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Abstract

In this paper we propose a new multivariate GARCH model with time-varying conditional correlation structure. The approach adopted here is based on the decomposition of the covariances into correlations and standard deviations. The time-varying conditional correlations change smoothly between two extreme states of constant correlations according to an endogenous or exogenous transition variable. An LM test is derived to test the constancy of correlations and LM and Wald tests to test the hypothesis of partially constant correlations. Analytical expressions for the test statistics and the required derivatives are provided to make computations feasible. An empirical example based on daily return series of five frequently traded stocks in the Standard & Poor 500 stock index completes the paper. The model is estimated for the full five-dimensional system as well as several subsystems and the results discussed in detail.

JEL classification: C12; C32; C51; C52; G1

Key words: Multivariate GARCH; Constant conditional correlation; Dynamic conditional correlation; Return comovement; Variable correlation GARCH model; Volatility model evaluation

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1 Introduction

"...During major market events, correlations change dramatically ..." Bookstaber (1997)

Financial decision makers usually deal with many financial assets simultaneously. Modelling individual time series separately is thus an insufficient method as it leaves out information about comovements and interactions between the instruments of interest. Investors are facing risks that affect the assets in their portfolio in various ways which encourages them to find a position that allows to hedge against losses. In practice, this is often done by trying to diversify, possibly internationally, on many stock markets. When forming an efficient or optimal portfolio, correlations among, say, international stock returns are needed to determine gains from international portfolio diversification, and also the calculation of minimum variance hedge ratio needs updated correlations between assets in the hedge. Evidence that the correlations between national stock markets increase during financial crises but remain more or less unaffected during other times can be found for instance in King and Wadhwani (1990), Lin, Engle, and Ito (1994), de Santis and Gerard (1997), and Longin and Solnik (2001). As further examples, options depending on many underlying assets are very sensitive to correlations among those assets, and asset pricing models as well as some risk measures need measures of covariance between the assets in a portfolio. It is clear that there is an obvious need for a fexible and accurate model that can incorporate the information of possible comovements between the assets.

Volatility in multivariate financial data has been typically modelled applying the concept of conditional heteroskedasticity originally introduced by Engle (1982); see Bauwens, Laurent, and Rombouts (in press) for a recent review. In the multivariate context, one also has to model the conditional covariances, not only the conditional variances. One possibility is to model the former directly, VEC and BEKK as well as F–GARCH, O–GARCH, GO–GARCH, and GDC–GARCH models may serve as examples. Another alternative is to model them through conditional correlations. One of the most frequently used multivariate GARCH models is the Constant Conditional Correlation (CCC) GARCH model of Bollerslev (1990). In this model comovements between heteroskedastic time series are modelled by allowing each series to follow a separate GARCH process while restricting the conditional correlations between the GARCH processes to be constant. The estimation of parameters of the CCC–GARCH model is relatively simple and the model has thus become popular among practitioners. An extension to CCC–GARCH model allowing dynamic interactions between the conditional variance equations, which creates a richer and more flexible autocorrelation structure than the one in the standard CCC–GARCH model, was introduced by Jeantheau (1998), and its moment structure was considered by He and Teräsvirta (2004).

In practice, the assumption of constant conditional correlations has often been found too restrictive, especially in the context of asset returns. Tests developed by Tse (2000) and Bera and Kim (2002) often reject the constancy of conditional correlations. There is evidence that the correlations are not only dependent on time but also on the state of uncertainty in the markets. The conditional correlations can thus fluctuate over time and, in particular, they have been reported to increase during periods of market turbulence.

Tse and Tsui (2002) and Engle (2002) defined dynamic conditional correlation GARCH models (VC–GARCH and DCC–GARCH, respectively) that impose GARCH-type dynamics on the conditional correlations as well as on the conditional variances. These models are flexible enough to capture many kinds of heteroskedastic behaviour in multivariate series. The number of parameters in Engle's DCC–GARCH model remains relatively low because all conditional correlations are generated by first-order GARCH processes with identical parameters. In this model, the GARCH-type correlation processes are not linked to the individual GARCH processes of the return series. Recently, Kwan, Li, and Ng (2005) proposed an extension to the VC–GARCH model using a threshold approach.

Pelletier (in press) recently proposed a model with a regime-switching correlation structure driven by an unobserved state variable that follows a two-dimensional first-order Markov chain. The regime-switching model asserts that the correlations remain constant in each regime and the change between the states is abrupt and governed by transition probabilities. This model is motivated by the empirical finding that the correlations among asset returns tend to increase during periods of distress whereas the series behave in a more independent manner in tranquil periods.

In this paper we introduce another way of modelling comovements in the returns. The Smooth Transition Conditional Correlation (STCC) GARCH model allows the conditional correlations to change smoothly from one state to another as a function of a transition variable. This continuous variable may be a combination of observable stochastic variables, or a function of a lagged error term or terms. The empirical performance of the STCC-GARCH model thus depends on the ability of the transition variable to represent the forces affecting the conditional correlations.

A distinguishing feature of the STCC–GARCH model is that there is interaction between the volatility and correlation structures of the model. The model also has the appealing feature that it provides a framework in which constancy of the correlations and thus the adequacy of the model can be tested in a straightforward fashion. A special case of the STCC–GARCH model was independently introduced by Berben and Jansen (2005). Their model is bivariate, and the variable controlling the transition between the extreme regimes is simply the time.

The paper is organized as follows. In Section 2 the model is introduced and the estimation of its parameters considered. Section 3 is devoted to tests of constant correlations and partially constant correlations. Results of simulation experiments are reported in Section 4, and an application to illustrate the capabilities of the model is discussed in Section 5. Section 6 concludes. Technical derivations of the test statistics presented in the paper can be found in the Appendix.

2 The Smooth Transition Conditional Correlation (STCC) GARCH model

2.1 The general multivariate GARCH model

Consider the following stochastic N-dimensional vector process with the standard representation

$$\boldsymbol{y}_t = E\left[\boldsymbol{y}_t \mid \mathcal{F}_{t-1}\right] + \boldsymbol{\varepsilon}_t \qquad t = 1, 2, \ldots, T \tag{1}$$

where $\mathcal{F}_{t-1}$ is the sigma-field generated by all the information until time $t-1$. Each of the univariate error processes has the specification

$$\varepsilon_{it} = h_{it}^{1/2} z_{it},$$

where the errors z_{it} form a sequence of independent random variables with mean zero and variance one, for each $i = 1, \ldots, N$. The conditional variance h_{it} follows a univariate GARCH process

$$h_{it} = \alpha_{i0} + \sum_{j=1}^{q} \alpha_{ij} \varepsilon_{i,t-j}^2 + \sum_{j=1}^{p} \beta_{ij} h_{i,t-j} \tag{2}$$

with the non-negativity and stationarity restrictions imposed. The first and second conditional moments of the vector $\boldsymbol{z}_t$ are given by

$$\begin{aligned} E\left[\boldsymbol{z}_t \mid \mathcal{F}_{t-1}\right] &= \mathbf{0}, \\ E\left[\boldsymbol{z}_t\boldsymbol{z}_t' \mid \mathcal{F}_{t-1}\right] &= \boldsymbol{P}_t. \end{aligned} \tag{3}$$

Furthermore, the standardized errors $\boldsymbol{\eta}_t = \boldsymbol{P}_t^{-1/2}\boldsymbol{z}_t \sim iid(\mathbf{0}, \boldsymbol{I}_N)$. Since z_{it} has unit variance for all i, $\boldsymbol{P}_t = [\rho_{ij,t}]_{i,j=1,\ldots,N}$ is the conditional correlation matrix for the $\boldsymbol{\varepsilon}_t$ where

$$\begin{aligned} \rho_{ij,t} &= E\left[z_{it}z_{jt} \mid \mathcal{F}_{t-1}\right] = \frac{E\left[z_{it}z_{jt} \mid \mathcal{F}_{t-1}\right]}{\sqrt{E\left[z_{it}^2 \mid \mathcal{F}_{t-1}\right] E\left[z_{jt}^2 \mid \mathcal{F}_{t-1}\right]}} \\ &= \frac{E\left[\varepsilon_{it}\varepsilon_{jt} \mid \mathcal{F}_{t-1}\right]}{\sqrt{E\left[\varepsilon_{it}^2 \mid \mathcal{F}_{t-1}\right] E\left[\varepsilon_{jt}^2 \mid \mathcal{F}_{t-1}\right]}} = Corr\left[\varepsilon_{it}, \varepsilon_{jt} \mid \mathcal{F}_{t-1}\right]. \end{aligned} \tag{4}$$

The correlations $\rho_{ij,t}$ are allowed to be time-varying in a manner that will be defined later on. It will, however, be assumed that $\boldsymbol{P}_t \in \mathcal{F}_{t-1}$.

To establish the connection to the approach often used in context of conditional correlation models, let us denote the conditional covariance matrix of $\boldsymbol{\varepsilon}_t$ as

$$E\left[\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t' \mid \mathcal{F}_{t-1}\right] = \boldsymbol{H}_t = \boldsymbol{S}_t\boldsymbol{P}_t\boldsymbol{S}_t$$

where $\boldsymbol{P}_t$ is the conditional correlation matrix as in equation (3) and $\boldsymbol{S}_t = diag(h_{1t}^{1/2}, .., h_{Nt}^{1/2})$ with elements defined in (2). For the positive definiteness of $\boldsymbol{H}_t$ it is sufficient to require the correlation matrix $\boldsymbol{P}_t$ to be positive definite at each point in time. It follows that the error process in (1) can be written as

$$\boldsymbol{\varepsilon}_t = \boldsymbol{H}_t^{1/2}\boldsymbol{\eta}_t, \qquad \boldsymbol{\eta}_t \sim iid\left(\mathbf{0}, \boldsymbol{I}_N\right).$$

2.2 Smooth transitions in conditional correlations

In order to complete the definition of the model we have to specify the time-varying structure of the conditional correlations in (4). We propose the Smooth Transition Conditional Correlation GARCH (STCC–GARCH) model, in which the conditional correlations are assumed to change smoothly over time depending on a transition variable. In the simplest case there are two extreme states of nature with state-specific constant correlations among the variables. The correlation structure changes smoothly between the two extreme states of constant correlations as a function of the transition variable. More specifically, the conditional correlation matrix $\boldsymbol{P}_t$ is defined as follows:

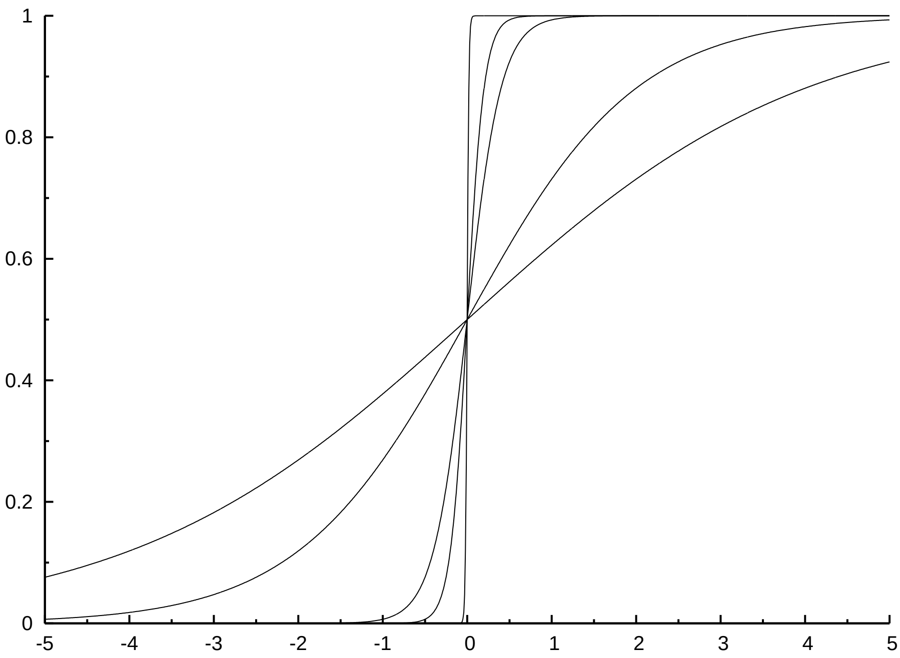
$$\boldsymbol{P}_t = \left(1 - G_t\right)\boldsymbol{P}_1 + G_t\boldsymbol{P}_2 \tag{5}$$

where $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are positive definite correlation matrices. Furthermore, G_t is a transition function whose values are bounded between 0 and 1. This structure ensures $\boldsymbol{P}_t$ to be positive definite with probability one, because it is a convex combination of two positive definite matrices.

The transition function is chosen to be the logistic function

$$G_t = \left(1 + e^{-\gamma(s_t - c)}\right)^{-1}, \quad \gamma > 0 \tag{6}$$

Figure 1: The logistic transition function for $\gamma = 0.5$, 1, 5, 10, and 100 and location $c = 0$.



where s_t is the transition variable, c determines the location of the transition and $\gamma > 0$ the slope of the function, that is, the speed of transition. The typical shape of the transition function is illustrated in Figure 1. Increasing γ, increases the speed of transition from 0 to 1 as a function of s_t, and the transition between the two extreme correlation states becomes abrupt as $\gamma \to \infty$. For simplicity, the parameters c and γ are assumed to be the same for all correlations. This assumption may sometimes turn out to be restrictive, but letting different parameters control the location and the speed of transition in correlations between different series may cause conceptual difficulties. This is because then $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ being positive definite does not imply the positive definiteness of every $\boldsymbol{P}_t$.

The choice of transition variable s_t depends on the process to be modelled. An important feature of the STCC–GARCH model is that the investigator can choose s_t to fit his research problem. In some cases, economic theory proposals may determine the transition variable, in some others available empirical information may be used for the purpose. Possible choices include time as in Berben and Jansen (2005), or functions of past values of one or more of the return series. Yet another option would be to use an exogenous variable, which is a natural idea for example when co-movements of individual stock returns are linked to the behaviour of the stock market itself. In that case, s_t could be a function of lagged values of the whole index. One could use the past conditional variance of the index returns, which Lanne and Saikkonen (2005) suggested when they constructed a univariate smooth transition GARCH model.

Another point worth considering in this context is the number of parameters. It increases rapidly with the number of series in the model, although the current parameterization is still quite parsimonious. However, if one wishes to model the dynamic behaviour between the series, a very small number of parameters may not be enough. Simplifications that are too radical are likely to lead to models that do not capture the behaviour that was to be modelled in the first place. It is possible to simplify the STCC–GARCH model at least to some extent such that it may still be useful in certain applications. As an example, one may restrict one of the two extreme correlation states to be that of complete independence ($\boldsymbol{P}_j = \boldsymbol{I}_N$, $j = 1$ or 2). This is a special case of a model where $\boldsymbol{P}_j = [\rho_{ij}]$ such that $\rho_{ij} = \rho$, $i \neq j$. Another possibility is to allow some of the conditional correlations to be time-varying, while the others remain constant over time. Examples of this will be discussed both in connection with testing and in the empirical application.

2.3 Estimation of the STCC–GARCH model

For the maximum likelihood estimation of parameters we assume joint conditional normality of the errors:

$$\boldsymbol{z}_t \mid \mathcal{F}_{t-1} \sim N\left(\boldsymbol{0}, \boldsymbol{P}_t\right).$$

Denoting by $\boldsymbol{\theta}$ the vector of all the parameters in the model, the log-likelihood for observation t is

$$l_t\left(\boldsymbol{\theta}\right) = -\frac{N}{2}\log\left(2\pi\right) - \frac{1}{2}\sum_{i=1}^{N}\log h_{it} - \frac{1}{2}\log\left|\boldsymbol{P}_t\right| - \frac{1}{2}\boldsymbol{z}_t'\boldsymbol{P}_t^{-1}\boldsymbol{z}_t, \quad t = 1, \ldots, T \tag{7}$$

and maximizing $\sum_{t=1}^{T} l_t(\boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}$ yields the maximum likelihood estimator $\hat{\boldsymbol{\theta}}_T$.

