\Delta_{t_n}^3 + \frac{1}{6}L^1L^1b\,\left\{(\Delta W_{t_n})^2 - 3\,\Delta_{t_n}\right\}\,\Delta W_{t_n}, \end{aligned} \tag{4.11}$$

and

$$Y_{t_{n+1}} = Y_{t_{n+1-}} + c(Y_{t_{n+1-}})\{J(t_{n+1}) - J(t_{n+1-})\}, \tag{4.12}$$

which achieves weak order of convergence $\beta = 3$. In (4.11) L^0 and L^1 are differential operators defined by

$$L^0 = a\,\frac{\partial}{\partial x} + \frac{1}{2}b^2\,\frac{\partial^2}{\partial x^2} \quad \text{and} \quad L^1 = b\,\frac{\partial}{\partial x}, \tag{4.13}$$

and the Gaussian random variable ΔZ_{t_n} is defined as in (3.19).

To implement the higher order schemes (4.8)–(4.9) and (4.11)–(4.12) one needs to evaluate several derivatives of the SDE coefficients. To avoid the computation of derivatives it is possible to design derivative free schemes which replace the derivatives by appropriate difference ratios. We present, as an example, the *jump-adapted order 2 derivative free scheme*, given by

$$\begin{aligned} Y_{t_{n+1-}} &= Y_{t_n} + \frac{1}{2}\Big(a(\overline{Y}_{t_n}) + a\Big)\Delta_{t_n} + \frac{1}{4}\Big(b(\overline{Y}^{+}_{t_n}) + b(\overline{Y}^{-}_{t_n}) + 2b\Big)\Delta W_{t_n} \\ &\quad + \frac{1}{4}\Big(b(\overline{Y}^{+}_{t_n}) - b(\overline{Y}^{-}_{t_n})\Big)\Big(\,(\Delta W_{t_n})^2 - \Delta_{t_n}\Big)\Big(\Delta_{t_n}\Big)^{-\frac{1}{2}} \end{aligned} \tag{4.14}$$

and

$$Y_{t_{n+1}} = Y_{t_{n+1-}} + c(Y_{t_{n+1-}})\{J(t_{n+1}) - J(t_{n+1-})\}, \tag{4.15}$$

with supporting values

$$\overline{Y}_{t_n} = Y_{t_n} + a\Delta_{t_n} + b\Delta W_{t_n}, \tag{4.16}$$

and

$$\overline{Y}^{\pm}_{t_n} = Y_{t_n} + a\Delta_{t_n} \pm b\sqrt{\Delta_{t_n}}, \tag{4.17}$$

which achieves weak order of convergence $\beta = 2$.

As noticed in Section 3, in some applications it is important to introduce implicitness in the scheme in order to enhance its numerical stability. Since in the context of weak approximations we can replace the Gaussian random variables with bounded random variables, as the two-point random variables $\Delta\widehat{W}_{t_n}$, it is also possible to introduce implicitness in the diffusion part of the scheme without incurring divisions by zero in the algorithm.

We present a *family of jump-adapted implicit Euler schemes* given by

$$Y_{t_{n+1-}} = Y_{t_n} + \big\{\zeta\bar{a}(Y_{t_{n+1-}}) + (1-\zeta)\bar{a}\big\}\Delta_{t_n} + \big\{\eta b(Y_{t_{n+1-}}) + (1-\eta)b\big\}\Delta\widehat{W}_{t_n} \tag{4.18}$$

and

$$Y_{t_{n+1}} = Y_{t_{n+1-}} + c(Y_{t_{n+1-}})\{J(t_{n+1}) - J(t_{n+1-})\}, \tag{4.19}$$

where $\bar{a} = a - \eta\, b\, b'$ is the corrected drift coefficient and $\zeta, \eta \in [0,1]$ are parameters that characterize the degree of implicitness in the drift and diffusion coefficients, respectively. The two-point distributed random variable $\Delta\widehat{W}_{t_n}$ is defined in (4.5)

and the scheme achieves weak order of convergence $\beta = 1$. It is possible to obtain higher order implicit schemes by using in the diffusion part (4.18) higher order weak implicit schemes for diffusions, see Kloeden & Platen (1999).

As previously mentioned, implicit schemes have an additional computational complexity, since they require, in general, the solution of an algebraic equation at each time step. It is possible to obtain a class of schemes, the so-called predictor-corrector schemes, which retain numerical stability properties similar to those of corresponding implicit schemes, but avoid the solution of an algebraic equation.

A *family of jump-adapted order 1 weak predictor-corrector schemes* is given by the corrector

$$Y_{t_{n+1-}} = Y_{t_n} + \Big\{\zeta \bar{a}(\bar{Y}_{t_{n+1-}}) + (1-\zeta)\,\bar{a}\Big\}\Delta_{t_n} + \Big\{\eta b(\bar{Y}_{t_{n+1-}}) + (1-\eta)\,b\Big\}\Delta\widehat{W}_{t_n}, \quad (4.20)$$

where $\bar{a} = a - \eta b b'$, the predictor

$$\bar{Y}_{t_{n+1-}} = Y_{t_n} + a\Delta_{t_n} + b\Delta\widehat{W}_{t_n}, \quad (4.21)$$

and

$$Y_{t_{n+1}} = Y_{t_{n+1-}} + c(Y_{t_{n+1-}})\{J(t_{n+1}) - J(t_{n+1-})\}, \quad (4.22)$$

with $\zeta, \eta \in [0, 1]$. This scheme achieves weak order of convergence $\beta = 1$. For higher order jump-adapted weak predictor-corrector schemes we refer to Bruti-Liberati & Platen (2005b)

5 Simulation in Finance under the Benchmark Approach

In this section we discuss some applications in finance involving simulation methods for SDEs with jumps, which will employ some of the strong and weak schemes presented in Section 3 and 4. We consider a general framework for financial modelling, known as the benchmark approach, presented in Platen & Heath (2006). The reader is referred to Platen & Heath (2006) for more details.

