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A FOURIER TRANSFORM ANALYSIS OF THE AMERICAN CALL OPTION ON ASSETS DRIVEN BY JUMP-DIFFUSION PROCESSES

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Abstract. This paper considers the Fourier transform approach to derive the implicit integral equation for the price of an American call option in the case where the underlying asset follows a jump-diffusion process. Using the method of Jamshidian (1992), we demonstrate that the call option price is given by the solution to an inhomogeneous integro-partial differential equation in an unbounded domain, and subsequently derive the solution using Fourier transforms. We also extend McKean's incomplete Fourier transform approach to solve the free boundary problem under Merton's framework, for a general jump size distribution. We show how the two methods are related to each other, and also to the Geske-Johnson compound option approach used by Gukhal (2001). The paper also derives results concerning the limit for the free boundary at expiry, and presents a numerical algorithm for solving the linked integral equation system for the American call price, delta and early exercise boundary. This scheme is applied to Merton's jump-diffusion model, where the jumps are log-normally distributed.

Keywords: American options, jump-diffusion, Volterra integral equation, free boundary problem.

JEL Classification: C61, D11.

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1. Introduction

The American option pricing problem has been explored in great depth in the option pricing literature. A recent survey by Barone-Adesi (2005) provides an overview of this research for the American put under the classical Brownian motion process for asset returns considered by Black & Scholes (1973) and Merton (1973). In practice, many assets, in particular foreign exchange rate, are found to have return distributions that are better represented by jump-diffusion processes. Examples of these findings are provided by Jarrow & Rosenfeld (1984), Ball & Torous (1985), Jorion (1988), Ahn & Thompson (1992), and Bates (1996). Merton (1976) provides a framework for pricing European options under jump-diffusion processes[1], and in this paper we explore the extension of this model to the pricing of American call options. We consider two approaches for deriving the linked system of integral equations for the price and early exercise boundary of an American call under Merton's jump-diffusion dynamics, focusing in particular on the use of integral transform techniques to solve the associated integro-partial differential equation (IPDE) for the American call price. We derive the limit of the early exercise boundary at maturity, and provide a numerical algorithm for solving the linked integral equation system based on the quadrature integration technique of Kallast & Kivinukk (2003). This algorithm generates estimates for the price, delta and early exercise boundary.

When deriving the integral equations for the price and early of American options, there are four particular approaches that can be used. The probabilistic method is demonstrated by Karatzas (1988) and Jacka (1991) for the pure-diffusion case, and has been generalised to jump-diffusion by Pham (1997). The discrete time approach using compound option theory is demonstrated by Geske & Johnson (1984), and Kim (1990) shows how to take the limit to provide the continuous time solution. Gukhal (2001) extends this solution technique to include Merton's jump-diffusion dynamics.

[1]Merton does not indicate how he obtained the solution he gives for the European call. In an appendix he verifies that the solution given satisfies the IPDE, but of course this procedure requires one to know the form of the solution.

The remaining two approaches focus on deriving solutions to the partial differential equation (PDE) for the American call price. McKean (1965) solves the homogeneous PDE in a restricted domain using an incomplete Fourier transform. An alternative approach presented by Jamshidian (1992) replaces the homogeneous PDE with an equivalent inhomoegenous PDE which must be solved in an unrestricted domain. The solution can then be derived using a standard Fourier transform, or through an application of Duhamel's principle. The extension of these solution methods to the jump-diffusion case has not been covered in the existing literature, and the first contribution of this paper is to provide this extension. In this paper we demonstrate how to use Fourier transform techniques to solve the IPDE for the American call option price and free boundary. The main advantage of the Fourier transform method is that it is broadly applicable to many option payoffs, so that a wider variety of American options such as puts, calls, butterflies, spread options and max-options can all be handled systematically. This is not the case for the approach based on the generalisation of the Geske-Johnson compound option solution of Gukhal (2001).

As in the standard Black-Scholes-Merton framework, there is no known closed-form solution for the American option price and early exercise boundary under jump-diffusion. Thus it is necessary to use numerical techniques to compute the option price and optimal exercise strategy. A range of numerical methods have been applied to the American option pricing problem under jump-diffusion. Tree methods are used by Amin (1993), Wu & Dai (2001) and Broadie & Yamamoto (2003). An algorithm using the Snell envelope for jump-diffusion is provided by Mullinaci (1996). Meyer (1998) generalises the method of lines to price American puts under Merton's jump-diffusion dynamics by using an time-explicit approximation for the integral term. In contrast to this, Matache, Schwab & Wihler (2004) use an time-implicit scheme for the integral term, in conjunction with a finite elements method. The resulting dense matrices are dense, and this is overcome by use of a wavelet compression technique, leading to sparse systems that are numerically efficient to evaluate.

The other large class of numerical solutions are based on the use of finite difference methods. Primary examples include Zhang (1997) and Carr & Hirsa (2003), in which the integral term is approximated explicitly to produce a tridiagonal system of equations. A similar methodology is applied in the fixed-point iteration method of d'Halluin, Forsyth & Vetzal (2005), and also the penalty method of d'Halluin, Forsyth & Labahn (2004). The main difference in these latter methods is the way in which the tridigagonal system is solved, and the use of fast Fourier transforms to approximate the integral component. A variation on this is used by Andersen & Andreasen (2000), in which they apply an alternating direction implicit (ADI) scheme to the integral term in order to keep the resulting system of difference equations tridiagonal. Briani, Chioma & Natalini (2004) provides a thorough review of the collection of finite difference solution methods for American options under jump-diffusion, and in particular they prove convergence for both time-explicit and time-implicit schemes applied to the integral term.

Despite the amount of existing literature on American options with jumps, there has been little work (to our knowledge) on the implementation of the integral equations for the price and free boundary of American options under jump-diffusion. While some authors such as Pham (1997) and Gukhal (2001) derive these integral equations, they do not discuss how they can be solved numerically. Here we extend the approach of Kallast & Kivinukk (2003) by applying a quadrature scheme to solve the linked integral equation system that arises for the American call and its free boundary in the case of jump-diffusion. While the focus of this paper is not on finding optimal numerical methods for American option prices with jumps, we are able to demonstrate that the proposed numerical integration scheme is able to accurately find the price, delta and early exercise boundary of American calls with log-normal jump sizes, and in particular, that the method is more efficient than a simple two-pass Crank-Nicolson finite difference scheme.

The remainder of this paper is structured as follows. Section 2 outlines the free boundary problem that arises from pricing an American call option under Merton's jump-diffusion model. Section 3 applies Jamshidian's method to derive an inhomogeneous IPDE for the American call price, which is then solved using Fourier transforms. Section 4 applies McKean's incomplete Fourier transform to solve the IPDE in terms of a transform variable. The transform is inverted, providing a McKean-style integral equation for the

American call price, and a corresponding integral equation for the call's early exercise boundary. We then demonstrate how to express the representation of McKean as Kim's representation, allowing us to relate our findings to the Jamshidian method and also Gukhal's (2001) solution method. A feature of the solution is that the integral equation for the call value and the integral equation for the free boundary are interdependent, so some approximation is required to use the two-pass procedure of the non-jump case. We derive the limit of the free boundary at expiry in Section 5 via two different approaches, and find that this limit is different to that found for pure-diffusion models. Section 6 analyses the integral equations in the case where the jump sizes follow a log-normal distribution, as suggested by Merton (1976). Section 7 outlines the numerical integration method used to solve the linked integral equation system for both the free boundary, price and delta of the American call. A selection of numerical results for the American call option and its early exercise boundary are also provided. Concluding remarks are presented in Section 8. Most of the lengthy mathematical derivations are given in appendices.

2. Problem Statement - Merton's Model

Let $C(S,\tau)$ be the price of an American option written on the underlying asset S at time to expiry $\tau = T$, and strike price K. We assume that S pays a continuous dividend yield of rate q. Let $a(\tau)$ denote the early exercise boundary at time to expiry τ, and assume S follows the jump-diffusion process

$$dS = (\mu - \lambda k)Sdt + \sigma S dW + (Y-1)Sd\bar{q}, \tag{1}$$

where t is the current time[2], μ is the instantaneous return per unit time, σ is the instantaneous volatility per unit time, W is a standard Wiener process and $\bar{q}$ is a Poisson process whose increments satisfy

$$d\bar{q} = \begin{cases} 1, & \text{with probability } \lambda dt, \\ 0, & \text{with probability } (1-\lambda dt). \end{cases}$$

[2]Note that $\tau = T - t$.

Let the jump size, Y, be a random variable whose probability measure we denote by $\mathbb{Q}$, with W, Y and $\bar{q}$ all independent. We use $G(Y)$ to denote the corresponding probability density function for Y. Thus the expected jump size, k, is given by

$$k = \mathbb{E}_{\mathbb{Q}}[Y-1] = \int_0^\infty (Y-1)G(Y)dY. \tag{2}$$

Following Merton's (1976) argument and assuming that the jump risk is fully diversifiable[3], it is known that C satisfies the integro-partial differential equation (IPDE)

$$\frac{\partial C}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - q - \lambda k) S \frac{\partial C}{\partial S} - rC + \lambda \int_0^\infty [C(SY,\tau) - C(S,\tau)]G(Y)dY, \tag{3}$$

in the region $0 \le \tau \le T$ and $0 \le S \le a(\tau)$, where r is the risk-free rate.

In the case of an American call option, the IPDE (3) is subject to the initial and boundary conditions

$$C(S,0) = \max(S-K,0), \quad 0 \le S < \infty \tag{4}$$

$$C(0,\tau) = 0, \quad \tau \ge 0, \tag{5}$$

$$C(a(\tau),\tau) = a(\tau) - K, \quad t \ge 0, \tag{6}$$

$$\lim_{S \to a(\tau)} \frac{\partial C}{\partial S} = 1, \quad \tau \ge 0. \tag{7}$$

Condition (4) is the payoff function for the call at expiry, and condition (5) ensures that the option is worthless if S falls to zero. The value-matching condition (6) forces the value of the call option to be equal to its payoff on the early exercise boundary, and the smooth-pasting condition (7) sets the delta of the American call to be continuous at the free boundary to guarantee arbitrage-free prices. For the finite call under consideration, we note that the standard arbitrage arguments that justify condition (7) are not readily applied under Merton's jump-diffusion model, since this depends upon the price process for S being continuous. The corresponding boundary conditions were proven by Pham (1997) for the American put case, and we shall assume here that this result for the put will extend naturally to the American call problem with a continuous dividend yield

[3]We make this assumption for convenience. The derivation that follows would carry through if we were to assume a constant market price of jump risk.

for S, as per Gukhal (2001). Figure 1 demonstrates the payoff, price profile and early exercise boundary for the American call under consideration.

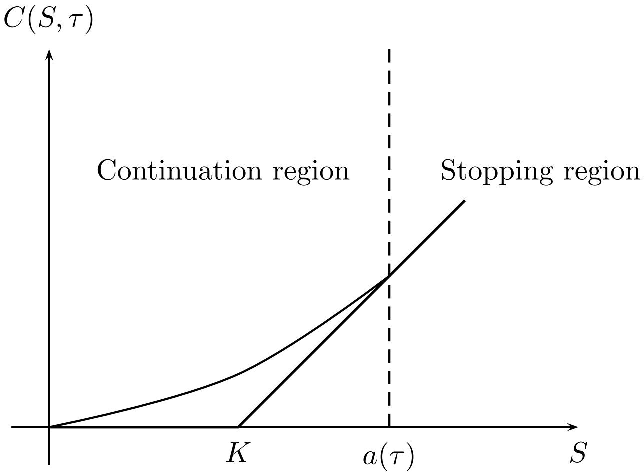


FIGURE 1. Continuation region for the American call option.

3. Jamshidian's Representation and Solution

In the pure-diffusion case, Jamshidian (1992) demonstrates that by evaluating the PDE for the American call price when $S > a(\tau)$, one can reformulate the free boundary problem in the restricted domain $0 \leq S \leq a(\tau)$ as an inhomogeneous PDE to be solved in the unrestricted domain $0 \leq S < \infty$. Here we show how to apply Jamshidian's formulation to American call options under Merton's (1976) jump-diffusion dynamics, given by (1).

Firstly, we note that the free boundary value problem given by (3) - (7) involves a homogeneous IPDE to be solved in the restricted asset price domain $0 \leq S \leq a(\tau)$. We highlight the fact that $C(S,\tau)$ and $\partial C/\partial S$ are continuous for $0 \leq S < \infty$, as given by the value-matching condition (6) and smooth-pasting condition (7). Jamshidian's approach is only certain to be applicable when such continuity holds. We now extend Jamshidian's results to allow for jumps in the asset price dynamics.

Proposition 3.1. *Solving the homogeneous IPDE (3) for $C(S,\tau)$ in the domain $0 \leq S \leq a(\tau)$ subject to the initial and boundary conditions (4) - (7) is equivalent to solving* *the inhomogeneous IPDE*

$$\begin{aligned}\frac{\partial C}{\partial \tau} &= \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - q - \lambda k) S \frac{\partial C}{\partial S} - rC + \lambda \int_0^\infty [C(SY,\tau) - C(S,\tau)] G(Y) dY \\ &\quad + H_1(S - a(\tau)) \left\{ qS - rK - \lambda \int_0^{a(\tau)/S} [C(SY,\tau) - (SY - K)] G(Y) dY \right\}, \quad (8)\end{aligned}$$

in the region $0 \leq \tau \leq T$, $0 \leq S < \infty$, *subject to the initial condition (4), where* $H_j(x)$ *is the Heaviside step function defined as*

$$H_j(x) = \begin{cases} 1, & x > 0, \\ \frac{1}{j}, & x = 0, \\ 0, & x < 0, \end{cases} \quad (9)$$

for $j = 1, 2$. ***Proof:*** *Refer to Appendix 1.*

□

There is a clear economic interpretation for the inhomogeneous term in equation (8), which has also been provided by Gukhal (2001). The $(qS - rK)$ term represents the net cash flows received from holding the portfolio $(S - K)$ whenever S is in the stopping region. This is already familiar from the pure-diffusion case (see for example Kim (1990)). The integral term arises entirely due to the introduction of jumps in the price process for S. Note that if no jumps are present (λ=0) then this term will be zero, and the inhomogeneous term becomes the same one presented by Jamshidian (1992). This additional term captures the rebalancing costs incurred by the option holder whenever the price of the underlying jumps down[4] from the stopping region into the continuation region. Figure 2 illustrates this effect in detail. If the holder of the option has observed that the underlying price is at $S_- > a(\tau)$, then the call will be optimally exercised. If an instant after exercising a jump of size Y occurs such that $S_+ = YS_- < a(\tau)$, then the portfolio $S - K$ held by the investor will now be worth less than the unexercised American call. This difference is the cost being captured by the integral in the inhomogeneous term in (8).

[4]Since $S \geq a(\tau)$, we know that $a(\tau)/S \leq 1$.

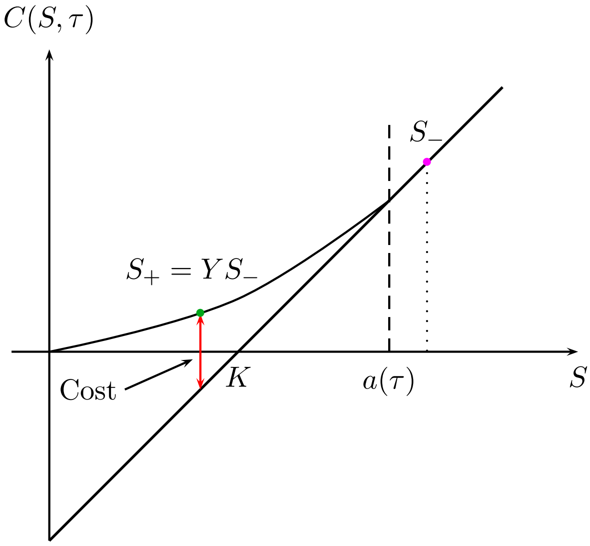


FIGURE 2. Cost incurred by the investor from downward jumps in S.

Having derived the inhomogeneous IPDE for $C(S,\tau)$ we now demonstrate how we can use Fourier transforms to find the solution. Our first step is to transform the IPDE to an equation with constant coefficients and a "standardised" strike of 1. Let $S = Ke^x$ and $C(S,t) = KV(x,\tau)$, with $b(\tau) = a(\tau)/K$. The transformed IPDE for V is then[5]

$$\begin{aligned}\frac{\partial V}{\partial \tau} &= \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} + \phi \frac{\partial V}{\partial x} - (r+\lambda)V + \lambda \int_0^\infty V(x+\ln Y,\tau)G(Y)dY \\ &\quad + H_1(x - \ln b(\tau)) \left\{ \lambda \int_0^{b(\tau)e^{-x}} [V(x+\ln Y,\tau) - (Ye^x - 1)]G(Y)dY \right\},\end{aligned} \tag{10}$$

where $\phi \equiv r - q - \lambda k - \frac{\sigma^2}{2}$. Equation (10) is to be solved in the region $0 \le \tau \le T$, $-\infty \le x \le \infty$, subject to the initial and boundary conditions

$$V(x,0) = \max(e^x - 1, 0), \quad -\infty < x < \infty, \tag{11}$$

$$\lim_{x\to-\infty} V(x,\tau) = 0, \quad \tau \ge 0. \tag{12}$$

It is worth noting that the value-matching and smooth-pasting conditions still hold, although we do not explicitly require them when solving (10) for $V(x,\tau)$.

[5]It should be noted that

$$\begin{aligned}C(SY,\tau) &= KV(\ln(\frac{SY}{K}),\tau) \\ &= KV(x+\ln Y,\tau).\end{aligned}$$

Since the x-domain is now $-\infty < x < \infty$, the Fourier transform of the inhomogeneous IPDE (10) can be found. Define the Fourier transform of V, $\mathcal{F}\{V(x,\tau)\}$, as

$$\mathcal{F}\{V(x,\tau)\} = \int_{-\infty}^{\infty} e^{i\eta x} V(x,\tau)dx, \tag{13}$$

with corresponding inversion

$$\mathcal{F}^{-1}\{\hat{V}(\eta,\tau)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\eta x}\hat{V}(\eta,\tau)d\eta, \tag{14}$$

where $i = \sqrt{-1}$. Applying this Fourier transform to (10), we can reduce the inhomogeneous IPDE to an inhomogeneous integro-differential equation, whose solution is readily found.

Proposition 3.2. *Using the initial and boundary conditions (11)-(12), the Fourier transform of the IPDE (10) with respect to x satisfies the integro-differential equation*

$$\frac{\partial \hat{V}}{\partial \tau} + \left[\frac{\sigma^2\eta^2}{2} + \phi i\eta + (r+\lambda) - \lambda A(\eta)\right]\hat{V} = \hat{F}_J(\eta,\tau) \tag{15}$$

where

$$\begin{aligned}\hat{F}_J(\eta,\tau) \equiv \mathcal{F}\Bigg\{ & H_1(x - \ln b(\tau))(qe^x - r) \\ & - H_1(x - \ln b(\tau))\lambda \int_0^{b(\tau)e^{-x}} [V(x+\ln Y,\tau) - (Ye^x - 1)]G(Y)dY\Bigg\},\end{aligned} \tag{16}$$

and

$$A(\eta) \equiv \int_0^{\infty} e^{-i\eta \ln Y} G(Y)dY. \tag{17}$$

Furthermore, the solution to the integro-differential equation (15) is given by

$$\begin{aligned}\hat{V}(\eta,\tau) = {} & \hat{V}(\eta,0)e^{-(\frac{1}{2}\sigma^2\eta^2 + \phi i\eta + (r+\lambda) - \lambda A(\eta))\tau} \\ & + \int_0^{\tau} e^{-(\frac{1}{2}\sigma^2\eta^2 + \phi i\eta + (r+\lambda) - \lambda A(\eta))(\tau-\xi)}\hat{F}_J(\eta,\xi)d\xi,\end{aligned} \tag{18}$$

where $\hat{V}(\eta,0) = \mathcal{F}\{V(x,0)\}$.

Proof: Refer to Appendix 2.

□

Now that $\hat{V}(\eta,\tau)$ has been found, we may invert the transform to recover $V(x,\tau)$, the American call price in the x-τ plane. By taking the inverse Fourier transform of (18), we have

$$\begin{aligned} V(x,\tau) &= \mathcal{F}^{-1}\left\{\hat{V}(\eta,0)e^{-(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta))\tau}\right\} \\ &\quad +\mathcal{F}^{-1}\left\{\int_0^\tau e^{-(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta))(\tau-\xi)}\hat{F}_J(\eta,\xi)d\xi\right\} \\ &\equiv V_E(x,\tau)+V_P(x,\tau) \\ &\equiv \frac{1}{K}\left[C_E(S,\tau)+C_P(S,\tau)\right] = \frac{1}{K}C(S,\tau) \end{aligned} \tag{19}$$

where $C_E(S,\tau) = KV_E(x,\tau)$ is the value of the corresponding European call written on S and $C_P(S,\tau) = KV_P(x,\tau)$ is the early exercise premium for $C(S,\tau)$. By performing the inversions, we can determine the analytic forms of C_E and C_P.

Proposition 3.3. *The price of the European call option, $C_E(S,\tau)$, in equation (19) is given by*

$$C_E(S,\tau) = \sum_{n=0}^{\infty}\frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\{C_{BS}[SX_ne^{-\lambda k\tau},K,K,r,q,\tau,\sigma^2]\}, \tag{20}$$

where

$$\begin{aligned} C_{BS}[S,K,\beta,r,q,\tau,\sigma^2] &= Se^{-q\tau}N[d_1(S,\beta,r,q,\tau,\sigma^2)] \\ &\quad -Ke^{-r\tau}N[d_2(S,\beta,r,q,\tau,\sigma^2)], \\ d_1(S,\beta,r,q,\tau,\sigma^2) &= \frac{\ln\frac{S}{\beta}+\left(r-q+\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \\ d_2(S,\beta,r,q,\tau,\sigma^2) &= d_1(S,\beta,r,q,\tau,\sigma^2)-\sigma\sqrt{\tau}, \\ N[\alpha] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\alpha}e^{-\frac{\eta^2}{2}}d\eta, \\ X_n &\equiv Y_1Y_2...Y_n; \quad X_0\equiv 1, \end{aligned}$$

and

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}^{(n)}\{f(X_n)\} &\equiv \int_0^\infty \int_0^\infty ... \int_0^\infty f(X_n)G(Y_1)G(Y_2)...G(Y_n)dY_1dY_2...dY_n \\ &= \int_0^\infty f(X_n)G(X_n)dX_n.\end{aligned}$$

The $Y_1, Y_2, ..., Y_n$ *are independent draws from the jump size distribution* $G(Y)$.

Proof: Refer to Appendix A3.1.

□

We note that equation (20) is in fact Merton's (1976) solution for a European call option under jump-diffusion, with general jump size density $G(Y)$. Next we shall determine the early exercise premium $C_P(S,\tau)$.

Proposition 3.4. *The early exercise premium,* $C_P(S,\tau)$, *in equation (19) is given by*

$$\begin{aligned}C_P(S,\tau) &= \sum_{n=0}^{\infty}\mathbb{E}_{\mathbb{Q}}^{(n)}\Bigg\{\int_0^\tau \frac{e^{-\lambda(\tau-\xi)}(\tau-\xi)^n}{n!} \\ &\qquad \times\Big[C_P^{(D)}[SX_ne^{-\lambda k(\tau-\xi)}, K, a(\xi), r, r, q, \tau-\xi, \sigma^2] \\ &\qquad -\lambda C_P^{(J)}[SX_ne^{-\lambda k(\tau-\xi)}, K, a(\xi), r, q, \tau-\xi, \sigma^2; C(S,\xi)]\Big]d\xi\Bigg\},\end{aligned} \tag{21}$$

where

$$\begin{aligned}&C_P^{(D)}[S, K, a(\xi), R, r, q, \tau, \sigma^2] \\ &\quad = qSe^{-q\tau}N\left[d_1\left(S, a(\xi), r, q, \tau, \sigma^2\right)\right] - RKe^{-r\tau}N\left[d_2\left(S, a(\xi), r, q, \tau, \sigma^2\right)\right],\end{aligned} \tag{22}$$

$$\begin{aligned}&C_P^{(J)}[S, K, a(\xi), r, q, \tau, \sigma^2; C(\cdot,\xi)] \\ &\quad = e^{-r\tau}\int_0^1 G(Y)\int_{a(\xi)}^{a(\xi)/Y}[C(\omega Y,\xi) - (\omega Y - K)]\kappa(S,\omega,r,q,\tau,\sigma^2)d\omega dY,\end{aligned} \tag{23}$$

and

$$\kappa(S,\omega,r,q,\tau,\sigma^2) \equiv \frac{1}{\omega\sigma\sqrt{2\pi\tau}}\exp\left\{-\frac{1}{2}d_2^2(S,\omega,r,q,\tau,\sigma^2)\right\}. \tag{24}$$

The operator $\mathbb{E}_{\mathbb{Q}}^{(n)}$ *and functions* N, d_1 *and* d_2 *have been defined in Proposition 3.3.*

Proof: Refer to Appendix A3.2.

□

Each of the linear terms in (21) represent discounted expected cash-flows incurred by the option holder when $S > a(\tau)$, as discussed previously for the interpretation of the inhomogeneous term in (8). Combining C_E and C_P, we can now write down that integral equation for the American call option price, $C(S, \tau)$.

Proposition 3.5. *Substituting (20) and (21) into equation (19), the American call price, $C(S, \tau)$ is given by*

$$\begin{aligned} C(S,\tau) &= \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \{ C_{BS}[SX_n e^{-\lambda k\tau}, K, K, r, q, \tau, \sigma^2] \}, \qquad (25) \\ &\quad + \sum_{n=0}^{\infty} \mathbb{E}_{\mathbb{Q}}^{(n)} \Bigg\{ \int_0^{\tau} \frac{e^{-\lambda(\tau-\xi)}[\lambda(\tau-\xi)]^n}{n!} \\ &\quad \times \Bigg[C_P^{(D)}[SX_n e^{-\lambda k(\tau-\xi)}, K, a(\xi), r, r, q, \tau-\xi, \sigma^2] \\ &\quad - \lambda C_P^{(J)}[SX_n e^{-\lambda k(\tau-\xi)}, K, a(\xi), r, q, \tau-\xi, \sigma^2; C(\cdot,\xi)] \Bigg] d\xi \Bigg\}, \end{aligned}$$

where C_{BS} is given by equation (21), and the functions C_P^D, and C_P^J are given by equations (22) and (23) respectively.

Proof: Direct substitution of (20) and (21) into (19) yields equation (25).