Asymptotic properties of the maximum likelihood estimators in the present case remain to be established. Bollerslev and Wooldridge (1992) provided a proof of consistency and asymptotic normality of the quasi maximum likelihood estimators in the context of general dynamic multivariate models. Recently, Ling and McAleer (2003) considered a class of vector ARMA–GARCH models and established strict stationarity and ergodicity as well as consistency and asymptotic normality of the QMLE under some reasonable moment conditions. In their model, however, the conditional correlations are assumed to be constant. Extending their results to cover the present situation would be interesting but is beyond the scope of this paper. For inference we merely assume that the asymptotic distribution of the ML-estimator is normal, that is,

$$\sqrt{T}\left(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0\right) \xrightarrow{d} N\left(\boldsymbol{0}, \mathfrak{I}^{-1}(\boldsymbol{\theta}_0)\right)$$

where $\boldsymbol{\theta}_0$ is the true parameter and $\mathfrak{I}(\boldsymbol{\theta}_0)$ the population information matrix evaluated at $\boldsymbol{\theta} = \boldsymbol{\theta}_0$.

Before estimating the STCC–GARCH model, however, it is necessary to first test the hypothesis that the conditional correlations are constant. The reason for this is that some of the parameters of the STCC–GARCH model are not identified if the true model has constant conditional correlations. Estimating an STCC–GARCH model without first testing the constancy hypothesis could thus lead to inconsistent parameter estimates. The same is true if one wishes to increase the number of transitions in an already estimated model. Testing constancy of conditional correlations will be discussed in the next section.

Maximization of the log-likelihood (7) with respect to all the parameters at once can be difficult due to numerical problems. For the DCC–GARCH model, Engle (2002) proposed a two-step estimation procedure based on the decomposition of the likelihood into a volatility and a correlation component. The univariate GARCH models are estimated first, independently of each other, and the correlations thereafter, conditionally on the GARCH parameter estimates. This implies that the dynamic behaviour of each return series, characterized by an individual GARCH process, is not linked to the time-varying correlation structure. Under this assumption, the parameter estimates of the DCC–GARCH model are consistent under reasonable regularity conditions; see Engle (2002) and Engle and Sheppard (2001) for discussion. For comparison, in the STCC–GARCH model the dynamic conditional correlations form a channel of interaction between the volatility processes. Parameter estimation accommodates this fact: the parameters are estimated simultaneously by conditional maximum likelihood.

Due to the large number of parameters in the model, estimation of the STCC-GARCH model is carried out iteratively by concentrating the likelihood. The parameters are divided into three sets: parameters in the GARCH equations, correlations, and parameters of the transition function, and the log-likelihood is maximized by sequential iteration over these sets. After the first completed iteration, the parameter estimates correspond to the estimates obtained by a two-step estimation procedure. Even if the parameter estimates do not change much during the sequence of iterations, the iterative method increases efficiency by yielding smaller standard errors than the two-step method. Furthermore, convergence is generally reached with a reasonable number of iterations.

It should be pointed out, however, that estimation requires care. The log-likelihood may have several local maxima, so estimation should be initiated from a set of different starting-values and achieved maxima compared before settling for final estimates.

3 Testing constancy of correlations

3.1 Test of constant conditional correlations

As already mentioned, the modelling of time-varying conditional correlations has to begin by testing the hypothesis of constant correlations. Tse (2000), Bera and Kim (2002), and Engle and Sheppard (2001) already proposed tests for this purpose. We shall present an LM–type test of constant conditional correlations against the STCC–GARCH alternative. A rejection of the null hypothesis supports the hypothesis of time-varying correlations or other types of misspecification but does not imply that the data have been generated from an STCC–GARCH model. For this reason our LM-type test can also be seen as a general misspecification test of the CCC–GARCH model.

In order to derive the test, consider an N-variate case where we wish to test the assumption of constant conditional correlations against conditional correlations that are time-varying with a simple transition of type (5) with a transition function defined by (6). For simplicity, assume that the conditional variance of each of the individual series follows a GARCH$(1, 1)$ process and let $\boldsymbol{\omega}_i = (\alpha_{i0}, \alpha_i, \beta_i)'$ be the vector of parameters for conditional variance h_{it}. Generalizing the test to other types of GARCH models for the individual series is straightforward. The STCC–GARCH model collapses into a constant correlation model under the null hypothesis

$$H_0: \quad \gamma = 0$$

in (6). When this restriction holds, however, some of the parameters of the model are not identified. To circumvent this problem, we follow Luukkonen, Saikkonen, and Teräsvirta (1988) and consider an approximation of the alternative hypothesis. It is obtained by a first-order Taylor approximation around $\gamma = 0$ to the transition function G_t:

$$\begin{aligned} G_t &= \left(1 + e^{-\gamma(s_t - c)}\right)^{-1} \\ &\simeq 1/2 + (1/4)(s_t - c)\,\gamma. \end{aligned} \tag{8}$$

Applying (8) to (5) linearizes the time-varying correlation matrix $\boldsymbol{P}_t$ as follows:

$$\boldsymbol{P}_t^* = \boldsymbol{P}_1^* - s_t \boldsymbol{P}_2^*$$

where

$$
\begin{aligned}
\boldsymbol{P}_1^* &= \frac{1}{2}(\boldsymbol{P}_1 + \boldsymbol{P}_2) + \frac{1}{4}c(\boldsymbol{P}_1 - \boldsymbol{P}_2)\gamma, \\
\boldsymbol{P}_2^* &= \frac{1}{4}(\boldsymbol{P}_1 - \boldsymbol{P}_2)\gamma. \qquad (9)
\end{aligned}
$$

If $\gamma = 0$, then $\boldsymbol{P}_2^* = 0$ and the correlations are constant. Thus we construct an auxiliary null hypothesis $H_0^{\text{aux}} : \boldsymbol{\rho}_2^* = 0$ where $\boldsymbol{\rho}_2^* = vecl\boldsymbol{P}_2^*$.[1]

This null hypothesis can be tested by an LM–test. Note that when H_0 holds, there is no approximation error because then $G_t \equiv 1/2$, and the standard asymptotic theory remains valid. Let $\boldsymbol{\theta} = (\boldsymbol{\omega}_1', \ldots, \boldsymbol{\omega}_N', \boldsymbol{\rho}_1^{*\prime}, \boldsymbol{\rho}_2^{*\prime})'$, where $\boldsymbol{\rho}_j^* = vecl\boldsymbol{P}_j^*$, $j = 1, 2$, be the vector of all parameters of the model. Under standard regularity conditions, the LM–statistic

$$
LM_{CCC} = T^{-1} \left(\sum_{t=1}^{T} \frac{\partial l_t(\hat{\boldsymbol{\theta}})}{\partial \boldsymbol{\rho}_2^{*\prime}} \right) \left[\hat{\mathfrak{I}}_T(\hat{\boldsymbol{\theta}}) \right]^{-1}_{(\boldsymbol{\rho}_2^*, \boldsymbol{\rho}_2^*)} \left(\sum_{t=1}^{T} \frac{\partial l_t(\hat{\boldsymbol{\theta}})}{\partial \boldsymbol{\rho}_2^*} \right), \qquad (10)
$$

evaluated at the maximum likelihood estimators under the restriction $\boldsymbol{\rho}_2^* = 0$, has an asymptotic χ^2 distribution with $N(N-1)/2$ degrees of freedom. In expression (10), $[\hat{\mathfrak{I}}_T(\hat{\boldsymbol{\theta}})]^{-1}_{(\boldsymbol{\rho}_2^*, \boldsymbol{\rho}_2^*)}$ is the south-east $\frac{N(N-1)}{2} \times \frac{N(N-1)}{2}$ block of the inverse of $\hat{\mathfrak{I}}_T$, where $\hat{\mathfrak{I}}_T$ is a consistent estimator for the asymptotic information matrix. For derivation and details of the statistic, as well as the suggested consistent estimator for the asymptotic information matrix, see the Appendix.

A straightforward extension is to test the constancy of conditional correlations against partially constant correlations:

$$
\begin{aligned}
H_0 &: \quad \gamma = 0 \\
H_1 &: \quad \rho_{ij,1} = \rho_{ij,2} \quad \text{for } (i,j) \in N_1
\end{aligned}
$$

where $N_1 \subset \{1, \ldots, N\} \times \{1, \ldots, N\}$. Under the null hypothesis we again face the identification problem which is solved by linearizing the transition function. Details are given in the Appendix.

These tests involve a particular transition variable. Thus a failure to reject the null of constant correlations is just an indication that there is no evidence of time-varying correlations, given this transition variable. Evidence of time-varying correlations may still be found in case of another indicator. This highlights the importance of choosing a relevant transition variable for the data at hand, and in practice it may be useful to consider several alternatives unless restrictions implied by economic theory make the choice unique.

It should be mentioned that Berben and Jansen (2005) have in a bivariate context coincidentally proposed a test of the correlations being invariant with respect to calendar time. Their test is derived using an approach similar to ours, but they choose a different estimator for the information matrix in (10). Based on our simulation experiments, the estimator they use is substantially less efficient in finite samples than ours, especially when the number of series in the model is large.

3.2 Test of partially constant conditional correlations

The LM–statistic (10) is designed to test the null hypothesis of constant conditional correlations against the STCC–GARCH model. After estimating the model, it is possible to test the

[1] The *vecl* operator stacks the columns of the strict lower diagonal (obtained by excluding the diagonal elements) of the square argument matrix.

constancy of conditional correlations between a subset of return series such that the other conditional correlations remain time-varying both under the null hypothesis and the alternative. We derive both a Lagrange multiplier and a Wald test for this purpose. Both have a partially constant STCC–GARCH model as the null hypothesis, meaning that some of the correlations are constrained to be constant under

$$H_0 : \quad \rho_{ij,1} = \rho_{ij,2} \quad \text{for} \;\; (i,j) \in N_0$$

where $N_0 \subset \{1,\ldots,N\} \times \{1,\ldots,N\}$. The alternative hypothesis is an unrestricted STCC–GARCH model. The identification problem encountered when testing whether the complete model has constant correlations is not present here. Let $\boldsymbol{\theta} = (\boldsymbol{\omega}_1',\ldots,\boldsymbol{\omega}_N',\boldsymbol{\rho}_1',\boldsymbol{\rho}_2',c,\gamma)'$, where $\boldsymbol{\rho}_1 = vecl\boldsymbol{P}_1$ and $\boldsymbol{\rho}_2 = vecl\boldsymbol{P}_2$. Under standard regularity conditions, the usual Wald-statistic

$$W_{PCCC} = T\boldsymbol{a}(\hat{\boldsymbol{\theta}})' \left(\boldsymbol{A} \left[\hat{\mathfrak{I}}_T(\hat{\boldsymbol{\theta}}) \right]^{-1}_{(\boldsymbol{\rho},\boldsymbol{\rho})} \boldsymbol{A}' \right)^{-1} \boldsymbol{a}(\hat{\boldsymbol{\theta}}), \tag{11}$$

evaluated at the maximum likelihood estimators of the full STCC model, has an asymptotic χ^2 distribution with degrees of freedom equal to the number of constraints to be tested. Furthermore, in expression (11), $\boldsymbol{A} = \partial \boldsymbol{a}/\partial \boldsymbol{\rho}'$ where $\boldsymbol{\rho} = (\boldsymbol{\rho}_1', \boldsymbol{\rho}_2')'$ and $\boldsymbol{a}$ is the vector of constraints, and $[\hat{\mathfrak{I}}_T(\hat{\boldsymbol{\theta}})]^{-1}_{(\boldsymbol{\rho},\boldsymbol{\rho})}$ is the block corresponding to the correlation parameters of the inverse of $\hat{\mathfrak{I}}_T$.

It is also possible to apply the LM principle to this problem. Under the null hypothesis, the statistic

$$LM_{PCCC} = T^{-1}\boldsymbol{q}(\hat{\boldsymbol{\theta}})' \left[\hat{\mathfrak{I}}_T(\hat{\boldsymbol{\theta}}) \right]^{-1}_{(\boldsymbol{\rho},\boldsymbol{\rho})} \boldsymbol{q}(\hat{\boldsymbol{\theta}}) \tag{12}$$

evaluated at the restricted maximum likelihood estimators, has an asymptotic χ^2 distribution with the number of degrees of freedom equal to the number of constraints to be tested. In (12), $\boldsymbol{q}$ is the block of the score vector corresponding to the correlation parameters, and $[\hat{\mathfrak{I}}_T(\hat{\boldsymbol{\theta}})]^{-1}_{(\boldsymbol{\rho},\boldsymbol{\rho})}$ is again the block corresponding to the correlation parameters of the inverse of $\hat{\mathfrak{I}}_T$. The derivation and details of the Wald and LM statistics as well as of the consistent estimator of the asymptotic information matrix are given in the Appendix.

The most important difference between the two test statistics is that the Wald statistic is evaluated at the estimates obtained from the estimation of the full STCC–GARCH model, whereas the LM–statistic makes use of restricted estimates. Choosing the test has implications when the computation time of the statistic is concerned. For example, when one half of the correlation parameters are restricted, the estimation time is reduced by more than one half, compared to the estimation of the unrestricted model. Furthermore, since the models in question are complicated and nonlinear, it is preferable to first estimate the restricted model and then evaluate the need for a more general specification.

A straightforward extension to these tests is to allow some of the correlations to be constant even in the alternative model. This leads to the following pair of hypotheses:

$$\begin{aligned} H_0 &: \quad \rho_{ij,1} = \rho_{ij,2} \quad \text{for} \;\; (i,j) \in N_0 \\ H_1 &: \quad \rho_{ij,1} = \rho_{ij,2} \quad \text{for} \;\; (i,j) \in N_1 \end{aligned}$$

where $N_1 \subset N_0 \subset \{1,\ldots,N\} \times \{1,\ldots,N\}$. This may be useful when the dimension of the model is high and constancy of at least some of the conditional correlations would be an appropriate initial simplification. The details for this case can be found in the Appendix.

4 Simulation experiments

4.1 Size study

We study the finite-sample properties of the test of constant conditional correlations in a small simulation experiment. We generate data from a bivariate GARCH$(1,1)$ model with normal errors and choose a transition variable external to the model. The values of this variable are generated from a univariate GARCH$(1,1)$ model. Furthermore, the strength of the constant conditional correlation between the two series in the model is varied throughout from -0.9 to 0.9. The three series are parameterized as follows:

i	α_{i0}	α_{i1}	β_{i1}
1	0.02	0.04	0.95
2	0.01	0.03	0.96
ext	0.002	0.06	0.91

where the values for the GARCH parameters are chosen to be representative for two stocks and the S&P500 index. The sample sizes are 1000, 2500, and 5000, and the number of replications is 5000. To eliminate initialization effects, the first 1000 observations are removed from the series before generating the actual observations.

Values of the test statistic (10) are calculated using the analytical expression for the estimator of the information matrix. The rejection frequencies at the asymptotic significance levels 0.05 and 0.10 are shown in Figure 2. The actual size of the test seems to be quite close to the nominal size already for the sample size of 1000.

We also carried out experiments with the partial constancy tests, albeit with fewer replications, because the computational burdens were substantial. The results were, however, quite satisfactory, suggesting that the tests do not suffer from size distortion.

4.2 Power study

There is no direct benchmark to which to compare our constancy test. It may be interesting, however, to find out how powerful the test is when the data are generated by the DCC-GARCH model of Engle (2002). The VC-GARCH model of Tse and Tsui (2002) would be a comparable choice of an alternative. In this case, it is natural to assume the transition variable to be a function of generated observations, because correlations generated by the DCC-GARCH model are not influenced by exogenous information. The power of our test depends on the choice of the transition variable, and for this reason choosing a variable that is not informative about the change in the correlations yields low or no power. This is confirmed by our simulations. The power turns out to be very close to the nominal size when the transition variable carries no information of the variability of the correlations even when the data are generated from the STCC–GARCH model.