Let us consider a market with $d \in \mathbb{N}$ sources of trading uncertainty. We model the continuous trading uncertainty by $m \in \{1, 2, \ldots, d\}$ independent $\underline{\mathcal{A}}$-adapted Wiener processes $W^k = \{W^k_t, t \in [0, T]\}$, $k \in \{1, 2, \ldots, m\}$. Moreover, we introduce $d - m$ $\underline{\mathcal{A}}$-adapted counting processes $p^k = \{p^k_t, t \in [0, T]\}$, whose intensities $h^k = \{h^k_t, t \in [0, T]\}$ are predictable, strictly positive processes with

$$\int_0^t h^k_s ds < \infty$$

almost surely, for $t \in [0, T]$ and $k \in \{1, 2 \ldots, d - m\}$. Thus, the event-driven uncertainties are specified by $d - m$ normalized jump martingales with stochastic differentials

$$dW^k_t = (dp^{k-m}_t - h^{k-m}_t dt)(h^{k-m}_t)^{-\frac{1}{2}}, \quad (5.1)$$

for $k \in \{m+1, \ldots, d\}$ and $t \in [0, T]$. Therefore, the total trading uncertainty is specified by the vector process of independent $(\underline{\mathcal{A}}, P)$-martingales $W = \{W_t = (W_t^1, \ldots, W_t^d)^\top\}$.

We consider $d+1$ primary security accounts. These comprise a locally riskless savings account $S^0 = \{S_t^0, t \in [0, T]\}$, which continuously accrues interest at an instantaneous rate r_t, as well as d nonnegative risky primary security accounts $S^j = \{S_t^j, t \in [0, T]\}$, determined by the SDE

$$dS_t^j = S_{t-}^j \Big(a_t^j dt + \sum_{k=1}^d b_t^{j,k} dW_t^k \Big), \tag{5.2}$$

for $j \in \{1, 2, \ldots, d\}$. The short rate process r, the appreciation rates a^j and the generalized volatility processes $b^{j,k}$ are assumed to be almost sure finite, predictable stochastic processes satisfying appropriate conditions to ensure the existence and uniqueness of a strong solution of the SDE (5.2). Moreover, to ensure nonnegativity of the primary securities, we assume that $b_t^{j,k} \geq -\sqrt{h_t^{k-m}}$, for all $j \in \{1, \ldots, d\}$, $k \in \{m+1, \ldots, d\}$ and $t \in [0, T]$. We also require the generalized volatility matrix $b = [b_t^{j,k}]_{j,k=1}^d$ to be invertible for Lebesgue-almost every $t \in [0, T]$. As a consequence, we can introduce the market price of risk vector

$$\theta_t = (\theta_t^1, \ldots, \theta_t^d)^\top = b_t^{-1}[a_t - r_t 1], \tag{5.3}$$

for all $t \in [0, T]$, where $a_t = (a_t^1, \ldots, a_t^d)^\top$ is the appreciation rate vector and $1 = (1, 1, \ldots, 1)^\top$ is the unit vector. We can then rewrite the SDE (5.2) as

$$dS_t^j = S_{t-}^j \Big(r_t dt + \sum_{k=1}^d b_t^{j,k} \big(\theta_t^k dt + dW_t^k \big) \Big), \tag{5.4}$$

for $j \in \{1, 2, \ldots, d\}$.

Now we consider portfolios of primary security accounts. We say that a predictable stochastic process $\delta = \{\delta_t = (\delta_t^0, \delta_t^1, \ldots, \delta_t^d)^\top, t \in [0, T]\}$ is a strategy if δ is appropriately integrable, see Protter (2004). The jth component of δ denotes the number of units of the jth primary security account held at time $t \in [0, T]$ in the corresponding portfolio S^δ, so that

$$S_t^\delta = \sum_{j=0}^d \delta_t^j S_t^j. \tag{5.5}$$

Moreover, we assume that the strategy is self-financing, that means

$$dS_t^\delta = \sum_{j=0}^d \delta_t^j dS_t^j. \tag{5.6}$$

Let us denote by $\mathcal{V}^+$ the set of strictly positive portfolios processes. Then for a strictly positive portfolio $S^\delta \in \mathcal{V}^+$ we can define the proportion $\pi^j_{\delta,t}$ of its value invested in the jth primary security account as

$$\pi^j_{\delta,t} = \delta^j_t \frac{S^j_t}{S^\delta_t} \tag{5.7}$$

for all $t \in [0,T]$ and $j \in \{1,\dots,d\}$. Therefore, by (5.6), (5.4) and (5.7) we obtain the SDE for the portfolio S^δ in the form

$$dS^\delta_t = S^\delta_{t-}\Big(r_t dt + \pi^\top_{\delta,t-} b_t\big(\theta_t dt + dW_t\big)\Big), \tag{5.8}$$

for all $t \in [0,T]$, where $\pi_{\delta,t} = (\pi^1_{\delta,t},\dots,\pi^d_{\delta,t})^\top$ and $dW_t = (dW^1_t,\dots,dW^d_t)^\top$. To guarantee a strictly positive portfolio, we require that

$$\sum_{j=1}^d \pi^j_{\delta,t} b^{j,k}_t > -\sqrt{h^{k-m}_t} \tag{5.9}$$

almost surely, for all $k \in \{m+1,\dots,d\}$ and $t \in [0,T]$.