□

The solution (25) is readily compared with that of Gukhal (2001), who derives (25) by generalising the compound option approach of Kim (1990) to the jump-diffusion case. The three additive components of the call value in equation (25) each have a clear economic interpretation, as outlined by Gukhal (2001). The first term, C_{BS}, represents the European component of the American call option's value, while the remaining two terms combine to form the total early exercise premium. The middle term is a natural extension of the early exercise premium that arises in the pure-diffusion case. More specifically, this term calculates the dividend received when holding the underlying, less the interest payable on a loan of size K. Thus C_P^D captures the potential income to the option holder should the option be exercised to buy the underlying by borrowing K at the risk-free rate. The third term, C_P^J, arises entirely due to the introduction of jumps in the price process for S, and captures the rebalancing costs incurred by the option holder whenever the price of the underlying jumps down from the stopping region into the continuation region (see Figure 2).

In equation (25), the value of the American call option is expressed as a function of the underlying asset price S, and time to maturity τ. As we have already noted, equation (25) also depends upon the unknown early exercise boundary, $a(\tau)$. By requiring the expression for $C(S,\tau)$ to satisfy the boundary condition (6), we can derive a similar integral equation for the value of $a(\tau)$. This integral equation is given by

$$\begin{aligned} a(\tau) - K \;=\; & \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \{ C_{BS}[a(\tau)X_n e^{-\lambda k\tau}, K, K, r, q, \tau, \sigma^2] \}, \qquad (26) \\ & + \sum_{n=0}^{\infty} \mathbb{E}_{\mathbb{Q}}^{(n)} \Bigg\{ \int_0^\tau \frac{e^{-\lambda(\tau-\xi)}[\lambda(\tau-\xi)]^n}{n!} \\ & \qquad \times \Bigg[C_P^{(D)}[a(\tau)X_n e^{-\lambda k(\tau-\xi)}, K, a(\xi), r, r, q, \tau-\xi, \sigma^2] \\ & \qquad - \lambda C_P^{(J)}[a(\tau)X_n e^{-\lambda k(\tau-\xi)}, K, a(\xi), r, q, \tau-\xi, \sigma^2; C(\cdot,\xi)] \Bigg] d\xi \Bigg\}, \end{aligned}$$

It is particularly crucial to note that the integral equation (26) depends upon the unknown call value $C(S,\tau)$, and this dependence arises entirely from integral terms that have been introduced by the presence of jumps in the dynamics for S.

The general form of the integral equation system consisting of (25) and (26) can be written as

$$C(S,\tau) = \Omega_C(S,\tau) + \int_0^\tau \Psi_C[C(S,\xi), a(\xi), \xi, \tau, S] d\xi, \qquad (27)$$

$$a(\tau) = \Omega_a(a(\tau),\tau) + \int_0^\tau \Psi_a[C(a(\tau),\xi), a(\xi), \xi, \tau, a(\tau)] d\xi, \qquad (28)$$

where the definitions of the functions (Ω_C, Ψ_C) and (Ω_a, Ψ_a) are implied by the right hand sides of equations (25) and (26) respectively. The interdependence of (27) and (28) is obvious, and it is this interdependence that makes numerical implementation more involved than for the corresponding no-jump problem[6]. Thus in order to implement these integral equations for the free boundary and call price, we need to develop numerical techniques to solve the linked integral equation system (25)-(26).

Before concluding this section, we present an alternative form for the double integral involving κ in equation (24).

Proposition 3.6. *By changing the order of integration, C_P^J in equation (23) can be rewritten as*

$$C_P^{(J)}[S, K, a(\xi), r, q, \tau, \sigma^2; C(S, \xi)] \\ = e^{-r\tau} \int_0^1 [C(a(\xi)z, \xi) - (a(\xi)z - K)] \int_0^z G(Y)\kappa(S/a(\xi), z, r, q, \tau, \sigma^2) dY dz. \quad (29)$$

Proof: Refer to Appendix A3.3.

□

While the modified representation in (29) is less economically intuitive than the original, we will show that it offers significant advantages when attempting to solve (26) numerically for specific values of $G(Y)$. In particular, we will demonstrate in Section 6 that when $G(Y)$ is the log-normal density function given by Merton (1976), the innermost integral in (29) can be evaluated analytically. In this way we are able to reduce (29) to a one-dimensional integral for certain realisations of the density $G(Y)$, which makes the task of numerically evaluating (29) much simpler.

4. McKean's Incomplete Fourier Transform

An alternative solution method for the free boundary problem (3)-(7) is provided by McKean (1965). In the pure-diffusion case, McKean extended the domain of the problem to $0 \leq S < \infty$ by assuming $C(S, \tau) = 0$ for $S > a(\tau)$. Under this assumption it is possible to solve the American call free boundary value problem using Fourier transforms, paying close attention to the behaviour of the solution at $S = a(\tau)$. Chiarella, Kucera & Ziogas (2004) give a detailed account of McKean's method for the pure-diffusion case. Here

[6]There the dependence is sequential i.e. first one solves for the free boundary which then feeds into an integral expression for the option price

we extend McKean's method to Merton's jump-diffusion model, demonstrating how the incomplete Fourier transform leads to McKean's representation of (25). We also demonstrate the equivalence between the two forms.

We begin by extending the domain of the problem using the Heaviside step function (9). Multiplying both sides of the IPDE (3) by $H_2(a(\tau) - S)$, we have

$$H_2(a(\tau)-S)\left(\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} + (r-q-\lambda k)S\frac{\partial C}{\partial S} - rC - \frac{\partial C}{\partial \tau}\right)$$
$$+H_2(a(\tau)-S)\left(\lambda\int_0^\infty [C(SY,\tau)-C(S,\tau)]G(Y)dY\right) = 0, \qquad (30)$$

to be solved in the region $0 \leq \tau \leq T$ and $0 \leq S < \infty$, subject to the initial and boundary conditions (4)-(7), and where $H_2(x)$ is defined by (9). It is important to note that since the boundary conditions and IPDE remain unchanged after multiplying (3) by $H_2(a(\tau) - S)$, then by the theorems of existence and uniqueness the solution to McKean's representation will be the same as the solution to the free boundary value problem given by (3)-(7).

Our first step is to again transform the IPDE to an equation with constant coefficients and a "standardised" strike of 1. Let $S = Ke^x$ and $C(S,t) = KV(x,\tau)$. The transformed IPDE for V is then

$$H_2(\ln b(\tau)-x)\left(\frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} + \phi\frac{\partial V}{\partial x} - rV - \frac{\partial V}{\partial \tau}\right)$$
$$+H_2(\ln b(\tau)-x)\left(\lambda\int_0^\infty [V(x+\ln Y,\tau)-V(x,\tau)]G(Y)dY\right) = 0, \quad (31)$$

to be solved in the region $0 \leq \tau \leq T$, $-\infty < x \leq \ln b(\tau)$, where ϕ and $b(\tau)$ are defined in Section 3. The initial and boundary conditions assume the form

$$V(x,0) = \max(e^x-1,0), \quad -\infty < x < \infty, \qquad (32)$$

$$\lim_{x\to-\infty} V(x,\tau) = 0, \quad \tau \geq 0, \qquad (33)$$

$$V(\ln b(\tau),\tau) = b(\tau)-1, \quad \tau \geq 0, \qquad (34)$$

$$\lim_{x\to\ln b(\tau)}\frac{\partial V}{\partial x} = b(\tau), \quad \tau \geq 0. \qquad (35)$$

For simplicity, we shall denote $b(\tau)$ by $b \equiv b(\tau)$ when it is clear at which time the free boundary is being evaluated.

To solve the free boundary problem defined by equations (31)-(35), we shall apply the Fourier transform technique to reduce the IPDE (31) to an ordinary differential equation (ODE). Note that the function V and its first two derivatives with respect to x tend to zero as $x \to -\infty$. This knowledge is required to eliminate limit terms that arise in integration by parts (see Appendix 4).

Since the x-domain is now $-\infty < x < \infty$, the Fourier transform of the IPDE can be found. Given the definition of the Fourier transform in equation (13), the transform of (31) is,

$$\begin{aligned} \mathcal{F}\left\{H_2(\ln b - x)\frac{\partial V}{\partial \tau}\right\} &= \frac{\sigma^2}{2}\mathcal{F}\left\{H_2(\ln b - x)\frac{\partial^2 V}{\partial x^2}\right\} + \phi\mathcal{F}\left\{H_2(\ln b - x)\frac{\partial V}{\partial x}\right\} \\ &\quad -(r+\lambda)\mathcal{F}\Big\{H_2(\ln b - x)V\Big\} \qquad (36) \\ &\quad +\lambda\mathcal{F}\left\{H_2(\ln b - x)\int_0^\infty V(x+\ln Y,\tau)G(Y)dY\right\}. \end{aligned}$$

By the definition of the Fourier transform, we have

$$\mathcal{F}\left\{H_2(\ln b - x)V(x,\tau)\right\} = \int_{-\infty}^{\ln b} e^{i\eta x}V(x,\tau)dx \equiv \mathcal{F}^b\{V(x,\tau)\} \equiv \hat{V}^b(\eta,\tau), \qquad (37)$$

and we refer to $\mathcal{F}^b$ as the incomplete Fourier transform of $V(x,\tau)$ with respect to x. The transform is called "incomplete" because it can be interpreted as a standard Fourier transform applied to $V(x,\tau)$ in the domain $-\infty < x < \ln b(\tau)$. We now apply (37) to carry out the transform operations in (36).

Proposition 4.1. *Using the initial and boundary conditions (32)-(35), the incomplete Fourier transform of the IPDE (31) with respect to x satisfies the integro-differential equation*

$$\frac{\partial \hat{V}^b}{\partial \tau} + \left[\frac{\sigma^2\eta^2}{2} + \phi i\eta + (r+\lambda) - \lambda A(\eta)\right]\hat{V}^b = \hat{F}^b_M(\eta,\tau) \qquad (38)$$

where

$$\hat{F}_M^b(\eta,\tau) \equiv e^{i\eta \ln x}\left[\frac{\sigma^2 b}{2} + \left(\frac{b'}{b} - \frac{\sigma^2 i\eta}{2} + \phi\right)(b-1)\right] + \lambda\Phi(\eta,\tau), \tag{39}$$

$$\Phi(\eta,\tau) \equiv \int_0^\infty e^{-i\eta \ln Y} G(Y)\left[\int_{\ln b}^{\ln bY} e^{i\eta z} V(z,\tau)dz\right]dY, \tag{40}$$

with $A(\eta)$ given by (17) and $b' \equiv db(\tau)/d\tau$. Furthermore, the solution to the integro-differential equation (38) is given by

$$\begin{aligned}\hat{V}^b(\eta,\tau) &= \hat{V}^b(\eta,0)e^{-(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta))\tau} \\ &\quad + \int_0^\tau e^{-(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta))(\tau-\xi)}\hat{F}_M^b(\eta,\xi)d\xi.\end{aligned} \tag{41}$$

Proof: Refer to Appendix 4.

□

Now that $\hat{V}^b(\eta,\tau)$ has been found, we may invert the transform using (14) to recover $V(x,\tau)$, the American call price in the x-τ plane. By taking the inverse Fourier transform of (41), we have

$$\begin{aligned}V(x,\tau) &= \mathcal{F}^{-1}\left\{\hat{V}^b(\eta,0)e^{-(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta))\tau}\right\} \\ &\quad + \mathcal{F}^{-1}\left\{\int_0^\tau e^{-(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta))(\tau-s)}\hat{F}_M^b(\eta,s)ds\right\} \\ &\equiv V_1(x,\tau)+V_2(x,\tau) \\ &\equiv \frac{1}{K}\left[C_1(S,\tau)+C_2(S,\tau)\right] = \frac{1}{K}C(S,\tau)\end{aligned} \tag{42}$$

where $-\infty < x < \ln b(\tau)$, and the forms of the functions $C_1(S,\tau)$ and $C_2(S,\tau)$ are given by Propositions 4.2, 4.3 and 4.4 below.

Proposition 4.2. *The function $C_1(S,\tau)$ in equation (42) is given by*

$$\begin{aligned}C_1(S,\tau) &= \sum_{n=0}^{\infty}\frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\{C_{BS}[SX_ne^{-\lambda k\tau},K,K,r,q,\tau,\sigma^2] \\ &\qquad - C_{BS}[SX_ne^{-\lambda k\tau},K,a(0^+),r,q,\tau,\sigma^2]\}\end{aligned} \tag{43}$$

where

$$a(0^+) \equiv \lim_{\tau \to 0^+} a(\tau), \tag{44}$$

with C_{BS}, X_n and $\mathbb{E}_{\mathbb{Q}}^{(n)}$ defined in Proposition 3.3.

Proof: Refer to Appendix A5.1.

□

Next we consider the more complicated function $V_2(x,\tau)$. The first step is to break the function down into two linear components that arise from the form of function $\hat{F}_M^b$ in equation (39). Thus we have

$$\begin{aligned}
V_2(x,\tau) &= \mathcal{F}^{-1}\left\{\int_0^\tau e^{-(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta))(\tau-\xi)}\hat{F}_M^b(\eta,\xi)d\xi\right\} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\eta x}\int_0^\tau e^{-(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta))(\tau-\xi)} \\
&\qquad \times e^{i\eta \ln b(\xi)}\left[\frac{\sigma^2 b(\xi)}{2}+\left(\frac{b'(\xi)}{b(\xi)}-\frac{\sigma^2 i\eta}{2}+\phi\right)(b(\xi)-1)\right]d\xi d\eta \\
&\quad +\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\eta x}\int_0^\tau e^{-(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta))(\tau-\xi)}\lambda\Phi(\eta,\xi)d\xi d\eta \\
&\equiv V_2^{(1)}(x,\tau)+V_2^{(2)}(x,\tau) \\
&\equiv \frac{1}{K}\left[C_2^{(1)}(S,\tau)+C_2^{(2)}(S,\tau)\right]=\frac{1}{K}C_2(S,\tau).
\end{aligned} \tag{45}$$

We start by considering the function $C_2^{(1)}(S,\tau)$.

Proposition 4.3. *The term $C_2^{(1)}(S,\tau)$ in equation (45) is given by*

$$\begin{aligned}
C_2^{(1)}(S,\tau) &= \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\left\{\int_0^\tau e^{-(r+\lambda)(\tau-\xi)}(\tau-\xi)^n(a(\xi)-K)\right. \\
&\qquad \times\left[\frac{\sigma^2 a(\xi)}{2}+\left(\frac{a'(\xi)}{a(\xi)}+\frac{1}{2}\left[\left(r-q-\frac{\sigma^2}{2}\right)-\frac{\ln[SX_n e^{-\lambda k(\tau-\xi)}/a(\xi)]}{\tau-\xi}\right]\right)\right] \\
&\qquad \left.\times\kappa(SX_n e^{-\lambda k(\tau-\xi)}/a(\xi),1,r,q,\tau-\xi,\sigma^2)d\xi\right\},
\end{aligned} \tag{46}$$

where κ is defined by equation (24), and the operator $\mathbb{E}_{\mathbb{Q}}^{(n)}$ has been defined in Proposition 3.3.

Proof: Refer to Appendix A5.2.

□

Equation (46) is a generalisation of the integral term in McKean's (1965) solution that allows for the presence of jumps. If we set $\lambda = 0$ in (46), we obtain the integral term found by McKean[7].

The last remaining term to be evaluated is $C_2^{(2)}(S,\tau)$, which is the extra term introduced into the expression for the American option price by the presence of jumps in the stochastic process for S.

Proposition 4.4. *The term* $C_2^{(2)}(S,\tau)$ *is given by*

$$\begin{aligned} C_2^{(2)}(S,\tau) &= -\lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \Bigg\{ \int_0^{\tau} (\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \\ &\quad \times \Bigg[\int_0^1 G(Y) \int_{a(\xi)}^{a(\xi)/Y} C(\omega Y,\xi) \kappa(S X_n e^{-\lambda k(\tau-\xi)}, \omega, r, q, \tau-\xi, \sigma^2) d\omega dY \\ &\quad - \int_1^{\infty} G(Y) \int_{a(\xi)/Y}^{a(\xi)} (\omega Y - K) \kappa(S X_n e^{-\lambda k(\tau-\xi)}, \omega, r, q, \tau-\xi, \sigma^2) d\omega dY \Bigg] d\xi \Bigg\} \end{aligned} \tag{47}$$

where κ *is defined by equation (24), and the operator* $\mathbb{E}_{\mathbb{Q}}^{(n)}$ *has been defined in Proposition 3.3.*

Proof: Refer to Appendix A5.3.

□

Now that we have derived the functions $C_1(S,\tau)$ and $C_2(S,\tau)$, we can provide McKean's integral equation for the price of the American call, $C(S,\tau)$, in the case where S follows a jump-diffusion processes.

[7]We also refer the reader to Chiarella et al. (2004) for a derivation of McKean's solution in the pure-diffusion case.

Proposition 4.5. *The integral equation for the price of the American call option, $C(S,\tau)$, is*

$$
\begin{aligned}
C(S,\tau) \quad = \quad & \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \{ C_{BS}[SX_n e^{-\lambda k\tau}, K, K, r, q, \tau, \sigma^2] \\
& \qquad - C_{BS}[SX_n e^{-\lambda k\tau}, K, a(0^+), r, q, \tau, \sigma^2] \} \\
& + \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \Big\{ \int_0^\tau e^{-(r+\lambda)(\tau-\xi)} (\tau-\xi)^n (a(\xi)-K) \\
& \times \left[\frac{\sigma^2 a(\xi)}{2} + \left(\frac{a'(\xi)}{a(\xi)} + \frac{1}{2} \left[\left(r - q - \frac{\sigma^2}{2} \right) - \frac{\ln[SX_n e^{-\lambda k(\tau-\xi)}/a(\xi)]}{\tau-\xi} \right] \right) \right] \\
& \qquad \times \kappa(SX_n e^{-\lambda k(\tau-\xi)}/a(\xi), 1, r, q, \tau-\xi, \sigma^2) d\xi \Big\} \\
& - \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \Big\{ \int_0^\tau (\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \\
& \times \Big[\int_0^1 G(Y) \int_{a(\xi)}^{a(\xi)/Y} C(\omega Y, \xi) \kappa(SX_n e^{-\lambda k(\tau-\xi)}, \omega, r, q, \tau-\xi, \sigma^2) d\omega dY \\
& - \int_1^\infty G(Y) \int_{a(\xi)/Y}^{a(\xi)} (\omega Y - K) \kappa(SX_n e^{-\lambda k(\tau-\xi)}, \omega, r, q, \tau-\xi, \sigma^2) d\omega dY \Big] d\xi \Big\}.
\end{aligned}
\tag{48}
$$

where the function C_{BS} and operator $\mathbb{E}_{\mathbb{Q}}^{(n)}$ have been defined in Proposition 3.3, and the function κ is given by equation (24).

Proof: Equation (48) follows from substituting equations (43), (46) and (47) into equation (42).

□

As was the case for the Jamshidian solution (25), equation (48) an integral equation rather than the integral expression obtained for the American call price in the no-jump case, because of the appearance of the option price in the integrals in the final summation term on the right-hand side. As we have pointed out previously, the presence of this term is due to the jump process. It should also be noted that, as in the no-jump option pricing case, in order to implement (48) we need to know the free boundary $a(\tau)$. An integral equation for this can be found by evaluating (48) at $S = a(\tau)$, and has the form[8]

$$\begin{aligned}
&\frac{a(\tau)-K}{2}\\
&= \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\{C_{BS}[a(\tau)X_n e^{-\lambda k\tau}, K, K, r, q, \tau, \sigma^2] \qquad (49)\\
&\qquad\qquad -C_{BS}[a(\tau)X_n e^{-\lambda k\tau}, K, a(0^+), r, q, \tau, \sigma^2]\}\\
&+\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\bigg\{\int_0^\tau e^{-(r+\lambda)(\tau-\xi)}(\tau-\xi)^n(a(\xi)-K)\\
&\quad\times\left[\frac{\sigma^2 a(\xi)}{2}+\left(\frac{a'(\xi)}{a(\xi)}+\frac{1}{2}\left[\left(r-q-\frac{\sigma^2}{2}\right)-\frac{\ln[a(\tau)X_n e^{-\lambda k(\tau-\xi)}/a(\xi)]}{\tau-\xi}\right]\right)\right]\\
&\quad\times\kappa(a(\tau)X_n e^{-\lambda k(\tau-\xi)}/a(\xi), 1, r, q, \tau-\xi, \sigma^2)d\xi\bigg\}\\
&-\lambda\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\bigg\{\int_0^\tau(\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)}\\
&\times\bigg[\int_0^1 G(Y)\int_{a(\xi)}^{a(\xi)/Y} C(\omega Y,\xi)\kappa(a(\tau)X_n e^{-\lambda k(\tau-\xi)},\omega,r,q,\tau-\xi,\sigma^2)d\omega dY\\
&-\int_1^\infty G(Y)\int_{a(\xi)/Y}^{a(\xi)}(\omega Y-K)\kappa(a(\tau)X_n e^{-\lambda k(\tau-\xi)},\omega,r,q,\tau-\xi,\sigma^2)d\omega dY\bigg]d\xi\bigg\}.
\end{aligned}$$

Equations (48)-(49) form a linked system of integral equations for the option price, $C(S,\tau)$, and the early exercise boundary $a(\tau)$. McKean's form for the integral equation system has two significant drawbacks relative to the equivalent Jamshidian form (25)-(26). Firstly, (48)-(49) depend upon the derivative of the early exercise boundary, $da(\tau)/d\tau$. This adds further complexity to the task of solving the system (48)-(49) since we will need to estimate $a'(\tau)$ in addition to $a(\tau)$. The second shortcoming is that the terms in in (48)-(49) are not readily interpreted in an economic context. While we can easily identify the European call price component, and thus determine the early exercise premium terms, the economic representation of those terms is not intuitive. Thus for the purposes of practical analysis and numerical solutions, we recommend the use of Jamshidian's form (25) over the McKean-style representation.

[8]The factor of $\frac{1}{2}$ on the left-hand side is a consequence of the definition of $H_j(x)$ from equation (9). McKean's form requires that $H_2(x)$ be used since we are taking the Fourier transform of $V(x,\tau)H(\ln a(\tau)-x)$ which is discontinuous at $x = \ln a(\tau)$. See Chiarella et al. (2004) for further details.

To conclude this section, we make a note of the equivalence between the Jamshidian solution form in Proposition 3.5 and the McKean solution in Proposition 4.5.

Proposition 4.6. *An application of integration by parts to (46) and algebraic manipulation of equation (48) reduces the McKean integral equation (48) for $C(S,\tau)$ to the form of the Jamshidian integral equation (25).*

Proof: Refer to Appendix 6.

□

Thus we have demonstrated how to extend the incomplete Fourier transform technique of McKean (1965) to Merton's (1976) jump-diffusion dynamics, and demonstrated that the resulting integral equations for the option price and free boundary are equivalent to those found by Gukhal (2001) using the compound option derivation approach of Kim (1990), and also Jamshidian's (1992) method whereby the homogeneous IPDE can be written as an inhomogeneous IPDE which is subsequently solved using the Fourier transform approach. Of the three approaches, we highlight that Jamshidian's technique is by far the simplest to resolve analytically, quickly leading us to an economically intuitive form for the early exercise premium of $C(S,\tau)$.

5. Limit of the Early Exercise Boundary at Expiry

Understanding the value of the free boundary just prior to expiry, at $\tau = 0^+$, is very important under jump-diffusion. Existing literature (e.g. Amin (1993) and Carr & Hirsa (2003)) give this limit as being identical to the pure-diffusion case. Here we show that this assumption is incorrect, using two different methods to derive the limit. We find that the presence of jumps does in fact have an impact on the early exercise boundary at expiry, and this difference can be expressed analytically, as stated in Proposition 5.1.

Proposition 5.1. *The limit of the early exercise boundary, $a(\tau)$, as $\tau \to 0^+$ is given by*

$$a(0^+) = K \max\left(1, \frac{r + \lambda \int_0^{K/a(0^+)} G(Y)dY}{q + \lambda \int_0^{K/a(0^+)} YG(Y)dY}\right). \tag{50}$$

Proof: Refer to Appendix 7.

□

It is worthwhile to observe that when $\lambda = 0$ equation (50) simplifies to the limit derived by Kim (1990) for the pure-diffusion American call free boundary. Note that (50) is an implicit expression for $a(0^+)$, but it can be solved quickly and accurately using standard root-finding techniques. Furthermore, as $q \to 0$ the solution to the implicit part of equation (50) increases without bound. Thus when $q = 0$, $a(0^+)$ becomes infinite, and we observe the well-known property that it is never optimal to exercise an American call option early in the absence of dividends.

Before concluding this section, we shall take a closer look at the properties of equation (50), specifically with a view to better understanding the solution to

$$a(0^+) = f(a(0^+)), \tag{51}$$

where

$$f(a(0^+)) = K \frac{r + \lambda \int_0^{K/a(0^+)} G(Y)dY}{q + \lambda \int_0^{K/a(0^+)} YG(Y)dY}.$$

Once (51) is solved, then the max( ) operator can be applied. Since the value of the underlying is always non-negative, we must consider the domain $a(0^+) \geq 0$ when finding the solution to (51). It is not possible to provide a simple, explicit summary of the behaviour of (51) for various values of $a(0^+)$, because the integral terms[9] depend upon $a(0^+)$, and the function $f(a(0^+))$ involves the parameters r, q and λ, as well as the jump-size density $G(Y)$.

Firstly, we see that is is simple to evaluate $f(a(0^+))$ at the limits of the domain. Specifically, we can show that

$$f(0) = K \frac{r + \lambda}{q + \lambda(k+1)} \geq 0, \tag{52}$$

and

$$\lim_{a(0^+) \to \infty} f(a(0^+)) \equiv f(\infty) = K \frac{r}{q}. \tag{53}$$

[9]While we note that these integral terms are expectations over the jump-size density $G(Y)$, this does not aid us when trying to provide a general analysis of $f(a(0^+)$.

Thus for $f(a(0^+))$ to be finite at each extremity of the domain, it is sufficient that we have $q > 0$. In this case, it is clear that $f(a(0^+))$ is continuous, and (51) will have at least one solution. Since $a(0^+)$ appears only in the limits of the integral terms over the density $G(Y)$ within $f(a(0^+))$, we can safely claim that the behaviour of $f(a(0^+))$ with respect to $a(0^+)$ will be bounded by the behaviour of $G(Y)$. Further exploration appears difficult without specifying the form of $G(Y)$, and as such we provide a more detailed analysis in Section 6.

6. American Call with Log-Normal Jumps

Before we begin exploring a numerical solution method for the integral equation system (25)-(26), we shall consider a specific example for the jump-size density, $G(Y)$. Here we consider a log-normal distribution for the jump sizes, Y, in accordance with a model suggested by Merton (1976). The probability density function for Y is given by

$$G(Y) = \frac{1}{Y\delta\sqrt{2\pi}} e^{-\frac{(\ln Y - (\gamma - \delta^2/2))^2}{2\delta^2}}, \tag{54}$$

where we set $\gamma \equiv \ln(1+k)$, and δ^2 is the variance of $\ln Y$. Furthermore we note that for this choice of $G(Y)$ we have $\mathbb{E}_{\mathbb{Q}}[Y] = e^{\gamma}$. Gukhal (2001) assumes that $\gamma = 0$ when deriving his equation (5.1) for the American call option price, but here we forego this assumption and provide a more general form of Gukhal's results.