Our first choice of a transition variable is a linear combination of lags of squared returns. The transition variable equals $s_t = \boldsymbol{w}'\bar{\boldsymbol{y}}_t^{(2)}$ where $\boldsymbol{w}$ is 5×1 vector of weights and $\bar{\boldsymbol{y}}_t^{(2)}$ is 5×1 vector whose kth element is $\frac{1}{N}\sum_{i=1}^{N} y_{i,t-k}^2$. The following weighting schemes are considered:

equal:	$\boldsymbol{w} = (0.2, 0.2, 0.2, 0.2, 0.2)'$
arithmetic:	$\boldsymbol{w} = (0.3, 0.25, 0.2, 0.15, 0.1)'$
geometric:	$\boldsymbol{w} = (0.5, 0.25, 0.125, 0.0625, 0.0625)'$

3–variate

i	α_{i0}	α_{i1}	β_{i1}
1	0.02	0.04	0.95
2	0.01	0.03	0.96
3	0.002	0.06	0.93

$\overline{R}$	y_1	y_2
y_2	0.6	
y_3	0.5	0.4

α	β
0.05	0.90

5–variate

i	α_{i0}	α_{i1}	β_{i1}
1	0.02	0.04	0.95
2	0.01	0.03	0.96
3	0.002	0.06	0.93
4	0.007	0.02	0.97
5	0.03	0.05	0.94

$\overline{R}$	y_1	y_2	y_3	y_4
y_2	0.8			
y_3	0.75	0.65		
y_4	0.7	0.6	0.5	
y_5	0.65	0.55	0.45	0.35

α	β
0.05	0.90

Table 1: Parameters for the DCC–GARCH model used in the power simulations: α_{i0}, α_{i1}, and β_{i1} are the GARCH parameters for series i, $\overline{R}$ is the unconditional correlation matrix, and α and β are the parameters governing the GARCH-type dynamics of the conditional correlations.

The power simulations are performed on three- and five-dimensional models parameterized as shown in Table 1. As in the size study, the sample sizes are 1000, 2500, and 5000 and the number of replications 5000. The rejection frequencies are presented in Table 2.

As can be seen, the power increases with the number of the series. This may not be surprising because the DCC–GARCH model imposes the same dynamic structure on all cross-products $z_{it}z_{jt}$, and the time-varying structure thus becomes more evident as the dimension of the model increases. The power results suggest that the average squared returns over the past five days are quite informative indicators of correlations generated by the DCC–GARCH model. An STCC-GARCH model with this transition variable could be a reasonable substitute for the (true) DCC-GARCH model.

We have also simulated the case in which the transition variable used in the test is a function of past returns such that it accounts for both the sign and the size of returns. In this experiment our test has low power when the data are generated by the DCC-GARCH model. Thus, if we believe that both the direction and the strength of the movement of the stock price jointly affect the conditional correlations, the DCC–GARCH model may not be our favourite model. If the effects moving the correlations are essentially 'market effects', they can be taken into account by the STCC–GARCH model.

5 Application to daily stock returns

The data set of our application consists of daily returns of five S&P 500 composite stocks traded at the New York Stock Exchange and the S&P 500 index itself. The main criterion for choosing the stocks is that they are frequently traded and that the trades are often large. The stocks are Ford, General Motors, Hewlett-Packard, IBM, and Texas Instruments, and the observation period begins January 3, 1984 and ends December 31, 2003. As usual, closing

3–variate		$T = 1000$		$T = 2500$		$T = 5000$	
		5%	10%	5%	10%	5%	10%
	equal	0.474	0.586	0.783	0.851	0.960	0.975
	arithmetic	0.461	0.571	0.771	0.841	0.952	0.974
	geometric	0.361	0.480	0.669	0.763	0.911	0.942
5–variate		5%	10%	5%	10%	5%	10%
	equal	0.803	0.873	0.987	0.991	1.000	1.000
	arithmetic	0.783	0.855	0.983	0.991	1.000	1.000
	geometric	0.641	0.744	0.963	0.976	0.998	0.999

Table 2: Rejection frequencies for the test of constant conditional correlations when the data generating processes are the DCC–GARCH models in Table 1.

prices are transformed into returns by taking natural logarithms, differencing, and multiplying by 100, which gives a total of 5038 observations for each of the series. To avoid problems in the estimation of the GARCH equations, the observations in the series are truncated such that extremely large negative returns are set to a common value of -10. This is preferred to removing them altogether, because we do not want to remove the information in comovements related to very large negative returns. It turns out that the truncated observations lie more than ten standard deviations below the mean. Descriptive statistics of the return series can be found in Table 3. To give an idea of how truncation affects the overall properties of the series, the values of the statistics have been tabulated both before and after truncation.

5.1 Choosing the transition variable

We consider the possibility that common shocks affect conditional correlations between daily returns. The transition variable in the transition function is a function of lagged returns of the S&P 500 index. As already discussed in Section 2.2, several choices are available. A question frequently investigated, see for instance Andersen, Bollerslev, Diebold, and Labys (2001) and Chesnay and Jondeau (2001), is whether comovements in the returns are stronger during general market turbulence than they are during more tranquil times. In that case, a lagged squared or absolute daily return, or a sum of lags of either ones, would be an obvious choice. Following Lanne and Saikkonen (2005), one could also consider the conditional variance of the S&P 500 returns. A model-based estimate of this quantity may be obtained by specifying and estimating an adequate GARCH model for the S&P 500 return series.

We restricted our attention to different functions of lagged squared and absolute returns of the index. Specifically, we considered the unweighted averages of the both lagged squared and lagged absolute returns over periods ranging from one to twenty days, and weighted averages of the same quantities with exponentially decaying weights with the discount ratios 0.9, 0.7, 0.5, and 0.3. The constant conditional correlations hypothesis was then tested using each of the 48 transition variables in the complete five-dimensional model as well as in every one of its submodels. The clearly strongest overall rejection occurred (these results are not reported here) when the transition variable was the seven-day average of lagged absolute returns. The graph of this transition variable is presented in Figure 3.

Table 4 contains the p-values of the constancy test based on this transition variable for all bivariate, trivariate, four-variable, and the full five-variable CCC–GARCH model. The test rejects the null hypothesis of constant correlations at significance level 0.01 for all bivariate, eight out of ten trivariate and four out of five four-variate models. The full five-variate CCC–GARCH model is rejected against STCC–GARCH as well. Generally, the rejections grow stronger as the dimension of the model increases, but the decrease of p-values is not monotonic.

If the interest lies in finding out whether the direction of the price movement as well as its strength affect conditional correlations, a function of lagged returns that preserves the sign of the returns is an appropriate transition variable. In order to accommodate this possibility, we considered the following two sets of lagged returns: $\{r_{t-j} : j = 1, \ldots, 10\}$ and $\{\sum_{i=1}^{j} r_{t-i} : j = 2, \ldots, 10\}$; note that $\sum_{i=1}^{j} r_{t-i} = 100(p_{t-1} - p_{t-(j+1)})$ where p_t is the log-price of the stock. The constant conditional correlations hypothesis was tested using these nineteen choices of transition variables. The strongest rejection most frequently occurred (results not reported here) when the transition variable was a lagged two-day return $100(p_{t-1} - p_{t-3})$. In this case, all CCC–GARCH models except the bivariate Ford-General Motors one, were rejected at the 0.01 level, most of them very strongly. The transition variable is depicted in Figure 3.

5.2 Effects of market turbulence on conditional correlations

We shall first investigate the case in which conditional correlations are assumed to fluctuate as a result of time-varying market distress which is measured by lagged seven-day averages of absolute S&P 500 returns. Four remarks are in order. First, we only consider first-order STCC–GARCH models. One may want to argue that more effort should be invested in the specification of the individual volatility models. However, in the context of financial returns data it is often found that the first-order GARCH model performs sufficiently well, and for the time being we settle for this the option. Second, estimation of parameters is carried out both by the iterative maximum likelihood and by the two-step procedure.[2] The standard errors of the parameter estimates of the STCC–GARCH model are calculated using numerical second derivatives for all estimates except the estimate of the velocity of transition parameter γ. This exception is due to the fact that in most of the cases the sequence of estimates of γ is converging towards some very large value, which causes numerical problems in calculating standard errors. As a consequence of slow convergence and the numerical problems encountered, the maximum value of γ is constrained to 100. This serves as an adequate approximation, since the transition function changes little beyond $\gamma \geq 100$, as Figure 1 indicates. Note, however, that in these cases the estimated standard errors are conditional on assuming $\gamma = 100$. Third, STCC–GARCH models are only estimated for data for which the constant correlations hypothesis is rejected. Finally, all computations have been performed using Ox, version 3.30; see Doornik (2002).

Instead of presenting all estimation and testing results we focus on specific examples that illustrate the behaviour of the dynamic interactive structure of the models. We begin by considering the bivariate models. Results from higher-dimensional models are discussed thereafter. The estimation and testing results for selected combinations of assets are presented in Tables 4 – 7.

When the transition variable is a function of lagged absolute S&P 500 returns, positive and negative returns of the same size have the same effect on the correlations and the absolute

[2]Without presenting the results, we note that the parameter estimates obtained from the two-step method differ somewhat from the estimates from the iterative method when an STCC–GARCH model is estimated, whereas the results are fairly similar under the assumption of constant correlations.

magnitude of the returns carries all the information of possible comovements in the returns. Small seven-day averages of absolute returns are associated with the conditional correlation matrix $\boldsymbol{P}_1$, whereas the large ones are related to $\boldsymbol{P}_2$.

In the F-GM model the estimated location of the transition is 0.24 which leaves 37% of the average absolute S&P 500 returns below the estimated location parameter c. In all the other estimated bivariate models the transition takes place around 0.5–0.8. This is a range such that 2–11% of the average absolute S&P 500 returns exceed the estimate of c. Except for the F-GM model, a feature common to every estimated bivariate model is that during market turbulence, indicated by a high value of the seven-day average absolute index return, the correlations are considerably higher than during calm periods. As to Ford and General Motors, the returns on these two stocks have fairly high conditional correlations in general, especially during periods when markets are *not* turbulent.

Turning to three-dimensional models in Table 6, the relationship between Ford and General Motors seems to be quite strong, and it dictates the estimated location to be around 0.24 as in the bivariate F-GM model. But then, in models that contain only one of the two automotive companies, the transition is estimated to take place around 0.7. The estimated correlations in the trivariate models behave in the same way as their counterparts in bivariate models whenever the estimated location of the transition coincides with the corresponding estimates from the two bivariate models. If this not the case, one pair of correlations dominates and the constancy of the remaining ones may not be rejected by a the partial constancy test. This is obvious from results reported in Table 6. Table 7 shows that the same pattern repeats itself in the four-variate models as well as in the five-variate model. In the five-variate model many of the correlations seem to be constant. Tests of partial constancy do not reject the null hypothesis that all the correlations except the ones between Ford and General Motors and Ford and Hewlett-Packard, respectively, are constant. The p-values for joint LM- and Wald tests of this hypothesis against the full STCC–GARCH model are 0.12 and 0.13, respectively. The parameter estimates of the restricted STCC–GARCH model can also be found in Table 7.

These correlations are quite different from the ones obtained from four-variate models with only one automotive company (Ford is excluded from the model given in Table 7). In the four-variate model, the correlations increase with the degree of market turbulence. It seems that the F-GM relationship is so strong that it prevents the investigator from seeing other interesting details in the data. As a whole, comparing the four- and five-variate models suggests that a single transition function may not be enough when one wants to characterize time-variation in these correlations. Extensions to the model are possible but, however, less parsimonious than the original model. Besides, the standard STCC-GARCH model ensures each correlation matrix to be positive definite as long as $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are positive definite matrices. This property becomes difficult to retain if the number of transitions exceeds one, unless strong restrictions, such as assuming $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ block diagonal, are placed on these matrices.

The estimated time-varying conditional correlations from the five-variate model appear in Figure 4. Note that most of the correlations in this model are constant around 0.3–0.4. The F-GM correlations are considerably higher when the market is calm than when it is turbulent, whereas the other two remain almost constant over time at the level of about 0.3. These correlations are in marked contrast with the correlations from the four-variate GM-HPQ-IBM-TXN model shown in Figure 5. When the markets are calm, the correlations are constant around 0.3–0.4. When there is strong turbulence, the correlations increase substantially. The results from this model support the notion of strong comovement of returns during times of distress.

5.3 Effects of shock asymmetry on conditional correlations

In order to investigate how possible asymmetry in the way S&P 500 returns affect the correlations $100(p_{t-1}-p_{t-3})$ is selected to be the transition variable in the model. The results of the constancy tests appear in Table 4. The tests of constant correlations reject constancy for each model except for the bivariate F-GM one. An STCC–GARCH$(1,1)$ model is thus estimated for all the other combinations. The S&P 500 two-day returns that are lower than the estimated location imply a correlation state approaching that of $\boldsymbol{P}_1$, whereas the returns greater than the estimated location result in correlations closer to the other extreme state, $\boldsymbol{P}_2$.

The results for the bivariate models can be found in Table 8. The models not involving Texas Instruments have the location of transition such that less than 1% of the S&P 500 two-day returns have values below the estimate of the location parameter c. In these models large negative shocks induce strongly positive conditional correlations between the returns; otherwise the correlations remain positive but are less strong. When Texas Instruments is combined with Ford or General Motors the transition is estimated to take place close to zero. Negative two-day index returns induce correlations slightly higher than positive two-day index returns. Combining Texas Instruments with Hewlett-Packard or IBM results in models where the estimated location is such that 30% of the S&P 500 two-day returns are larger than the estimated parameter c. In those two models the correlations are slightly weaker but still positive when the two-day returns of the index exceed the estimated location.

Modelling higher-dimensional combinations is complicated by the restriction that only one location for change is allowed in the model. A selection of the results from the three-variate models appear in Table 9. The strongest relationship typically determines the estimate of the location parameter in the transition function, and the estimated correlations are adapted to that location. For example, the estimated location in the trivariate HPQ-IBM-TXN model is close to the corresponding estimates from the HPQ-TXN and IBM-TXN models and repeats the correlation patterns in those models. This means that the correlations between HPQ and IBM are now based on a transition function that is different from the one in the bivariate model for these two return series. Similarly, in the F-HPQ-TXN and GM-HPQ-TXN models the strongest relationship appears to be the one between F and HPQ in the former and GM and HPQ in the latter model as the location parameter obtains about the same estimate as in these bivariate models. Yet another outcome is the one where none of the relationships is much stronger than the others. This may result in a completely new location for the transition. Consider the F-GM-TXN model. It is seen from Table 9 that the corresponding bivariate models have very different correlation structures. Besides, constancy of the conditional correlations between F and GM was not rejected when tested. The estimated location for the transition equals 0.17, a value different from any other location estimate in bivariate and trivariate models. It can be seen from Table 10 that c is estimated close to this value in the five-variate model as well. This is also true for all four-variate models involving TXN (not reported in the table). Furthermore, there is a local maximum in the likelihood of the F-GM-HPQ-IBM model corresponding to that value of the location parameter. It the second highest maximum found: the estimates corresponding to the highest maximum are reported in the table.

We again graph some of the conditional correlations. Figure 6 contains the correlations from the trivariate HPQ-IBM-TXN model. They are positive and fluctuate between 0.3 and 0.5. A completely different picture emerges from Figure 7 where F has been added to this trivariate combination to form a four-variate model. The correlations mostly remain unchanged, expect for a few occasions when the market has received a very strong negative shock. Note, however, that the F-GM correlation fluctuates very little, which is in line with the fact that the constancy of this conditional correlation was not rejected when tested in the bivariate framework. In Figure 8, the patterns of estimated correlations from the complete five-variate model are close to the ones shown in the trivariate model of Figure 6.

In theory, as a solution to the 'multilocation problem' one could generalize the STCC–GARCH model such that it would allow different slope and location parameters for each pair of correlations. However, as already mentioned, such an extension entails the statistical problem of ensuring positive definiteness of the correlation matrix at each point of time.