We can now introduce the central object in the benchmark approach, namely the *growth-optimal portfolio* (GOP), see Kelly (1956), Long (1990), Karatzas & Shreve (1998) and Platen & Heath (2006). It is defined as the portfolio that maximizes expected logarithmic utility from terminal wealth. One can show that this is equivalent to maximizing the growth rate, which is the drift of $\log(S^\delta_t)$, over all positive portfolios $S^\delta \in \mathcal{V}^+$.

Let us also assume that $\sqrt{h^{k-m}_t} > \theta^k_t$, for all $t \in [0,T]$ and $k \in \{m+1,\dots,d\}$, thereby excluding portfolios with infinite growth rate which explode. This may be interpreted as a kind of no-arbitrage condition. Under these assumptions it can be shown that a GOP exists, see Platen & Heath (2006). Furthermore, this portfolio is unique for given initial value. The dynamics of the GOP S^{δ_*} are given by the following SDE

$$dS^{\delta_*}_t = S^{\delta_*}_{t-}\Big(r_t dt + c^\top_t\big(\theta_t dt + dW_t\big)\Big) \tag{5.10}$$

for all $t \in [0,T]$, with $S^{\delta_*}_0 > 0$. Here the predictable process $c_t = (c^1_t,\dots,c^d_t)^\top$ is given by

$$c^k_t := \begin{cases} \theta^k_t & \text{for } k \in \{1,\dots,m\} \\ \frac{\theta^k_t}{1-\theta^k_t (h^{k-m}_t)^{-\frac{1}{2}}} & \text{for } k \in \{m+1,\dots,d\}. \end{cases} \tag{5.11}$$

From (5.8) and (5.10) we can also identify the optimal fractions $\pi_{\delta_*,t} = (\pi^1_{\delta_*,t},\dots,\pi^d_{\delta_*,t})^\top = (c^\top_t b^{-1}_t)^\top$ of the GOP S^{δ_*}.

Recall that the GOP is defined as the strictly positive portfolio that maximizes the growth rate among all strictly positive portfolios. It also possesses other

outstanding properties. For instance, it is the portfolio with the largest long term growth rate among all strictly positive portfolios $S^\delta \in \mathcal{V}^+$, that is

$$g^\delta := \limsup_{T\to\infty} \frac{1}{T} \ln\Big(\frac{S_T^\delta}{S_0^\delta}\Big) \le g^{\delta_*}. \tag{5.12}$$

The benchmark approach uses the GOP as numeraire or reference unit. For any portfolio S^δ we introduce its benchmarked value $\hat{S}^\delta$, given by

$$\hat{S}_t^\delta = \frac{S_t^\delta}{S_t^{\delta_*}} \tag{5.13}$$

for all $t \in [0,T]$. By Itô's formula, (5.2) and (5.10) the benchmarked portfolio process $\hat{S}^\delta = \{\hat{S}_t^\delta, t \in [0,T]\}$ satisfies the SDE

$$\begin{aligned} d\hat{S}_t^\delta &= \sum_{k=1}^m \Big(\sum_{j=1}^d \delta_t^j \hat{S}_t^j b_t^{j,k} - \hat{S}_t^\delta \theta_t^k\Big) dW_t^k \\ &\quad + \sum_{k=m+1}^d \Big(\Big(\sum_{j=1}^d \delta_t^j \hat{S}_{t-}^j b_t^{j,k}\Big)\Big(1 - \frac{\theta_t^k}{\sqrt{h_t^{k-m}}}\Big) - \hat{S}_{t-}^\delta \theta_t^k\Big) dW_t^k \end{aligned} \tag{5.14}$$

for all $t \in [0,T]$. Note that the SDE (5.14) is driftless. If we restrict our attention to the class of nonnegative portfolios $\mathcal{V}$, then we obtain the following result.

Theorem 5.1 *Any nonnegative benchmarked portfolio process $\hat{S}^\delta$ is an $(\underline{\mathcal{A}}, P)$-supermartingale, whence*

$$\hat{S}_t^\delta \ge E\Big(\hat{S}_\tau^\delta | \mathcal{A}_t\Big) \tag{5.15}$$

for all $\tau \in [0,T]$ and $t \in [0,\tau]$.

It is important to introduce an appropriate notion of arbitrage and to verify that the benchmark framework precludes it. We consider only nonnegative portfolios, since we assume that market participants are not permitted to trade if their total tradable wealth becomes negative.

Definition 5.2 *A nonnegative portfolio $S^\delta \in \mathcal{V}$ permits arbitrage if it starts at the level zero, which is $S_0^\delta = 0$ almost surely, and is at a later stopping time $\tau \in [0,T]$ strictly positive with strictly positive probability, that is,*

$$P(S_\tau^\delta > 0) > 0.$$

Because of the supermartingale property of nonnegative benchmarked portfolios in Theorem 5.1, the benchmark framework precludes arbitrage in the sense of Definition 5.2, see Platen & Heath (2006).

We have seen in Theorem 5.1 that any nonnegative portfolio process is an $(\underline{\mathcal{A}}, P)$-supermartingale when benchmarked by using the GOP as numeraire. We call a value process *fair* if its benchmarked value is an $(\underline{\mathcal{A}}, P)$-martingale. Its current benchmarked value is then the best forecast of its future benchmarked value.