Proposition 6.1. *In the case where $G(Y)$ is given by equation (54), the integral equation for $C(S,\tau)$ in (25) becomes*

$$\begin{aligned} C(S,\tau) \;=\; & \sum_{n=0}^{\infty} \frac{e^{-\lambda'\tau}(\lambda'\tau)^n}{n!} C_{BS}[S,K,K,r_n(\tau),q,\tau,v_n^2(\tau)], \\ & + \sum_{n=0}^{\infty} \Bigg\{ \int_0^{\tau} \frac{e^{-\lambda'(\tau-\xi)}[\lambda'(\tau-\xi)]^n}{n!} \\ & \times \Bigg[C_P^{(D)}[S,K,a(\xi),r,r_n(\tau-\xi),q,\tau-\xi,v_n^2(\tau-\xi)] \\ & - \lambda C_P^{(J)}[S,K,a(\xi),r_n(\tau-\xi),q,\tau-\xi,v_n^2(\tau-\xi);C(\cdot,\xi)] \Bigg] d\xi \Bigg\}, \end{aligned} \tag{55}$$

where $\lambda' = \lambda(1+k)$, $r_n(\tau) = r - \lambda k + n\gamma/\tau$ and $v_n^2(\tau) = \sigma^2 + n\delta^2/\tau$.

Proof: Refer to Appendix 8.

□

While equation (55) has incorporated the distribution for Y, the last term, which involves a double-integral, should be further simplified before attempting to implement (55) numerically.

Proposition 6.2. *By use of Proposition 3.6, the term* $C_P^{(J)}$ *in Proposition 6.1 can be expressed as*

$$\begin{aligned} & C_P^{(J)}[S, K, a(\xi), r_n(\tau), q, \tau, v_n^2(\tau); C(\cdot, \xi)] \\ & = e^{-r_n(\tau)\tau} \int_0^1 [C(a(\xi)z, \xi) - (a(\xi)z - K)] \kappa(S/a(\xi), z, r_{n+1}(\tau), q, \tau, v_{n+1}^2(\tau)) \\ & \qquad \times N[D(S/a(\xi), z, r_n(\tau), q, v_n(\tau), v_{n+1}(\tau), \tau, \gamma, \delta)] dz, \end{aligned} \tag{56}$$

where

$$\begin{aligned} & D(S/a(\xi), z, r_n(\tau), q, v_n(\tau), v_{n+1}(\tau), \tau, \gamma, \delta) \\ & \qquad \equiv \frac{\delta^2 \ln \frac{S}{a(\xi)z} + \left[(\ln z)\, v_{n+1}^2(\tau) + \delta^2 [r_n(\tau) - q] - \gamma v_n^2(\tau) \right] \tau}{v_n(\tau) v_{n+1}(\tau) \delta \tau}. \end{aligned} \tag{57}$$

Proof: Refer to Appendix 9.

□

We draw the reader's attention to the fact that in the form (56) the $C_P^{(J)}$ term in (55) now only involves a single integral which will result in a considerable saving in computational effort.

By evaluating (55) at $S = a(\tau)$ and using the value-matching condition (6), the integral equation for the early exercise boundary, $a(\tau)$, in the case of log-normal jump sizes, is given by

$$\begin{aligned}a(\tau)-K &= \sum_{n=0}^{\infty}\frac{e^{-\lambda'\tau}(\lambda'\tau)^n}{n!}C_{BS}[a(\tau),K,K,r_n(\tau),q,\tau,v_n^2(\tau)], \qquad (58)\\ &+\sum_{n=0}^{\infty}\Bigg\{\int_0^{\tau}\frac{e^{-\lambda'(\tau-\xi)}[\lambda'(\tau-\xi)]^n}{n!}\\ &\times\Bigg[C_P^{(D)}[a(\tau),K,a(\xi),r,r_n(\tau-\xi),q,\tau-\xi,v_n^2(\tau-\xi)]\\ &-\lambda C_P^{(J)}[a(\tau),K,a(\xi),r_n(\tau-\xi),q,\tau-\xi,v_n^2(\tau-\xi);C(\cdot,\xi)]\Bigg]d\xi\Bigg\},\end{aligned}$$

where $C_P^{(D)}$ and $C_P^{(J)}$ are given by (22) and (56) respectively.

6.1. **Delta for the American Call.** We now provide two important results regarding the delta of the American call option, $\Delta_C(S,\tau)$, which we will use in Section 7. Recall that the delta is defined as $\Delta_C(S,\tau)\equiv\partial C(S,\tau)/\partial S$. Firstly, we note that by differentiating both the IPDE (3) and boundary conditions (4)-(6) with respect to S, we can derive an IPDE for $Delta_C(S,\tau)$. The IPDE for the American call delta is given by

$$\frac{\partial\Delta_C}{\partial\tau}=\frac{1}{2}\sigma^2S^2\frac{\partial^2\Delta_C}{\partial S^2}+(r-q-\lambda k+\sigma^2)S\frac{\partial\Delta_C}{\partial S}-q\Delta_C+\lambda\int_0^{\infty}[\Delta_C(SY,\tau)-\Delta_C(S,\tau)]G(Y)dY, \qquad (59)$$

which is solved in the region $0\le\tau\le T$ and $0\le S\le a(\tau)$, subject to the initial and boundary conditions

$$\begin{aligned}\Delta_C(S,0) &= H_1(S-K),\quad 0\le S<\infty \qquad (60)\\ \Delta_C(0,\tau) &= 0,\quad \tau\ge 0, \qquad (61)\\ \Delta_C(a(\tau),\tau) &= 1,\quad t\ge 0. \qquad (62)\end{aligned}$$

Note that the free boundary, $a(\tau)$, is determined by solving (3) for $C(S,\tau)$, and as such $a(\tau)$ will already be known when solving (59) for $\Delta_C(S,\tau)$.

The second result we provide here is an integral equation for the American call delta. By differentiating (55) with respect to S, we find that

$$\begin{aligned}\Delta_C(S,\tau) &= \sum_{n=0}^{\infty}\frac{e^{-\lambda'\tau}(\lambda'\tau)^n}{n!}\Delta_{BS}[S,K,K,r_n(\tau),q,\tau,v_n^2(\tau)], \\ &\quad+\sum_{n=0}^{\infty}\Bigg\{\int_0^{\tau}\frac{e^{-\lambda'(\tau-\xi)}[\lambda'(\tau-\xi)]^n}{n!} \\ &\qquad\times\Bigg[\Delta_P^{(D)}[S,K,a(\xi),r,r_n(\tau-\xi),q,\tau-\xi,v_n^2(\tau-\xi)] \\ &\qquad-\lambda\Delta_P^{(J)}[S,K,a(\xi),r_n(\tau-\xi),q,\tau-\xi,v_n^2(\tau-\xi);C(\cdot,\xi)]\Bigg]d\xi\Bigg\},\end{aligned} \tag{63}$$

where

$$\begin{aligned}&\Delta_P^{(D)}[S,K,a(\xi),r,r_n(\tau),q,\tau,v_n^2(\tau)] \\ &\quad= e^{-q\tau}\Bigg\{\frac{1}{v_n(\tau)\sqrt{\tau}}N'\left[d_1\left(S,a(\xi),r_n(\tau),q,\tau,v_n^2(\tau)\right)\right](q-r) \\ &\qquad\qquad+qN\left[d_1\left(S,a(\xi),r_n(\tau),q,\tau,v_n^2(\tau)\right)\right]\Bigg\},\end{aligned} \tag{64}$$

$$\begin{aligned}&\Delta_P^{(J)}[S,K,a(\xi),r_n(\tau),q,\tau,v_n^2(\tau);C(\cdot,\xi)] \\ &\quad= e^{-r_n(\tau)\tau}\int_0^1\frac{[C(a(\xi)z,\xi)-(a(\xi)z-K)]}{Sv_n(\tau)\sqrt{\tau}}\kappa(S/a(\xi),z,r_{n+1}(\tau),q,\tau,v_{n+1}^2(\tau)) \\ &\qquad\times\Bigg[\frac{\delta}{v_{n+1}(\tau)\sqrt{\tau}}N'[D(S/a(\xi),z,r_n(\tau),q,v_n(\tau),v_{n+1}(\tau),\tau,\gamma,\delta)] \\ &\qquad\qquad-d_2(S/a(\xi),z,r_{n+1}(\tau),q,\tau,v_{n+1}^2(\tau))\Bigg]dz,\end{aligned} \tag{65}$$

with

$$N'[x]=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}.$$

6.2. **Properties of the Free Boundary at Expiry.** Since we are now considering a specific form for $G(Y)$, we return to the topic of analysing the behaviour of the early exercise boundary, $a(\tau)$, as $\tau\to 0^+$. Firstly we evaluate (50) for the log-normal density $G(Y)$.

Proposition 6.3. *When $G(Y)$ is given by (54), the limit of the early exercise boundary $a(\tau)$ as $\tau \to 0^+$ becomes*

$$a(0^+) = K \max\left(1, \frac{r + \lambda N[\{\ln K/a(0^+) - (\gamma - \frac{\delta^2}{2})\}/\delta]}{q + \lambda' N[\{\ln K/a(0^+) - (\gamma + \frac{\delta^2}{2})\}/\delta]}\right). \tag{66}$$

Proof: Evaluate the integral terms in (50) using $G(Y)$ from (54).

□

To develop an understanding of the case where $a(0^+) > K$, we shall explore some numerical realisations of the equation

$$b(0^+) = f(b(0^+)), \tag{67}$$

where

$$f(b(0^+)) = \frac{r + \lambda N[\{-\ln b(0^+) - (\gamma - \frac{\delta^2}{2})\}/\delta]}{q + \lambda' N[\{-\ln b(0^+) - (\gamma + \frac{\delta^2}{2})\})/\delta]},$$

and we recall that $b(\tau) = a(\tau)/K$. It is not possible to provide a simple, explicit summary of the behaviour of (67) for various values of $b(0^+)$ because the cumulative normal density functions depend upon $b(0^+)$, and the function $f(b(0^+))$ involves the parameters r, q, λ, γ and δ, all of which have a significant impact on the value of $f(b(0^+))$. Nevertheless, we can use numerical examples to offer some additional insight into the nature of (67).

For log-normal jump-sizes we can show that

$$f(0) = \frac{r + \lambda}{q + \lambda e^{\gamma}} \geq 0, \tag{68}$$

and

$$\lim_{b(0^+) \to \infty} f(b(0^+)) \equiv f(\infty) = \frac{r}{q}. \tag{69}$$

When $q > 0$ it is clear that $f(b(0^+))$ is continuous, and (67) will have at least one solution. We shall demonstrate by example that $f(b(0^+))$ is not monotonic, nor is it strictly bounded by the end values (68)-(69). This makes it difficult to prove that for $q > 0$ equation (67) has at most one solution. Since $b(0^+)$ appears only inside cumulative normal functions within $f(b(0^+))$, the behaviour of $f(b(0^+))$ with respect to $b(0^+)$ will be bounded by the behaviour of $N(\ln x)$. In particular, we recall that $0 \leq N(\ln x) \leq 1$, and that $N(\ln x)$ is a smooth, continuous function of x, where $x \geq 0$. From this we postulate that the function $f(b(0^+))$ will not display any oscillating features within the domain under consideration.

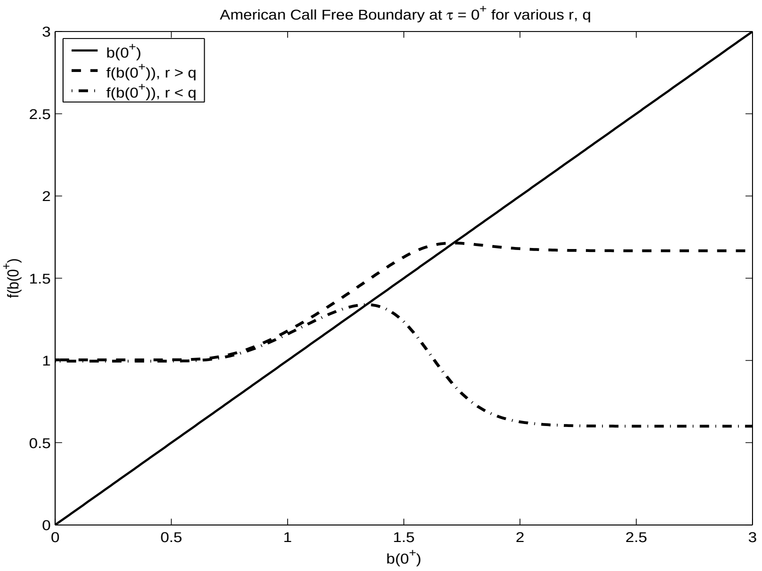


FIGURE 3. Behaviour of equation (67) when $\lambda = 5$, $\gamma = 0$ and $\delta = 0.2$. When $r > q$ we set $r = 0.05$, $q = 0.03$, and $r = 0.03$, $q = 0.05$ when $r < q$.

To provide evidence in support of our claims regarding equation (67), we now present some numerical examples. Firstly, we demonstrate the limits (68)-(69) for varying values of r and q. Setting $\lambda = 5$, $\gamma = 0$ and $\delta = 0.2$, we plot the functions $y = b(0^+)$ and $y = f(b(0^+))$ for various values of r and q, as shown in Figure 3. When $r = 0.05$ and $q = 0.03$, we can see that $f(0) < f(\infty)$. On the other hand, when $r = 0.03$ and $q = 0.05$, we now have $f(0) > f(\infty)$. In both cases it is clear that $f(b(0^+))$ is not bounded by these endpoint values, and we can see that the relative values of r and q directly influence the values of $f(0)$ and $f(\infty)$.

Since it is difficult to appreciate the impact of the jump-parameters on $f(b(0^+))$ using comparative statics, we again provide numerical examples to highlight the properties of $f(b(0^+))$. In all cases we set $r = 0.03$ and $q = 0.05$, with default jump-parameter values as used in generating Figure 3. In Figure 4 we see how $f(b(0^+))$ is affected by

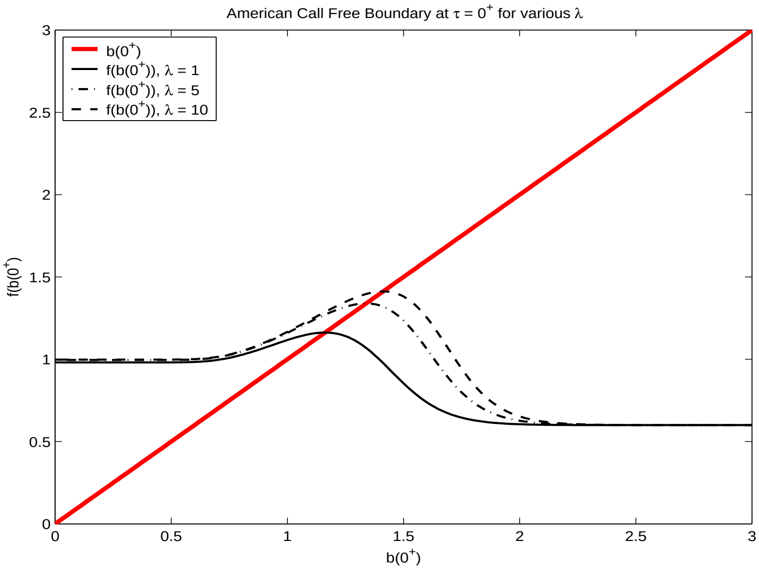


FIGURE 4. Behaviour of equation (67) when $r = 0.03$, $q = 0.05$, $\gamma = 0$ and $\delta = 0.2$, for various values of λ.

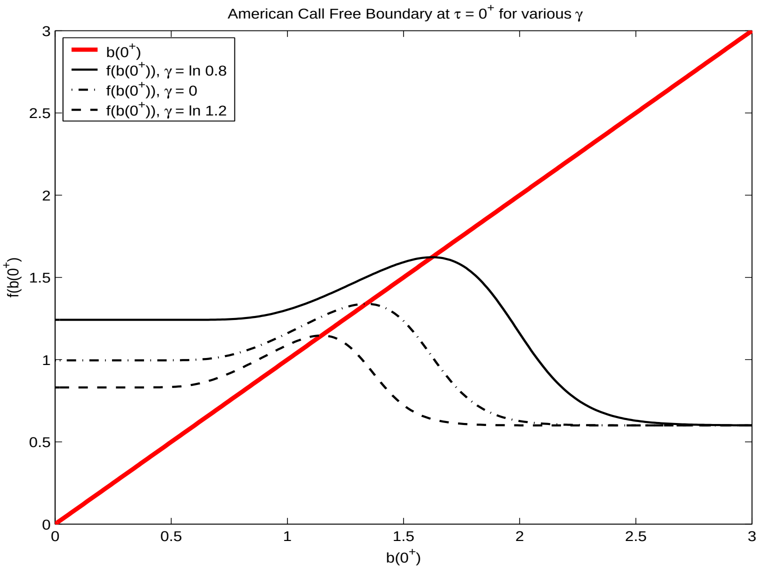


FIGURE 5. Behaviour of equation (67) when $r = 0.03$, $q = 0.05$, $\lambda = 5$ and $\delta = 0.2$, for various values of γ.

changes in λ. Aside from the obvious impact this has on $f(0)$, we can see that as λ increases, the peak of $f(b(0^+))$ also increases. Next we vary γ to produce Figure 5. In addition to varying the value of $f(0)$, changes in γ affect the size and location of the "hump" in $f(b(0^+))$. As γ is increased, the "hump" feature reduces in size and shifts towards the origin. Finally we observe the impact of varying δ values in Figure 6. We

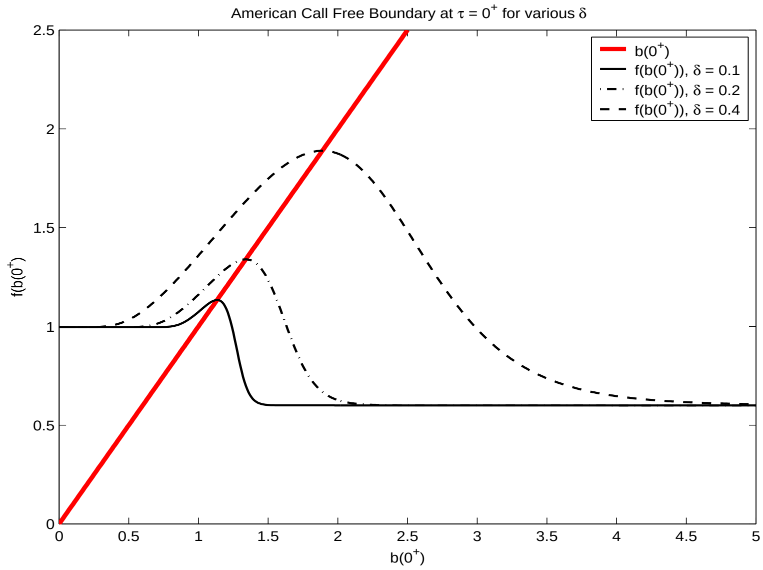


FIGURE 6. Behaviour of equation (67) when $r = 0.03$, $q = 0.05$, $\lambda = 5$ and $\gamma = 0$, for various values of δ

see that as δ increases, the width of the "hump" feature in $f(b(0^+))$ increases. Thus the jump-parameters primarily influence the shape and location of the non-linear features of $f(b(0^+))$, with λ and γ also affecting the value of $f(0)$. The r and q parameters only affect the endpoint values of $f(b(0^+))$. It should be noted that in all the cases presented thus far, there is clearly only one solution to equation (67), given by the intercept of $y = b(0^+)$ and $y = f(b(0^+))$.

The last and most important scenario to consider is when $q = 0$. In this case, $f(\infty)$ is no longer finite, instead increasing without bound as $b(0^+) \to \infty$. Figure 7 demonstrates the behaviour of $f(b(0^+))$ with $q = 0$ for a selection of additional parameter values. It is clear from the plot that there is no solution for $b(0^+) = f(b(0^+))$. Furthermore, the only way that equation (67) will be satisfied when $q = 0$ is by taking the limit as $b(0^+) \to \infty$, in which case both sides of (67) will increase without bound. Thus we infer that when $q = 0$, the free boundary at $\tau = 0^+$ becomes infinite, and it is never optimal to exercise an American call early in the absence of dividends.

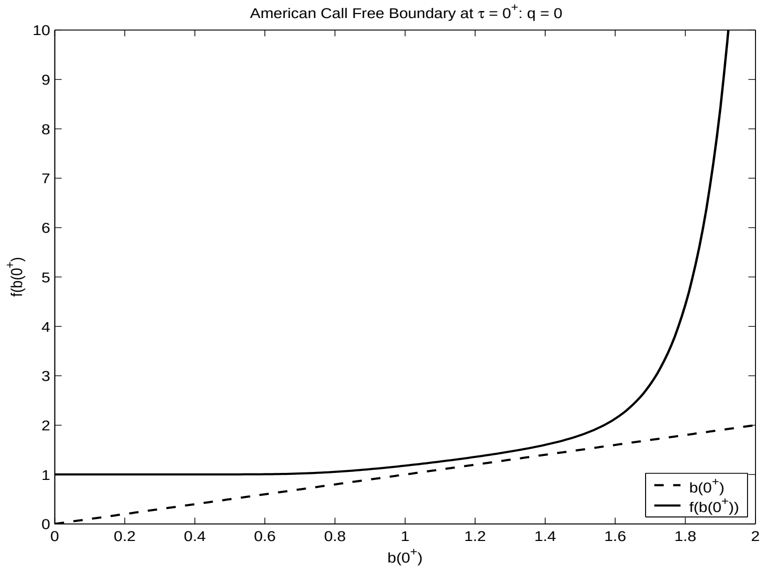


FIGURE 7. Behaviour of equation (67) when $q = 0$. Other parameter values are $r = 0.03$, $\lambda = 10$, $\gamma = 0$ and $\delta = 0.2$.

7. Numerical Implementation and Results

We now provide a numerical scheme with which to evaluate the linked integral equation system formed by (56) and (58). The proposed method is an extension to the jump-diffusion case of the quadrature scheme detailed by Kallast & Kivinukk (2003), and here we focus on the necessary adjustments that are needed to deal with the introduction of jumps in the dynamics for S. We firstly discretise the time variable, τ, into N equally spaced intervals of length h. Thus $\tau = ih$ for $i = 0, 1, 2, ..., N$, and $h = T/N$. We denote the call price profile at time step i by $C(S, ih) = C_i(S)$, and similarly the free boundary at time step i by $a(ih) = a_i$. Using the standard numerical technique that is applied to Volterra integral equations, we can solve the system (56)-(58) for increasing values of i, until eventually the entire free boundary and price profile are found. When calculating the infinite summations, we continue adding terms until the size of the Poisson coefficient for a given value of n is less than 10^{-20}. For the parameter values considered here, this typically results in the use of around 30 terms for the summations. In order to start the algorithm we require the initial value of $C_0(S)$, which is simply the payoff function, and also a_0, where $a_0 \equiv a(0^+)$, which is given by equation (66).

Since (56) depends upon $C(S,\tau)$, an approximation will be needed for $C_i(S)$ at each time step. A suitable approximation is given by $C_{i-1}(S)$, which is simply the American call price at the previous time step. By using this approximation in the integral term involving $C_i(S)$, we are able to reduce the integral equation (56) to an integral expression for the free boundary. This approximation introduces a very small degree of error into the values for the free boundary and option price. One can correct this error using an iterative scheme as follows. Having approximated the boundary a_i and price profile $C_i(S)$, the solution process is repeated for time step i using the most recent approximation for $C_i(S)$ in the integral term involving $C_i(S)$. In practice this iterative scheme converges rapidly (typically within 2-3 iterations for the parameters we considered), and the improvement in accuracy was very minor, relative to the Crank-Nicolson benchmark we used for the true solution. Thus we chose not to make use of an iterative scheme to correct for these approximation errors in the integral over $C_i(S)$, and the scheme used is otherwise identical to that proposed by Kallast & Kivinukk (2003), save that we must determine and store the option prices at each time step after the free boundary has been computed.

The price at the $(i-1)th$ time step is calculated for a suitably large number of evenly-spaced S values. Here we used 50 points in the range $0 \leq S \leq 250$. All necessary interpolation was conducted using cubic splines fitted locally through 7 values of $C_i(S)$. We then use Newton's method to solve for the early exercise boundary, as in Kallast & Kivinukk (2003), with two necessary additions. The first addition addresses the evaluation of the inner integral over the interval $[0,1]$. This is computed using Gaussian integration for moments, with parameter $\alpha = -0.5$. Full details for this Gauss-quadrature scheme can be found in Abramowitz & Stegun (1970). The second addition relates to finding the derivative of (58) with respect to $a(\tau)$ for use in Newton's method. This is given by (63), evaluated at $S = a(\tau)$, in continuous time, although it is difficult to determine the limit of the integrands when $\xi = \tau$. As per Kallast & Kivinukk (2003), we find these limits by first taking the limits of the integrand for the option price at $\xi = \tau$, and subsequently differentiate these with respect to $a(\tau)$. These limits are all finite, including the new limit required for the jump-related integral term, and the required derivatives are easily determined. Since we need to evaluate (63) for use in Newton's method, there is no significant computation involved in evaluating the American call delta once the free boundary has been estimated.

Having determined the discretised forms for the price and delta of $C_i(S)$, we then use Newton's method to solve for a_i. Before proceeding to the next time step, we use a_i to calculate a new approximation for $C_i(S)$, which is required when evaluating the double integral term at all subsequent time steps. This update for C_i is essential to ensure that the estimated free boundary remains monotonic. The algorithm *American Call - Integration* in Appendix 10 outlines how the procedure is carried out for each i. Note that as the value of i increases, the computational burden will also increase at a "faster than linear" rate, since the integration at step i depends on all values of a_j and $C_j(S)$ for $j = 0, 1, 2, ...i - 1$.

To explore the efficiency of the proposed numerical integration method, we compare it with a finite difference solution for the IPDE (3). We apply the Crank-Nicolson scheme to all terms except for the integral. We initially estimate the integral term by approximating $C_i(S)$ with $C_{i-1}(S)$, as in Carr & Hirsa (2003). We then evaluate the integral using the Hermite Gauss-quadrature scheme, which can be found in Abramowitz & Stegun (1970). The resulting tridiagonal matrix is inverted using LU-decomposition, and the early exercise condition is then applied to the solution at each time step. An evenly spaced grid is used, and the free boundary is estimated at each time step using cubic spline interpolation of the price profile, combined with the bisection method.