5.4 Comparison

We conclude this section with a brief informal comparison of the time-varying correlations implied by the STCC– and DCC–GARCH models. The former models are the four- and five-dimensional ones reported in the previous subsections. The corresponding five-dimensional DCC–GARCH$(1, 1)$ model is estimated using the two-step estimation method of Engle (2002). The estimated GARCH equations in the DCC–GARCH model differ slightly from the ones in the STCC–GARCH models due to the two-step procedure, whereas the correlation dynamics are very persistent (for conciseness we do not present the estimation results).

As can be seen from Figure 5, in the four-variate STCC–GARCH model periods of strongly volatile periods are reflected by increased correlation levels. For example, during the stock market crash in November 1987 the correlations are high for a short period but return quickly to lower values. These shifts are also visible in Figure 7, indicating that they are, more precisely, resulting from large negative price movements rather than just high volatility. Figures 6 and 8 show that the higher the index returns, the lower the correlations. The rather turbulent period beginning in the late 1990s results in correlations that are more often in the 'high regime' than the previous ones.

Contrary to these results, the DCC–GARCH model suggests a very persistent response of correlations to large shocks. For example, the events of November 1987 lead to increased correlations, but the return to lower levels is very slow. Another notable fact is that the turbulent period from the late 1990s onwards does not seem to have much of an effect on the correlations, except that the correlations generally display an upward tendency at the very end of the observation period.

These two approaches thus lead to rather different conclusions about the conditional correlations between the return series. Since these correlations cannot be observed, it is not possible to decide whether the STCC–GARCH or the DCC–GARCH models yield results that are closer to the 'truth'. In theory, testing the models against each other may be possible but at the same time computationally quite demanding. These models may also be compared by investigating their out-of-sample forecasting performance, which has not been done in this study.

6 Conclusions

We have proposed a new multivariate conditional correlation model with time-varying correlations, the STCC–GARCH model. The conditional correlations are changing smoothly between two extreme states according to a transition variable that can be exogenous to the system. These correlations are weighted averages of two sets of constant correlations, which means that the corresponding time-varying correlation matrices are always positive definite on the condition that the two constant correlation matrices are positive definite.

The transition variable controlling the time-varying correlations can be chosen quite freely, depending on the modelling problem at hand. The STCC–GARCH model may thus be used for investigating the effects of numerous potential factors, endogenous as well as exogenous, on conditional correlations. In this respect the model differs from most other dynamic conditional correlation models such as the ones proposed by Tse and Tsui (2002), Engle (2002), and Pelletier (in press).

The STCC–GARCH model is applied to up to a five-variable set of daily returns of frequently traded stocks included in the S&P 500 index. When using the seven-day lagged average of the daily absolute return of the index as the transition variable we find that the conditional correlations are substantially higher during periods of high volatility than otherwise. Asymmetric response of correlations to shocks is examined using the one-day lag of the two-day index returns. In that case very large negative returns on the index imply very high conditional correlations between the volatilities.

In its present form the model allows for a single transition with location and smoothness parameters common to all series. In theory this restriction can be relaxed, but finding a useful way of doing it is left for future work. The model may be further refined by allowing specifications of the univariate GARCH equations beyond the standard GARCH$(1, 1)$ model. An extension in the spirit of the multivariate CCC–GARCH model of Jeantheau (1998) would be an interesting alternative. Then the squared returns would be linked not only through the conditional correlations but also through the GARCH equations. Another point worth considering is incorporating higher-frequency data into the model. Recent research has emphasized the importance of information that is present in the high-frequency data but lost in aggregation. One such possibility would be to use the realized volatility or bipower variation of stock index returns over a day or a number of days as the transition variable in a model for stock returns. This possibility is left for future research.

Figure 2: Actual size of the test of constant correlation plotted against different values of correlation using simulated data (5000 replicates), and sample sizes 1000, 2500 and 5000. Rejection frequencies are in percentages. The test is calculated with covariance estimator (14).

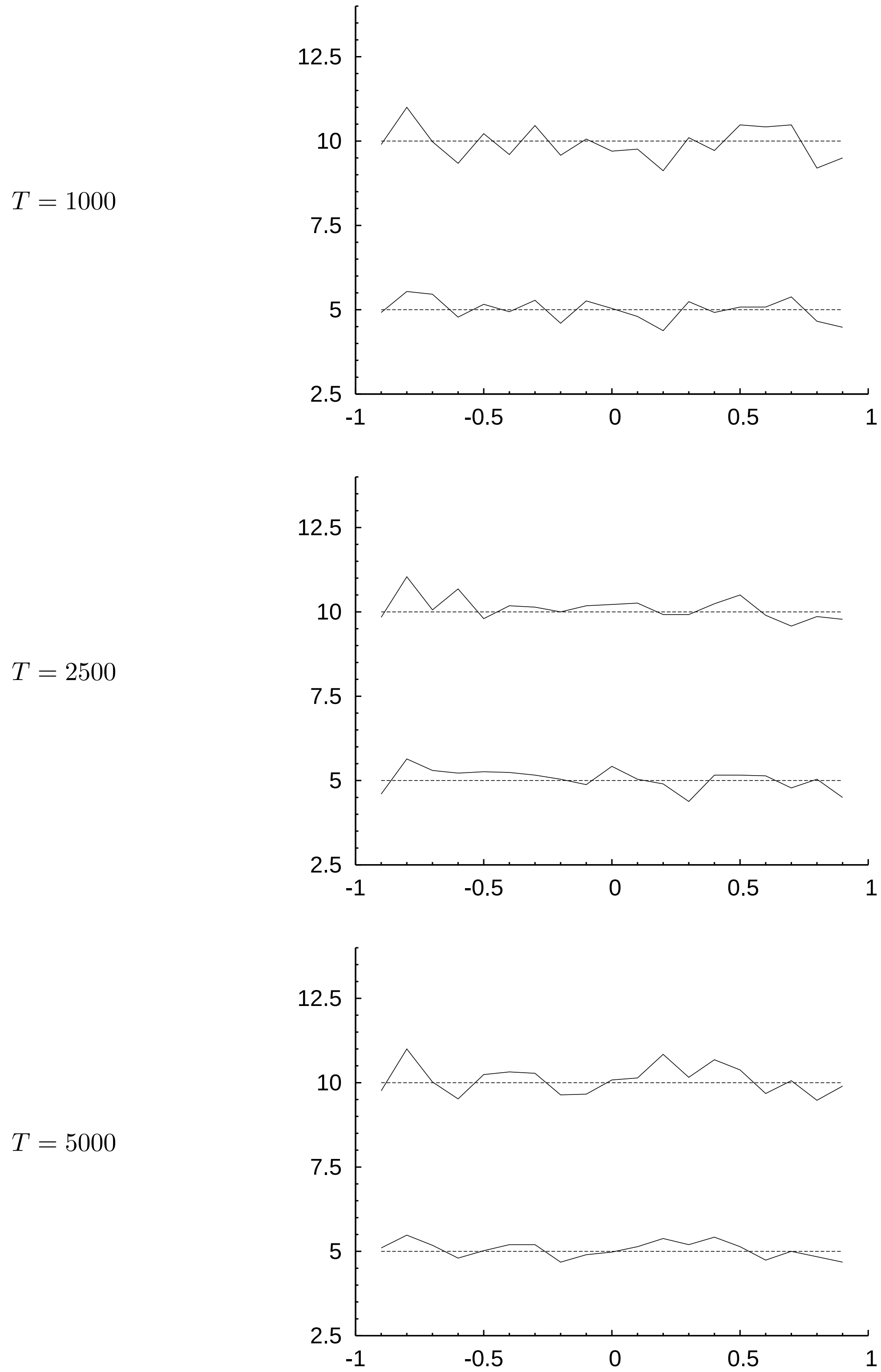


Figure 3: The S&P 500 returns from January 3, 1984 to December 31, 2003. The upper panel shows the returns (one observation falls outside the presented range), the middle panel shows the average of the absolute value returns over seven days (two observations fall outside the presented range), and the lower panel shows the log of the price difference over two days, or return over two days (five observations fall outside the presented range).

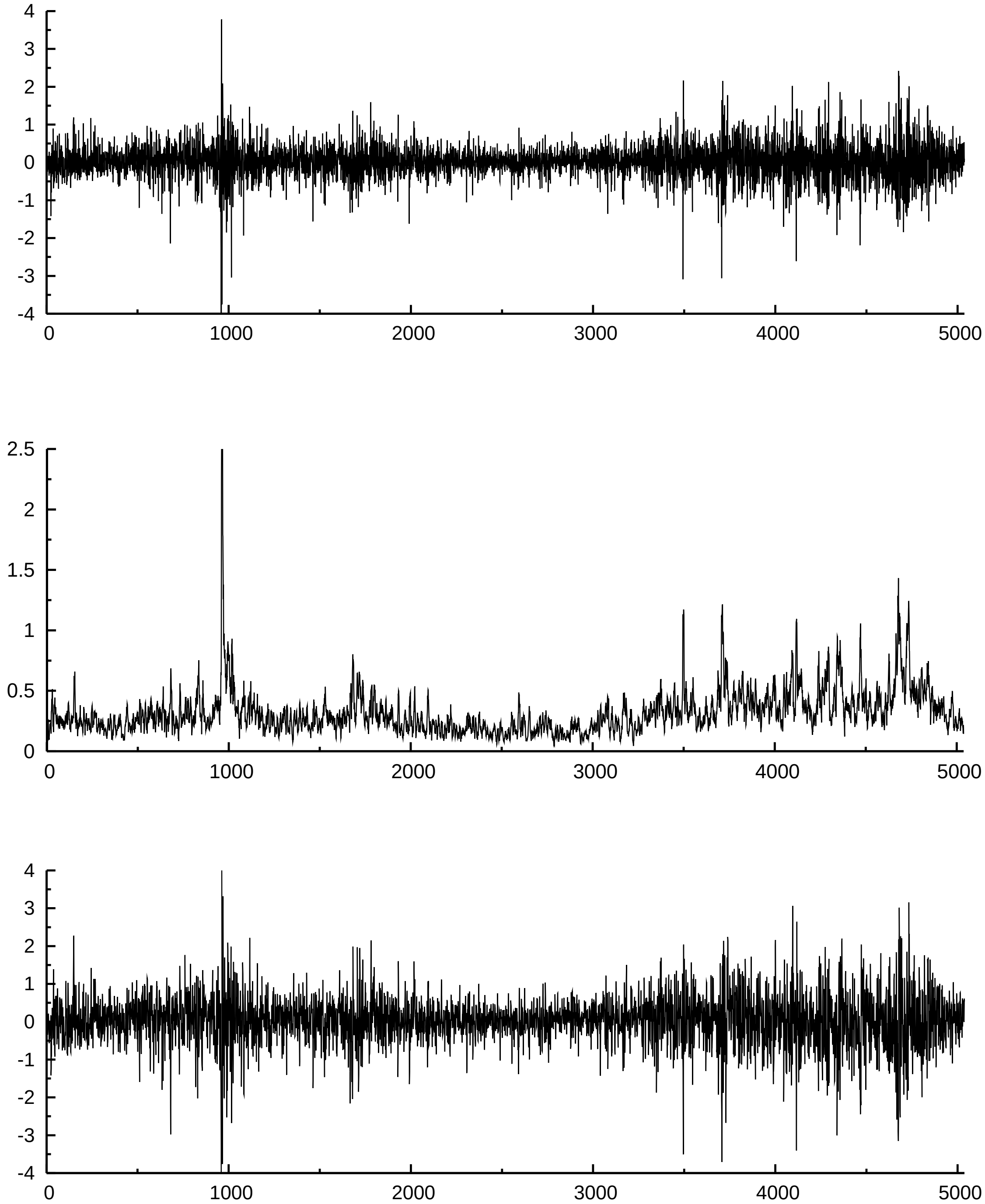


Figure 4: The estimated time-varying conditional correlations from the five-variable STCC–model when the transition variable is lagged absolute S&P 500 returns averaged over seven days, where some of the correlations restricted constant, see Table 7. The estimated location is $\hat{c} = 0.24$.

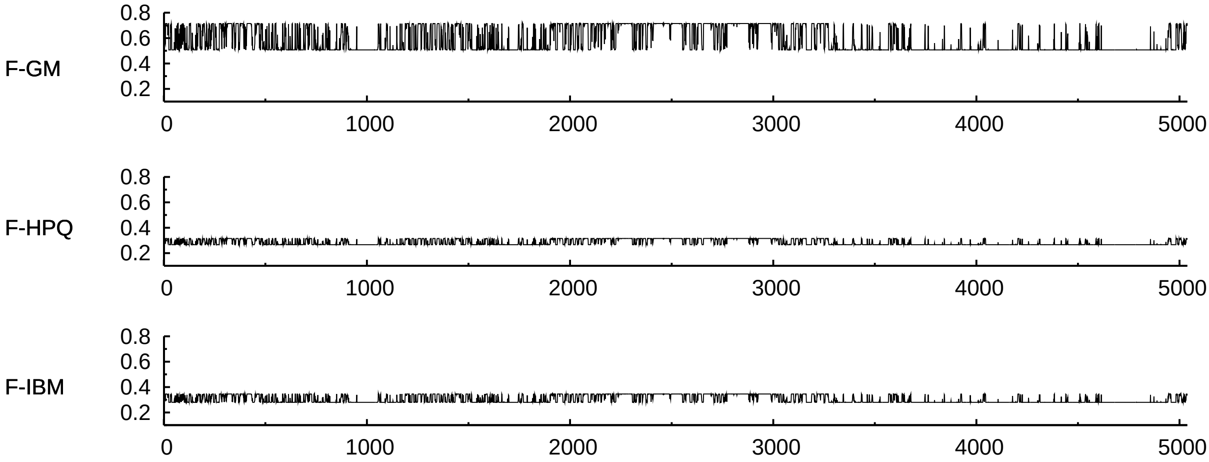


Figure 5: The estimated conditional correlations from the four-variate (GM-HPQ-IBM-TXN) STCC–GARCH model model from Table 7 when the transition variable is lagged absolute S&P 500 returns averaged over seven days. The estimated location is $\hat{c} = 0.71$.

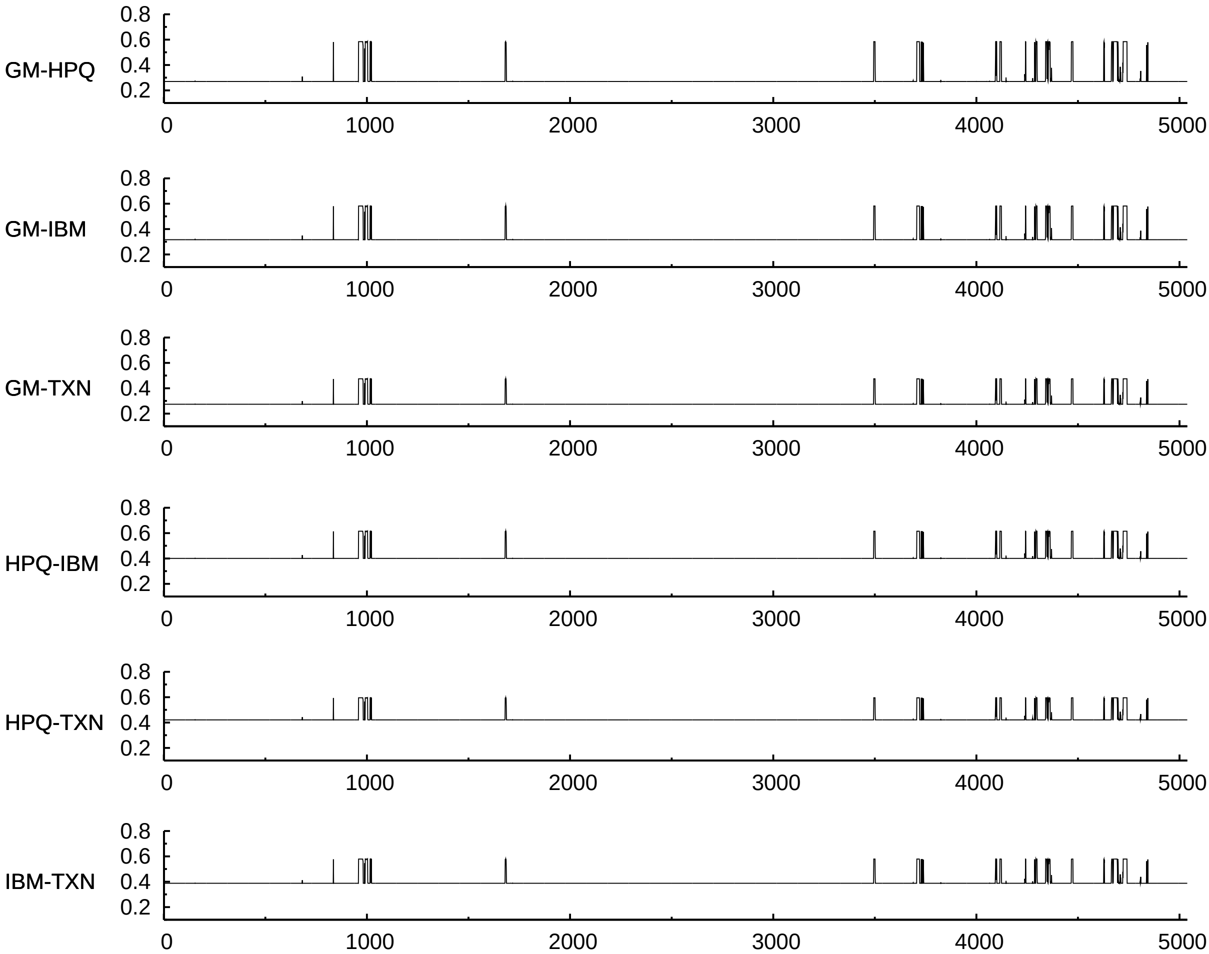


Figure 6: The estimated conditional correlations from the three-variable (HPQ-IBM-TXN) STCC–model when the transition variable is a lagged S&P 500 return over two days, see Table 9. The estimated location is $\hat{c} = 0.26$. The time period covers the years from 1986 to 1989.