We call an $\underline{\mathcal{A}}_\tau$-measurable payoff H_τ, maturing at a stopping time $\tau \in [0, T]$, a *contingent claim* if

$$E\Big(\frac{|H_\tau|}{S_\tau^{\delta_*}}\Big) < \infty.$$

For a contingent claim H_τ its benchmarked conditional expectation $\hat{U}_{H_\tau} = \{\hat{U}_{H_\tau}(t), t \in [0, T]\}$, given by

$$\hat{U}_{H_\tau}(t) = E\Big(\frac{|H_\tau|}{S_\tau^{\delta_*}}|\mathcal{A}_t\Big),$$

is an $\underline{\mathcal{A}}_\tau$-martingale. The value process $U_{H_\tau} = \{U_{H_\tau}(t), t \in [0, T]\}$, with

$$U_{H_\tau}(t) = \hat{U}_{H_\tau}(t)\, S_t^{\delta_*},$$

is thus fair. The fair value $U_{H_\tau}(t)$ at time t of a contingent claim H_τ is uniquely determined by the *real world pricing formula*

$$U_{H_\tau}(t) = S_t^{\delta_*} E\Big(\frac{H_\tau}{S_\tau^{\delta_*}}|\mathcal{A}_t\Big). \tag{5.16}$$

Note that the martingale property of benchmarked fair values is formulated with respect to the real world probability measure P and no change of measure is required. The real world pricing formula (5.16) is the key pricing formula under the benchmark approach. It generalizes standard risk neutral pricing as well as actuarial pricing. However, under the benchmark approach one can use models which do not admit an equivalent risk neutral probability measure. Indeed, some realistic models proposed in the financial literature do not admit an equivalent risk neutral probability measure and, therefore, the standard risk neutral pricing methodology cannot be applied for these models, see Platen & Heath (2006). When an equivalent risk neutral probability measure exists, the real world pricing formula (5.16) leads to the same price as the standard risk neutral pricing formula. It is also interesting to note that when there are several nonnegative portfolio processes that replicate a payoff H_τ, then the fair portfolio provides the minimal replicating portfolio. When several equivalent risk neutral probability measures exist, the real world pricing formula coincides with the price obtained under the so-called minimal equivalent martingale measure, see Föllmer & Schweizer (1991). The real world pricing formula (5.16) also emerges from utility indifference pricing for payoffs that cannot be replicated, see Davis (1997) and Platen & Heath (2006). In jump-diffusion models for credit risk and other areas the benchmark approach has a clear advantage over the risk neutral methodology in the statistical estimation of jump intensities. The estimation of the jump intensity from historical prices under the risk neutral measure is almost impossible

as the change of measure modifies the jump intensity. On the contrary, under the benchmark approach we can estimate directly the jump intensity by observing historical events since we work only with the real world probability measure P.

An important step towards the practical applicability of the real world pricing formula (5.16) and other results of the benchmark approach is provided by the diversification theorem presented in Platen & Heath (2006). This theorem states, without major modelling assumptions, that any diversified portfolio approximates the GOP in a realistic market. Statistical analysis of historical data supports the conjecture that diversified global portfolios approximate the GOP well. Therefore, as a proxy for the GOP we can use any global diversified index.

One observes on historical data that the inverse of a discounted diversified global index decreases systematically over long time. In a risk neutral setting this inverse should reflect the natural Radon-Nikodym derivative of the candidate risk neutral probability measure. The systematic decline of this process signals an inconsistency in the risk neutral approach, because the Radon-Nikodym derivative needs to be an $(\underline{\mathcal{A}}, P)$-martingale under the prevailing risk neutral theory. Under the benchmark approach there is no problem with the above mentioned empirical stylized fact. Real world pricing can still be applied.

We will now present some numerical examples of scenario simulation and Monte Carlo simulation under the benchmark approach. For ease of presentation, we consider a simple jump-diffusion market with only two risky securities, that is $d = 2$. Here the continuous trading uncertainty is driven by one Wiener process W^1 and the event driven trading uncertainty by the jump martingale W^2. Moreover, the short rate process, the market prices of risk and the generalized volatilities are assumed to be constant. Therefore, the risky primary security accounts S^1 and S^2 follow the SDE

$$dS_t^j = S_{t-}^j \Big(r dt + \sum_{k=1}^{2} b^{j,k} \big(\theta^k dt + dW_t^k \big) \Big), \tag{5.17}$$

for $j \in \{1, 2\}$. According to (5.10), the GOP satisfies the SDE

$$dS_t^{\delta_*} = S_{t-}^{\delta_*} \Big(r dt + \theta^1 \big(\theta^1 dt + dW_t^1 \big) + \frac{\theta^2}{1 - \theta^2 (h^{-\frac{1}{2}})} \big(\theta^2 dt + dW_t^2 \big) \Big). \tag{5.18}$$

We rewrite the dynamics (5.17)–(5.18) by using a Poisson process N instead of the jump martingale W^2, as presented in Section 3. The risky primary security accounts, thus, follow the SDEs

$$dS_t^j = S_{t-}^j \Big(a^{S^j} dt + b^{S^j} dW_t + c^{S^j} dN_t \Big), \tag{5.19}$$

with

$$a^{S^j} = r + b^{j,1} \theta^1 + b^{j,2} (\theta^2 - \sqrt{h}), \quad b^{S^j} = b^{j,1} \quad \text{and} \quad c^{S^j} = \frac{b^{j,2}}{\sqrt{h}},$$

for $j \in \{1, 2\}$. The GOP dynamics are given by

$$dS_t^{\delta_*} = S_{t_-}^{\delta_*}\Big(a^{S^{\delta_*}}dt + b^{S^{\delta_*}}dW_t + c^{S^{\delta_*}}dN_t\Big), \tag{5.20}$$

with

$$a^{S^{\delta_*}} = r + (\theta^1)^2 + \frac{\theta^2(\theta^2 - \sqrt{h})}{1 - \theta^2(h^{-\frac{1}{2}})}, \quad b^{S^{\delta_*}} = \theta^1 \quad \text{and} \quad c^{S^{\delta_*}} = \frac{\theta^2}{\sqrt{h} - \theta^2}.$$

These simple examples of SDEs with jumps have explicit solutions, given by

$$S_T^j = S_0^j e^{(a^{S^j} - \frac{(b^j)^2}{2})T + b^{S^j}W_T}(1 + c^{S^j})^{N_T}, \tag{5.21}$$

for $j \in \{1, 2, \delta^*\}$. This will help us performing error estimations in the simulation studies that follow.