To improve the accuracy of the Crank-Nicolson solution, we use a two-step procedure at each time step. After determining an initial solution at time step i, denoted here as $C_i^{(1)}(S)$, using the estimate of $C_{i-1}(S)$ in the integral term, we then find an updated estimate by repeating the process using the $C_i^{(1)}(S)$ values in the integral estimate to produce $C_i^{(2)}(S)$. In practice we find that $C_i^{(1)}$ typically converges from below, whilst $C_i^{(2)}$ converges from above. Thus we take $C_i(S) = C_i^{(1)}(S)/2 + C_i^{(1)}(S)/2$ for the final Crank-Nicolson solution. This appears to greatly improve the convergence rate for the Crank-Nicolson scheme, although we do not report details of the convergence of $C^{(1)}$

and $C^{(2)}$ here[10]. In all cases we set the S domain to be $0 \leq S \leq 250$. We also calculate the American call delta by taking a central difference approximation using the price estimates.

In assessing the efficiency of the numerical integration method, we use a Crank-Nicolson solution with 5000 time steps and 5000 space steps for the true solution. Since the numerical integration scheme requires evaluation of the option delta as part of the solution, it is also of value to consider the efficiency with which delta is calculated. For the true delta we solve the IPDE (59) with boundary conditions (60)-(61), using the same Crank-Nicolson scheme applied to (3). The free boundary and call price are found first, and then the free boundary is used to solve the IPDE for delta. We again use 5000 time steps and space steps when computing delta.

To compare the efficiency of the numerical integration and finite-difference methods, we compute the price and delta of an American call option with 6-months to maturity, and a strike price of $K = 100$. The global volatility, s^2, is set equal to 30.64%. The jump intensity is set to $\lambda = 5$, and the jump volatility is $\delta^2 = 0.05$. We then consider six different parameter sets, specifically $\mathbb{E}_{\mathbb{Q}}[Y] = e^{\gamma}$ values of 0.95, 1.00 and 1.04, along with the combinations $r = 8\%$, $q = 12\%$, and $r = 12\%$, $q = 8\%$. The diffusion volatility σ^2 is chosen such that the global volatility was preserved for varying values of γ. Table 1 summarises the values of σ^2 used to ensure that the global volatility was the same for each combination of γ and λ.

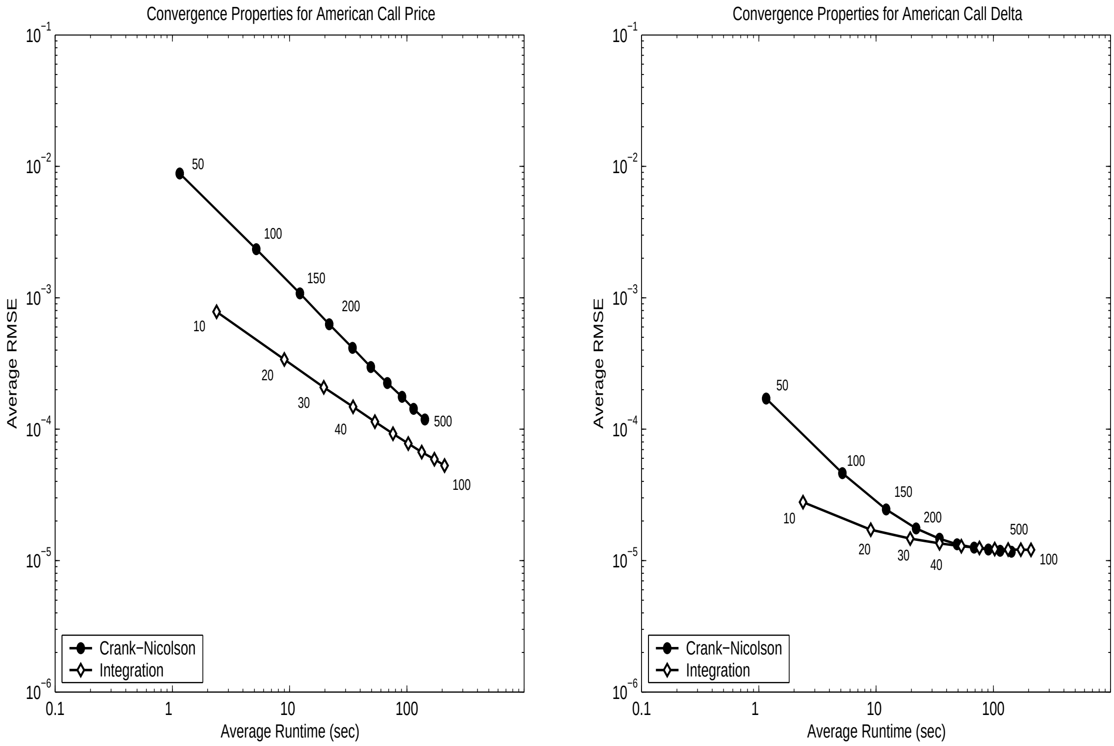
σ^2	λ	e^{γ}	δ^2
0.3064	0	-	-
0.0625	5.00	0.95	0.05
0.0500	5.00	1.00	0.05
0.0211	5.00	1.04	0.05

TABLE 1. Parameter values used for the diffusion volatility and jump component. The global volatility is fixed at $s^2 = 30.64\%$, determined by $s^2 = \sigma^2 + \lambda[e^{2\gamma+\delta^2} - 2e^{\gamma} + 1]$.

[10]Briani et al. (2004) note that it is unclear how to select the stopping criteria when using iterative finite difference solutions for (3). Since we observe greater accuracy by using the average of the first and second iteration results than using the second iteration alone, the averaging scheme we use here is clearly more efficient than using a stopping criteria that involves three or more iterations.

We compute the root mean square error (RMSE) using option prices and deltas with $S = 80, 90, 100, 110$ and 120. This is repeated for each of the six parameter sets, from which the average runtime and RMSE is then calculated. Note that in all cases the runtimes include the time required to find the free boundary, price and delta for the American call. For the integration method we use 20 integration points for the Gauss-quadrature scheme, and consider a sequence of 10 different time step values, with $N = 10, 20, \ldots, 90, 100$. For the Crank-Nicolson method the integral term is approximated using 50 integration points, and we again use 10 time step values, with $N = 50, 100, \ldots, 450, 500$. We set the number of space steps equal to double the number of time steps. The code for both methods is implemented using LAHEY™FORTRAN 95 running on a PC with a Pentium 4 2.40 GHz processer, 512MB of RAM, and running the Windows XP Professional operating system.

The relative efficiency for each method is shown in Figure 8, with 8(a) showing the average RMSE error for the American call price, and 8(b) displaying the same information for the delta. Note that the average runtimes for each discretisation level are the same in 8(a) and 8(b) since the price and delta were found using a single algorithm. Firstly, we find that for the parameters and discretisations considered, the numerical integration method consistently displays greater efficiency than the Crank-Nicolson scheme. Although the improved efficiency diminishes for smaller time step sizes, when the step sizes are large, the numerical integration scheme provides an improved accuracy of roughly one order of magnitude over the Crank-Nicolson scheme for the same runtime. While we also observe a greater efficiency when finding delta using the numerical integration scheme, the benefits are less substantial than for the price, and in particular, there is little difference between the accuracy of either method for runtimes beyond 40 seconds. Figure 8 indicates that, especially for large time step sizes, the numerical integration scheme is both faster and more accurate than the Crank-Nicolson scheme under consideration, for computing both the American call price and delta. While this improved efficiency in pricing persists out to runtimes of 100 seconds, the differences between the two methods diminishes for increased discretisation levels, and is less distinct in the case of delta.



(a) (b)

FIGURE 8. Comparing the efficiency of numerical integration and Crank-Nicolson for the price and delta of American call options with log-normal jump sizes. Fixed parameters are $K = 100$, $T - t = 0.50$ and $\lambda = 5$. RMSE is found using $S = 80, 90, 100, 110$ and 120. Average RMSE and runtimes found using 6 parameter sets, with $r = 8\%$, $q = 12\%$, and $r = 12\%$, $q = 8\%$, along with $e^{\gamma} = 0.95, 1.00$ and 1.04. Figure 8(a) displays the price efficiency, and Figure 8(b) shows the delta efficiency.

Numbers on the plot indicate the time steps associated with a given point. Crank-Nicolson space steps are set equal to double the number of time steps. Note that the reported runtimes indicate the total time required to find the free boundary, price and delta for the American call. Both axes are given in log-scale.

Next we present sample free boundary profiles for the American call option. In Figure 9 we consider the case where $r < q$, and in Figure 10 we set $r > q$. We again consider three different values of e^{γ} (the same values used to generate Figure 8), and compare the resulting boundaries with the pure-diffusion case of $\lambda = 0$. The diffusion volatility σ was again adjusted in each case as detailed in Table 1. The most obvious feature of these results is the dramatic effect the presence of jumps has on the profile for the free boundary. Close to expiry, the free boundary with jumps is significantly larger than in the pure-diffusion case. This follows from the increased probability of large price movements near expiry, made possible by the presence of jumps within the return dynamics. Thus the holder of the call is less likely to exercise near expiry under the jump-diffusion model to best minimise the potential costs from downward jumps.

As time to expiry increases, we see that the pure-diffusion boundary increases more rapidly compared with the jump-diffusion examples, since the jump component becomes less dominant within the underlying dynamics for large time intervals. While jumps are more likely to be observed over longer time intervals, they become less influential overall, since there are sufficient opportunities for the jumps to be reversed, either by jumps in the opposite direction or through the diffusion term. Therefore when far from maturity the holder of the call is more likely to exercise early under jump-diffusion than in the pure-diffusion case. These findings coincide with those of Amin (1993), who also notes that for a sufficiently large time to expiry, the probability density for the underlying converges under both models, such that there is no clear distinction between pure-diffusion and jump-diffusion.

We also point out that Amin does not provide any formal evidence relating to the limit of the free boundary at expiry, although his numerical results are consistent with the limiting value given here by equation (66). In particular, our Figure 9 is closely related to Figure 6 in Amin (1993). We find that, for the parameter values used by Amin, the limit (66) correctly identifies the value of the free boundary at $\tau = 0^+$, and thus our limit result for $a(\tau)$ is in keeping with the numerical results of Amin.

One further observation we can make from Figure 9 is the impact of the value of γ on the free boundary. As γ increases, the value of the early exercise boundary decreases. This is attributable to the potential for the option holder to incur a rebalancing cost when the price jumps from the stopping region back down into the continuation region. Recall that $\gamma > 0$ implies upward jumps on average, thus making the expected cost of downward jumps quite small. When $\gamma < 0$, we expect downward jumps on average, and the holder will therefore require that S be even larger before exercising the call early.

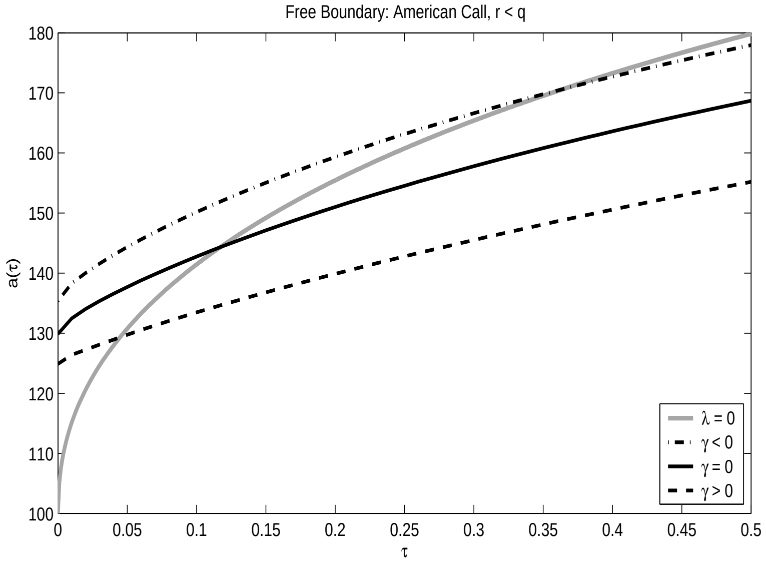


FIGURE 9. Early exercise boundaries for the American call option, for a range of γ values, compared with the pure-diffusion case of $\lambda = 0$. The numerical integration scheme uses 100 time steps, with 20 integration points for the Gauss-quadrature component. Other parameter values are $K = 100$, $T = 0.5$, $r = 8\%$, $q = 12\%$, $\lambda = 5$ and $\delta^2 = 0.05$. See Table 1 for further details.

Finally, we demonstrate the impact of jumps on the American call price, relative to the pure-diffusion case. In figures 11 and 12 we plot the price differences between the pure-diffusion and jump-diffusion American call prices for the same three values of e^{γ}. All other parameter values are the same as those used to generate the free boundaries in figures 9 and 10. Positive (negative) differences indicate that the jump-diffusion price is greater than (less than) the pure-diffusion price. Figure 11 shows the results for $r < q$ and 12 uses $r > q$.

While the shapes of the plots vary somewhat depending on the relative values of r and q, this mostly occurs deep in-the-money, and is related to the impact that r and q have on the value of the free boundary. In general we observe that when the call is at-the-money ($K = 100$) or close to at-the-money, the jump-diffusion price is consistently less than the pure-diffusion price. Furthermore, when the call is deep out-of-the-money, the jump-diffusion price is generally larger than for pure-diffusion.

For deep in-the-money American calls, there are a number of factors that affect the price differences. First we note that the early exercise feature will always reduce this

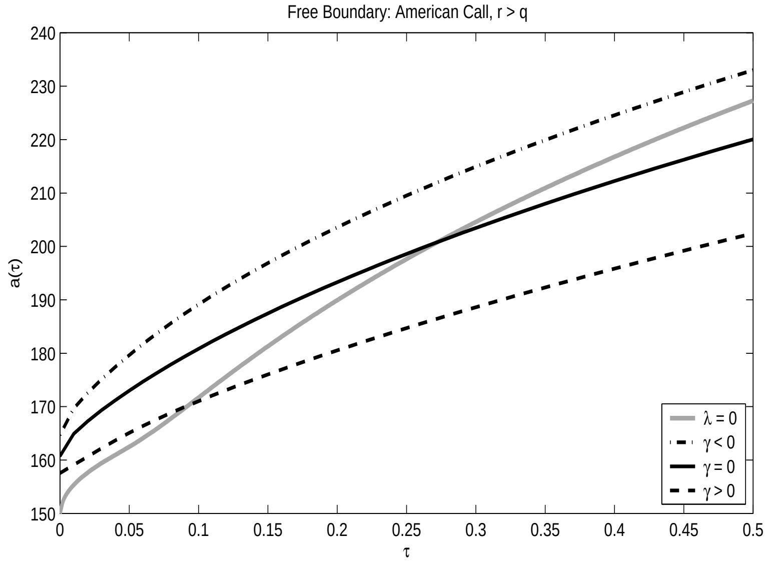


FIGURE 10. Early exercise boundaries for the American call option, for a range of γ values, compared with the pure-diffusion case of $\lambda = 0$. The numerical integration scheme uses 100 time steps, with 20 integration points for the Gauss-quadrature component. Other parameter values are $K = 100$, $T = 0.5$, $r = 12\%$, $q = 8\%$, $\lambda = 5$ and $\delta^2 = 0.05$. See Table 1 for further details.

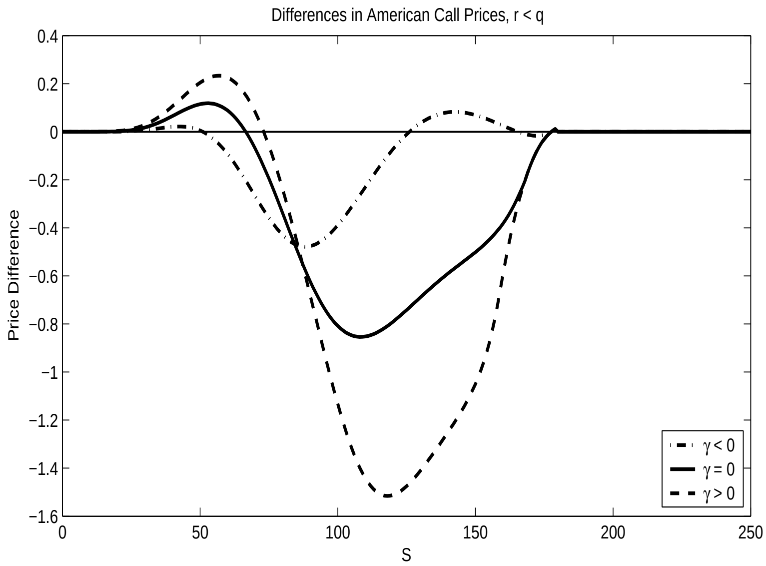


FIGURE 11. Price differences between the pure-diffusion American call and the corresponding contract under jump-diffusion, for various values of γ. Other parameter values are $K = 100$, $T = 0.5$, $r = 8\%$, $q = 12\%$, $\lambda = 5$ and $\delta^2 = 0.05$. See Table 1 for further details.

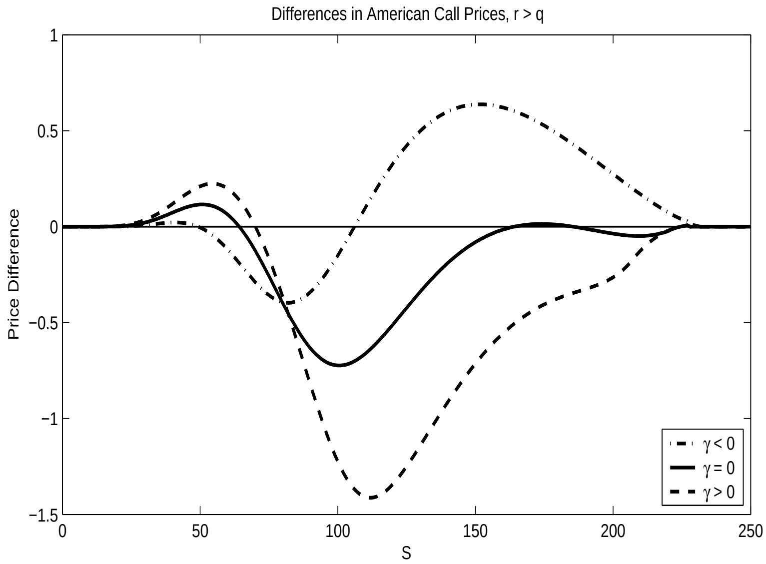


FIGURE 12. Price differences between the pure-diffusion American call and the corresponding contract under jump-diffusion, for various values of γ. Other parameter values are $K = 100$, $T = 0.5$, $r = 12\%$, $q = 8\%$, $\lambda = 5$ and $\delta^2 = 0.05$. See Table 1 for further details.

difference to zero for large values of S. When $\gamma < 0$, the difference is mostly positive for S values just below the free boundary, while the opposite is true when $\gamma = 0$ and $\gamma > 0$. For the European call we would expect to see greater prices under jump-diffusion for large values of S, but for American options this depends upon the value of γ, at least in part. Since $\gamma < 0$ indicates downward jumps are expected on average, this will increase the likelihood of the option holder incurring rebalancing costs, and could provide some of the reason for the increased call value relative to the pure-diffusion case. Otherwise, the early exercise feature dominates the price profile for large values of S, and thus we do not observe the same behaviour as we would for European calls. Nevertheless, the leptokurtic features introduced into the return dynamics for S are clearly represented by the increased call prices out-of-the-money, and the reduced prices in a region around the strike. This implies that the jump-diffusion model is able to reflect the basic volatility smile structure observed in market option prices. We have elected not to demonstrate this result using Black-Scholes implied volatilities, as this procedure only makes theoretical sense in the case of European options. It is clear, however, from the relative price differences that the jump-diffusion dynamics have the potential to capture volatility smile behaviour.

8. Conclusion

This paper explores the pricing of American call options in the case where the underlying asset follows a jump-diffusion process, as originally proposed by Merton (1976). We consider two approaches for solving the IPDE for the American call price. The first uses the method of Jamshidian (1992) to find an inhomogeneous IPDE for the American call price in an unrestricted domain, which we then solve using Fourier transforms. This leads us to recover Gukhal's (2001) results, which he derives via the compound option method. The second solution method is an extension of McKean's (1965) incomplete Fourier transform approach for American calls under pure-diffusion. We generalise his solution to the jump-diffusion case, and derive an alternative integral equation for the option price that involves the derivative of the free boundary. Referring to Kim (1990), we demonstrate that the solutions found using the McKean and Jamshidian approaches are equivalent, and in this way provide a more complete understanding of how the various solution techniques for American call options are applied in the jump-diffusion setting.

There are two significant contributions made regarding the integral equation system for the American call price and free boundary. Firstly, we derive the limit of the free boundary as the time to expiry tends to zero. In particular, we show that the limit is clearly dependent upon the jump intensity and jump-size distribution, a fact not reported in existing literature on American option pricing with jumps. This limit is particularly useful when solving numerically for the free boundary, since it provides a more accurate starting point. The second contribution is to express the integral term for the expected costs incurred from downward jumps in a form that is more tractable for numerical integration purposes. In particular, in the case where the jump sizes are log-normally distributed, we are able to reduce the term from a triple integral to a double integral involving the cumulative normal density, a task far easier to implement with high levels of accuracy.

The other main result of this paper concerns the use of numerical integration to solve for the free boundary, price and delta of the American call with jumps. We propose a quadrature integration scheme based on the pure-diffusion case in Kallast & Kivinukk (2003). We address the double integral term, and provide a fast, accurate means of evaluating this, along with a simple way to overcome the implicit dependency of the integral equation on the unknown option price. We compare this numerical integration solution with a suitable Crank-Nicolson scheme, and find that the proposed numerical integration is more efficient than the finite difference approach, both for computing the option price and delta. This improved efficiency is most prominent for large time step sizes, and diminishes as the step size reduces.

We use this integration scheme to demonstrate the impact of jumps on the free boundary of the American call, relative to the pure-diffusion case with equivalent global volatility. The results presented here correspond with the tree methods used by Amin (1993). In particular, option holders are less likely to exercise early close to expiry, and more likely to exercise further from expiry when jumps are introduced. The relative values of time to expiry where these differences occur depends upon the jump parameter value, and in particular we show how different values for the mean jump-size impact on the free boundary. We also demonstrate the price differences between jump-diffusion and pure-diffusion American calls, and as expected, find that the call premium is smaller in a region around the strike price when jumps are present, but larger when the option is deep out-of-the-money. For deep in-the-money options, the early exercise feature causes the American call price to rapidly tend towards the payoff function.

While the numerical results presented consider only log-normal jump sizes, the numerical integration approach is readily applicable to a range of jump size distributions. One avenue for future research is to explore these alternatives, and in particular observe what difficulties are encountered when trying to simplify and evaluate the triple integral term for other jump size densities. We have only considered the call option here, and a broader range of payoff functions can be explored, in particular those with more complex stopping and continuation regions, such as those that arise with American option portfolios such as strangles and butterflies. Merton's model for the jump process assumes that jump risk is fully diversifiable, an assumption we have chosen to retain for simplicity. This assumption could be relaxed within the Fourier transform framework, but only certain kinds of jump risk could be catered for. The numerical algorithm presented has only been compared with the Crank-Nicolson scheme. There are numerous other numerical methods that have not been considered, such as the tree methods of Amin (1993) and Broadie & Yamamoto (2003), the method of lines used by Meyer (1998), and various finite difference scheme implementations, including Andersen & Andreasen (2000) and d'Halluin et al. (2004). A detailed analysis of the efficiency of these various numerical methods is planned as a future research project.

Appendix 1. Deriving the Inhomogeneous IPDE

Whenever S is in the stopping region, it is optimal to exercise the American call option, and hence the call option price is given by $C(S,\tau) = S - K$ for all $S \geq a(\tau)$. Although $C(S,\tau)$ only satisfies the IPDE (3) for $0 \leq S \leq a(\tau)$, we can introduce an inhomogeneous term in (3) such that C satisfies the IPDE for all $S \geq 0$. Jamshidian (1992) demonstrated that this was possible under pure-diffusion dynamics, and here we extend his result to the jump-diffusion case.

To derive the required inhomogeneous term, we evaluate (3) when $C(S,\tau) = S - K$. Thus we have

$$\begin{aligned}\Psi(S,\tau) \quad \equiv \quad & H_1(S-a(\tau))\left\{\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} + (r-q-\lambda k)S\frac{\partial C}{\partial S} - rC - \frac{\partial C}{\partial \tau}\right. \\ & \left. +\lambda\int_0^\infty [C(SY,\tau) - C(S,\tau)]G(Y)dY\right\} \\ = \quad & H_1(S-a(\tau))\left\{K(r+\lambda) - S(q+\lambda[k+1]) + \lambda\int_0^\infty C(SY,\tau)G(Y)dY\right\},\end{aligned}$$

where H_1 is the Heaviside step function given by (9). The Heaviside function is used to denote that Ψ is only valid for $S \geq a(\tau)$. Since $C(SY,\tau) = S - K$ when $SY \geq a(\tau)$, we can express $\Psi(S,\tau)$ as

$$\begin{aligned}\Psi(S,\tau) &= H_1(S-a(\tau))\Bigg\{K(r+\lambda)-S(q+\lambda[k+1]) \\ &\quad +\lambda\left(\int_0^{a(\tau)/S}C(SY,\tau)G(Y)dY+\int_{a(\tau)/S}^{\infty}(SY-K)G(Y)dY\right)\Bigg\} \\ &= H_1(S-a(\tau))\Bigg\{K(r+\lambda)-S(q+\lambda[k+1])+\lambda\int_0^{a(\tau)/S}C(SY,\tau)G(Y)dY \\ &\quad +\lambda\left(\int_0^{\infty}(SY-K)G(Y)dY-\int_0^{a(\tau)/S}(SY-K)G(Y)dY\right)\Bigg\}.\end{aligned}$$

Recalling that $k=\mathbb{E}^{Q_Y}[Y-1]$, we have

$$\Psi(S,\tau)=H_1(S-a(\tau))\left\{rK-qS+\lambda\int_0^{a(\tau)/S}[C(SY,\tau)-(SY-K)]G(Y)dY\right\}. \quad (70)$$

Since (70) is the value of the IPDE (3) when $S\geq a(\tau)$, we can rewrite the IPDE as

$$\begin{aligned}\frac{\partial C}{\partial\tau} &= \frac{1}{2}\sigma^2S^2\frac{\partial^2C}{\partial S^2}+(r-q-\lambda k)S\frac{\partial C}{\partial S}-rC+\lambda\int_0^{\infty}[C(SY,\tau)-C(S,\tau)]G(Y)dY \\ &\quad +H_1(S-a(\tau))\left\{qS-rK-\lambda\int_0^{a(\tau)/S}[C(SY,\tau)-(SY-K)]G(Y)dY\right\},\end{aligned}$$

which is equation (8) from Proposition 3.1. Note that it is easy to verify that (8) is satisfied by $C(S,\tau)$ for $0\leq S<\infty$.