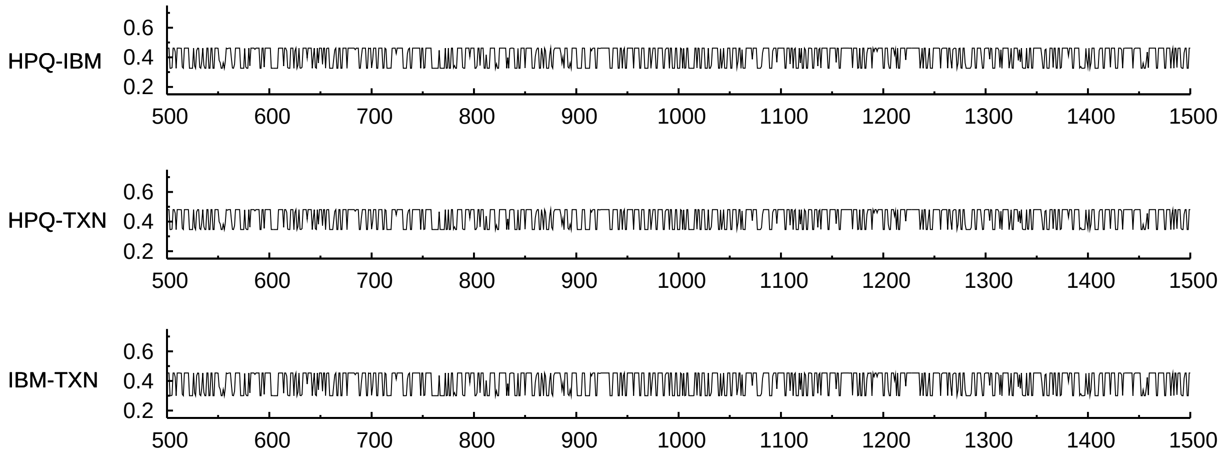


Figure 7: The estimated conditional correlations from the four-variable (F-GM-HPQ-IBM) STCC–model when the transition variable is a lagged S&P 500 return over two days, see Table 10. The estimated location is $\hat{c} = -1.97$. The time period covers the years from 1986 to 1989.

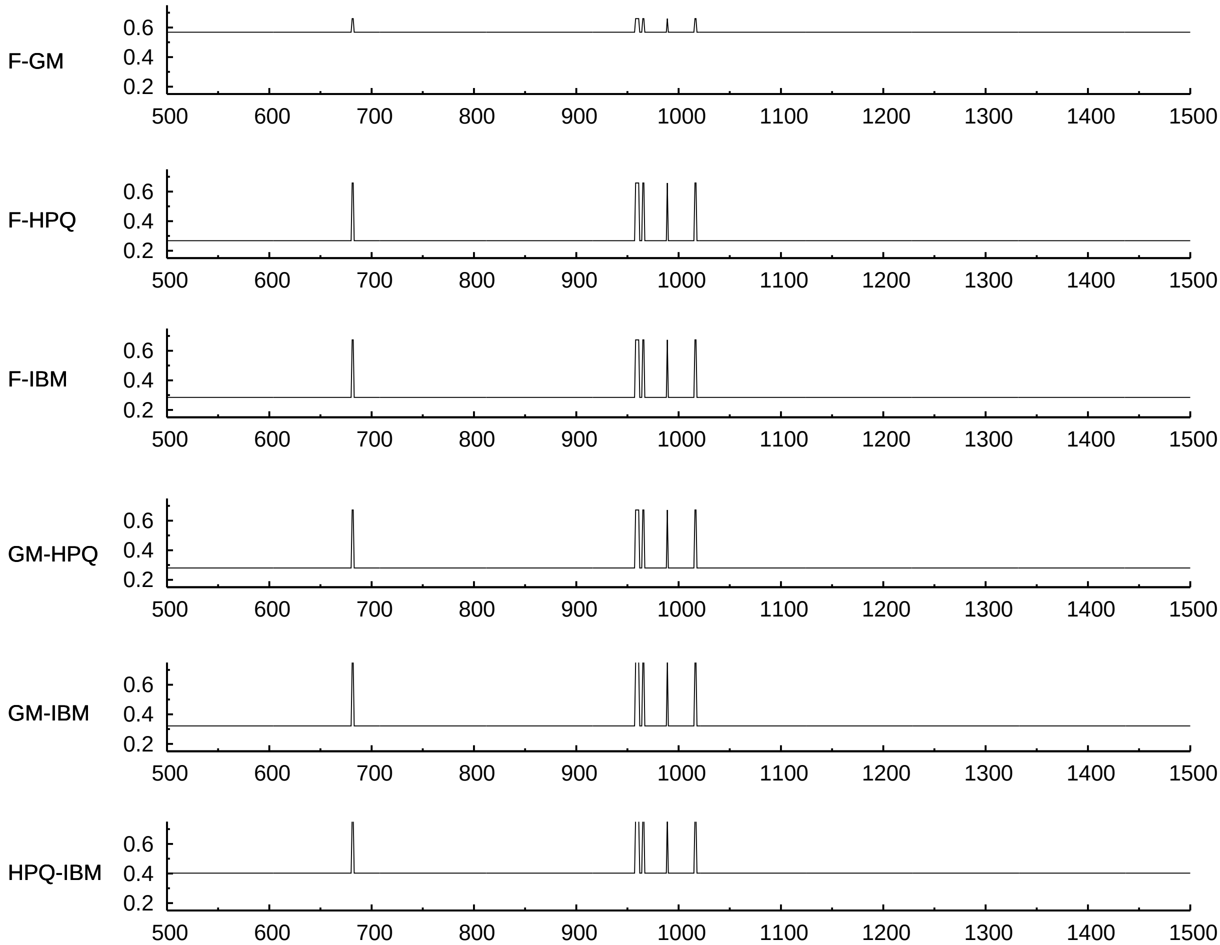


Figure 8: The estimated conditional correlations from the five-variable STCC–model when the transition variable is a lagged S&P 500 return over two days, see Table 10. The estimated location is $\hat{c} = 0.19$. The time period covers the years from 1986 to 1989.

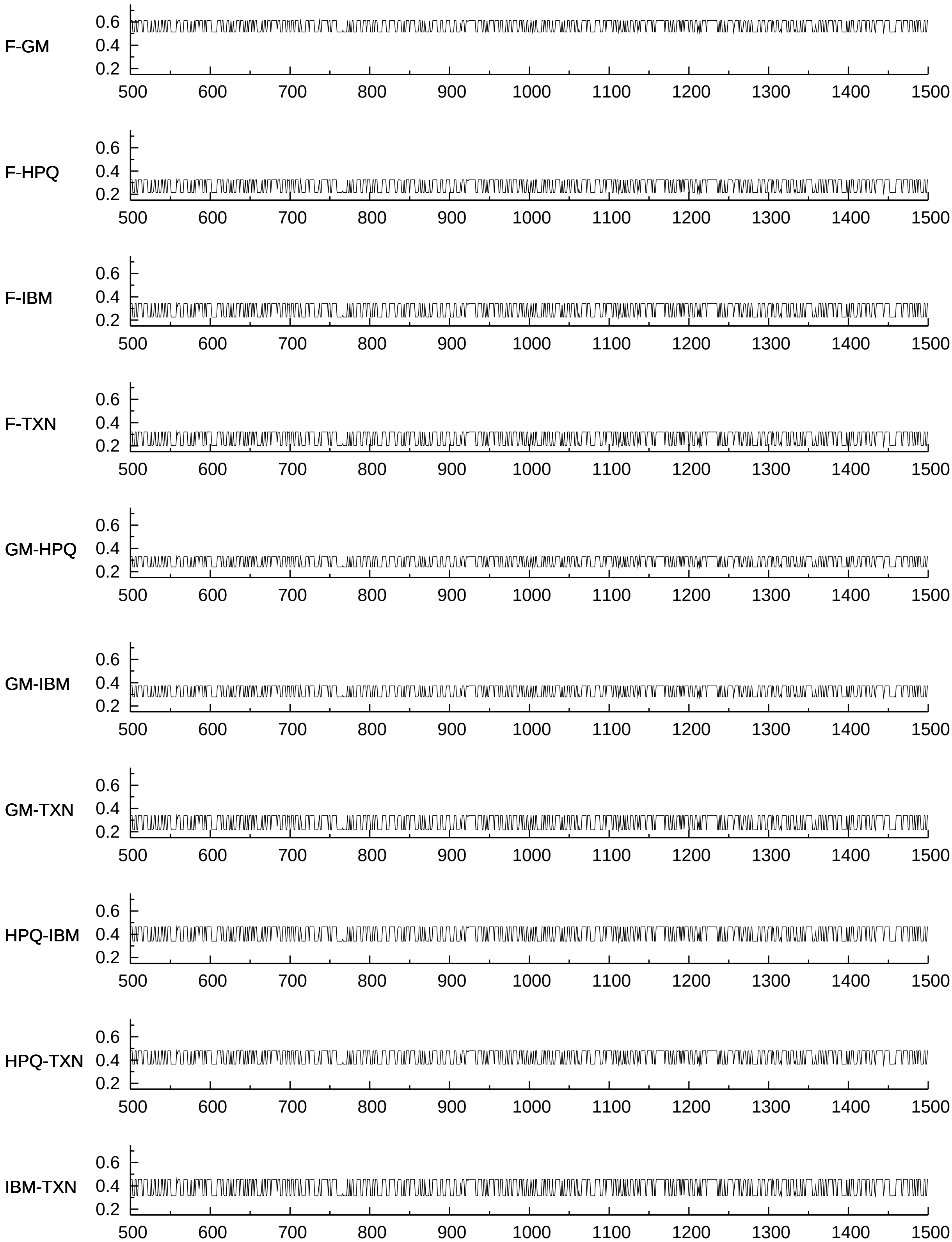
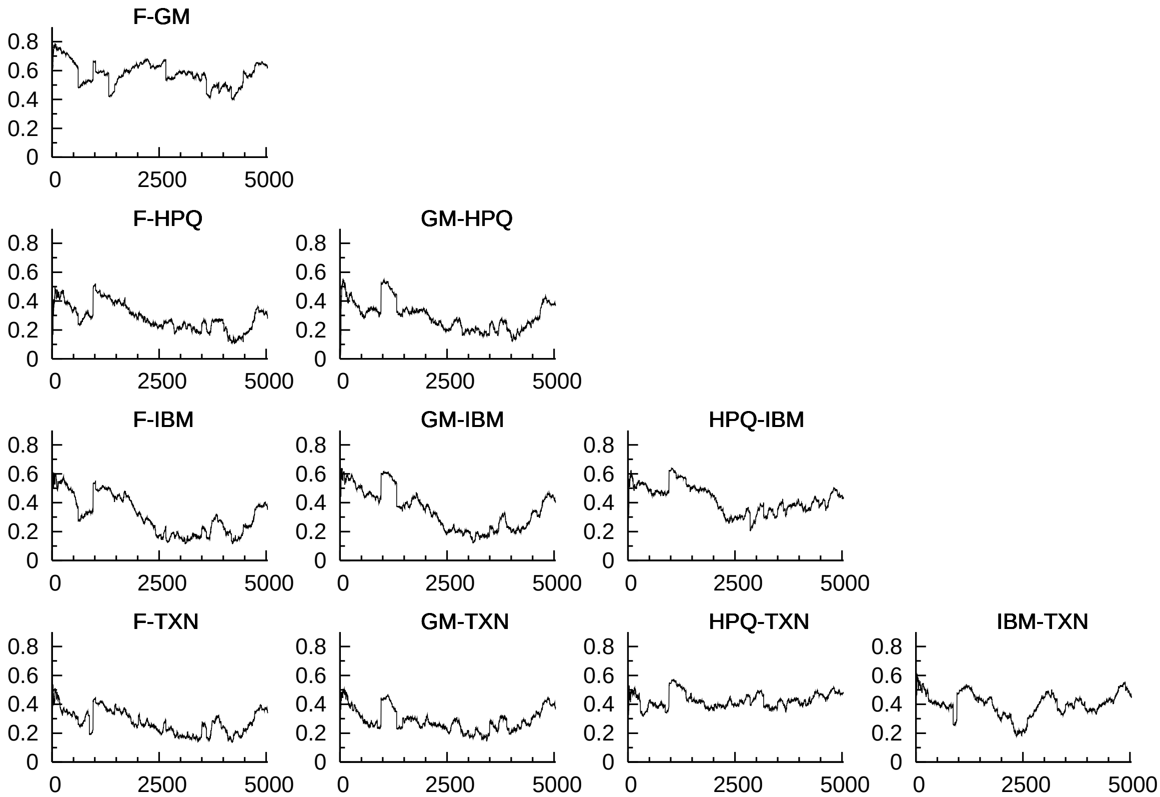


Figure 9: The estimated conditional correlations from the five-variable DCC–model.



	abbr.	min	max	mean (b.t.)	mean (a.t.)	st.dev (b.t.)	st.dev (a.t.)	skewness (b.t.)	skewness (a.t.)	kurtosis (b.t.)	kurtosis (a.t.)	no. of trunc.
Ford	F	−30.44	6.30	−0.0084	0.0062	1.173	0.956	−9.076	−1.162	215.176	18.220	5
General Motors	GM	−30.23	5.93	−0.0028	0.0016	0.950	0.860	−6.720	−0.739	210.671	13.704	2
Hewlett-Packard	HPQ	−29.23	6.93	−0.0053	0.0046	1.317	1.154	−5.841	−0.672	125.323	11.433	3
IBM	IBM	−30.89	5.37	−0.0023	0.0061	1.031	0.854	−10.404	−0.919	306.696	19.480	3
Texas Instruments	TXN	−48.37	9.36	−0.0115	0.0150	1.704	1.308	−9.794	−0.393	230.166	9.379	6
S&P500 Index	–	−9.94	3.78	0.0164	–	0.473	–	−2.012	–	43.194	–	–
Transition variable 1	$s_t^{(1)}$	0	3.39	0.3202	–	0.198	–	4.707	–	53.354	–	–
Transition variable 2	$s_t^{(2)}$	−12.20	6.00	0.0328	–	0.673	–	−1.584	–	31.173	–	–

Table 3: Descriptive statistics of the asset returns. The mean, standard deviation, skewness and kurtosis are reported both before (b.t.) and after (a.t.) removing the extreme negative returns. The cutoff value for truncation corresponds roughly to 10 standard deviations. The transition variables are $s_t^{(1)}$, the lagged absolute S&P 500 index return averaged over past seven days, and $s_t^{(2)}$, the lagged S&P 500 index return over past two days.