5.1 Scenario Simulation

In this section we apply the strong schemes presented in Section 3 to perform a scenario simulation for the two risky primary accounts S^1 and S^2, as well as the GOP S^{δ_*}.

We choose the following parameters: $r = 0.05$, $\theta^1 = 0.15$, $\theta^2 = 0.1$, $b^{1,1} = 0.2$, $b^{1,2} = 0.15$, $b^{2,1} = 0.3$, $b^{2,2} = 0.2$ and $h = 0.2$. Moreover, we choose at first a large time step size $\Delta = 1$ and sample the Wiener increments $\Delta W_n = W_{t_{n+1}} - W_{t_n}$ and the Poisson increments $\Delta N = N_{t_{n+1}} - N_{t_n}$ at each time step over 20 years.

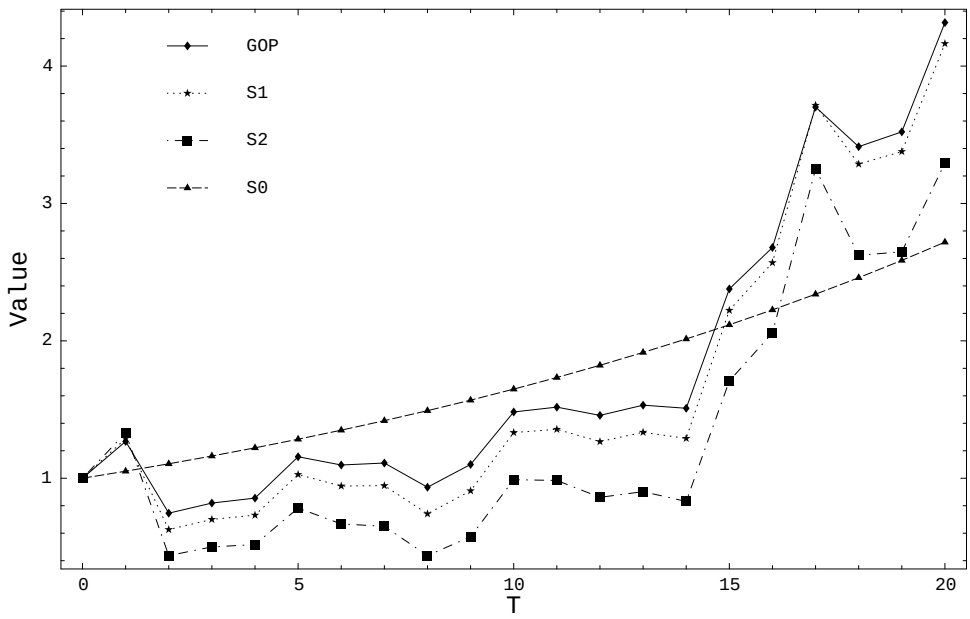

Figure 5.1: Sample path of GOP, S^0, S^1 and S^2.

In Figure 5.1 we plot sample paths of the GOP, the risk free primary security account S^0 and the risky primary security accounts S^1 and S^2, using the explicit solution (5.21) and the increments ΔW_n and ΔN_n.

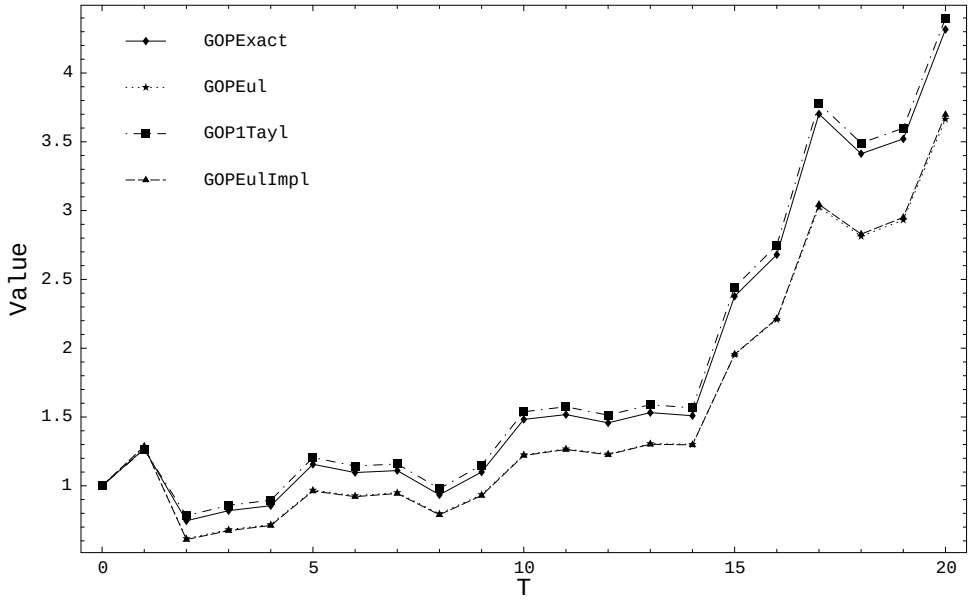

Figure 5.2: Explicit solution, Euler, 1Taylor and implicit Euler scheme for the GOP.