Appendix 2. Proof of Proposition 3.2

When taking the Fourier transform of (10), we note that $V(x,\tau)$ and $\partial V/\partial x$ do not approach zero as $x\to\infty$. Lewis (2000) proves that the Fourier transform is still valid in this case, although one must instead take the complex Fourier transform in a strip of the complex plane. Since the end result is equivalent to assuming that $V(x,\tau)$ and $\partial V/\partial x$ tend to zero as $x\to\infty$, we shall simply apply this assumption and suppress the finer details involved.

According to Chiarella et al. (2004), from the pure-diffusion case (i.e. the model with no jumps) we know that

$$\mathcal{F}\left\{\frac{\partial V}{\partial x}\right\} = -i\eta\hat{V}, \quad \mathcal{F}\left\{\frac{\partial^2 V}{\partial x^2}\right\} = -\eta^2\hat{V}, \quad \text{and} \quad \mathcal{F}\left\{\frac{\partial V}{\partial \tau}\right\} = \frac{\partial \hat{V}}{\partial \tau}. \tag{71}$$

For the inhomogeneous term, we let

$$\hat{F}_J(\eta,\tau) \equiv \mathcal{F}\left\{F_J(x,\tau)\right\}, \tag{72}$$

where

$$\begin{aligned} F_J(x,\tau) \equiv \; & H_1(x-\ln b(\tau))(qe^x - r) \\ & -H_1(x-\ln b(\tau))\lambda\int_0^{b(\tau)e^{-x}} [V(x+\ln Y,\tau)-(Ye^x-1)]G(Y)dY. \end{aligned} \tag{73}$$

This leaves one term to be evaluated, namely

$$\mathcal{F}\left\{\int_0^\infty V(x+\ln Y,\tau)G(Y)dY\right\} = \int_{-\infty}^\infty e^{i\eta x}\int_0^\infty V(x+\ln Y,\tau)G(Y)dY\,dx. \tag{74}$$

Using the change of variable $z = x+\ln Y$, equation (74) becomes

$$\begin{aligned} \mathcal{F}\left\{\int_0^\infty V(x+\ln Y,\tau)G(Y)d(Y)\right\} &= \int_0^\infty G(Y)e^{-i\eta\ln Y}\int_{-\infty}^\infty e^{i\eta z}V(z,\tau)dzdY \\ &= A(\eta)\hat{V}(\eta,\tau), \end{aligned}$$

where

$$A(\eta) \equiv \int_0^\infty e^{-i\eta\ln Y}G(Y)dY.$$

Hence, our IPDE is transformed into the integro-differential equation

$$\frac{\partial\hat{V}}{\partial\tau} + \left[\frac{\sigma^2\eta^2}{2} + \phi i\eta + (r+\lambda) - \lambda A(\eta)\right]\hat{V} = \hat{F}_J(\eta,\tau).$$

The solution to this integro-differential equation is given by

$$\begin{aligned} \hat{V}(\eta,\tau) = \; & \hat{V}(\eta,0)e^{-(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta))\tau} \\ & + \int_0^\tau e^{-(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta))(\tau-\xi)}\hat{F}_J(\eta,\xi)d\xi, \end{aligned}$$

where $\mathcal{F}\{V(x,0)\} \equiv \hat{V}(\eta,0)$.

Appendix 3. Derivation of the American Call Integral Equations - Jamshidian Method

A3.1. **Proof of Proposition 3.3.** Consider the function $V_E(x,\tau)$, given by

$$V_E(x,\tau) = \mathcal{F}^{-1}\left\{\hat{V}(\eta,0)e^{-\left(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta)\right)\tau}\right\}. \tag{75}$$

To evaluate this inversion, recall the convolution result for Fourier transforms given by

$$\mathcal{F}\left\{\int_{-\infty}^{\infty} f((x-u),\tau_1)g(u,\tau_2)du\right\} = \hat{F}(\eta,\tau_1)\hat{G}(\eta,\tau_2), \tag{76}$$

where $\hat{F}$ and $\hat{G}$ are the Fourier transforms, with respect to x, of $f(x,\tau_1)$ and $g(x,\tau_2)$ respectively. If we let

$$\hat{F}(\eta,\tau_1) = e^{-\left(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta)\right)\tau},$$

then $f(x,\tau_1)$ is given by

$$\begin{aligned} f(x,\tau_1) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta)\right)\tau}e^{-i\eta x}d\eta \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\lambda A(\eta)\tau}e^{-\left[\frac{1}{2}\sigma^2\eta^2+i[\phi\tau+x]\eta+(r+\lambda)\tau\right]}d\eta. \end{aligned}$$

Furthermore, let

$$\hat{G}(\eta,\tau_2) = \hat{V}(\eta,0).$$

Hence $g(x,\tau_2)$ will simply be the payoff function, $g(x,\tau_2) = \max(e^x - 1, 0)$.

Using a Taylor series expansion, the expression for $f(x,\tau_1)$ becomes

$$f(x,\tau_1) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda\tau)^n A(\eta)^n}{n!}e^{-\left[\frac{1}{2}\sigma^2\eta^2+i[\phi\tau+x]\eta+(r+\lambda)\tau\right]}d\eta. \tag{77}$$

Note that by definition

$$\begin{aligned} A(\eta)^n &= \left\{\int_0^\infty e^{-i\eta\ln Y} G(Y)dY\right\}^n \\ &= \int_0^\infty e^{-i\eta\ln Y_1} G(Y_1)dY_1 ... \int_0^\infty e^{-i\eta\ln Y_n} G(Y_n)dY_n \\ &= \int_0^\infty \int_0^\infty ... \int_0^\infty G(Y_1)G(Y_2)...G(Y_n)e^{-i\eta\ln(Y_1Y_2...Y_n)}dY_1dY_2...dY_n \\ &= \int_0^\infty \int_0^\infty ... \int_0^\infty G(Y_1)G(Y_2)...G(Y_n)e^{-i\eta\ln(Y_1Y_2...Y_n)}dY_1dY_2...dY_n, \\ &= \mathbb{E}_{\mathbb{Q}}^{(n)}\left\{e^{-i\eta\ln X_n}\right\}, \end{aligned}$$

where

$$\mathbb{E}_{\mathbb{Q}}^{(n)}\{(\cdot)\} = \int_0^\infty \int_0^\infty ... \int_0^\infty (\cdot)G(Y_1)G(Y_2)...G(Y_n)dY_1dY_2...dY_n,$$

with $\mathbb{E}_{\mathbb{Q}}^{(0)}\{(\cdot)\} \equiv (\cdot)$, and $X_n \equiv Y_1Y_2...Y_n$, with $X_0 \equiv 1$.

Substituting for $A(\eta)^n$, $f(x,\tau_1)$ becomes

$$f(x,\tau_1) = \sum_{n=0}^{\infty} \frac{(\lambda\tau)^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(r+\lambda)\tau}e^{-\left(\frac{1}{2}\sigma^2\eta^2+i[\phi\tau+x+\ln X_n]\eta\right)}d\eta\right\}.$$

By completing the square with respect to η and changing the integration variable, we find that

$$f(x,\tau_1) = \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\left\{\frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}\exp\left\{-\frac{[x+\ln X_n+\phi\tau]^2}{2\sigma^2\tau}\right\}\right\}. \tag{78}$$

Thus, by use of the convolution theorem (76) we have

$$V_E(x,\tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\left\{\frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}\int_0^\infty (e^u-1)\exp\left\{-\frac{[x-u+\ln X_n+\phi\tau]^2}{2\sigma^2\tau}\right\}du\right\},$$

which, in terms of S is

$$C_E(S,\tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\left\{I_1(S,\tau)-I_2(S,\tau)\right\}, \tag{79}$$

where we set

$$I_1(S,\tau) \equiv \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}\int_0^\infty Ke^u\exp\left\{-\frac{[\ln(SX_n/K)-u+\phi\tau]^2}{2\sigma^2\tau}\right\}du, \tag{80}$$

and

$$I_2(S,\tau) \equiv \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_0^{\infty} K \exp\left\{-\frac{[\ln(SX_n/K) - u + \phi\tau]^2}{2\sigma^2\tau}\right\} du. \tag{81}$$

Beginning with I_1, we have

$$I_1(S,\tau) = \frac{Ke^{-r\tau}}{\sigma\sqrt{2\pi\tau}} e^{-\beta^2/(2\sigma^2\tau)} \int_0^{\infty} \exp\left\{-\frac{u^2 - 2(\beta + \sigma^2\tau)u}{2\sigma^2\tau}\right\} du,$$

where $\beta \equiv \ln(SX_n/K) + \phi\tau$. Completing the square with respect to u and changing the integration variable, we find that[11]

$$I_1(S,\tau) = SX_n e^{-\lambda k\tau} e^{-q\tau} N[d_1(SX_n e^{-\lambda k\tau}, K, r, q, \tau, \sigma^2)], \tag{82}$$

where $N[\cdot]$ is the cumulative normal density function, and d_1 is given in Proposition 3.3. For I_2, a suitable change of integration variable gives

$$I_2(S,\tau) = Ke^{-r\tau} N[d_2(SX_n e^{-\lambda k\tau}, K, r, q, \tau, \sigma^2)], \tag{83}$$

where d_2 is as stated in Proposition 3.3. Finally, substituting I_1 and I_2 into (79), we find that

$$C_E(S,\tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ C_{BS}[SX_n e^{-\lambda k\tau}, K, K, r, q, \tau, \sigma^2] \right\}, \tag{84}$$

where C_{BS} is the solution the Black-Scholes-Merton solution for a European call option. Note that (84) is the solution provided by Merton (1976) for the price of a European call option under jump-diffusion.

A3.2. **Proof of Proposition 3.4.** We begin this proof by examining the function $V_P(x,\tau)$, which is given by

$$V_P(x,\tau) = \int_0^{\tau} \mathcal{F}^{-1} \left\{ \hat{F}_J(\eta,\xi) e^{-(\frac{\sigma^2\eta^2}{2} + i\phi\eta + (r+\lambda) - \lambda A(\eta))(\tau-\xi)} \right\} d\xi.$$

[11]Recall that $\phi = r - q - \lambda k - \sigma^2/2$.

Using equation (16) we recall that

$$\begin{aligned}\mathcal{F}^{-1}\left\{\hat{F}_J(\eta,\xi)\right\} &= H_1(x-\ln b(\xi))(qe^x-r)\\ &\quad -H_1(x-\ln b(\xi))\lambda\int_0^{b(\xi)e^{-x}}[V(x+\ln Y,\xi)-(Ye^x-1)]G(Y)dY.\end{aligned}$$

Furthermore, following the derivation for (78) we have

$$\begin{aligned}&\mathcal{F}^{-1}\left\{e^{-(\frac{\sigma^2\eta^2}{2}+i\phi\eta+(r+\lambda)-\lambda A(\eta))(\tau-\xi)}\right\}\\ &= \sum_{n=0}^{\infty}\frac{e^{-\lambda(\tau-\xi)}[\lambda(\tau-\xi)]^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\left\{\frac{e^{-r(\tau-\xi)}}{\sigma\sqrt{2\pi(\tau-\xi)}}\exp\left\{-\frac{[x+\ln X_n+\phi(\tau-\xi)]^2}{2\sigma^2(\tau-\xi)}\right\}\right\}.\end{aligned}$$

Thus by use of the convolution theorem (76), we obtain

$$\begin{aligned}V_P(x,\tau) &= \int_0^{\tau}\sum_{n=0}^{\infty}\left[\frac{e^{-\lambda(\tau-\xi)}[\lambda(\tau-\xi)]^n}{n!}\int_{-\infty}^{\infty}H(u-\ln b(\xi))\right.\\ &\quad\times\left[qe^u-r-\lambda\int_0^{b(\xi)e^{-x}}[V(u+\ln Y,\xi)-(Ye^u-1)]G(Y)dY\right]\\ &\quad\left.\times\mathbb{E}_{\mathbb{Q}}^{(n)}\left\{\frac{e^{-r(\tau-\xi)}}{\sigma\sqrt{2\pi(\tau-\xi)}}\exp\left\{-\frac{[x-u+\ln X_n+\phi(\tau-\xi)]^2}{2\sigma^2(\tau-\xi)}\right\}\right\}dud\xi\right],\end{aligned}$$

which in terms of S is

$$C_P(S,\tau)=\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\int_0^{\tau}(\tau-\xi)^n e^{-\lambda(\tau-\xi)}\left[I_3(S,\tau)-I_4(S,\tau)-I_5(S,\tau)\right]d\xi, \tag{85}$$

where

$$I_3(S,\tau) \equiv \frac{qe^{-r(\tau-\xi)}}{\sigma\sqrt{2\pi(\tau-\xi)}}\int_{\ln b(\xi)}^{\infty}Ke^u\exp\left\{-\frac{[\ln(SX_n/K)-u+\phi(\tau-\xi)]^2}{2\sigma^2(\tau-\xi)}\right\}du \tag{86}$$

$$I_4(S,\tau) \equiv \frac{re^{-r(\tau-\xi)}}{\sigma\sqrt{2\pi(\tau-\xi)}}\int_{\ln b(\xi)}^{\infty}K\exp\left\{-\frac{[\ln(SX_n/K)-u+\phi(\tau-\xi)]^2}{2\sigma^2(\tau-\xi)}\right\}du, \tag{87}$$

and

$$\begin{aligned}I_5(S,\tau) &\equiv \lambda\frac{e^{-r(\tau-\xi)}}{\sigma\sqrt{2\pi(\tau-\xi)}}\int_{\ln b(\xi)}^{\infty}\exp\left\{-\frac{[\ln(SX_n/K)-u+\phi(\tau-\xi)]^2}{2\sigma^2(\tau-\xi)}\right\}\\ &\quad\times\int_0^{b(\xi)e^{-u}}[C(KYe^u,\xi)-(KYe^u-K)]G(Y)dY\,du.\end{aligned} \tag{88}$$

To simplify I_3 and I_4, we make use of the results for I_1 and I_2 in Appendix A3.1. Firstly, we note that (86) is simply (80) with τ replaced by $(\tau - \xi)$. Thus from (82) we have

$$I_3(S,\tau) = qSX_n e^{-\lambda k(\tau-\xi)} e^{-q(\tau-\xi)} N[d_1(SX_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau-\xi, \sigma^2)]. \qquad (89)$$

Similarly for I_2, we can use (83) to show that (87) is

$$I_4(S,\tau) = rKe^{-r(\tau-\xi)} N[d_2(SX_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau-\xi, \sigma^2)]. \qquad (90)$$

For I_5, we change the order of integration using Fubini's theorem, and make the change of integration variable $\omega = Ke^u$, which gives

$$\begin{aligned} I_5(S,\tau) &= \lambda e^{-r(\tau-\xi)} \int_0^1 G(Y) \int_{a(\xi)}^{a(\xi)/Y} [C(\omega Y,\xi) - (\omega Y - K)] \\ &\qquad \times \kappa(SX_n e^{-\lambda k(\tau-\xi)}, \omega, r, q, \tau-\xi, \sigma^2) d\omega dY, \end{aligned}$$

where κ is defined by (24). Finally, substituting I_3, I_4 and I_5 into (85) gives equation (21) from Proposition 3.4.

A3.3. **Alternative Representation for $C_P^{(J)}$.** The representation for $C_P^{(J)}$ in (23) cannot be further simplified without explicit knowledge of the density $G(Y)$. In cases where the density is known, however, it may be possible to complete the integration with respect to Y analytically. Here we change the order of integration to develop a form for the double integral that will be easier to evaluate using numerical integration methods.

Recall from (23) that

$$\begin{aligned} & C_P^{(J)}[S, K, a(\xi), r, q, \tau, \sigma^2; C(\cdot,\xi)] \\ &\quad = e^{-r\tau} \int_0^1 G(Y) \int_{a(\xi)}^{a(\xi)/Y} [C(\omega Y,\xi) - (\omega Y - K)] \kappa(S,\omega,r,q,\tau,\sigma^2) d\omega dY. \end{aligned}$$

Making the change of integration variable $z = \omega Y/a(\xi)$ for the integral with respect to ω, $C_P^{(J)}$ becomes

$$\begin{aligned}
&C_P^{(J)}[S, K, a(\xi), r, q, \tau, \sigma^2; C(\cdot, \xi)] \\
&\quad = e^{-r\tau} \int_0^1 \int_Y^1 G(Y)[C(a(\xi)z, \xi) - (a(\xi)z - K)]\kappa(S/a(\xi), z, r, q, \tau, \sigma^2)dzdY.
\end{aligned}$$

Finally, changing the order of integration using Fubini's theorem, we obtain

$$\begin{aligned}
&C_P^{(J)}[S, K, a(\xi), r, q, \tau, \sigma^2; C(S, \xi)] \\
&\quad = e^{-r\tau} \int_0^1 [C(a(\xi)z, \xi) - (a(\xi)z - K)] \int_0^z G(Y)\kappa(S/a(\xi), z, r, q, \tau, \sigma^2)dY dz.
\end{aligned}$$

Appendix 4. Properties of the Incomplete Fourier Transform

According to Chiarella et al. (2004), from the pure-diffusion case (i.e. the model with no jumps) we know that

$$\mathcal{F}^b\left\{\frac{\partial V}{\partial x}\right\} = (b-1)e^{i\eta \ln b} - i\eta \hat{V}, \tag{91}$$

$$\mathcal{F}^b\left\{\frac{\partial^2 V}{\partial x^2}\right\} = e^{i\eta \ln b}(b - i\eta(b-1)) - \eta^2 \hat{V}, \tag{92}$$

$$\text{and} \quad \mathcal{F}^b\left\{\frac{\partial V}{\partial \tau}\right\} = \frac{\partial \hat{V}}{\partial \tau} - \frac{b'}{b}e^{i\eta \ln b}(b-1), \tag{93}$$

where $b' \equiv db(\tau)/d\tau$. This leaves one term to be evaluated, namely

$$\begin{aligned}
&\mathcal{F}\left\{H_2(\ln b - x)\int_0^\infty V(x + \ln Y, \tau)G(Y)dY\right\} \\
&\qquad = \int_{-\infty}^{\ln b} e^{i\eta x} \int_0^\infty V(x + \ln Y, \tau)G(Y)dY dx.
\end{aligned} \tag{94}$$

Using the change of variable $z = x + \ln Y$, equation (94) becomes

$$
\begin{aligned}
&\mathcal{F}\left\{H_2(\ln b - x)\int_0^\infty V(x+\ln Y,\tau)G(Y)d(Y)\right\} \\
&= \int_0^\infty \int_{-\infty}^{\ln b+\ln Y} e^{i\eta(z-\ln Y)}V(z,\tau)G(Y)dzdY \\
&= \int_0^\infty \left[\int_{-\infty}^{\ln b} e^{i\eta(z-\ln Y)}V(z,\tau)G(Y)dz\right. \\
&\quad \left. + \int_{\ln b}^{\ln Y+\ln b} e^{i\eta(z-\ln Y)}V(z,\tau)G(Y)dz\right] dY \\
&= \int_0^\infty e^{-i\eta\ln Y}G(Y)dY\int_{-\infty}^{\ln b}V(z,\tau)e^{i\eta z}dz \\
&\quad + \int_0^\infty e^{-i\eta\ln Y}G(Y)\left[\int_{\ln b}^{\ln Y+\ln b} e^{i\eta z}V(z,\tau)dz\right]dY \\
&= A(\eta)\hat{V}(\eta,\tau) + \Phi(\eta,\tau),
\end{aligned}
$$

where

$$
A(\eta) \equiv \int_0^\infty e^{-i\eta\ln Y}G(Y)dY,
$$

and

$$
\Phi(\eta,\tau) \equiv \int_0^\infty e^{-i\eta\ln Y}G(Y)\left[\int_{\ln b}^{\ln Y+\ln b} e^{i\eta\ln z}V(z,\tau)dz\right]dY.
$$

Hence, our IPDE is transformed into the integro-differential equation

$$
\frac{\partial\hat{V}}{\partial\tau} + \left[\frac{\sigma^2\eta^2}{2} + \phi i\eta + (r+\lambda) - \lambda A(\eta)\right]\hat{V} = \hat{F}_M(\eta,\tau),
$$

where

$$
\hat{F}_M(\eta,\tau) \equiv e^{i\eta\ln b}\left[\frac{\sigma^2 b}{2} + \left(\frac{b'}{b} - \frac{\sigma^2 i\eta}{2} + \phi\right)(b-1)\right] + \lambda\Phi(\eta,\tau).
$$

The solution to this integro-differential equation is given by

$$
\begin{aligned}
\hat{V}(\eta,\tau) \quad = \quad & \hat{V}(\eta,0)e^{-(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta))\tau} \\
& + \int_0^\tau e^{-(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta))(\tau-\xi)}\hat{F}_M(\eta,\xi)d\xi,
\end{aligned}
$$

where $\mathcal{F}^b\{V(x,0)\} \equiv \hat{V}(\eta,0)$.

Appendix 5. Derivation of the American Call Integral Equations - McKean's Method

A5.1. **Proof of Proposition 4.2.** Consider the function $V_1(x,\tau)$, given by

$$V_1(x,\tau) = \mathcal{F}^{-1}\left\{\hat{V}^b(\eta,0)e^{-\left(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta)\right)\tau}\right\}. \tag{95}$$

We can evaluate this inversion using the convolution theorem (76). In particular, we note that the inversion (95) is equivalent to (75) with $\hat{G}(\eta,\tau_2)$ replaced by $\hat{V}^b(\eta,0)$. Hence $g(x,\tau_2)$ will simply be the payoff function in the continuation region, given by

$$g(x,\tau_2) = H_2(\ln b(0^+) - x)\max(e^x - 1, 0) = H_2(\ln b(0^+) - x)H_2(x)(e^x - 1),$$

where $b(0^+)$ denotes the limit of $b(\tau)$ as $\tau \to 0^+$. Thus, using the notation and results from Appendix A3.1, $V_1(x,\tau)$ becomes

$$\begin{aligned} V_1(x,\tau) &= \sum_{n=0}^{\infty}\frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \\ &\quad\times\mathbb{E}_{\mathbb{Q}}^{(n)}\left\{\frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}\int_0^{\ln b(0^+)}(e^u - 1)\exp\left\{-\frac{[x-u+\ln X_n+\phi\tau]^2}{2\sigma^2\tau}\right\}du\right\}, \end{aligned}$$

which, in terms of S is

$$C_1(S,\tau) = \sum_{n=0}^{\infty}\frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\left\{I_1(S,\tau) - I_2(S,\tau) - [J_1(S,\tau) - J_2(S,\tau)]\right\}, \tag{96}$$

where I_1 and I_2 are given by (82) and (83) respectively, and we set

$$J_1(S,\tau) \equiv \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}\int_{\ln b(0^+)}^{\infty}Ke^u\exp\left\{-\frac{[\ln(SX_n/K) - u + \phi\tau]^2}{2\sigma^2\tau}\right\}du, \tag{97}$$

and

$$J_2(S,\tau) \equiv \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}\int_{\ln b(0^+)}^{\infty}K\exp\left\{-\frac{[\ln(SX_n/K) - u + \phi\tau]^2}{2\sigma^2\tau}\right\}du. \tag{98}$$

By comparing J_1 with I_1, and J_2 with I_2, we can readily show that

$$J_1(S,\tau) = SX_ne^{-\lambda k\tau}e^{-q\tau}N[d_1(SX_ne^{-\lambda k\tau}, a(0^+), r, q, \tau, \sigma^2)], \tag{99}$$

and

$$J_2(S,\tau) = Ke^{-r\tau}N[d_2(SX_ne^{-\lambda k\tau}, a(0^+), r, q, \tau, \sigma^2)]. \quad (100)$$

Finally , by substituting (82), (83), (99) and (100) into equation (96), we find that

$$\begin{aligned} C_1(S,\tau) &= \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\{C_{BS}[SX_ne^{-\lambda k\tau}, K, K, r, q, \tau, \sigma^2] \\ &\qquad -C_{BS}[SX_ne^{-\lambda k\tau}, K, a(0^+), r, q, \tau, \sigma^2]\}, \end{aligned}$$

where C_{BS} is the Black-Scholes-Merton solution given by (21).

A5.2. **Proof of Proposition 4.3.** We begin this proof by examining the function $V_2^{(1)}(x,\tau)$.

$$\begin{aligned} V_2^{(1)}(x,\tau) &= \frac{1}{2\pi}\int_0^\tau e^{-(r+\lambda)(\tau-\xi)}\int_{-\infty}^{\infty} e^{-\frac{\sigma^2\eta^2}{2}(\tau-\xi)-i\eta[\phi(\tau-\xi)+x-\ln b(\xi)]}e^{\lambda A(\eta)(\tau-\xi)} \\ &\qquad \times \left[\frac{\sigma^2 b(\xi)}{2} + \left(\frac{b'(\xi)}{b(\xi)} - \frac{\sigma^2 i\eta}{2} + \phi\right)(b(\xi)-1)\right] d\xi d\eta. \end{aligned}$$

We let

$$\begin{aligned} f_1(\xi) &= \frac{\sigma^2 b(\xi)}{2} + \left(\frac{b'(\xi)}{b(\xi)} + \phi\right)(b(\xi)-1), \\ f_2(\xi) &= \frac{\sigma^2 i}{2}(b(\xi)-1), \end{aligned}$$

$\hat{p}(\xi) = \sigma^2(\tau-\xi)/2$, and $\hat{q}(X_n,\xi) = i[x + \ln(X_n) + \phi(\tau-\xi) - \ln b(\xi)]$, where X_n is as defined in Appendix A3.1. Applying a Taylor series expansion to $e^{\lambda A(\eta)(\tau-\xi)}$, we can rewrite $V_2^{(1)}(x,\tau)$ as

$$\begin{aligned} V_2^{(1)}(x,\tau) &= \frac{1}{2\pi}\int_0^\tau e^{-(r+\lambda)(\tau-\xi)}\left[\int_{-\infty}^{\infty} e^{-\hat{p}(\xi)\eta^2 - i\eta[\phi(\tau-\xi)+x-\ln b(\xi)]}\sum_{n=0}^{\infty}\frac{\lambda(\tau-\xi)^n A(\eta)^n}{n!}\right. \\ &\qquad \left.\times\ \{f_1(\xi) - \eta f_2(\xi)\}d\eta\right] d\xi \\ &= \sum_{n=0}^{\infty}\frac{1}{2\pi}\frac{\lambda^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\left\{\int_0^\tau e^{-(r+\lambda)(\tau-\xi)}(\tau-\xi)^n\right. \\ &\qquad \left.\times\left[\int_{-\infty}^{\infty} e^{-\hat{p}(\xi)\eta^2 - \hat{q}(X_n,\xi)\eta}\{f_1(\xi) - \eta f_2(\xi)\}d\eta\right] d\xi\right\}, \end{aligned}$$

where $\mathbb{E}_{\mathbb{Q}}^{(n)}$ is as stated in Appendix A3.1.