	$s_t^{(1)}$		$s_t^{(2)}$	
	LM_{CCC}	p–value	LM_{CCC}	p–value
F – GM	36.65	1×10^{-9}	2.80	0.0943
F – HPQ	9.12	0.0025	20.95	5×10^{-6}
F – IBM	8.21	0.0042	45.59	1×10^{-11}
F – TXN	14.66	0.0001	22.01	3×10^{-6}
GM – HPQ	14.74	0.0001	30.34	4×10^{-8}
GM – IBM	24.80	6×10^{-7}	53.43	3×10^{-13}
GM – TXN	14.12	0.0002	35.35	3×10^{-9}
HPQ – IBM	11.81	0.0008	37.12	1×10^{-9}
HPQ – TXN	7.96	0.0048	21.45	4×10^{-6}
IBM – TXN	13.24	0.0003	48.31	4×10^{-12}
F – GM – HPQ	64.06	8×10^{-14}	30.15	1×10^{-6}
F – GM – IBM	66.57	2×10^{-14}	61.72	3×10^{-13}
F – GM – TXN	75.49	3×10^{-16}	29.71	2×10^{-6}
F – HPQ – IBM	10.98	0.0118	67.54	1×10^{-14}
F – HPQ – TXN	11.20	0.0107	36.88	5×10^{-8}
F – IBM – TXN	16.08	0.0011	77.08	1×10^{-16}
GM – HPQ – IBM	24.47	2×10^{-5}	76.99	1×10^{-16}
GM – HPQ – TXN	14.59	0.0022	50.99	5×10^{-11}
GM – IBM – TXN	26.29	8×10^{-6}	88.13	6×10^{-19}
HPQ – IBM – TXN	11.97	0.0075	63.40	1×10^{-13}
F – GM – HPQ – IBM	77.52	1×10^{-14}	78.67	7×10^{-15}
F – GM – HPQ – TXN	73.64	7×10^{-14}	44.27	7×10^{-8}
F – GM – IBM – TXN	80.47	3×10^{-15}	87.07	1×10^{-16}
F – HPQ – IBM – TXN	8.22	0.2221	83.39	7×10^{-16}
GM – HPQ – IBM – TXN	17.24	0.0085	94.43	4×10^{-18}
F – GM – HPQ – IBM – TXN	81.40	3×10^{-13}	90.24	5×10^{-15}

Table 4: Test of constant conditional correlation against STCC–GARCH model for all combinations of assets. The transition variables are $s_t^{(1)}$, the lagged absolute S&P 500 index returns averaged over seven days, and $s_t^{(2)}$, a lagged S&P 500 index return over two days.

model	α_0	α	β	ρ_1	ρ_2	c	γ
F	0.0071	0.0166	0.9760	0.7157	0.5075	0.2395	100
	(0.0013)	(0.0020)	(0.0026)	(0.0116)	(0.0127)	(0.0042)	
GM	0.0246	0.0380	0.9291				
	(0.0059)	(0.0057)	(0.0129)				
F	0.0059	0.0198	0.9744	0.2636	0.4539	0.7221	100
	(0.0012)	(0.0025)	(0.0029)	(0.0138)	(0.0378)	(0.0182)	
HPQ	0.0111	0.0194	0.9719				
	(0.0034)	(0.0039)	(0.0062)				
F	0.0065	0.0220	0.9719	0.2855	0.4332	0.7154	100
	(0.0014)	(0.0027)	(0.0032)	(0.0136)	(0.0403)	(0.0325)	
IBM	0.0058	0.0703	0.9277				
	(0.0017)	(0.0089)	(0.0093)				
F	0.0065	0.0210	0.9727	0.2560	0.3568	0.5076	100
	(0.0014)	(0.0027)	(0.0032)	(0.0146)	(0.0297)	(0.0184)	
TXN	0.0102	0.0324	0.9612				
	(0.0023)	(0.0043)	(0.0050)				
GM	0.0264	0.0474	0.9162	0.2688	0.6000	0.7142	100
	(0.0063)	(0.0066)	(0.0144)	(0.0136)	(0.0350)	(0.0120)	
HPQ	0.0171	0.0253	0.9614				
	(0.0059)	(0.0059)	(0.0101)				
GM	0.0218	0.0440	0.9263	0.3179	0.6823	0.8413	100
	(0.0047)	(0.0055)	(0.0110)	(0.0130)	(0.0364)	(0.0173)	
IBM	0.0061	0.0706	0.9270				
	(0.0017)	(0.0085)	(0.0090)				
GM	0.0259	0.0467	0.9178	0.2687	0.5054	0.6628	100
	(0.0058)	(0.0063)	(0.0133)	(0.0138)	(0.0363)	(0.0121)	
TXN	0.0125	0.0352	0.9579				
	(0.0028)	(0.0047)	(0.0057)				
HPQ	0.0271	0.0351	0.9443	0.3989	0.6503	0.6867	100
	(0.0104)	(0.0092)	(0.0166)	(0.0123)	(0.0321)	(0.0178)	
IBM	0.0081	0.0762	0.9187				
	(0.0020)	(0.0092)	(0.0100)				
HPQ	0.0270	0.0333	0.9458	0.4205	0.6044	0.6827	100
	(0.0097)	(0.0085)	(0.0154)	(0.0121)	(0.0334)	(0.0215)	
TXN	0.0155	0.0361	0.9548				
	(0.0034)	(0.0051)	(0.0065)				
IBM	0.0093	0.0767	0.9162	0.3871	0.6001	0.6966	100
	(0.0022)	(0.0094)	(0.0105)	(0.0124)	(0.0351)	(0.0168)	
TXN	0.0144	0.0369	0.9550				
	(0.0030)	(0.0047)	(0.0058)				

Table 5: Estimation results for all bivariate STCC–GARCH models (standard errors in parentheses) when the transition variable is a lagged absolute S&P 500 index returns averaged over seven days.

model	α_0	α	β		$\boldsymbol{P}_1$		$\boldsymbol{P}_2$		c	γ
F	0.0071	0.0169	0.9758		F	GM	F	GM	0.2376	100
	(0.0012)	(0.0020)	(0.0026)						(0.0043)	
GM	0.0203	0.0337	0.9392	GM	0.7169		0.5101			
	(0.0044)	(0.0046)	(0.0099)		(0.0116)		(0.0126)			
IBM	0.0053	0.0678	0.9310	IBM	0.3161^r	0.3037^r	0.2914^r	0.3575^r		
	(0.0015)	(0.0085)	(0.0088)		(0.0235)	(0.0230)	(0.0157)	(0.0152)		
F	0.0071	0.0163	0.9762		F	GM	F	GM	0.2394	100
	(0.0012)	(0.0019)	(0.0026)						(0.0043)	
GM	0.0230	0.0352	0.9335	GM	0.7149		0.5068			
	(0.0052)	(0.0050)	(0.0114)		(0.0117)		(0.0127)			
TXN	0.0116	0.0327	0.9608	TXN	0.2726^r	0.2943^r	0.2755^r	0.2898^r		
	(0.0016)	(0.0044)	(0.0053)		(0.0249)	(0.0235)	(0.0159)	(0.0160)		
F	0.0065	0.0212	0.9725		F	IBM	F	IBM		
	(0.0013)	(0.0026)	(0.0031)							
IBM	0.0069	0.0688	0.9267	IBM	0.2858		0.4130		0.7010	100
	(0.0018)	(0.0089)	(0.0095)		(0.0136)		(0.0413)		(0.0176)	
TXN	0.0130	0.0354	0.9572	TXN	0.2610	0.3875	0.3928	0.5826		
	(0.0027)	(0.0043)	(0.0053)		(0.0139)	(0.0124)	(0.0394)	(0.0374)		
GM	0.0218	0.0426	0.9276		GM	IBM	GM	IBM		
	(0.0046)	(0.0054)	(0.0108)							
IBM	0.0069	0.0688	0.9265	IBM	0.3157		0.5811		0.6874	100
	(0.0018)	(0.0084)	(0.0092)		(0.0132)		(0.0381)		(0.0167)	
TXN	0.0144	0.0369	0.9549	TXN	0.2716	0.3863	0.4954	0.5851		
	(0.0029)	(0.0046)	(0.0056)		(0.0138)	(0.0124)	(0.0385)	(0.0376)		

Table 6: Estimation results for selected combinations of trivariate STCC–GARCH models (standard errors in parentheses) when the transition variable is a lagged absolute S&P 500 index returns averaged over seven days. The first two models failed to reject the hypothesis of partially constant correlations with respect to the parameters indicated by a superscript r.

model	α_0	α	β		$\boldsymbol{P}_1$				$\boldsymbol{P}_2$				c	γ
GM	0.0225 (0.0049)	0.0426 (0.0054)	0.9265 (0.0113)		GM	HPQ	IBM		GM	HPQ	IBM		0.7052 (0.0141)	100
HPQ	0.0253 (0.0094)	0.0314 (0.0080)	0.9489 (0.0149)	HPQ	0.2701 (0.0135)				0.5841 (0.0372)					
IBM	0.0070 (0.0018)	0.0684 (0.0083)	0.9267 (0.0091)	IBM	0.3162 (0.0131)	0.4002 (0.0122)			0.5819 (0.0382)	0.6153 (0.0368)				
TXN	0.0168 (0.0033)	0.0375 (0.0048)	0.9525 (0.0061)	TXN	0.2736 (0.0136)	0.4209 (0.0120)	0.3877 (0.0124)		0.4748 (0.0399)	0.5955 (0.0349)	0.5784 (0.0376)			
F	0.0072 (0.0012)	0.0160 (0.0019)	0.9765 (0.0025)		F	GM	HPQ	IBM	F	GM	HPQ	IBM		
GM	0.0207 (0.0046)	0.0326 (0.0046)	0.9398 (0.0100)	GM	0.7146 (0.0117)				0.5068 (0.0128)				0.2396 (0.0044)	100
HPQ	0.0199 (0.0084)	0.0244 (0.0072)	0.9603 (0.0133)	HPQ	0.3399 (0.0244)	0.3312 (0.0234)			0.2585 (0.0157)	0.2826 (0.0156)				
IBM	0.0064 (0.0017)	0.0648 (0.0083)	0.9315 (0.0089)	IBM	0.3293 (0.0228)	0.3178 (0.0223)	0.4173 (0.0202)		0.2857 (0.0159)	0.3523 (0.0154)	0.4144 (0.0145)			
TXN	0.0153 (0.0030)	0.0335 (0.0043)	0.9575 (0.0055)	TXN	0.2763 (0.0246)	0.2945 (0.0232)	0.4352 (0.0199)	0.3714 (0.0201)	0.2742 (0.0159)	0.2890 (0.0160)	0.4300 (0.0140)	0.4193 (0.0147)		
F	0.0072 (0.0012)	0.0161 (0.0019)	0.9764 (0.0025)		F	GM	HPQ	IBM	F	GM	HPQ	IBM		
GM	0.0207 (0.0046)	0.0326 (0.0046)	0.9397 (0.0101)	GM	0.7147 (0.0117)				0.5068 (0.0127)				0.2394 (0.0042)	100
HPQ	0.0193 (0.0083)	0.0247 (0.0073)	0.9604 (0.0134)	HPQ	0.3153 (0.0198)	0.2982^R (0.0130)			0.2659 (0.0148)	0.2982^R (0.0130)				
IBM	0.0065 (0.0017)	0.0649 (0.0083)	0.9314 (0.0090)	IBM	0.3456 (0.0180)	0.3409^R (0.0126)	0.4169^R (0.0117)		0.2794 (0.0152)	0.3403^R (0.0126)	0.4169^R (0.0117)			
TXN	0.0153 (0.0030)	0.0334 (0.0043)	0.9575 (0.0055)	TXN	0.2760^R (0.0130)	0.2935^R (0.0128)	0.4346^R (0.0115)	0.4026^R (0.0119)	0.2760^R (0.0130)	0.2935^R (0.0128)	0.4346^R (0.0115)	0.4026^R (0.0119)		

Table 7: Estimation results for one four-variate and the five-variate STCC–GARCH models (standard errors in parentheses) when the transition variable is a lagged absolute S&P 500 index returns averaged over seven days. The superscript R indicates that the correlation is restricted to be constant.

model	α_0	α	β	ρ_1	ρ_2	c	γ
F	0.0060 (0.0012)	0.0193 (0.0024)	0.9747 (0.0029)	0.6992 (0.0441)	0.2662 (0.0134)	−1.9766 (0.0292)	100
HPQ	0.0104 (0.0030)	0.0184 (0.0034)	0.9734 (0.0054)				
F	0.0065 (0.0013)	0.0209 (0.0026)	0.9726 (0.0031)	0.7159 (0.0442)	0.2829 (0.0132)	−2.1172 (0.0582)	100
IBM	0.0059 (0.0017)	0.0707 (0.0086)	0.9272 (0.0091)				
F	0.0067 (0.0014)	0.0209 (0.0027)	0.9726 (0.0032)	0.3347 (0.0176)	0.2145 (0.0189)	0.0139 (0.0225)	100
TXN	0.0102 (0.0023)	0.0320 (0.0042)	0.9625 (0.0050)				
GM	0.0263 (0.0060)	0.0463 (0.0062)	0.9174 (0.0136)	0.7160 (0.0430)	0.2777 (0.0133)	−1.9667 (0.0329)	100
HPQ	0.0142 (0.0046)	0.0218 (0.0047)	0.9670 (0.0079)				
GM	0.0205 (0.0042)	0.0414 (0.0048)	0.9302 (0.0098)	0.8898 (0.0459)	0.3048 (0.0145)	−2.2036 (0.2217)	1.92
IBM	0.0058 (0.0016)	0.0694 (0.0082)	0.9284 (0.0087)				
GM	0.0250 (0.0055)	0.0462 (0.0061)	0.9199 (0.0126)	0.4290 (0.0300)	0.1995 (0.0268)	−0.1817 (0.1552)	3.95
TXN	0.0116 (0.0026)	0.0334 (0.0044)	0.9602 (0.0054)				
HPQ	0.0204 (0.0074)	0.0276 (0.0067)	0.9563 (0.0122)	0.7846 (0.0325)	0.4012 (0.0121)	−1.7490 (0.0185)	100
IBM	0.0085 (0.0021)	0.0779 (0.0094)	0.9166 (0.0103)				
HPQ	0.0280 (0.0113)	0.0336 (0.0097)	0.9450 (0.0178)	0.4786 (0.0128)	0.3372 (0.0218)	0.2946 (0.0123)	100
TXN	0.0149 (0.0033)	0.0336 (0.0048)	0.9576 (0.0062)				
IBM	0.0098 (0.0023)	0.0775 (0.0097)	0.9150 (0.0109)	0.4468 (0.0131)	0.2845 (0.0241)	0.3514 (0.0120)	100
TXN	0.0136 (0.0028)	0.0333 (0.0043)	0.9588 (0.0053)				

Table 8: Estimation results for all bivariate STCC–GARCH models (standard errors in parentheses) when the transition variable is a lagged S&P 500 index return over two days.