Using the same sample paths of W and N, Figure 5.2 plots the GOP obtained from the explicit solution, the Euler scheme, the implicit Euler scheme and order 1.0 strong Taylor scheme; labelled as "GOPExact","GOPEul", "GOPEulImpl" and "GOP1Tayl", respectively. For the implicit Euler scheme we have chosen the implicitness parameter $\zeta = 1$. We can clearly see the higher accuracy of the order 1.0 Taylor scheme. After two years the Euler and the implicit Euler schemes produce a significant error that becomes higher at the end of the 20 years. If we have more information about the Wiener and the Poisson process at finer time increments all the schemes become more accurate. However, to ensure that the approximate solution is close to the true solution until the end of the period it is recommended to use the order 1.0 strong scheme.

In Figures 5.3 and 5.4 we show similar plots when approximating the risky primary securities S^1 and S^2. In this case one also notices the higher accuracy of the order 1.0 strong Taylor scheme. In Figure 5.4 we can see that the Euler and the implicit Euler schemes lead to approximations that even become negative, while the approximation corresponding to the order 1.0 strong Taylor scheme remains positive and close to the true dynamics. These results give only an indication of the accuracy achieved by these schemes, which is here based on a single scenario. To carefully analyze the strong order of convergence, one could run a simulation of the errors of several simulations with different time increments and check that the schemes achieve the strong orders of convergence predicted by the convergence theorems, see Bruti-Liberati & Platen (2005a). We leave such a study for the next section, when we use weak approximations.

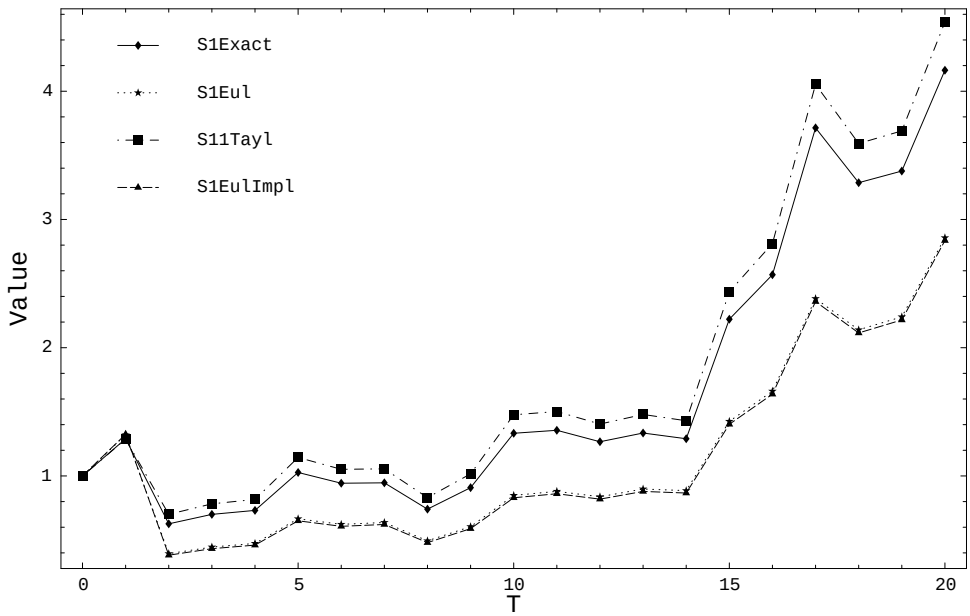

Figure 5.3: Explicit solution, Euler, 1Taylor and implicit Euler scheme for S^1.

5.2 Monte Carlo Simulation

In this section we show some numerical results for the evaluation of the expectation of a function g of the GOP at a terminal time T. Specifically, we approximate $E\Big(g(S_T^{\delta_*})\Big)$ using Monte Carlo simulation. According to the definition of the weak error (1.2), we are now considering a weak problem and the weak schemes presented in Section 4 should be used. Let us note that the convergence theorems in the literature assume some smoothness and growth conditions on the function g, as we have required in the definition of the weak error (1.2). These are also the usual assumptions made in the case of pure diffusions. There exist only few results with weaker assumptions on the function g, which are limited to pure diffusion SDEs and to the Euler scheme, see Bally & Talay (1996a, 1996b) and Guyon (2006). For this reason we will first consider the expectation of a smooth function of the GOP and then later the expectation of a non differentiable function of the GOP. In the first case we will estimate the second moment of the GOP and in the second case we will value a call option. Another important application that involves a smooth payoff is the evaluation of expected utility.

We assume that the GOP follows the SDE (5.20) with the same parameters as in Section 5.1, and terminal time $T = 0.5$. We now compute by Monte Carlo simulation the second moment of the GOP, $E\Big(\big(S_T^{\delta_*}\big)^2\Big)$, at time T. Since the SDE (5.20) admits the explicit solution (5.21), we obtain for the second moment of the GOP the closed form solution

$$E\Big(\big(S_T^{\delta_*}\big)^2\Big) = \big(S_0^{\delta_*}\big)^2 e^{(2a^{S^{\delta_*}}+(b^{S^{\delta_*}})^2)T+hT\left((c^{S^{\delta_*}})^2+2c^{S^{\delta_*}}\right)}.$$

Therefore, the weak error $\varepsilon_w(\Delta)$, defined in (1.2), can be computed for the Monte