Following the details in Chiarella et al. (2004) we can readily show that

$$\begin{aligned} V_2^{(1)}(x,\tau) &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \Bigg\{ \int_0^{\tau} e^{-(r+\lambda)(\tau-\xi)} (\tau-\xi)^n (b(\xi)-1) \\ & \qquad \times \left[\frac{\sigma^2 b(\xi)}{2} + \left(\frac{b'(\xi)}{b(\xi)} + \frac{1}{2} \left[\phi - \frac{x + \ln(X_n) - \ln b(\xi)}{\tau-\xi} \right] \right) \right] \\ & \qquad \times \kappa(X_n e^x e^{-\lambda k(\tau-\xi)}/b(\xi), 1, r, q, \tau-\xi, \sigma^2) d\xi \Bigg\}, \end{aligned}$$

where κ is given by (24). Finally, we return to the original state variable, S, by using $C_2^{(1)}(S,\tau) = KV_2^{(1)}(x,\tau)$, with $S = Ke^x$. This results in

$$\begin{aligned} C_2^{(1)}(S,\tau) &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \Bigg\{ \int_0^{\tau} e^{-(r+\lambda)(\tau-\xi)} (\tau-\xi)^n (a(\xi)-K) \\ & \qquad \times \left[\frac{\sigma^2 a(\xi)}{2} + \left(\frac{a'(\xi)}{a(\xi)} + \frac{1}{2} \left[\left(r - q - \frac{\sigma^2}{2} \right) - \frac{\ln[SX_n e^{-\lambda k(\tau-\xi)}/a(\xi)]}{\tau-\xi} \right] \right) \right] \\ & \qquad \times \kappa(SX_n e^{-\lambda k(\tau-\xi)}/a(\xi), 1, r, q, \tau-\xi, \sigma^2) d\xi \Bigg\}, \end{aligned}$$

A5.3. **Proof of Proposition 4.4.** The term $V_2^{(2)}(x,\tau)$ is given by

$$V_2^{(2)}(x,\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^{\tau} e^{-\left(\frac{1}{2}\sigma^2\eta^2 + \phi i\eta + (r+\lambda) - \lambda A(\eta)\right)(\tau-\xi)} e^{-i\eta x} \lambda \Phi(\eta,\xi) d\xi d\eta,$$

where we recall that

$$\Phi(\eta,\xi) = \int_0^{\infty} e^{-i\eta \ln Y} G(Y) \left[\int_{\ln b(\xi)}^{\ln Y b(\xi)} e^{i\eta z} V(z,\xi) dz \right] dY.$$

We begin by changing the order of integration within $V_2^{(2)}(x,\tau)$, which gives

$$\begin{aligned} V_2^{(2)}(x,\tau) &= \lambda \int_0^{\tau} e^{-(r+\lambda)(\tau-\xi)} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\lambda A(\eta)(\tau-\xi)} e^{-\frac{1}{2}\sigma^2\eta^2(\tau-\xi) - i\eta(\phi(\tau-\xi)+x)} \\ & \qquad \times \int_0^{\infty} e^{-i\eta \ln Y} G(Y) \int_{\ln b(\xi)}^{\ln Y b(\xi)} e^{i\eta z} V(z,\xi) dz dY d\eta d\xi. \end{aligned}$$

Applying a Taylor series expansion to $e^{\lambda A(\eta)(\tau-\xi)}$, we can rewrite $V_2^{(2)}(x,\tau)$ as

$$\begin{aligned} V_2^{(2)}(x,\tau) &= \sum_{n=0}^{\infty}\frac{\lambda^{n+1}}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\bigg\{\int_0^{\tau}(\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)}\int_0^{\infty}G(Y) \\ &\quad\times\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\frac{1}{2}\sigma^2\eta^2(\tau-\xi)-i\eta(\phi(\tau-\xi)+x+\ln X_n Y)} \\ &\quad\times\int_{\ln b(\xi)}^{\ln Y b(\xi)}e^{i\eta z}V(z,\xi)dzd\eta dYd\xi\bigg\}, \end{aligned}$$

where the operator $\mathbb{E}_{\mathbb{Q}}^{(n)}$ is defined in Proposition 3.3, and its source outlined in Appendix A3.1. Consider the two innermost integrals, $I(x,\tau,Y,\xi)$, defined as

$$\begin{aligned} I(x,\tau,Y,\xi) &\equiv \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\frac{1}{2}\sigma^2\eta^2(\tau-\xi)-i\eta(\phi(\tau-\xi)+x+\ln Y\ln X_n)} \\ &\quad\times\int_{\ln b(\xi)-\ln Y}^{\ln b(\xi)}e^{i\eta(x+\ln Y)}V(x+\ln Y,\xi)dxd\eta, \end{aligned}$$

where the integral with respect to x has been derived using the change of variable $z = x+\ln Y$. To evaluate I, we shall express it as the inverse Fourier transform

$$\begin{aligned} I(x,\tau,Y,\xi) &= -\mathcal{F}^{-1}\bigg\{e^{-\frac{1}{2}\sigma^2\eta^2(\tau-\xi)-i\eta(\phi(\tau-\xi)+\ln X_n)} \\ &\quad\times\int_{\ln b(\xi)}^{\ln b(\xi)-\ln Y}e^{i\eta x}V(x+\ln Y,\xi)dx\bigg\}. \end{aligned} \tag{101}$$

Since we know that $0<Y<\infty$, we must now consider two separate cases to evaluate equation (101), the first case being when $0<Y<1$. We can rewrite (101) as

$$\begin{aligned} I(x,\tau,Y,\xi) &= -\mathcal{F}^{-1}\bigg\{e^{-\frac{1}{2}\sigma^2\eta^2(\tau-\xi)-i\eta(\phi(\tau-\xi)+\ln X_n)} \\ &\quad\times\int_{-\infty}^{\infty}H_2(\ln b(\xi)-\ln Y-x)H_2(x-\ln b(\xi))e^{i\eta x}V(x+\ln Y,\xi)dx\bigg\}. \end{aligned}$$

To evaluate this inversion, we again refer to the convolution result for Fourier transforms given by equation (76). Let

$$\hat{F}(\eta,\xi) = e^{-\frac{1}{2}\sigma^2\eta^2(\tau-\xi)-i\eta(\phi(\tau-\xi)+\ln X_n)},$$

so that $f(x,\xi)$ is given by

$$f(x,\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\hat{p}\eta^2-\hat{q}\eta}d\eta = \sqrt{\frac{\pi}{\hat{p}}}e^{\frac{\hat{q}^2}{4\hat{p}}},$$

with $\hat{p} = \sigma^2(\tau-\xi)/2$ and $\hat{q} = i(\phi(\tau-\xi)+\ln X_N + x)$. Thus $f(x,\xi)$ evaluates to

$$f(x,\xi) = \frac{1}{\sigma\sqrt{2\pi(\tau-\xi)}}\exp\left\{-\frac{[\phi(\tau-\xi)+\ln X_n + x]^2}{2\sigma^2(\tau-\xi)}\right\}.$$

For the second part of the convolution, let

$$\hat{G}(\eta,\xi) = \int_{-\infty}^{\infty} H_2(\ln b(\xi) - \ln Y - x)H_2(x - \ln b(\xi))e^{i\eta x}V(x+\ln Y,\xi)dx.$$

Therefore $g(x,\xi)$ is simply

$$g(x,\xi) = H_2(\ln b(\xi) - \ln Y - x)H_2(x - \ln b(\xi))V(x+\ln Y,\xi).$$

Combining f and g, the inverse Fourier transform, I, becomes

$$\begin{aligned} I(x,\tau,Y,\xi) &= -\int_{\ln b(\xi)}^{\ln b(\xi)-\ln Y} V(u+\ln Y,\xi) \\ &\quad \times\kappa(X_n e^{x-u}e^{-\lambda k(\tau-\xi)},1,r,q,\tau-\xi,\sigma^2)du, \end{aligned} \tag{102}$$

where κ is given by (24).

In the second case we have $1 < Y < \infty$, and we can rewrite equation (101) as

$$\begin{aligned} I(x,\tau,Y,\xi) &= \mathcal{F}^{-1}\Big\{e^{-\frac{\sigma^2\eta^2}{2}(\tau-\xi)-i\eta(\phi(\tau-\xi)+\ln X_n)} \\ &\quad \times\int_{-\infty}^{\infty} H_2(\ln b(\xi) - x)H_2(x - \ln b(\xi) + \ln Y)e^{i\eta x}V(x+\ln Y,\xi)dx\Big\}. \end{aligned}$$

Following the same method as used in the first case, we find that

$$\begin{aligned} I(x,\tau,Y,\xi) &= \int_{\ln b(\xi)-\ln Y}^{\ln b(\xi)} V(u+\ln Y,\xi) \\ &\quad \times\kappa(X_n e^{x-u}e^{-\lambda k(\tau-\xi)},1,r,q,\tau-\xi,\sigma^2)du. \end{aligned} \tag{103}$$

Since the results (102) and (103) depend entirely upon the relevant value of Y, we can integrate piecewise over the Y-domain, and thereby express $V_2^{(2)}(x,\tau)$ as

$$\begin{aligned} V_2^{(2)}(x,\tau) &= \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ \int_0^\tau (\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \int_0^\infty G(Y) I(x,\tau,Y,\xi) dY d\xi \right\} \\ &= \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ \int_0^\tau (\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \right. \\ &\quad \times \left[-\int_0^1 G(Y) \int_{\ln b(\xi)}^{\ln b(\xi)-\ln Y} V(u+\ln Y,\xi) \right. \\ &\quad \times \kappa(X_n e^{x-u} e^{-\lambda k(\tau-\xi)}, 1, r, q, \tau-\xi, \sigma^2) du dY \\ &\quad + \int_1^\infty G(Y) \int_{\ln b(\xi)-\ln Y}^{\ln b(\xi)} V(u+\ln Y,\xi) \\ &\quad \left. \left. \times \kappa(X_n e^{x-u} e^{-\lambda k(\tau-\xi)}, 1, r, q, \tau-\xi, \sigma^2) du dY \right] d\xi \right\}. \end{aligned}$$

Using $C_2^{(2)}(S,\tau) = KV_2^{(2)}(x,\tau)$, we have

$$\begin{aligned} C_2^{(2)}(S,\tau) &= -\lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ \int_0^\tau (\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \right. \\ &\quad \times \left[\int_0^1 G(Y) \int_{\ln[a(\xi)/K]}^{\ln[a(\xi)/KY]} C(Ke^u Y,\xi) \right. \\ &\quad \times \kappa(SX_n e^{-u} e^{-\lambda k(\tau-\xi)}, 1, r, q, \tau-\xi, \sigma^2) du dY \\ &\quad - \int_1^\infty G(Y) \int_{\ln[a(\xi)/KY]}^{\ln[a(\xi)/K]} C(Ke^u Y,\xi) \\ &\quad \left. \left. \times \kappa(SX_n e^{-u} e^{-\lambda k(\tau-\xi)}, 1, r, q, \tau-\xi, \sigma^2) du dY \right] d\xi \right\}. \end{aligned}$$

We introduce a change of integration variable to simplify the expression for $C_2^{(2)}(S,\tau)$. Letting $\omega = Ke^u$, we have

$$\begin{aligned}C_2^{(2)}(S,\tau) &= -\lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \Bigg\{ \int_0^{\tau} (\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \\ &\quad \times \Bigg[\int_0^1 G(Y) \int_{a(\xi)}^{a(\xi)/Y} C(\omega Y,\xi) \\ &\quad \times \kappa(SX_n e^{-\lambda k(\tau-\xi)}, \omega, r, q, \tau-\xi, \sigma^2) d\omega dY \\ &\quad - \int_1^{\infty} G(Y) \int_{a(\xi)/Y}^{a(\xi)} C(\omega,\xi) \\ &\quad \times \kappa(SX_n e^{-\lambda k(\tau-\xi)}, \omega, r, q, \tau-\xi, \sigma^2) d\omega dY \Bigg] d\xi \Bigg\}.\end{aligned}$$

Finally, we analyse the domains for the integrals with respect to ω. For the first integral, the domain for ωY is $Ya(\xi) < \omega Y < a(\xi)$. Thus ωY lies in the continuation region, and the value of $C(\omega Y,\xi)$ is unknown. For the second integral, the domain for ωY is $a(\xi) < \omega Y < Ya(\xi)$. Since ωY lies in the stopping region, the value of $C(\omega Y,\xi)$ is known to be $C(\omega Y,\xi) = \omega Y - K$, where $\omega > K/Y$. Thus $C_2^{(2)}(S,\tau)$ can be written as shown in equation (47).

Appendix 6. Equivalence of Propositions 3.5 and 4.5

There are two main steps that we must perform to prove that equation (48) of Proposition 4.5 is equivalent to (25) from Proposition 3.5. The first involves applying integration by parts to $C_2^{(1)}(S,\tau)$ from (46), as detailed by Kim (1990). The second requires extensive algebraic manipulation of (48), with a particular focus on the term $C_2^{(2)}(S,\tau)$ from (47).

A6.1. **Integration by Parts.** We now aim to simplify the expression for $C_2^{(1)}(S,\tau)$ using the methods of Kim (1990). The first step is to rewrite d_2 in κ as

$$\begin{aligned}d_2(SX_n e^{-\lambda k(\tau-\xi)}/a(\xi), 1, r, q, \tau-\xi, \sigma^2) &= \frac{\ln(SX_n e^{-\lambda k(\tau-\xi)}/a(\xi)) + \left(r - q - \frac{\sigma^2}{2}\right)(\tau-\xi)}{\sigma\sqrt{\tau-\xi}} \\ &= \frac{y_n - Q(\xi)}{\sqrt{\tau-\xi}},\end{aligned}$$

where

$$y_n \equiv \frac{\ln SX_n + \left(r - q - \lambda k - \frac{\sigma^2}{2}\right)\tau}{\sigma},$$

and

$$Q(\xi) \equiv \frac{\ln a(\xi) + \left(r - q - \lambda k - \frac{\sigma^2}{2}\right)\xi}{\sigma}.$$

It is important to note that the derivative of $Q(\xi)$ with respect to ξ is given by

$$Q'(\xi) = \frac{1}{\sigma}\left(\frac{a'(\xi)}{a(\xi)} + \left(r - q - \lambda k - \frac{\sigma^2}{2}\right)\right).$$

Using these results, $C_2^{(1)}(S,\tau)$ becomes

$$\begin{aligned}
C_2^{(1)}(S,\tau) &= \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\bigg\{\int_0^\tau \frac{(\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)-\frac{[y_n-Q(\xi)]^2}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)}} \\
&\quad \times\left[\frac{\sigma a(\xi)}{2} + \frac{1}{\sigma}\left(\frac{a'(\xi)}{a(\xi)} + \left(r-q-\lambda k-\frac{\sigma^2}{2}\right) - \left(r-q-\lambda k-\frac{\sigma^2}{2}\right)\right.\right. \\
&\quad \left.\left. + \frac{1}{2}\left[\left(r-q-\lambda k-\frac{\sigma^2}{2}\right) - \frac{\ln\frac{SX_n}{a(\xi)}}{\tau-\xi}\right]\right)(a(\xi)-K)\right]d\xi\bigg\} \\
&= \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\bigg\{\int_0^\tau e^{-(r+\lambda)(\tau-\xi)}(\tau-\xi)^n\frac{e^{-\frac{[y_n-Q(\xi)]^2}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)}} \\
&\quad \times\Bigg[\ \frac{\sigma a(\xi)}{2} + \left(Q'(\xi) - \frac{1}{\sigma}\left(r-q-\lambda k-\frac{\sigma^2}{2}\right)\right. \\
&\quad \left. - \ \frac{1}{2\sigma}\left[\frac{\ln SX_n - \ln a(\xi) - \left(r-q-\lambda k-\frac{\sigma^2}{2}\right)(\tau-\xi)}{\tau-\xi}\right]\right) \\
&\quad \times (a(\xi)-K)\Bigg]d\xi \\
&= \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\bigg\{\int_0^\tau e^{-(r+\lambda)(\tau-\xi)}(\tau-\xi)^n\frac{e^{-\frac{[y_n-Q(\xi)]^2}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)}} \\
&\quad \times\left[\frac{\sigma a(\xi)}{2} + \left(Q'(\xi) - \frac{y_n-Q(\xi)}{2(\tau-\xi)}\right)(a(\xi)-K)\right]d\xi\bigg\}.
\end{aligned}$$

Thus we arrive at a new expression for $C_2^{(1)}(S,\tau)$, given by

$$\begin{aligned} C_2^{(1)}(S,\tau) \quad = \quad & \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \bigg\{ \int_0^\tau a(\xi) e^{-(r+\lambda)(\tau-\xi)} (\tau-\xi)^n \frac{e^{-\frac{[y_n - Q(\xi)]^2}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)}} \\ & \times \left[\frac{\sigma}{2} + Q'(\xi) - \frac{y_n - Q(\xi)}{2(\tau-\xi)} \right] d\xi \\ & - K \int_0^\tau e^{-(r+\lambda)(\tau-\xi)} (\tau-\xi)^n \frac{e^{-\frac{[y_n - Q(\xi)]^2}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)}} \\ & \times \left[Q'(\xi) - \frac{y_n - Q(\xi)}{2(\tau-\xi)} \right] d\xi \bigg\}. \end{aligned} \tag{104}$$

In order to simplify equation (104), we must derive two results. The first result we require is

$$\begin{aligned} & (\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} G(\xi) \frac{e^{-\frac{[y_n - Q(\xi)]^2}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)}} \left[\frac{\sigma}{2} + Q'(\xi) - \frac{[y_n - Q(\xi)]}{2(\tau-\xi)} \right] \\ & = (\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \frac{a(\xi)}{\sqrt{\tau-\xi}} \left[\frac{\sigma(\tau-\xi) + 2Q'(\xi)(\tau-\xi) - y_n + Q(\xi)}{2(\tau-\xi)} \right] \\ & \quad \times \frac{1}{\sqrt{2\pi}} e^{-\frac{[y_n - Q(\xi) + \sigma(\tau-\xi)]^2}{2(\tau-\xi)}} e^{\frac{\sigma^2}{2}(\tau-\xi) + \sigma(y_n - Q(\xi))} \\ & = -(\tau-\xi)^n S X_n e^{-\lambda k(\tau-\xi)} e^{-(q+\lambda)(\tau-\xi)} \frac{1}{\sqrt{2\pi}} e^{-\frac{[y_n - Q(\xi) + \sigma(\tau-\xi)]^2}{2(\tau-\xi)}} \\ & \quad \times \left[\frac{\frac{1}{2}\frac{1}{\sqrt{\tau-\xi}}(y_n - Q(\xi) + \sigma(\tau-\xi)) - (Q'(\xi) + \sigma)\sqrt{\tau-\xi}}{(\sqrt{\tau-\xi})^2} \right] \\ & = -(\tau-\xi)^n S X_n e^{-\lambda k(\tau-\xi)} e^{-(q+\lambda)(\tau-\xi)} \frac{\partial}{\partial \xi} N\left(\frac{y_n - Q(\xi) + \sigma(\tau-\xi)}{\sqrt{\tau-\xi}} \right). \end{aligned} \tag{105}$$

For the second result we have

$$\begin{aligned} & (\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \frac{e^{-\frac{[y_n - Q(\xi)]^2}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)}} \left[Q'(\xi) - \frac{[y_n - Q(\xi)]}{2(\tau-\xi)} \right] \\ & = -(\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \frac{1}{\sqrt{2\pi}} e^{-\frac{[y_n - Q(\xi)]^2}{2(\tau-\xi)}} \left[\frac{-Q'(\xi)\sqrt{\tau-\xi} + \frac{1}{2}(y_n - Q(\xi))\frac{1}{\sqrt{\tau-\xi}}}{(\sqrt{\tau-\xi})^2} \right] \\ & = -(\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \frac{\partial}{\partial \xi} N\left(\frac{y_n - Q(\xi)}{\sqrt{\tau-\xi}} \right). \end{aligned} \tag{106}$$

Using equations (105) and (106) in (104), $C_2^{(1)}(S,\tau)$ becomes

$$\begin{aligned} C_2^{(1)}(S,\tau) &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\Big\{ -\int_0^{\tau} (\tau-\xi)^n S X_n e^{-\lambda k(\tau-\xi)} e^{-(q+\lambda)(\tau-\xi)} \\ &\quad \times \frac{\partial}{\partial \xi} N\left(\frac{y_n - Q(\xi) + \sigma(\tau-\xi)}{\sqrt{\tau-\xi}}\right) d\xi \\ &\quad + K\int_0^{\tau} (\tau-\xi)^n e^{-(\lambda+r)(\tau-\xi)} \frac{\partial}{\partial \xi} N\left(\frac{y_n - Q(\xi)}{\sqrt{\tau-\xi}}\right) d\xi. \end{aligned} \tag{107}$$

By applying integration by parts, we can evaluate equation (107) as

$$\begin{aligned} C_2^{(1)}(S,\tau) &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\Big\{ S X_n e^{-\lambda k\tau} e^{-(q+\lambda)\tau} \tau^n N\left(\frac{y_n - Q(0^+) + \sigma\tau}{\sqrt{\tau}}\right) \\ &\quad + \int_0^{\tau} S X_n e^{-\lambda k(\tau-\xi)} e^{-(q+\lambda)(\tau-\xi)} (\tau-\xi)^{n-1} [(\lambda[k+1]+q)(\tau-\xi) - n] \\ &\quad \times N\left(\frac{y_n - Q(\xi) + \sigma(\tau-\xi)}{\sqrt{\tau-\xi}}\right) d\xi \\ &\quad - K e^{-(r+\lambda)\tau} \tau^n N\left(\frac{y_n - Q(0^+)}{\sqrt{\tau}}\right) \\ &\quad - \int_0^{\tau} K(\tau-\xi)^{n-1} e^{-(r+\lambda)(\tau-\xi)} [(\tau-\xi)(r+\lambda) - n] \\ &\quad \times N\left(\frac{y_n - Q(\xi)}{\sqrt{\tau-\xi}}\right) d\xi \Big\} + (K-S)\mathbf{1}_{S=a(\tau)}, \end{aligned}$$

where we define

$$\mathbf{1}_{S=a(\tau)} \equiv \begin{cases} \frac{1}{2}, & S = a(\tau), \\ 0 & \text{otherwise.} \end{cases}$$

In terms of the original variables, $C_2^{(1)}(S,\tau)$ is

$$\begin{aligned} C_2^{(1)}(S,\tau) \;=\; & \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ C_{BS}\left[SX_n e^{-\lambda k\tau}, K, a(0^+), r, q, \tau, \sigma^2 \right] \right\} \\ & +(K-S)\mathbf{1}_{S=a(\tau)} \\ & + \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ \int_0^{\tau} (\tau-\xi)^{n-1} \left[SX_n e^{-\lambda k(\tau-\xi)} e^{-(q+\lambda)(\tau-\xi)} \right.\right. \\ & \qquad \times [(\lambda[k+1]+q)(\tau-\xi)-n] \\ & \qquad \times N\left[d_1\left(SX_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau-\xi, \sigma^2 \right) \right] \\ & -Ke^{-(r+\lambda)(\tau-\xi)}[(\tau-\xi)(r+\lambda)-n] \\ & \left.\left. \times N\left[d_2\left(SX_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau-\xi, \sigma^2 \right) \right] \right] d\xi \right\}. \end{aligned} \tag{108}$$

Before proceeding further, we note that if we combine $C_1(S,\tau)$ from (43) with $C_2^{(1)}(S,\tau)$ in (108), some of the terms will cancel, giving us

$$\begin{aligned} & H(a(\tau)-S)C(S,\tau) + (S-K)\mathbf{1}_{S=a(\tau)} \\ & = \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \{ C_{BS}[SX_n e^{-\lambda k\tau}, K, K, r, q, \tau, \sigma^2] \} \\ & + \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ \int_0^{\tau} (\tau-\xi)^{n-1} SX_n e^{-\lambda k(\tau-\xi)} e^{-(q+\lambda)(\tau-\xi)} \right. \\ & \qquad \times [(\lambda[k+1]+q)(\tau-\xi)-n] \\ & \qquad \left. \times N\left[d_1\left(SX_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau-\xi, \sigma^2 \right) \right] d\xi \right\} \\ & - \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ \int_0^{\tau} (\tau-\xi)^{n-1} K e^{-(r+\lambda)(\tau-\xi)} [(\tau-\xi)(r+\lambda)-n] \right. \\ & \qquad \left. \times N\left[d_2\left(SX_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau-\xi, \sigma^2 \right) \right] d\xi \right\} \\ & -\lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ \int_0^{\tau} (\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \right. \\ & \times \left[\int_0^1 G(Y) \int_{a(\xi)}^{a(\xi)/Y} C(\omega Y,\xi)\kappa(SX_n e^{-\lambda k(\tau-\xi)}, \omega, r, q, \tau-\xi, \sigma^2) d\omega dY \right. \\ & \left.\left. - \int_1^{\infty} G(Y) \int_{a(\xi)/Y}^{a(\xi)} (\omega Y - K)\kappa(SX_n e^{-\lambda k(\tau-\xi)}, \omega, r, q, \tau-\xi, \sigma^2) d\omega dY \right] d\xi \right\}. \end{aligned} \tag{109}$$

A6.2. **Simplification of the Integral Terms in (109).** The next step is to simplify (109) in order to demonstrate that it is equivalent to (25). Firstly, we note that when $S = a(\tau)$, the left-hand side of (109) becomes

$$C(a(\tau), \tau) = a(\tau) - K,$$

and thus we can remove the indicator function and Heaviside step function from the expression for (109). We now rewrite equation (109) as

$$\begin{aligned} C(S,\tau) &= \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ C_{BS}[SX_n e^{-\lambda k\tau}, K, K, r, q, \tau, \sigma^2] \right\} \\ &\quad + (A_1^{(1)} + A_1^{(2)} - A_1^{(3)}) - (A_2^{(1)} + A_2^{(2)} - A_2^{(3)}) - (J_1 - J_2) \end{aligned}$$

where

$$\begin{aligned} A_1^{(1)} &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ q \int_0^{\tau} (\tau-\xi)^n f(SX_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau-\xi, \sigma^2) d\xi \right\}, \\ A_1^{(2)} &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ \lambda[k+1] \int_0^{\tau} (\tau-\xi)^n f(SX_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau-\xi, \sigma^2) d\xi \right\}, \\ A_1^{(3)} &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ n \int_0^{\tau} (\tau-\xi)^{n-1} f(SX_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau-\xi, \sigma^2) d\xi \right\}, \end{aligned}$$

$$\begin{aligned} A_2^{(1)} &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ r \int_0^{\tau} (\tau-\xi)^n K g(SX_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau-\xi, \sigma^2) d\xi \right\}, \\ A_2^{(2)} &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ \lambda \int_0^{\tau} (\tau-\xi)^n K g(SX_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau-\xi, \sigma^2) d\xi \right\}, \\ A_2^{(3)} &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ n \int_0^{\tau} (\tau-\xi)^{n-1} K g(SX_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau-\xi, \sigma^2) d\xi \right\}, \\ J_1 &= \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ \int_0^{\tau} (\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \right. \\ &\quad \left. \times \left[\int_0^1 G(Y) \int_{a(\xi)}^{a(\xi)/Y} C(\omega Y, \xi) \kappa(SX_n e^{-\lambda k(\tau-\xi)}, \omega, r, q, \tau-\xi, \sigma^2) d\omega dY \right] d\xi \right\}, \end{aligned}$$

and

$$J_2 = \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \Big\{ \int_0^\tau (\tau - \xi)^n e^{-(r+\lambda)(\tau-\xi)} \times \left[\int_1^\infty G(Y) \int_{a(\xi)/Y}^{a(\xi)} (\omega Y - K) \kappa(S X_n e^{-\lambda k(\tau-\xi)}, \omega, r, q, \tau - \xi, \sigma^2) d\omega dY \right] d\xi \Big\},$$

with

$$f(SX_n e^{-\lambda(\tau-\xi)}, Ka(\xi), r, q, \tau - \xi, \sigma^2) \equiv S X_n e^{-\lambda k(\tau-\xi)} e^{-(q+\lambda)(\tau-\xi)} N[d_1(SX_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau - \xi, \sigma^2)],$$

and

$$g(SX_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau - \xi, \sigma^2) \equiv e^{-(r+\lambda)(\tau-\xi)} N[d_2(SX_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau - \xi, \sigma^2)].$$

Rearrange $C(S, \tau)$ to obtain the form

$$C(S,\tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ C_{BS}[SX_n e^{-\lambda k \tau}, K, K, r, q, \tau, \sigma^2] \right\} + [A_1^{(1)} - A_2^{(1)}] - J_1 + \{A_1^{(2)} - A_1^{(3)} - (A_2^{(2)} - A_2^{(3)}) + J_2\}.$$

We shall now simplify the linear expression $\{A_1^{(2)} - A_1^{(3)} - (A_2^{(2)} - A_2^{(3)}) + J_2\}$.