model	α_0	α	β		$\boldsymbol{P}_1$		$\boldsymbol{P}_2$		c	γ
F	0.0067	0.0176	0.9755		F	GM	F	GM	0.1725	100
	(0.0012)	(0.0020)	(0.0025)						(0.0283)	
GM	0.0184	0.0340	0.9407	GM	0.6118		0.5130			
	(0.0036)	(0.0042)	(0.0085)		(0.0108)		(0.0158)			
TXN	0.0113	0.0327	0.9611	TXN	0.3212	0.3409	0.2049	0.2161		
	(0.0025)	(0.0043)	(0.0051)		(0.0162)	(0.0160)	(0.0214)	(0.0163)		
F	0.0062	0.0192	0.9745		F	HPQ	F	HPQ	-1.9711	100
	(0.0013)	(0.0024)	(0.0029)						(0.0198)	
HPQ	0.0127	0.0196	0.9702	HPQ	0.7007		0.2662			
	(0.0038)	(0.0039)	(0.0066)		(0.0447)		(0.0134)			
TXN	0.0135	0.0327	0.9591	TXN	0.5253	0.7732	0.2629	0.4228		
	(0.0028)	(0.0066)	(0.0055)		(0.0561)	(0.0391)	(0.0134)	(0.0118)		
GM	0.0266	0.0454	0.9177		GM	HPQ	GM	HPQ	-1.9632	100
	(0.0059)	(0.0060)	(0.0132)						(0.0240)	
HPQ	0.0175	0.0237	0.9623	HPQ	0.7092		0.2785			
	(0.0059)	(0.0054)	(0.0097)		(0.0450)		(0.0132)			
TXN	0.0155	0.0346	0.9559	TXN	0.5730	0.7687	0.2785	0.4237		
	(0.0032)	(0.0047)	(0.0061)		(0.0536)	(0.0410)	(0.0133)	(0.0117)		
HPQ	0.0337	0.0369	0.9372		HPQ	IBM	HPQ	IBM	0.2559	100
	(0.0168)	(0.0131)	(0.0254)						(0.0283)	
IBM	0.0095	0.0757	0.9170	IBM	0.4619		0.3261			
	(0.0022)	(0.0095)	(0.0107)		(0.0136)		(0.0216)			
TXN	0.0160	0.0338	0.9566	TXN	0.4811	0.4536	0.3459	0.3003		
	(0.0032)	(0.0044)	(0.0058)		(0.0132)	(0.0138)	(0.0219)	(0.0223)		

Table 9: Estimation results for selected combinations of trivariate STCC–GARCH models (standard errors in parentheses) when the transition variable is a lagged S&P 500 index return over two days.

model	α_0	α	β		$\boldsymbol{P}_1$				$\boldsymbol{P}_2$				c	γ
F	0.0062 (0.0011)	0.0171 (0.0019)	0.9764 (0.0023)		F	GM	HPQ		F	GM	HPQ		−1.9731 (0.0205)	100
GM	0.0172 (0.0033)	0.0312 (0.0037)	0.9446 (0.0077)	GM	0.6576^r (0.0496)				0.5683^r (0.0097)					
HPQ	0.0117 (0.0034)	0.0187 (0.0036)	0.9718 (0.0060)	HPQ	0.6319 (0.0588)	0.6475 (0.0532)			0.2679 (0.0133)	0.2800 (0.0132)				
IBM	0.0050 (0.0014)	0.0607 (0.0075)	0.9366 (0.0081)	IBM	0.6470 (0.0498)	0.7440 (0.0394)	0.7379 (0.0442)		0.2843 (0.0132)	0.3208 (0.0129)	0.4026 (0.0120)			
F	0.0067 (0.0011)	0.0172 (0.0019)	0.9759 (0.0024)		F	GM	HPQ	IBM	F	GM	HPQ	IBM		
GM	0.0170 (0.0034)	0.0322 (0.0040)	0.9444 (0.0079)	GM	0.6113 (0.0108)				0.5133 (0.0159)				0.1850 (0.0083)	100
HPQ	0.0154 (0.0059)	0.0218 (0.0056)	0.9662 (0.0099)	HPQ	0.3248 (0.0161)	0.3302 (0.0160)			0.2155 (0.0211)	0.2409 (0.0207)				
IBM	0.0065 (0.0017)	0.0650 (0.0084)	0.9312 (0.0092)	IBM	0.3430 (0.0155)	0.3730 (0.0151)	0.4638 (0.0137)		0.2264 (0.0218)	0.2779 (0.0210)	0.3408 (0.0194)			
TXN	0.0144 (0.0028)	0.0327 (0.0040)	0.9587 (0.0051)	TXN	0.3198 (0.0160)	0.3399 (0.0157)	0.4795 (0.0136)	0.4565 (0.0138)	0.2041 (0.0217)	0.2165 (0.0214)	0.3651 (0.0187)	0.3152 (0.0203)		

Table 10: Estimation results for one four-variate and the five-variate STCC–GARCH model (standard errors in parentheses) when the transition variable is a lagged S&P 500 index return over two days. When testing the hypothesis of partially constant correlations in the four-variate model, Wald test fails to reject and LM-test barely rejects the constancy of the parameters indicated by a superscript r (p-value of the Wald statistic W_{PCCC} is 0.2646 and of the LM-statistic LM_{PCCC} is 0.0095).

Appendix

Construction of LM(/Wald)-statistic

Let $\boldsymbol{\theta}_0$ be the vector of true parameters. Under suitable assumptions and regularity conditions,

$$\sqrt{T}^{-1}\frac{\partial l(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}} \xrightarrow{d} N(0, \mathfrak{I}(\boldsymbol{\theta}_0)). \tag{13}$$

To derive LM-statistics of the constant conditional correlation hypothesis $\boldsymbol{\rho}_2^* = 0$ consider the following quadratic form:

$$T^{-1}\frac{\partial l(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}'}\mathfrak{I}(\boldsymbol{\theta}_0)^{-1}\frac{\partial l(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}} = T^{-1}\left(\sum_{t=1}^{T}\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}'}\right)\mathfrak{I}(\boldsymbol{\theta}_0)^{-1}\left(\sum_{t=1}^{T}\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}}\right)$$

and evaluate it at the maximum likelihood estimators under the restriction $\boldsymbol{\rho}_2^* = 0$. The limiting information matrix $\mathfrak{I}(\boldsymbol{\theta}_0)$ is replaced by the consistent estimator

$$\hat{\mathfrak{I}}_T(\boldsymbol{\theta}_0) = T^{-1}\sum_{t=1}^{T}E\left[\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}}\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}'} \mid \mathcal{F}_{t-1}\right]. \tag{14}$$

The following derivations are straightforward implications of the definitions and elementary rules of matrix algebra. Results in Anderson (2003) and Lütkepohl (1996) are heavily relied upon.

Test of constant conditional correlations against an STCC–GARCH model

To construct the test statistic we introduce some simplifying notation. Let $\boldsymbol{\omega}_i = (\alpha_{i0}, \alpha_i, \beta_i)'$, $i = 1, \ldots, N$, denote the parameter vectors of the GARCH equations, and $\boldsymbol{\rho}^* = (\boldsymbol{\rho}_1^{*\prime}, \boldsymbol{\rho}_2^{*\prime})'$, where $\boldsymbol{\rho}_1^* = vecl\boldsymbol{P}_1^*$ and $\boldsymbol{\rho}_2^* = vecl\boldsymbol{P}_2^*$ are the vectors holding all the unique off-diagonal elements in the two matrices $\boldsymbol{P}_1^*$ and $\boldsymbol{P}_2^*$, respectively. The notation $vecl\boldsymbol{P}$ is used to denote the vec-operator applied to the strictly lower triangular part of the matrix $\boldsymbol{P}$. Let $\boldsymbol{\theta} = (\boldsymbol{\omega}_1', \ldots, \boldsymbol{\omega}_N', \boldsymbol{\rho}^{*\prime})'$ be the full parameter vector and $\boldsymbol{\theta}_0$ the corresponding vector of true parameters under the null. The linearized time-varying correlation matrix is $\boldsymbol{P}_t^* = \boldsymbol{P}_1^* - s_t\boldsymbol{P}_2^*$ as defined in (9). Furthermore, let $\boldsymbol{v}_{it} = (1, y_{it}^2, h_{it})'$, $i = 1, \ldots, N$, and $\boldsymbol{v}_{\boldsymbol{\rho}^*t} = (1, -s_t)'$. Symbols $\otimes$ and $\odot$ represent the Kronecker and Hadamard products of two matrices, respectively. Let 1_i be a $N \times 1$ vector of zeros with ith element equal to one and $\mathbf{1}_n$ be a $n \times n$ matrix of ones. The identity matrix $\boldsymbol{I}$ is of size N unless otherwise indicated by a subscript.

Consider the log-likelihood function for observation t as defined in (7) with linearized time-varying correlation matrix:

$$l_t(\boldsymbol{\theta}) = -\frac{N}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{N}\log(h_{it}) - \frac{1}{2}\log|\boldsymbol{P}_t^*| - \frac{1}{2}\boldsymbol{z}_t'\boldsymbol{P}_t^{*-1}\boldsymbol{z}_t.$$

The first order derivatives of the log-likelihood function with respect to the GARCH and correlation parameters are

$$\begin{aligned}\frac{\partial l_t(\boldsymbol{\theta})}{\partial\boldsymbol{\omega}_i} &= -\frac{1}{2h_{it}}\frac{\partial h_{it}}{\partial\boldsymbol{\omega}_i}\left\{1 - z_{it}1_i'\boldsymbol{P}_t^{*-1}\boldsymbol{z}_t\right\}, \quad i = 1, \ldots, N,\\ \frac{\partial l_t(\boldsymbol{\theta})}{\partial\boldsymbol{\rho}^*} &= -\frac{1}{2}\frac{\partial(vec\boldsymbol{P}_t^*)'}{\partial\boldsymbol{\rho}^*}\left\{vec\boldsymbol{P}_t^{*-1} - \left(\boldsymbol{P}_t^{*-1}\otimes\boldsymbol{P}_t^{*-1}\right)(\boldsymbol{z}_t\otimes\boldsymbol{z}_t)\right\},\end{aligned}$$

where

$$\begin{aligned}\frac{\partial h_{it}}{\partial\boldsymbol{\omega}_i} &= \boldsymbol{v}_{i,t-1} + \beta_i\frac{\partial h_{i,t-1}}{\partial\boldsymbol{\omega}_i}, \quad i = 1, \ldots, N,\\ \frac{\partial(vec\boldsymbol{P}_t^*)'}{\partial\boldsymbol{\rho}^*} &= \boldsymbol{v}_{\boldsymbol{\rho}^*t}\otimes\boldsymbol{U}'.\end{aligned}$$

The matrix $\boldsymbol{U}$ is an $N^2 \times \frac{N(N-1)}{2}$ matrix of zeros and ones, whose columns are defined as

$$\left[vec\left(1_i1_j' + 1_j1_i'\right)\right]_{i=1,\ldots N-1, j=i+1,\ldots,N}$$

and the columns appear in the same order from left to right as the indices in $vecl\boldsymbol{P}_t$. Under the null hypothesis $\boldsymbol{\rho}_2^* = 0$, and thus the derivatives at the true parameter values under the null can be written as

$$\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\omega}_i} = -\frac{1}{2h_{it}}\frac{\partial h_{it}(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\omega}_i}\left\{1 - z_{it}1_i'\boldsymbol{P}_1^{*-1}\boldsymbol{z}_t\right\}, \quad i = 1, \ldots, N, \tag{15}$$

$$\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\rho}^*} = -\frac{1}{2}\frac{\partial\left(vec\boldsymbol{P}_t^*(\boldsymbol{\theta}_0)\right)'}{\partial \boldsymbol{\rho}^*}\left\{vec\boldsymbol{P}_1^{*-1} - \left(\boldsymbol{P}_1^{*-1}\otimes\boldsymbol{P}_1^{*-1}\right)(\boldsymbol{z}_t\otimes\boldsymbol{z}_t)\right\}. \tag{16}$$

Taking conditional expectations of the cross products of (15) and (16) yields, for $i, j = 1, \ldots, N$,

$$\begin{aligned}
E_{t-1}\left[\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\omega}_i}\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\omega}_i'}\right] &= \frac{1}{4h_{it}^2}\frac{\partial h_{it}(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\omega}_i}\frac{\partial h_{it}(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\omega}_i'}\left(1 + 1_i'\boldsymbol{P}_1^{*-1}1_i\right), \\
E_{t-1}\left[\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\omega}_i}\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\omega}_j'}\right] &= \frac{1}{4h_{it}h_{jt}}\frac{\partial h_{it}(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\omega}_i}\frac{\partial h_{jt}(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\omega}_j'}\left(\rho_{1,ij}^*1_i'\boldsymbol{P}_1^{*-1}1_j\right), \quad i \neq j \\
E_{t-1}\left[\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\rho}^*}\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\rho}^{*\prime}}\right] &= \frac{1}{4}\frac{\partial\left(vec\boldsymbol{P}_t^*(\boldsymbol{\theta}_0)\right)'}{\partial \boldsymbol{\rho}^*}\left(\boldsymbol{P}_1^{*-1}\otimes\boldsymbol{P}_1^{*-1} + \left(\boldsymbol{P}_1^{*-1}\otimes\boldsymbol{I}\right)\boldsymbol{K}\left(\boldsymbol{P}_1^{*-1}\otimes\boldsymbol{I}\right)\right)\frac{\partial vec\boldsymbol{P}_t^*(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\rho}^{*\prime}}, \\
E_{t-1}\left[\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\omega}_i}\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\rho}^{*\prime}}\right] &= \frac{1}{4h_{it}}\frac{\partial h_{it}(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\omega}_i}\left(1_i'\boldsymbol{P}_1^{*-1}\otimes 1_i' + 1_i'\otimes 1_i'\boldsymbol{P}_1^{*-1}\right)\frac{\partial vec\boldsymbol{P}_t^*(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\rho}^{*\prime}},
\end{aligned} \tag{17}$$

where

$$\boldsymbol{K} = \begin{bmatrix} 1_1 1_1' & \cdots & 1_N 1_1' \\ \vdots & \ddots & \vdots \\ 1_1 1_N' & \cdots & 1_N 1_N' \end{bmatrix}. \tag{18}$$

Expressions (17) for the conditional expectations follow from the fact that for a model with general correlation matrix $\boldsymbol{P}_t$,

$$\begin{aligned}
E_{t-1}\left[\boldsymbol{z}_t\boldsymbol{z}_t'\otimes\boldsymbol{z}_t\boldsymbol{z}_t'\right] &= \left(\boldsymbol{I}_{N^2} + \boldsymbol{K}\right)(\boldsymbol{P}_t\otimes\boldsymbol{P}_t) + vec\boldsymbol{P}_t\,(vec\boldsymbol{P}_t)' \\
&= (\boldsymbol{P}_t\otimes\boldsymbol{P}_t) + (\boldsymbol{I}\otimes\boldsymbol{P}_t)\boldsymbol{K}(\boldsymbol{I}\otimes\boldsymbol{P}_t) + vec\boldsymbol{P}_t\,(vec\boldsymbol{P}_t)'
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
E_{t-1}\left[z_{it}\boldsymbol{z}_t'\otimes\boldsymbol{z}_t\boldsymbol{z}_t'\right] &= E_{t-1}\left[\boldsymbol{z}_t\boldsymbol{z}_t'\otimes\boldsymbol{z}_t\boldsymbol{z}_t'\right](1_i\otimes\boldsymbol{I}), \\
E_{t-1}\left[z_{it}z_{jt}'\otimes\boldsymbol{z}_t\boldsymbol{z}_t'\right] &= \left(1_i'\otimes\boldsymbol{I}\right)E_{t-1}\left[\boldsymbol{z}_t\boldsymbol{z}_t'\otimes\boldsymbol{z}_t\boldsymbol{z}_t'\right](1_j\otimes\boldsymbol{I}),
\end{aligned}$$

see Anderson (2003). In the present case $\boldsymbol{P}_t$ is replaced with $\boldsymbol{P}_1^*$.