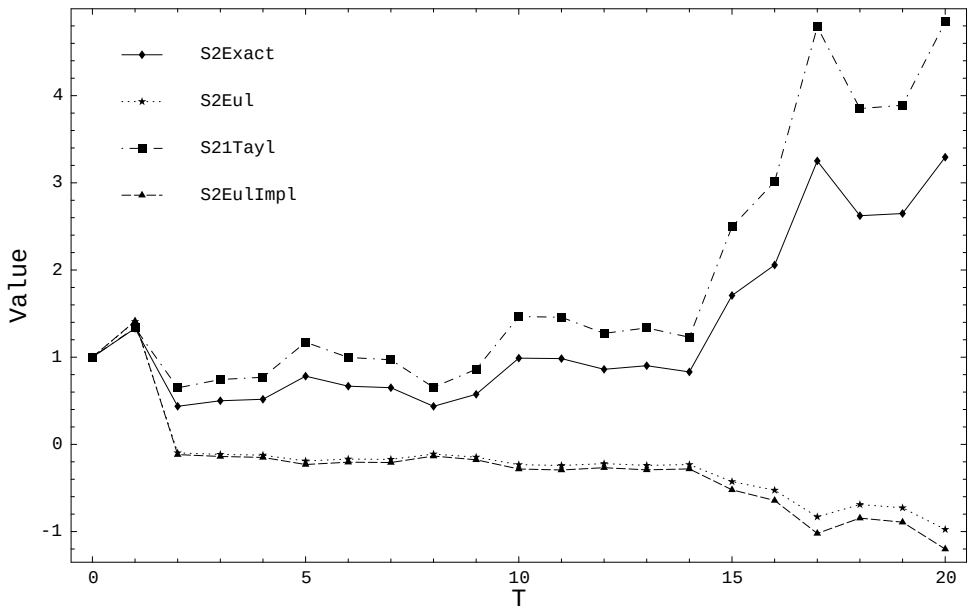

Figure 5.4: Explicit solution, Euler, 1Taylor and implicit Euler scheme for S^2.

Carlo simulations.

In Figure 5.5 we show a log-log plot of the logarithm $\log_2(\varepsilon_w(\Delta))$ of the weak error, as defined in (1.2), versus the logarithm $\log_2(\Delta)$ of the time step size for the Euler, the jump-adapted Euler, the jump-adapted predictor-corrector and the order 2 weak Taylor schemes. These are labelled "Eul", "EulJA", "PredCorrJA" and "2Taylor", respectively. The achieved numerical orders of convergence are given by the slopes of the lines in the log-log plots. To analyze the discretization error, we run sufficiently many simulations to render the statistical error negligible.

The numerical experiments confirm that the four schemes above achieve their theoretically predicted weak orders of convergence. Moreover, comparing the Euler scheme with the jump-adapted Euler scheme, we see that the jump-adapted scheme is more accurate, even though they both achieve the same weak order of convergence $\beta = 1$. This is due to the simulation of the impact of jumps at the correct jump times for jump-adapted schemes. The jump-adapted predictor-corrector scheme is the most accurate among the first order schemes analyzed. As explained in Section 4, this is a rather useful scheme, since it retains the stability properties of implicit schemes without requiring the solution of an additional algebraic equation in each time step. Finally, the order 2 weak Taylor scheme achieves a weak order of convergence $\beta = 2$, as seen from the graph in Figure 5.5. It is also the most accurate scheme for the time step sizes analyzed.

As an example with a non-smooth payoff, we compute now the price of a European call option on a diversified world stock index. As explained in Section 5, the GOP is a good proxy for such an index. Therefore, we regard the option under consideration as a call on the GOP. Its payoff is $H_T = (S_T^{\delta_*} - K)^+$, where K is

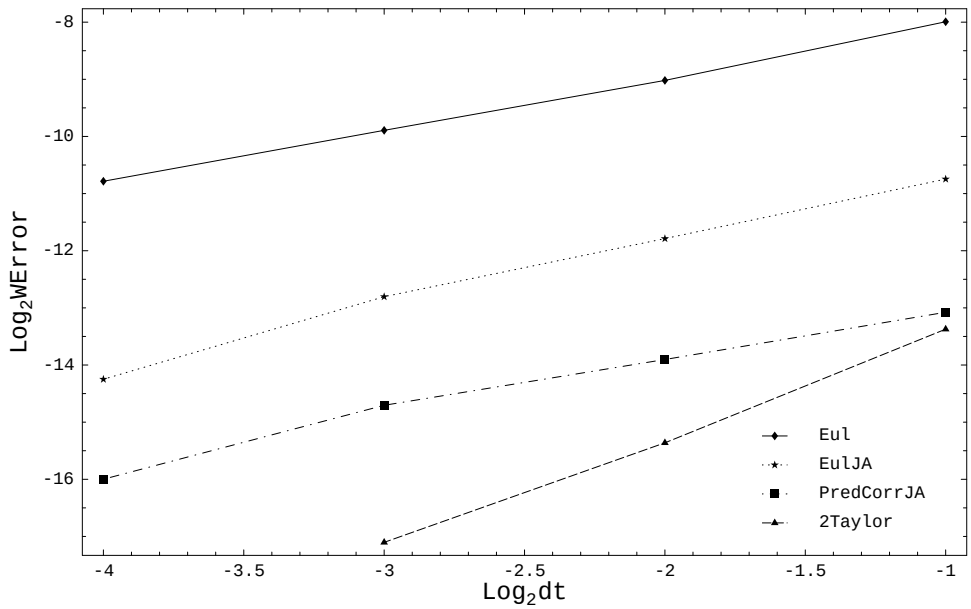

Figure 5.5: Weak error for Euler, jump-adapted Euler, jump-adapted predictor-corrector and 2Taylor schemes.