A6.2.1. *Simplifying J_2.* Consider the integral

$$I(Y, \xi) = \int_{a(\xi)/Y}^{a(\xi)} (\omega Y - K) \kappa(SX_n e^{-\lambda k(\tau-\xi)}, \omega, r, q, \tau - \xi, \sigma^2) d\omega,$$

where

$$\kappa(SX_n e^{-\lambda k(\tau-\xi)}, \omega, r, q, \tau - \xi, \sigma^2) = \frac{1}{\omega\sigma\sqrt{2\pi(\tau-\xi)}} \exp\left\{ \frac{-[(r - q - \lambda k - \frac{\sigma^2}{2})(\tau - \xi) + \ln \frac{SX_n}{\omega}]^2}{2\sigma^2(\tau - \xi)} \right\}.$$

Making the change of variable $\omega = e^u$, $I(Y,\xi)$ becomes

$$\begin{aligned} I(Y,\xi) &= \frac{1}{\sigma\sqrt{2\pi(\tau-\xi)}} \\ &\quad\times\left[\int_{\ln[a(\xi)/Y]}^{\ln a(\xi)} Ye^u \exp\left\{\frac{-[(r-q-\lambda k-\frac{\sigma^2}{2})(\tau-\xi)+\ln SX_n-u]^2}{2\sigma^2(\tau-\xi)}\right\}du\right. \\ &\quad\left.-\int_{\ln[a(\xi)/Y]}^{\ln a(\xi)} K\exp\left\{\frac{-[(r-q-\lambda k-\frac{\sigma^2}{2})(\tau-\xi)+\ln SX_n-u]^2}{2\sigma^2(\tau-\xi)}\right\}du\right], \end{aligned}$$

which simplifies to

$$\begin{aligned} I(Y,\xi) &= YSX_ne^{-\lambda k(\tau-\xi)}e^{r(\tau-\xi)}e^{-q(\tau-\xi)} \\ &\quad\times\Big\{N\left[-d_1\left(SX_ne^{-\lambda k(\tau-\xi)},a(\xi),r,q,\tau-\xi,\sigma^2\right)\right] \\ &\quad -N\left[-d_1\left(YSX_ne^{-\lambda k(\tau-\xi)},a(\xi),r,q,\tau-\xi,\sigma^2\right)\right]\Big\} \\ &\quad -K\Big\{N\left[-d_2\left(SX_ne^{-\lambda k(\tau-\xi)},a(\xi),r,q,\tau-\xi,\sigma^2\right)\right] \\ &\quad -N\left[-d_2\left(YSX_ne^{-\lambda k(\tau-\xi)},a(\xi),r,q,\tau-\xi,\sigma^2\right)\right]\Big\}. \end{aligned}$$

Using the relationship $N(-x) = 1 - N(x)$, the expression for J_2 becomes

$$\begin{aligned} J_2 &= \lambda\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\Big\{\int_0^{\tau}(\tau-\xi)^n e^{-\lambda(\tau-\xi)} \\ &\quad\times\int_1^{\infty}G(Y)\Big[J_2^{(1)}(Y,\xi,X_n)-J_2^{(2)}(Y,\xi,X_n) \\ &\quad -(J_2^{(3)}(Y,\xi,X_n)-J_2^{(4)}(Y,\xi,X_n))\Big]dYd\xi\Big\}, \end{aligned}$$

where

$$\begin{aligned} J_2^{(1)}(Y,\xi,X_n) &= YSX_ne^{-\lambda k(\tau-\xi)}e^{-q(\tau-\xi)}N\left[d_1\left(YSX_ne^{-\lambda k(\tau-\xi)},a(\xi),r,q,\tau-\xi,\sigma^2\right)\right], \\ J_2^{(2)}(Y,\xi,X_n) &= YSX_ne^{-\lambda k(\tau-\xi)}e^{-q(\tau-\xi)}N\left[d_1\left(SX_ne^{-\lambda k(\tau-\xi)},a(\xi),r,q,\tau-\xi,\sigma^2\right)\right], \\ J_2^{(3)}(Y,\xi,X_n) &= Ke^{-r(\tau-\xi)}N\left[d_2\left(YSX_ne^{-\lambda k(\tau-\xi)},a(\xi),r,q,\tau-\xi,\sigma^2\right)\right] \end{aligned}$$

and

$$J_2^{(4)}(Y,\xi,X_n) = Ke^{-r(\tau-\xi)}N\left[d_2\left(SX_ne^{-\lambda k(\tau-\xi)},a(\xi),r,q,\tau-\xi,\sigma^2\right)\right].$$

A6.2.2. *Simplifying* $A_1^{(2)}$ *and* $A_2^{(2)}$. First recall that

$$k \equiv \int_0^\infty (Y-1)G(Y)dY = \int_0^\infty YG(Y)dY - 1.$$

Substituting this into $A_1^{(2)}$, we have

$$\begin{aligned}
A_1^{(2)} &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ \lambda \left(\int_0^\infty YG(Y)dY \right) \right. \\
&\qquad \left. \times \int_0^\tau (\tau-\xi)^n f(SX_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau-\xi, \sigma^2), d\xi \right\} \\
&= \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ \int_0^\tau (\tau-\xi)^n e^{-\lambda(\tau-\xi)} \int_0^\infty G(Y) Y S X_n e^{-\lambda k(\tau-\xi)} e^{-q(\tau-\xi)} \right. \\
&\qquad \left. \times N[d_1(SX_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau-\xi, \sigma^2)] dY d\xi \right\} \\
&= \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ \int_0^\tau (\tau-\xi)^n e^{-\lambda(\tau-\xi)} \int_0^1 G(Y) J_2^{(2)}(Y,\xi,X_n) dY d\xi \right\} \\
&\qquad + \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ \int_0^\tau (\tau-\xi)^n e^{-\lambda(\tau-\xi)} \int_1^\infty G(Y) J_2^{(2)}(Y,\xi,X_n) dY d\xi \right\}.
\end{aligned}$$

For $A_2^{(2)}$, we note that by the properties of $G(Y)$,

$$\int_0^\infty A_2^{(2)} G(Y)dY = A_2^{(2)},$$

since $A_2^{(2)}$ does not involve Y. Hence $A_2^{(2)}$ can be rewritten as

$$\begin{aligned}
A_2^{(2)} &= \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ \int_0^\tau (\tau-\xi)^n \right. \\
&\qquad \left. \times \int_0^\infty KG(Y) g(SX_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau-\xi, \sigma^2) dY d\xi \right\} \\
&= \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ \int_0^\infty (\tau-\xi)^n e^{-\lambda(\tau-\xi)} \int_0^\infty G(Y) K e^{-r(\tau-\xi)} \right. \\
&\qquad \left. \times N[d_2(SX_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau-\xi, \sigma^2)] dY d\xi \right\} \\
&= \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ \int_0^\tau (\tau-\xi)^n e^{-\lambda(\tau-\xi)} \int_0^1 G(Y) J_2^{(4)}(Y,\xi,X_n) dY d\xi \right\} \\
&\qquad + \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ \int_0^\tau (\tau-\xi)^n e^{-\lambda(\tau-\xi)} \int_1^\infty G(Y) J_2^{(4)}(Y,\xi,X_n) dY d\xi \right\}.
\end{aligned}$$

A6.2.3. *Simplifying* $A_1^{(3)}$ *and* $A_2^{(3)}$. Since the first term in the summations for $A_1^{(3)}$ and $A_2^{(3)}$ are zero[12], we can rewrite these as sums beginning at $n=1$. First, $A_1^{(3)}$ becomes

$$\begin{aligned}
A_1^{(3)} &= \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ n \int_0^\tau (\tau-\xi)^{n-1} f(SX_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau-\xi, \sigma^2) d\xi \right\} \\
&= \lambda \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} \mathbb{E}_{\mathbb{Q}}^{(n-1)} \left\{ \int_0^\infty G(Y) \right. \\
&\qquad \left. \times \int_0^\tau (\tau-\xi)^{n-1} f(YSX_{n-1} e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau-\xi, \sigma^2) d\xi dY \right\} \\
&= \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ \int_0^\tau (\tau-\xi)^n e^{-\lambda(\tau-\xi)} \int_0^\infty G(Y) SX_n Y e^{-\lambda k(\tau-\xi)} e^{-q(\tau-\xi)} \right. \\
&\qquad \left. \times N\left[d_1\left(YSX_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau-\xi, \sigma^2 \right) \right] dY d\xi \right\} \\
&= \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ \int_0^\tau (\tau-\xi)^n e^{-\lambda(\tau-\xi)} \int_0^1 G(Y) J_2^{(1)}(Y, \xi, X_n) dY d\xi \right\} \\
&\quad + \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ \int_0^\tau (\tau-\xi)^n e^{-\lambda(\tau-\xi)} \int_1^\infty G(Y) J_2^{(1)}(Y, \xi, X_n) dY d\xi \right\}.
\end{aligned}$$

Similarly for $A_2^{(3)}$,

$$\begin{aligned}
A_2^{(3)} &= \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ n \int_0^\tau (\tau-\xi)^{n-1} g(SX_n e^{-\lambda(\tau-\xi)}, a(\xi), r, q, \tau-\xi, \sigma^2) d\xi \right\} \\
&= \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ \int_0^\tau (\tau-\xi)^n e^{-\lambda(\tau-\xi)} \int_0^\infty G(Y) K e^{-r(\tau-\xi)} \right. \\
&\qquad \left. \times N\left[d_2\left(YSX_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau-\xi, \sigma^2 \right) \right] dY d\xi \right\} \\
&= \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ \int_0^\tau (\tau-\xi)^n e^{-\lambda(\tau-\xi)} \int_0^1 G(Y) J_2^{(3)}(Y, \xi, X_n) dY d\xi \right\} \\
&\quad + \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ \int_0^\tau (\tau-\xi)^n e^{-\lambda(\tau-\xi)} \int_1^\infty G(Y) J_2^{(3)}(Y, \xi, X_n) dY d\xi \right\}.
\end{aligned}$$

A6.2.4. *Obtaining Equation (25).* Combining the results from Sections A6.2.1-A6.2.3, the integral equation for the American call price becomes

$$C(S,\tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ C_{BS}[SX_n e^{-\lambda k\tau}, K, K, r, q, \tau, \sigma^2] \right\} + [A_1^{(1)} - A_2^{(1)}] - J_1 + \Psi,$$

[12]The first term in each sum is multiplied by n, and the sums begin with $n=0$.

where

$$\Psi \equiv \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \Bigg\{ \int_0^{\tau} (\tau - \xi)^n e^{-\lambda(\tau-\xi)} \times \int_0^1 G(Y) \Big[J_2^{(2)}(Y, \xi, X_n) - J_2^{(1)}(Y, \xi, X_n) - (J_2^{(4)}(Y, \xi, X_n) - J_2^{(3)}(Y, \xi, X_n)) \Big] dY d\xi \Bigg\}.$$

To complete the proof for Proposition 4.6, we re-express the term $[J_2^{(2)} - J_2^{(1)} - (J_2^{(4)} - J_2^{(3)})]$ as an integral over the kernel κ. Comparing the expression for Ψ with the simplification of J_2 in Section A6.2.1, we can readily conclude that

$$\Psi = \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \Bigg\{ \int_0^{\tau} (\tau - \xi)^n e^{-(r+\lambda)(\tau-\xi)} \times \left[\int_0^1 G(Y) \int_{a(\xi)}^{a(\xi)/Y} (\omega Y - K) \kappa(S X_n e^{-\lambda k(\tau-\xi)}, \omega, r, q, \tau - \xi, \sigma^2) d\omega dY \right] d\xi \Bigg\}.$$

Finally, substituting for $A_1^{(1)}$, $A_2^{(1)}$, J and Ψ, the integral equation for $C(S, \tau)$ becomes

$$\begin{aligned} C(S, \tau) = & \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \{ C_{BS}[S X_n e^{-\lambda k \tau}, K, K, r, q, \tau, \sigma^2] \} \\ & + \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \Bigg\{ \int_0^{\tau} (\tau - \xi)^n e^{-\lambda(\tau-\xi)} q S X_n e^{-\lambda k(\tau-\xi)} e^{-q(\tau-\xi)} \\ & \qquad \times N \left[d_1 \left(S X_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau - \xi, \sigma^2 \right) \right] d\xi \Bigg\} \\ & - \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \Bigg\{ \int_0^{\tau} (\tau - \xi)^n e^{-\lambda(\tau-\xi)} r K e^{-r(\tau-\xi)} \\ & \qquad \times N \left[d_2 \left(S X_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau - \xi, \sigma^2 \right) \right] d\xi \Bigg\} \\ & - \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \Bigg\{ \int_0^{\tau} (\tau - \xi)^n e^{-(r+\lambda)(\tau-\xi)} \\ & \qquad \times \int_0^1 G(Y) \int_{a(\xi)}^{a(\xi)/Y} [C(\omega Y, \xi) - (\omega Y - K)] \\ & \qquad \times \kappa(S X_n e^{-\lambda k(\tau-\xi)}, \omega, r, q, \tau - \xi, \sigma^2) d\omega dY d\xi \Bigg\}, \end{aligned}$$

which is the result given in Proposition 3.5.

Appendix 7. Value of the Free Boundary at Expiry

Here we provide a means of deriving the limit of $a(\tau)$ as $\tau \to 0^+$. This derivation is based on the analysis of Wilmott, Dewynne & Howison (1993) for the pure-diffusion American call, and this simple, intuitive method is taken from Chiarella et al. (2004).

Wilmott et al. (1993) demonstrate how to determine the limit of the early exercise boundary by performing a local analysis of the PDE for small time to maturity. Chiarella et al. (2004) demonstrate that this is equivalent to setting the inhomogeneous term in Jamshidian's (1992) form for the PDE to zero, setting $\tau = 0$, $S = a(0^+)$, and solving for the free boundary. Referring to the inhomogeneous IPDE (8), the inhomogeneous term of interest is[13]

$$H_1(S - a(\tau)) \left\{ qS - rK - \lambda \int_0^\infty [C(SY,\tau) - (SY - K)] G(Y) dY \right\}. \qquad (110)$$

Setting (110) equal to zero and evaluating at $\tau = 0$ with $S = a(0^+)$ we have

$$qa(0^+) - rK - \lambda \int_0^\infty [C(a(0^+)Y, 0) - (a(0^+)Y - K)] G(Y) dY = 0. \qquad (111)$$

Given that $C(S,0) = \max(S - K, 0)$, equation (111) becomes

$$qa(0^+) - rK - \lambda \int_0^\infty [\max(a(0^+)Y - K) - (a(0^+)Y - K)] G(Y) dY = 0. \qquad (112)$$

Since $a(\tau) \geq K$ for all $\tau \geq 0$ the integral term is zero for $Y \geq K/a(0^+)$, and hence

$$qa(0^+) - rK + \lambda \int_0^{K/a(0^+)} (a(0^+)Y - K) G(Y) dY = 0, \qquad (113)$$

which we can rearrange to give

$$a(0^+) = K \frac{r + \lambda \int_0^{K/a(0^+)} G(Y) dY}{q + \lambda \int_0^{K/a(0^+)} Y G(Y) dY}. \qquad (114)$$

By noting again that $a(\tau) \geq K$ must hold for all $\tau \geq 0$, we arrive at the result in Proposition 5.1.

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[13]Note that since $C(SY,\tau) = SY - K$ for $Y \geq a(\tau)/S$, we have

$$\int_0^{a(\tau)/S} [C(SY,\tau) - (SY - K)] dY = \int_0^\infty [C(SY,\tau) - (SY - K)] dY.$$

.

A7.1. **Alternative Derivation: Kim's Method.** An alternative approach to the derivation of the limit (50) is given by Kim (1990), in which he takes the limit of the integral equation for the free boundary as $\tau \to 0^+$. Here we shall extend this method to the jump-diffusion case, and thereby use equation (26) to determine $a(0^+)$.

We recall that $a(\tau)$ is the solution to

$$\begin{aligned} a(\tau) - K \quad = \quad & \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \{C_{BS}[a(\tau)X_n e^{-\lambda k\tau}, K, K, r, q, \tau, \sigma^2]\} \qquad (115) \\ & + \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \bigg\{ \int_0^{\tau} (\tau-\xi)^n e^{-\lambda(\tau-\xi)} q a(\tau) X_n e^{-\lambda k(\tau-\xi)} e^{-q(\tau-\xi)} \\ & \qquad \times N\left[d_1\left(a(\tau)X_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau-\xi, \sigma^2\right)\right] d\xi \bigg\} \\ & - \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \bigg\{ \int_0^{\tau} (\tau-\xi)^n e^{-\lambda(\tau-\xi)} r K e^{-r(\tau-\xi)} \\ & \qquad \times N\left[d_2\left(a(\tau)X_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau-\xi, \sigma^2\right)\right] d\xi \bigg\} \\ & - \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \bigg\{ \int_0^{\tau} (\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \\ & \qquad \times \int_0^1 G(Y) \int_{a(\xi)}^{a(\xi)/Y} [C(\omega Y, \xi) - (\omega Y - K)] \\ & \qquad \times \kappa(a(\tau)X_n e^{-\lambda k(\tau-\xi)}, \omega, r, q, \tau-\xi, \sigma^2) d\omega dY d\xi \bigg\}, \end{aligned}$$

Following Kim (1990), we factorise (115) to produce

$$a(\tau) - K = \sum_{n=0}^{\infty} \left[a(\tau) f_n^{(1)}(a(\tau), \tau) - \left(f_n^{(2)}(a(\tau), \tau) - g_n(a(\tau), \tau) \right) \right], \qquad (116)$$

where

$$\begin{aligned} f_n^{(1)}(a(\tau), \tau) \quad \equiv \quad & \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ X_n e^{-\lambda k\tau} e^{-q\tau} N[d_1(a(\tau)X_n e^{-\lambda k\tau}, K, r, q, \tau, \sigma^2(\tau))] \right\} \\ & + \frac{\lambda^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \bigg\{ \int_0^{\tau} (\tau-\xi)^n e^{-\lambda(\tau-\xi)} q X_n e^{-\lambda k(\tau-\xi)} e^{-q(\tau-\xi)} \\ & \qquad \times N[d_1(a(\tau)X_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau-\xi, \sigma^2)] d\xi \bigg\}, \end{aligned}$$

$$\begin{aligned} f_n^{(2)}(a(\tau),\tau) &\equiv \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\left\{Ke^{-r\tau}N[d_2(a(\tau)X_ne^{-\lambda k\tau},K,r,q,\tau,\sigma^2(\tau))]\right\} \\ &\quad +\frac{\lambda^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\bigg\{\int_0^\tau(\tau-\xi)^ne^{-\lambda(\tau-\xi)}rKe^{-r(\tau-\xi)} \\ &\qquad \times N[d_2(a(\tau)X_ne^{-\lambda k(\tau-\xi)},a(\xi),r,q,\tau-\xi,\sigma^2)]d\xi\bigg\}, \end{aligned}$$

and

$$\begin{aligned} g_n(a(\tau),\tau) &\equiv \frac{\lambda^{n+1}}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\bigg\{\int_0^\tau(\tau-\xi)^ne^{-(r+\lambda)(\tau-\xi)} \\ &\qquad \times\int_0^1G(Y)\int_{a(\xi)}^{a(\xi)/Y}[C(\omega Y,\xi)-(\omega Y-K)] \\ &\qquad \times\kappa(a(\tau)X_ne^{-\lambda k(\tau-\xi)},\omega,r,q,\tau-\xi,\sigma^2)d\omega dYd\xi\bigg\}, \end{aligned}$$

where we make use of the definition of C_{BS} from Proposition 3.3, and note that all other necessary definitions can be found in Proposition 3.3, with the exception of the kernel, κ, which is given by equation (24).

Rearranging the factorised expression (116), and taking the limit as $\tau\to0^+$ we have

$$a(0^+)=\lim_{\tau\to0^+}\frac{K-\sum_{n=0}^\infty\left[f_n^{(2)}(a(\tau),\tau)+g_n(a(\tau),\tau)\right]}{1-\sum_{n=0}^\infty f_n^{(1)}(a(\tau),\tau)}. \tag{117}$$

We now seek to evaluate the limit in (117). Since we know that $a(\tau)\geq K$ for all $\tau\geq0$, we shall consider separately the cases $a(0^+)=K$ and $a(0^+)>K$.

Firstly, for the $a(0^+)=K$ case, we can readily show that

$$\lim_{\tau\to0^+}\sum_{n=0}^\infty f_n^{(1)}(a(\tau),\tau)=N[0]=\frac{1}{2},$$

$$\lim_{\tau\to0^+}\sum_{n=0}^\infty f_n^{(2)}(a(\tau),\tau)=N[0]=\frac{1}{2},$$

and

$$\lim_{\tau\to0^+}\sum_{n=0}^\infty g_n(a(\tau),\tau)=0.$$

The last of these limits follows from the fact that the integrand is well behaved for small values of τ, as we know that the option price is finite near expiry for a given value of S.

Thus we have

$$\lim_{\tau\to 0^+} a(\tau) = K\frac{1-\frac{1}{2}}{1-\frac{1}{2}} = K,$$

and $a(0^+) = K$ satisfies equation (117), making this one possible solution for the free boundary at expiry.