the strike price. According to (5.16), the price of this instrument is given by

$$\begin{aligned} C_t &= S_t^{\delta_*} E\Big(\frac{\big(S_T^{\delta_*} - K\big)^+}{S_T^{\delta_*}} | \mathcal{A}_t\Big) \\ &= S_t^{\delta_*} E\Big((1 - \frac{K}{S_T^{\delta_*}})^+ | \mathcal{A}_t\Big). \end{aligned} \tag{5.22}$$

Thanks to the particular dynamics (5.20) that we have assumed for the GOP, we obtain the following closed form solution for the call price

$$C_t = \sum_{n=0}^{\infty} \frac{e^{-hT}\,(hT)^n}{n!} f_n, \tag{5.23}$$

where

$$f_n = S_t^{\delta_*} N(d1_n) - \, K\, N(d2_n)\, (1 + c^{S^{\delta_*}})^{-n}\, e^{-\left(a^{S^{\delta_*}} - (b^{S^{\delta_*}})^2\right)T}, \tag{5.24}$$

$$d1_n = \frac{\log(\frac{S_t^{\delta_*}\,(1+c^{S^{\delta_*}})^n}{K}) + (a^{S^{\delta_*}} - \frac{(b^{S^{\delta_*}})^2}{2})\,T}{b^{S^{\delta_*}}\sqrt{T}}, \tag{5.25}$$

and $d2_n = d1_n - b^{S^{\delta_*}}\sqrt{T}$. In (5.24) $N(\cdot)$ denotes the probability distribution of a standard Gaussian random variable.

In Figure 5.6 we plot the weak error resulting from the Euler, jump-adapted Euler, jump-adapted predictor-corrector and order 2 weak Taylor schemes for the call price, on a log-log scale. The strike price K is set equal to 1.2. Among

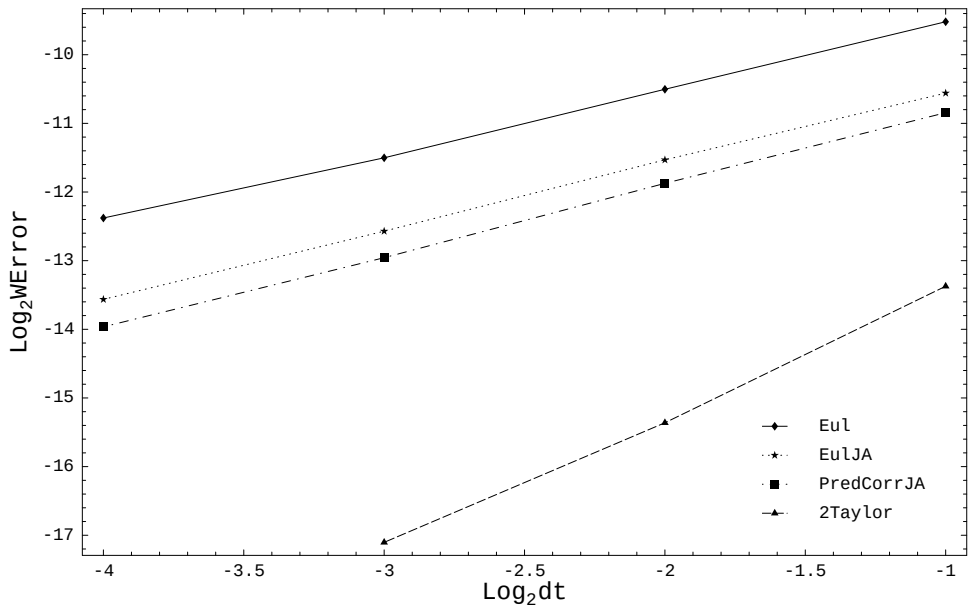

Figure 5.6: Weak error for Euler, jump-adapted Euler, jump-adapted predictor-corrector and 2Taylor schemes.

the first order schemes, the jump-adapted predictor-corrector scheme is the most accurate, while the Euler scheme is the least accurate, as we already noticed in Figure 5.5 for a smooth payoff function. Moreover, the order 2 weak Taylor scheme is more accurate than the first order schemes. In Figure 5.6 the order 2 weak Taylor scheme seems to numerically achieve second order of convergence. However, we report that in other simulations with different parameters, while the order 2 weak Taylor scheme is still the most accurate, it does not achieve the steepness of the slope of a second order weak scheme. One should notice that in this case the second order of weak convergence is not guaranteed by weak convergence theorems, since the required smoothness conditions are violated by the non-differentiable payoff.

6 Conclusions

In this paper we have presented an introductory survey on the numerical solution of SDEs with jumps and discussed some applications under the benchmark approach. Discrete time approximations can be divided into two main classes: strong schemes and weak schemes. Strong schemes are pathwise approximations and are more demanding to construct and to run. They are appropriate for problems such as filtering, scenario analysis and hedge simulation. A strong scheme generates a path that is aimed to be close to the path of the exact solution. Weak schemes, on the other hand, provide approximations of the probability measure of the exact solution. They are appropriate for problems such as moment estimation, derivative pricing or the evaluation of risk measures and expected utilities.

Since only an approximation of the probability distribution of the solution of the SDE is sought, for the construction of weak schemes one has much freedom in the choice of the random variables appearing in the approximations. The so-called simplified schemes exploit this possibility by using simple multi-point distributed random variables.

Numerical approximations of jump-diffusion SDEs can be also divided into jump-adapted schemes and schemes that do not include jump times in their discretization. Jump-adapted approximations are in general much simpler to derive and implement. However, by construction their computational complexity depends on the jump intensity. Derivative free and predictor-corrector schemes have been found to be rather efficient.

To illustrate applications in finance, a brief introduction to the benchmark approach has been given. Strong schemes are required for scenario simulations. The simpler weak schemes are sufficient in Monte Carlo simulations for the evaluation of moments and option prices.

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