Secondly, when $a(0^+) > K$, we now find that

$$\lim_{\tau\to 0^+} \sum_{n=0}^{\infty} f_n^{(1)}(a(\tau),\tau) = N[\infty] = 1;$$

$$\lim_{\tau\to 0^+} \sum_{n=0}^{\infty} f_n^{(2)}(a(\tau),\tau) = N[\infty] = 1;$$

$$\lim_{\tau\to 0^+} \sum_{n=0}^{\infty} g_n(a(\tau),\tau) = 0.$$

Thus

$$\lim_{\tau\to 0^+} a(\tau) = \frac{1-1}{1-1} = \frac{0}{0},$$

an indeterminate form which can be resolved using L'Hopitals's rule. Applying L'Hopital's rule, we find that

$$\begin{aligned} a(0^+) &= \lim_{\tau\to 0} \frac{-\frac{\partial}{\partial\tau}\left(\sum_{n=0}^{\infty}[f_n^{(2)}(a(\tau),\tau) - g_n(a(\tau),\tau)]\right)}{-\frac{\partial}{\partial\tau}\left(\sum_{n=0}^{\infty} f_n^{(1)}(a(\tau),\tau)\right)} \\ &= \lim_{\tau\to 0} \frac{\frac{\partial}{\partial\tau}f_0^{(2)}(a(\tau),\tau) + \frac{\partial}{\partial\tau}g_0(a(\tau),\tau) + \sum_{n=1}^{\infty}\left[\frac{\partial}{\partial\tau}f_n^{(2)}(a(\tau),\tau) + \frac{\partial}{\partial\tau}g_n(a(\tau),\tau)\right]}{\frac{\partial}{\partial\tau}f_0^{(1)}(a(\tau),\tau) + \sum_{n=1}^{\infty}\frac{\partial}{\partial\tau}f_n^{(1)}(a(\tau),\tau)}. \end{aligned} \tag{118}$$

We now consider the six linear terms within (118) individually, finding limits for each. For the first term[14] in the denominator,

$$\begin{aligned}\frac{\partial}{\partial\tau}f_0^{(1)}(a(\tau),\tau) &= e^{-(\lambda[1+k]+q)\tau}N'[d_1(a(\tau)e^{-\lambda k\tau},K,r,q,\tau,\sigma^2)]\\ &\quad\times\frac{\partial}{\partial\tau}[d_1(a(\tau)e^{-\lambda k\tau},K,r,q,\tau,\sigma^2)]\\ &\quad-(\lambda[1+k]+q)e^{-(\lambda[1+k]+q)\tau}N[d_1(a(\tau)e^{-\lambda k\tau},K,r,q,\tau,\sigma^2)]\\ &\quad+qN[0]\int_0^\tau\frac{\partial}{\partial\tau}\Big[qe^{-(\lambda[1+k]+q)(\tau-\xi)}\\ &\qquad\times N[d_1(a(\tau)e^{-\lambda k(\tau-\xi)},a(\xi),r,q,\tau-\xi,\sigma^2)]\Big]d\xi.\end{aligned}$$

Hence taking the limit we have

$$\begin{aligned}\lim_{\tau\to0^+}\frac{\partial}{\partial\tau}f_0^{(1)}(a(\tau),\tau) &= -(\lambda[k+1]+q)N[\infty]+\frac{q}{2}\\ &= -\left(\lambda[1+k]+\frac{q}{2}\right).\end{aligned}$$

Next we consider $f_n^{(1)}$ for $n\geq 1$. We have

$$\begin{aligned}\frac{\partial}{\partial\tau}f_n^{(1)}(a(\tau),\tau) &= \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\Bigg\{X_ne^{-(q+\lambda k)\tau}N'[d_1(a(\tau)X_ne^{-\lambda k\tau},K,r,q,\tau,\sigma^2)]\\ &\quad\times\frac{\partial}{\partial\tau}[d_1(a(\tau)X_ne^{-\lambda k\tau},K,r,q,\tau,\sigma^2)]\\ &\qquad-X_n(q+\lambda k)e^{-(q+\lambda k)\tau}N[d_1(a(\tau)X_ne^{-\lambda k\tau},K,r,q,\tau,\sigma^2)]\Bigg\}\\ &\quad+\mathbb{E}_{\mathbb{Q}}^{(n)}\left\{X_ne^{-(q+\lambda k)\tau}N[d_1(a(\tau)X_ne^{-\lambda k\tau},K,r,q,\tau,\sigma^2)]\right\}\\ &\quad\times\frac{1}{n!}[e^{-\lambda\tau}n(\lambda'\tau)^{n-1}\lambda'-(\lambda\tau)^ne^{-\lambda\tau}(\lambda)]\\ &\quad+\frac{\lambda^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\Bigg\{\int_0^\tau\frac{\partial}{\partial\tau}\{(\tau-\xi)^ne^{-(\lambda[k+1]+q)(\tau-\xi)}qX_n\\ &\quad\times N[d_1(a(\tau)X_ne^{-\lambda k(\tau-\xi)},a(\xi),r,q,\tau-\xi,\sigma^2)]\}d\xi\Bigg\}.\end{aligned}$$

We can safely infer that the integral term will tend to zero as $\tau\to0^+$ for all applicable n values, since all terms under the integral sign are bounded. When $n>1$ it is clear

[14]We recall that $N'(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$

that

$$\lim_{\tau\to 0^+}\frac{\partial}{\partial\tau}f_n^{(1)}(a(\tau),\tau)=0.$$

For $n=1$ the result is more complicated, as the derivative of $f_1^{(1)}$ with respect to τ is given by

$$\frac{\partial}{\partial\tau}f_1^{(1)}(a(\tau),\tau)=\int_0^\infty G(Y)Y\lambda N[d_1(a(\tau)Ye^{-\lambda k\tau},K,r,q,\tau,\sigma^2)]dY.$$

Since

$$\lim_{\tau\to 0^+}N[d_1(a(\tau)Ye^{-\lambda k\tau},K,r,q,\tau,\sigma^2)]=H_2(Ya(0^+)-K),$$

where $H_2(x)$ is the Heaviside step function given by (9), we find that

$$\begin{aligned}\lim_{\tau\to 0^+}\frac{\partial}{\partial\tau}f_1^{(1)}(a(\tau),\tau) &= \lambda\int_{K/a(0^+)}^\infty YG(Y)dY\\ &= \lambda[k+1]-\lambda\int_0^{K/a(0^+)}YG(Y)dY.\end{aligned}$$

The next term we consider is

$$\begin{aligned}\frac{\partial}{\partial\tau}f_0^{(2)}(a(\tau),\tau) &= Ke^{-(r+\lambda)\tau}N'[d_2(a(\tau)e^{-\lambda k\tau},K,r,q,\tau,\sigma^2)]\\ &\qquad\times\frac{\partial}{\partial\tau}[d_2(a(\tau)e^{-\lambda k\tau},K,r,q,\tau,\sigma^2)]\\ &\quad -K(r+\lambda)e^{-(r+\lambda)\tau}N[d_2(a(\tau)e^{-\lambda k\tau},K,r,q,\tau,\sigma^2)]\\ &\quad +rKN[0]\int_0^\tau\frac{\partial}{\partial\tau}\Big[rKe^{-(r+\lambda)(\tau-\xi)}\\ &\qquad\qquad\times N[d_2(a(\tau)e^{-\lambda k(\tau-\xi)},a(\xi),r,q,\tau-\xi,\sigma^2)]\Big]d\xi.\end{aligned}$$

We find that

$$\begin{aligned}\lim_{\tau\to 0^+}\frac{\partial}{\partial\tau}f_0^{(2)}(a(\tau),\tau) &= -K(r+\lambda)N[\infty]+K\frac{r}{2}\\ &= -K\left(\lambda+\frac{r}{2}\right).\end{aligned}$$

In the case where $n \geq 1$ we have the term

$$\begin{aligned}
\frac{\partial}{\partial\tau} f_n^{(2)}(a(\tau),\tau) &= \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\Bigg\{Ke^{-r\tau}N'[d_2(a(\tau)X_n e^{-\lambda k\tau},K,r,q,\tau,\sigma^2)] \\
&\qquad \times\frac{\partial}{\partial\tau}[d_2(a(\tau)X_n e^{-\lambda k\tau},K,r,q,\tau,\sigma^2)] \\
&\qquad\qquad -rKe^{-r\tau}N[d_2(a(\tau)X_n e^{-\lambda k\tau},K,r,q,\tau,\sigma^2)]\Bigg\} \\
&\quad +\mathbb{E}_{\mathbb{Q}}^{(n)}\left\{Ke^{-r\tau}N[d_2(a(\tau)X_n e^{-\lambda k\tau},K,r,q,\tau,\sigma^2)]\right\} \\
&\qquad \times\frac{1}{n!}[e^{-\lambda\tau}n(\lambda'\tau)^{n-1}\lambda' - (\lambda\tau)^n e^{-\lambda\tau}(\lambda)] \\
&\quad +\frac{\lambda^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\Bigg\{\int_0^\tau \frac{\partial}{\partial\tau}\{(\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)}rK \\
&\qquad \times N[d_2(a(\tau)X_n e^{-\lambda k(\tau-\xi)},a(\xi),r,q,\tau-\xi,\sigma^2)]\}d\xi\Bigg\}.
\end{aligned}$$

As with $f_n^{(1)}$, we readily find that for $n > 1$ we have

$$\lim_{\tau\to 0^+}\frac{\partial}{\partial\tau} f_n^{(2)}(a(\tau),\tau) = 0,$$

and when $n = 1$ we obtain

$$\frac{\partial}{\partial\tau} f_1^{(2)}(a(\tau),\tau) = \int_0^\infty G(Y)K\lambda N[d_2(a(\tau)Ye^{-\lambda k\tau},K,r,q,\tau,\sigma^2)]dY.$$

Since

$$\lim_{\tau\to 0^+} N[d_2(a(\tau)Ye^{-\lambda k\tau},K,r,q,\tau,\sigma^2)] = H_2(Ya(0^+)-K),$$

we conclude that

$$\begin{aligned}
\lim_{\tau\to 0^+}\frac{\partial}{\partial\tau} f_1^{(2)}(a(\tau),\tau) &= \lambda K\int_{K/a(0^+)}^\infty G(Y)dY \\
&= \lambda K - \lambda K\int_0^{K/a(0^+)} G(Y)dY.
\end{aligned}$$

Having found the limits for the $f_n^{(1)}$ and $f_n^{(2)}$, we now consider the g_n terms. Firstly, for $n = 0$ we have

$$\begin{aligned}\frac{\partial}{\partial\tau}g_0(a(\tau),\tau) &= \lim_{\alpha\to 0}\lambda\int_0^1 G(Y)\int_{a(\tau)}^{a(\tau)/Y}[C(\omega Y,\tau)-(\omega Y-K)] \\ &\qquad\times\kappa(a(\tau)Y,\omega,r,q,\alpha,\sigma^2)d\omega dY \\ &\quad+\lambda\int_0^\tau\frac{\partial}{\partial\tau}\Bigg\{e^{-(r+\lambda)(\tau-\xi)}\int_0^1 G(Y)\int_{a(\xi)}^{a(\xi)/Y}[C(\omega Y,\xi)-(\omega Y-K)] \\ &\qquad\times\kappa(a(\tau)Ye^{-\lambda k(\tau-\xi)},\omega,r,q,\tau-\xi,\sigma^2)d\omega dY\Bigg\}d\xi.\end{aligned}$$

Since the integral with respect to ξ will be finite as $\tau\to 0^+$, we find that

$$\begin{aligned}\lim_{\tau\to 0^+}\frac{\partial}{\partial\tau}g_0(a(\tau),\tau) &= \lim_{\alpha\to 0}\lambda\int_0^1 G(Y)\int_{a(0^+)}^{a(0^+)/Y}[\max(\omega Y-K,0)-(\omega Y-K)] \\ &\qquad\times\kappa(a(0^+)Y,\omega,r,q,\alpha,\sigma^2)d\omega dY.\end{aligned}$$

This limit will only be non-zero when $\omega < K/Y$. Since we know that $a(0+) > K$, the upper limit for the integral with respect to ω can be replaced by K/Y. Similarly, we require that $Y < K/\omega$ for the limit to be non-zero. Thus when $\omega = a(0^+)$ we require that $Y < K/a(0^+) < 1$, and this provides us with a new upper limit for the integral with respect to Y. Using these new integration limits, we have

$$\begin{aligned}\lim_{\tau\to 0^+}\frac{\partial}{\partial\tau}g_0(a(\tau),\tau) &= \lim_{\alpha\to 0}\lambda\int_0^{K/a(0^+)} G(Y)\int_{a(0^+)}^{K/Y}(K-\omega Y) \\ &\qquad\times\kappa(a(0^+)Y,\omega,r,q,\alpha,\sigma^2)d\omega dY \qquad (119)\\ &= \lambda\int_0^{K/a(0^+)}G(Y)[I_1(Y)-I_2(Y)]dY, \qquad (120)\end{aligned}$$

where we define

$$I_1(Y)\equiv\lim_{\alpha\to 0}\int_{a(0^+)}^{K/Y}\frac{K}{\omega\sqrt{2\pi\alpha}}\exp\left\{-\frac{[\ln a(0^+)-\ln\omega+(r-q-\frac{\sigma^2}{2})\alpha]^2}{2\sigma^2\alpha}\right\}d\omega,$$

and

$$I_2(Y)\equiv\lim_{\alpha\to 0}\int_{a(0^+)}^{K/Y}\frac{\omega Y}{\omega\sqrt{2\pi\alpha}}\exp\left\{-\frac{[\ln a(0^+)-\ln\omega+(r-q-\frac{\sigma^2}{2})\alpha]^2}{2\sigma^2\alpha}\right\}d\omega.$$

We can readily express I_1 and I_2 in terms of $N[\cdot]$, and find that

$$\begin{aligned} I_1(Y) &= \lim_{\alpha\to 0} K\left\{N\left[\frac{\ln(K/a(0^+)Y)-(r-q-\frac{\sigma^2}{2})\alpha}{\sigma\sqrt{\alpha}}\right]-N\left[-\frac{(r-q-\frac{\sigma^2}{2})\sqrt{\alpha}}{\sigma}\right]\right\} \\ &= K\left\{H_2(K-Ya(0^+))-\frac{1}{2}\right\}, \end{aligned}$$

and

$$\begin{aligned} I_2(Y) &= \lim_{\alpha\to 0} a(0^+)e^{(r-q)\alpha} \\ &\quad\times\left\{N\left[\frac{\ln(K/a(0^+)Y)-(r-q+\frac{\sigma^2}{2})\alpha}{\sigma\sqrt{\alpha}}\right]-N\left[-\frac{(r-q+\frac{\sigma^2}{2})\sqrt{\alpha}}{\sigma}\right]\right\} \\ &= a(0^+)\left\{H_2(K-Ya(0^+))-\frac{1}{2}\right\}. \end{aligned}$$

By substituting these expressions for I_1 and I_2 into (120) we find that

$$\begin{aligned} \lim_{\tau\to 0^+}\frac{\partial}{\partial\tau}g_0(a(\tau),\tau) &= \lambda\int_0^{K/a(0^+)} G(Y)[K-a(0^+)Y]\left\{H_2(K-Ya(0^+))-\frac{1}{2}\right\}dY \\ &= \frac{\lambda}{2}\left[K\int_0^{K/a(0^+)} G(Y)dY - a(0^+)\int_0^{K/a(0^+)} YG(Y)dY\right]. \end{aligned}$$

Lastly, we consider g_n for $n \geq 1$. The derivative with respect to τ is

$$\begin{aligned} \frac{\partial}{\partial\tau}g_n(a(\tau),\tau) &= \frac{\lambda^{n+1}}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\Bigg\{\int_0^\tau \frac{\partial}{\partial\tau}\Bigg[(\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \\ &\qquad\times\int_0^1 G(Y)\int_{a(\xi)}^{a(\xi)/Y}[C(\omega Y,\xi)-(\omega Y-K)] \\ &\qquad\times\kappa(a(\tau)Y,\omega,r,q,\tau-\xi,\sigma^2)d\omega dY\Bigg]d\xi\Bigg\}, \end{aligned}$$

whose limit is

$$\lim_{\tau\to 0^+}\frac{\partial}{\partial\tau}g_n(a(\tau),\tau) = 0.$$

Combining these six limits we find that

$$a(0^+) = \frac{K\left(r+\lambda\int_0^{K/a(0^+)}G(Y)dY\right)+a(0^+)\lambda\int_0^{K/a(0^+)}YG(Y)dY}{q+2\lambda\int_0^{K/a(0^+)}YG(Y)dY},$$

which can be be rearranged to give

$$a(0^+) = K \frac{r + \lambda \int_0^{K/a(0^+)} G(Y)dY}{q + \lambda \int_0^{K/a(0^+)} YG(Y)dY}.$$

Finally, since $a(0^+) \geq K$, we conclude that

$$a(0^+) = K \max\left(1, \frac{r + \lambda \int_0^{K/a(0^+)} G(Y)dY}{q + \lambda \int_0^{K/a(0^+)} YG(Y)dY}\right).$$

which is the result given in Proposition 5.1.

Appendix 8. American Call Evaluation for Log-Normal Jump Sizes

Consider the case where $G(Y)$ is given by

$$G(Y) = \frac{1}{Y\delta\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\ln Y - (\gamma - \frac{\delta^2}{2})}{\delta}\right)^2\right\},$$

which subsequently implies that

$$\mathbb{E}_{\mathbb{Q}}^{(n)}\{f(X_n)\} = \int_0^\infty f(X_n) \frac{1}{X_n\delta\sqrt{2\pi n}} \exp\left\{-\frac{1}{2}\left(\frac{\ln X_n - n(\gamma - \frac{\delta^2}{2})}{\delta\sqrt{n}}\right)^2\right\} dX_n.$$

We shall use this to evaluate all of the $\mathbb{E}_{\mathbb{Q}}^{(n)}$ operators in equation (25).

A8.1. **European Component.** Using the results from Merton (1976), the European component becomes

$$\begin{aligned} &\sum_{n=0}^\infty \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)}\left\{C_{BS}[SX_n e^{-\lambda k\tau}, K, K, r, q, \tau, \sigma^2]\right\} \\ &= \sum_{n=0}^\infty \frac{e^{-\lambda'\tau}(\lambda'\tau)^n}{n!} C_{BS}[S, K, K, r_n(\tau), q, \tau, v_n^2(\tau)], \end{aligned}$$

where $\lambda' = \lambda(1+k)$, $r_n(\tau) = r - \lambda k + n\gamma/\tau$, and $v_n^2(\tau) = \sigma^2 + n\delta^2/\tau$, with C_{BS} as defined in Proposition 3.3.

A8.2. **Early Exercise Premium - First Term.** Consider the first part of the early exercise premium, given by

$$C_P^{(1)}(S,\tau) = \sum_{n=0}^{\infty}\left\{\int_0^{\tau}\frac{e^{-\lambda(\tau-\xi)}[\lambda(\tau-\xi)]^n}{n!}\right.$$
$$\left.\times\mathbb{E}_{\mathbb{Q}}^{(n)}\{C_P^{(D)}[SX_ne^{-\lambda k(\tau-\xi)},K,a(\xi),r,r,q,\tau-\xi,\sigma^2]\}d\xi\right\}.$$

After referring to equations (22) and (54) for the definitions of $C_P^{(D)}$ and $G(Y)$ respectively, we can show that

$$\mathbb{E}_{\mathbb{Q}}^{(n)}\{X_nN[d_1(SX_ne^{-\lambda k(\tau-\xi)},a(\xi),r,q,\tau-\xi,\sigma^2)]\}$$
$$= e^{n\gamma}N[d_1(S,a(\xi),r_n(\tau-\xi),q,\tau-\xi,v_n^2(\tau-\xi))],$$

and

$$\mathbb{E}_{\mathbb{Q}}^{(n)}\{N[d_2(SX_ne^{-\lambda k(\tau-\xi)},a(\xi),r,q,\tau-\xi,\sigma^2)]\}$$
$$= N[d_2(S,a(\xi),r_n(\tau-\xi),q,\tau-\xi,v_n^2(\tau-\xi))].$$

Noting that $e^{n\gamma}=(k+1)^n$, $C_P^{(1)}$ becomes

$$C_P^{(1)}(S,\tau) = \sum_{n=0}^{\infty}\left\{\int_0^{\tau}\frac{e^{-\lambda'(\tau-\xi)}[\lambda'(\tau-\xi)]^n}{n!}\right.$$
$$\left.\times C_P^{(D)}[S,K,a(\xi),r,r_n(\tau-\xi),q,\tau-\xi,v_n^2(\tau-\xi)]d\xi\right\}.$$

A8.3. **Cost Term from Downward Jumps.** The final term to consider is the cost incurred when S jumps from the stopping region into the continuation region. This term is given by

$$C_P^{(2)}(S,\tau) = \lambda\sum_{n=0}^{\infty}\left\{\int_0^{\tau}\frac{e^{-\lambda(\tau-\xi)}[\lambda(\tau-\xi)]^n}{n!}\right.$$
$$\left.\times\mathbb{E}_{\mathbb{Q}}^{(n)}\{C_P^{(J)}[SX_ne^{-\lambda k(\tau-\xi)},K,a(\xi),r,q,\tau-\xi,\sigma^2;C(\cdot,\xi)]\}d\xi\right\}.$$

Referring to (23) and (24) for the definitions of $C_P^{(J)}$ and κ respectively, we find that in order to evaluate the the $\mathbb{E}_{\mathbb{Q}}^{(n)}$ operator, we must consider the expectation

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}^{(n)}\{\kappa(SX_n e^{-\lambda k(\tau-\xi)},\omega,r,q,\tau-\xi,\sigma^2)\}\\&\qquad=\int_0^\infty \frac{1}{X_n\delta\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\frac{[\ln X_n-n(\gamma-\frac{\delta^2}{2})]^2}{\delta^2 n}\right\}\frac{1}{\omega\sigma\sqrt{2\pi(\tau-\xi)}}\\&\qquad\quad\times\exp\left\{\frac{-[(r-q-\lambda k-\frac{\sigma^2}{2})(\tau-\xi)+\ln\frac{SX_n}{\omega}]^2}{2\sigma^2(\tau-\xi)}\right\}dX_n.\end{aligned}$$

Making the change of variable $x_n=\ln X_n$, this expectation evaluates to

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}^{(n)}\{\kappa(SX_n e^{-\lambda k(\tau-\xi)},\omega,r,q,\tau-\xi,\sigma^2)\}\\&=\frac{1}{\omega\sqrt{2\pi(\tau-\xi)v_n^2(\tau-\xi)}}\\&\qquad\times\exp\left\{-\frac{[\ln\frac{S}{\omega}+(r_n(\tau-\xi)-q-\frac{v_n^2(\tau-\xi)}{2})(\tau-\xi)]^2}{2v_n^2(\tau-\xi)(\tau-\xi)}\right\}.\end{aligned}$$

Finally, using the definitions for λ' and $r_n(\tau)$ we can rewrite $C_P^{(3)}$ as

$$\begin{aligned}C_P^{(2)}(S,\tau) &= \lambda\sum_{n=0}^{\infty}\left\{\int_0^\tau \frac{e^{-\lambda'(\tau-\xi)}[\lambda'(\tau-\xi)]^n}{n!}\right.\\&\qquad\left.\times C_P^{(J)}[S,K,a(\xi),r_n(\tau-\xi),q,\tau-\xi,v_n^2(\tau-\xi);C(\cdot,\xi)]\Big]d\xi\right\}.\end{aligned}$$

A8.4. **Final Result - Proposition 6.1.** Combining the results from sections A8.1-A8.3, we find that the integral equation for $C(S,\tau)$ in the case of log-normal jumps is given by

$$C(S,\tau)=\sum_{n=0}^{\infty}\frac{e^{-\lambda'\tau}(\lambda'\tau)^n}{n!}C_{BS}[S,K,K,r_n(\tau),q,\tau,V_n^2(\tau)]+C_P^{(1)}(S,\tau)-C_P^{(2)}(S,\tau),$$

which is equation (55) in Proposition 6.1.

Appendix 9. The Simplified Cost Term for Log-Normal Jump Sizes

Referring to the result in Proposition 3.6, the integral term in equation (25) that we seek to evaluate is

$$\begin{aligned} I(S,z,\tau,\xi) &\equiv \mathbb{E}_{\mathbb{Q}}^{(n)}\left\{\int_0^z G(Y)\kappa(SYX_n e^{-\lambda k(\tau-\xi)}/a(\xi),z,r,q,\tau-\xi,\sigma^2)\right\}dY \\ &= \frac{1}{\delta\sqrt{2\pi}}\int_0^z \frac{1}{Y}\exp\left\{-\frac{1}{2}\left[\frac{\ln Y-(\gamma-\frac{\delta^2}{2})}{\delta}\right]^2\right\}J(S,z,\tau,\xi,Y)dY, \end{aligned}$$

where

$$\begin{aligned} J(S,z,\tau,\xi,Y) &\equiv \frac{1}{z\sigma\sqrt{2\pi(\tau-\xi)}}\frac{1}{\delta\sqrt{2\pi n}}\int_0^\infty \frac{1}{X_n}\exp\left\{-\frac{1}{2}\left[\frac{\ln X_n-n(\gamma-\frac{\delta^2}{2})}{\delta\sqrt{n}}\right]^2\right\} \\ &\quad\times\exp\left\{-\frac{1}{2}\left[\frac{\ln\frac{SYX_n}{a(\xi)z}+(r-q-\lambda k-\frac{\sigma^2}{2})(\tau-\xi)}{\sigma\sqrt{\tau-\xi}}\right]^2\right\}dX_n. \end{aligned}$$

To evaluate $I(S,z,\tau,\xi)$ we need to make use of the following integration result. Let α_1, α_2, β_1, β_2 and z be real-valued functions independent of the integration variable ω. Then by completing the square in the exponent, we can prove that

$$\begin{aligned} &\int_0^z \frac{1}{\omega}\exp\left\{-\frac{[\ln\omega+\beta_1]^2}{\alpha_1}-\frac{[\ln\omega+\beta_2]^2}{\alpha_2}\right\}d\omega \\ &\qquad= \sqrt{\frac{\alpha_1\alpha_2\pi}{\alpha_1+\alpha_2}}\exp\left\{-\frac{(\beta_1-\beta_2)^2}{\alpha_1+\alpha_2}\right\}N[f(z)], \end{aligned} \tag{121}$$

where $N[\cdot]$ is the cumulative normal density function, and

$$f(z)=\sqrt{\frac{2}{\alpha_1\alpha_2}}\left(\frac{(\alpha_1+\alpha_2)\ln z+\alpha_1\beta_2+\alpha_2\beta_2}{\sqrt{\alpha_1+\alpha_2}}\right).$$

Applying (121) to $J(S,z,\tau,\xi)$ we find that

$$\begin{aligned} J(S,z,\tau,\xi,Y) &= \frac{1}{zv_n(\tau-\xi)\sqrt{2\pi(\tau-\xi)}} \\ &\qquad\times\exp\left\{-\frac{[\ln\frac{SY}{a(\xi)z}+(r_n(\tau-\xi)-q-\frac{v_n^2(\tau-\xi)}{2})(\tau-\xi)]^2}{2v_n^2(\tau-\xi)(\tau-\xi)}\right\}, \end{aligned}$$

where $r_n(\tau)$ and $v_n(\tau)$ are given by Proposition 6.1, and thus $I(S,z,\tau,\xi)$ becomes

$$\begin{aligned} I(S,z,\tau,\xi) &= \frac{1}{zv_n(\tau-\xi)\sqrt{2\pi(\tau-\xi)}} \\ &\quad \times \frac{1}{\delta\sqrt{2\pi}}\int_0^z \frac{1}{Y}\exp\left\{-\frac{[\ln Y-(\gamma-\frac{\delta^2}{2})]^2}{2\delta^2}\right\} \\ &\quad \times\exp\left\{-\frac{[\ln\frac{SY}{a(\xi)z}+(r_n(\tau-\xi)-q-\frac{v_n^2(\tau-\xi)}{2})(\tau-\xi)]^2}{2v_n^2(\tau-\xi)(\tau-\xi)}\right\} dY. \end{aligned}$$

Finally, we again apply (121) to $I(S,z,\tau,\xi)$ and obtain

$$\begin{aligned} I(S,z,\tau,\xi) &= \frac{1}{zv_{n+1}(\tau-\xi)\sqrt{2\pi(\tau-\xi)}}N[f(z)] \\ &\quad \times\exp\left\{-\frac{[\ln\frac{S}{a(\xi)z}+(r_{n+1}(\tau-\xi)-q-\frac{v_{n+1}^2(\tau-\xi)}{2})(\tau-\xi)]^2}{2v_{n+1}^2(\tau-\xi)(\tau-\xi)}\right\}, \end{aligned}$$

where

$$\begin{aligned} f(z) &= \frac{\delta^2\ln\frac{S}{za(\xi)}+\left[(\ln z)v_{n+1}^2(\tau-\xi)+\delta^2[r_n(\tau-\xi)-q]-\gamma v_n^2(\tau-\xi)\right](\tau-\xi)}{v_n(\tau-\xi)v_{n+1}(\tau-\xi)\delta(\tau-\xi)} \\ &\equiv D(S/a(\xi),z,r_n(\tau-\xi),q,v_n(\tau-\xi),v_{n+1}(\tau-\xi),\tau-\xi,\gamma,\delta). \end{aligned}$$

Substituting for $I(S,z,\tau,\xi)$ into (25) and combining this with the results in Proposition 6.1, we arrive at equation (56) of Proposition 6.2.

Appendix 10. Algorithm for Evaluating the American Call Option under Jump-Diffusion

Here we present the algorithm *American Call - Integration* which outlines the key steps in the proposed numerical integration scheme, presented in Section 7, for evaluating the price, delta and free boundary of an American call option under jump-diffusion with log-normal jump sizes.

Algorithm *American Call - Integration*

Input: S, r, q, σ, K, T (time to expiry), λ, γ, δ, TOL_n (tolerance for Poisson coefficients), N (number of time intervals), TOL_a (tolerance for Newton's method).

Output: C (American call price), Δ_C (American call delta), a (early exercise boundary).

1. use TOL_n to find the maximum number of terms needed for the infinite sums
2. solve equation (66) for a_0
3. $C_0(S) = \max(S - K, 0)$
4. **for** $i = 1$ **to** N
5. **do** let $C_i(S) = C_{i-1}(S)$ in right-hand side of (58)
6. set $a_i^{(0)} = a_{i-1}$; $j = 0$
7. solve equation (58) for a_i using Newton's method, making use of equation (63) evaluated at $S = a_i$
8. let $C_i(S) = C_{i-1}(S)$ in right-hand side of (56)
9. calculate new estimate for $C_i(S)$ using equation (56)and a_i
10. calculate $\Delta_C(S, \tau)$ using the equation (63)

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