The estimator for the information matrix is obtained by making use of the submatrices in (17). For a more compact expression, let $\boldsymbol{x}_t = (\boldsymbol{x}_{1t}', \ldots, \boldsymbol{x}_{Nt}')'$ where $\boldsymbol{x}_{it} = -\frac{1}{2h_{it}}\frac{\partial h_{it}}{\partial \boldsymbol{\omega}_i}$, and let $\boldsymbol{x}_{\boldsymbol{\rho}^*t} = -\frac{1}{2}\boldsymbol{v}_{\boldsymbol{\rho}^*t}\otimes\boldsymbol{U}'$, and let $\boldsymbol{x}_{it}^0$, $i = 1, \ldots, N, \boldsymbol{\rho}^*$, denote the corresponding expressions evaluated at the true values under the null hypothesis. Setting

$$\begin{aligned}
\boldsymbol{M}_1 &= T^{-1}\sum_{t=1}^{T}\boldsymbol{x}_t^0\boldsymbol{x}_t^{0\prime}\odot\left(\left(\boldsymbol{I} + \boldsymbol{P}_1^*\odot\boldsymbol{P}_1^{*-1}\right)\otimes\boldsymbol{1}_3\right), \\
\boldsymbol{M}_2 &= T^{-1}\sum_{t=1}^{T}\begin{bmatrix}\boldsymbol{x}_{1t}^0 & & 0 \\ & \ddots & \\ 0 & & \boldsymbol{x}_{Nt}^0\end{bmatrix}\begin{bmatrix}1_1'\boldsymbol{P}_1^{*-1}\otimes 1_1' + 1_1'\otimes 1_1'\boldsymbol{P}_1^{*-1} \\ \vdots \\ 1_N'\boldsymbol{P}_1^{*-1}\otimes 1_N' + 1_N'\otimes 1_N'\boldsymbol{P}_1^{*-1}\end{bmatrix}\boldsymbol{x}_{\boldsymbol{\rho}^*t}^{0\prime}, \\
\boldsymbol{M}_3 &= T^{-1}\sum_{t=1}^{T}\boldsymbol{x}_{\boldsymbol{\rho}^*t}^0\left(\boldsymbol{P}_1^{*-1}\otimes\boldsymbol{P}_1^{*-1} + \left(\boldsymbol{P}_1^{*-1}\otimes\boldsymbol{I}\right)\boldsymbol{K}\left(\boldsymbol{P}_1^{*-1}\otimes\boldsymbol{I}\right)\right)\boldsymbol{x}_{\boldsymbol{\rho}^*t}^{0\prime},
\end{aligned}$$

the information matrix $\mathfrak{I}(\boldsymbol{\theta}_0)$ is approximated by

$$\begin{aligned}
\hat{\mathfrak{I}}_T(\boldsymbol{\theta}_0) &= T^{-1}\sum_{t=1}^{T}E\left[\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\theta}}\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\theta}'}\mid\mathcal{F}_{t-1}\right] \\
&= \begin{bmatrix}\boldsymbol{M}_1 & \boldsymbol{M}_2 \\ \boldsymbol{M}_2' & \boldsymbol{M}_3\end{bmatrix}
\end{aligned}$$

The block corresponding to the correlation parameters of the inverse of $\hat{\Im}_T(\boldsymbol{\theta}_0)$ can be calculated as

$$\left(\boldsymbol{M}_3 - \boldsymbol{M}_2'\boldsymbol{M}_1^{-1}\boldsymbol{M}_2\right)^{-1}$$

from where the south-east $\frac{N(N-1)}{2} \times \frac{N(N-1)}{2}$ block corresponding to $\boldsymbol{\rho}_2^*$ can be extracted. Replacing the true unknown values with maximum likelihood estimators, the test statistic simplifies to

$$LM = T^{-1}\left(\sum_{t=1}^{T}\frac{\partial l_t(\hat{\boldsymbol{\theta}})}{\partial \boldsymbol{\rho}_2^{*\prime}}\right)\left[\hat{\Im}_T(\hat{\boldsymbol{\theta}})\right]^{-1}_{(\boldsymbol{\rho}_2^*,\boldsymbol{\rho}_2^*)}\left(\sum_{t=1}^{T}\frac{\partial l_t(\hat{\boldsymbol{\theta}})}{\partial \boldsymbol{\rho}_2^*}\right) \tag{20}$$

where $[\hat{\Im}_T(\hat{\boldsymbol{\theta}})]^{-1}_{(\boldsymbol{\rho}_2^*,\boldsymbol{\rho}_2^*)}$ is the block of the inverse of $\hat{\Im}_T$ corresponding to those correlation parameters that are set to zero under the null. It follows from (13) and consistency and asymptotic normality of ML estimators that the statistic (20) has an asymptotic $\chi^2_{\frac{N(N-1)}{2}}$ distribution when the null hypothesis is valid.

Test of constant conditional correlations against partially constant STCC–GARCH model

In this case the null model is a CCC–GARCH model, and the alternative model is partially constant STCC–GARCH model. Let there be k pairs of variables with constant correlations in the alternative model. The test is as above, but with the following changes to definitions and notations. The linearized time-varying correlation matrix is $\boldsymbol{P}_t^* = \boldsymbol{P}_1^* - s_t\boldsymbol{P}_2^{P*}$, where $\boldsymbol{P}_2^{P*}$ is as $\boldsymbol{P}_2^*$ but with the elements corresponding to the constant correlations under the alternative set to zero. The vector of correlation parameters is $\boldsymbol{\rho}^* = (\boldsymbol{\rho}_1^{*\prime}, \boldsymbol{\rho}_2^{P*\prime})'$, where $\boldsymbol{\rho}_2^{P*} = vecl\boldsymbol{P}_2^{P*}$ but with the elements corresponding to the constant correlations under the alternative being deleted. Furthermore,

$$\frac{\partial\left(vec\boldsymbol{P}_t^*\right)'}{\partial\boldsymbol{\rho}^*}$$

is as above, but with k rows deleted so that the remaining rows are corresponding to the elements in $\boldsymbol{\rho}^* = (\boldsymbol{\rho}_1^{*\prime}, \boldsymbol{\rho}_2^{P*\prime})'$. The same rows are also deleted from $\boldsymbol{x}_{\boldsymbol{\rho}^*t}$. With these modifications the test statistic is as in (20) above, and the asymptotic distribution under the null hypothesis is χ^2_R where R is the number of restrictions to be tested.

Test of partially constant correlations against an STCC–GARCH model

To construct the test statistic we introduce some simplifying notation. Let $\boldsymbol{\omega}_i = (\alpha_{i0}, \alpha_i, \beta_i)'$, $i = 1,\ldots,N$, denote the parameter vectors of the GARCH equations, $\boldsymbol{\rho} = (\boldsymbol{\rho}_1', \boldsymbol{\rho}_2')'$, where $\boldsymbol{\rho}_1 = vecl\boldsymbol{P}_1$ and $\boldsymbol{\rho}_2 = vecl\boldsymbol{P}_2$, and $\boldsymbol{\varphi} = (c,\gamma)'$. Let $\boldsymbol{\theta} = (\boldsymbol{\omega}_1',\ldots,\boldsymbol{\omega}_N',\boldsymbol{\rho}',\boldsymbol{\varphi}')'$ be the full parameter vector and $\boldsymbol{\theta}_0$ the corresponding vector of true parameters under the null. The time-varying correlation matrix is $\boldsymbol{P}_t = (1-G_t)\boldsymbol{P}_1 + G_t\boldsymbol{P}_2$ as defined in (5) and (6). Furthermore, let $\boldsymbol{v}_{it} = (1, y_{it}^2, h_{it})'$, $i = 1,\ldots,N$, $\boldsymbol{v}_{\boldsymbol{\rho}t} = (1-G_t, G_t)'$ and $\boldsymbol{v}_{\boldsymbol{\varphi}t} = (-\gamma, s_t - c)'$. Symbols $\otimes$ and $\odot$ represent the Kronecker and Hadamard products of two matrices, respectively. Let 1_i be a $1\times N$ vector of zeros with ith element equal to one and $\mathbf{1}_n$ be a $n \times n$ matrix of ones. The identity matrix $\boldsymbol{I}$ is of size N unless otherwise indicated by a subscript.

Consider the log-likelihood function for observation t as defined in (7):

$$l_t(\boldsymbol{\theta}) = -\frac{N}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{N}\log(h_{it}) - \frac{1}{2}\log|\boldsymbol{P}_t| - \frac{1}{2}\boldsymbol{z}_t'\boldsymbol{P}_t^{-1}\boldsymbol{z}_t.$$

The elements of the score for observation t are

$$\begin{aligned}
\frac{\partial l_t(\boldsymbol{\theta})}{\partial\boldsymbol{\omega}_i} &= -\frac{1}{2h_{it}}\frac{\partial h_{it}(\boldsymbol{\theta})}{\partial\boldsymbol{\omega}_i}\left\{1 - z_{it}1_i'\boldsymbol{P}_t^{-1}\boldsymbol{z}_t\right\}, \quad i = 1,\ldots,N,\\
\frac{\partial l_t(\boldsymbol{\theta})}{\partial\boldsymbol{\rho}} &= -\frac{1}{2}\frac{\partial\left(vec\boldsymbol{P}_t(\boldsymbol{\theta})\right)'}{\partial\boldsymbol{\rho}}\left\{vec\boldsymbol{P}_t^{-1} - \left(\boldsymbol{P}_t^{-1}\otimes\boldsymbol{P}_t^{-1}\right)(\boldsymbol{z}_t\otimes\boldsymbol{z}_t)\right\},\\
\frac{\partial l_t(\boldsymbol{\theta})}{\partial\boldsymbol{\varphi}} &= -\frac{1}{2}\frac{\partial\left(vec\boldsymbol{P}_t(\boldsymbol{\theta})\right)'}{\partial\boldsymbol{\varphi}}\left\{vec\boldsymbol{P}_t^{-1} - \left(\boldsymbol{P}_t^{-1}\otimes\boldsymbol{P}_t^{-1}\right)(\boldsymbol{z}_t\otimes\boldsymbol{z}_t)\right\},
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial h_{it}}{\partial \boldsymbol{\omega}_i} &= \boldsymbol{v}_{i,t-1} + \beta_i \frac{\partial h_{i,t-1}}{\partial \boldsymbol{\omega}_i}, \quad i = 1, \ldots, N, \\
\frac{\partial \left(vec\boldsymbol{P}_t\right)'}{\partial \boldsymbol{\rho}} &= \boldsymbol{v}_{\boldsymbol{\rho} t} \otimes \boldsymbol{U}', \\
\frac{\partial \left(vec\boldsymbol{P}_t\right)'}{\partial \boldsymbol{\varphi}} &= \boldsymbol{v}_{\boldsymbol{\varphi} t}(1-G_t)G_t vec\left(\boldsymbol{P}_1 - \boldsymbol{P}_2\right)'.
\end{aligned}$$

Next we evaluate the score at the true parameters under the null. Taking conditional expectations of the outer product of the score using (19) gives

$$\begin{aligned}
E_{t-1}\left[\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\omega}_i}\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\omega}_i'}\right] &= \frac{1}{4h_{it}^2}\frac{\partial h_{it}(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\omega}_i}\frac{\partial h_{it}(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\omega}_i'}\left(1 + 1_i'\boldsymbol{P}_t^{-1}1_i\right), \\
E_{t-1}\left[\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\omega}_i}\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\omega}_j'}\right] &= \frac{1}{4h_{it}h_{jt}}\frac{\partial h_{it}(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\omega}_i}\frac{\partial h_{jt}(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\omega}_j'}\left(\rho_{t,ij}1_i'\boldsymbol{P}_t^{-1}1_j\right), \quad i \neq j \\
E_{t-1}\left[\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\omega}_i}\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\rho}'}\right] &= \frac{1}{4h_{it}}\frac{\partial h_{it}(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\omega}_i}\left(1_i'\boldsymbol{P}_t^{-1} \otimes 1_i' + 1_i' \otimes 1_i'\boldsymbol{P}_t^{-1}\right)\frac{\partial vec\boldsymbol{P}_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\rho}'}, \\
E_{t-1}\left[\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\omega}_i}\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\varphi}'}\right] &= \frac{1}{4h_{it}}\frac{\partial h_{it}(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\omega}_i}\left(1_i'\boldsymbol{P}_t^{-1} \otimes 1_i' + 1_i' \otimes 1_i'\boldsymbol{P}_t^{-1}\right)\frac{\partial vec\boldsymbol{P}_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\varphi}'}, \\
E_{t-1}\left[\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\rho}}\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\rho}'}\right] &= \frac{1}{4}\frac{\partial \left(vec\boldsymbol{P}_t(\boldsymbol{\theta}_0)\right)'}{\partial \boldsymbol{\rho}}\left(\boldsymbol{P}_t^{-1} \otimes \boldsymbol{P}_t^{-1} + \left(\boldsymbol{P}_t^{-1} \otimes \boldsymbol{I}\right)\boldsymbol{K}\left(\boldsymbol{P}_t^{-1} \otimes \boldsymbol{I}\right)\right)\frac{\partial vec\boldsymbol{P}_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\rho}'}, \\
E_{t-1}\left[\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\rho}}\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\varphi}'}\right] &= \frac{1}{4}\frac{\partial \left(vec\boldsymbol{P}_t(\boldsymbol{\theta}_0)\right)'}{\partial \boldsymbol{\rho}}\left(\boldsymbol{P}_t^{-1} \otimes \boldsymbol{P}_t^{-1} + \left(\boldsymbol{P}_t^{-1} \otimes \boldsymbol{I}\right)\boldsymbol{K}\left(\boldsymbol{P}_t^{-1} \otimes \boldsymbol{I}\right)\right)\frac{\partial vec\boldsymbol{P}_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\varphi}'}, \\
E_{t-1}\left[\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\varphi}}\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\varphi}'}\right] &= \frac{1}{4}\frac{\partial \left(vec\boldsymbol{P}_t(\boldsymbol{\theta}_0)\right)'}{\partial \boldsymbol{\varphi}}\left(\boldsymbol{P}_t^{-1} \otimes \boldsymbol{P}_t^{-1} + \left(\boldsymbol{P}_t^{-1} \otimes \boldsymbol{I}\right)\boldsymbol{K}\left(\boldsymbol{P}_t^{-1} \otimes \boldsymbol{I}\right)\right)\frac{\partial vec\boldsymbol{P}_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\varphi}'}, \qquad (21)
\end{aligned}$$

where $\boldsymbol{K}$ is defined as in (18).

The estimator for the infomation matrix is obtained by using the submatrices in (21). To derive a more compact expression for the information matrix, let $\boldsymbol{x}_t = (\boldsymbol{x}_{1t}', \ldots, \boldsymbol{x}_{Nt}')'$ where $\boldsymbol{x}_{it} = -\frac{1}{2h_{it}}\frac{\partial h_{it}}{\partial \boldsymbol{\omega}_i}$, let $\boldsymbol{x}_{\boldsymbol{\rho} t} = -\frac{1}{2}\boldsymbol{v}_{\boldsymbol{\rho} t} \otimes \boldsymbol{U}'$, and $\boldsymbol{x}_{\boldsymbol{\varphi} t} = -\frac{1}{2}\boldsymbol{v}_{\boldsymbol{\varphi} t}(1-G_t)G_t vec(\boldsymbol{P}_1 - \boldsymbol{P}_2)'$ and let $\boldsymbol{x}_{it}^0$, $i = 1, \ldots, N, \boldsymbol{\rho}, \boldsymbol{\varphi}$, denote the corresponding expressions evaluated at the true values under the null hypothesis. Setting

$$\begin{aligned}
\boldsymbol{M}_1 &= T^{-1}\sum_{t=1}^{T}\boldsymbol{x}_t^0\boldsymbol{x}_t^{0\prime} \odot \left(\left(\boldsymbol{I} + \boldsymbol{P}_t \odot \boldsymbol{P}_t^{-1}\right) \otimes \boldsymbol{1}_3\right), \\
\boldsymbol{M}_2 &= T^{-1}\sum_{t=1}^{T}\begin{bmatrix}\boldsymbol{x}_{1t}^0 & & 0 \\ & \ddots & \\ 0 & & \boldsymbol{x}_{Nt}^0\end{bmatrix}\begin{bmatrix}1_1'\boldsymbol{P}_t^{-1} \otimes 1_1' + 1_1' \otimes 1_1'\boldsymbol{P}_t^{-1} \\ \vdots \\ 1_N'\boldsymbol{P}_t^{-1} \otimes 1_N' + 1_N' \otimes 1_N'\boldsymbol{P}_t^{-1}\end{bmatrix}\boldsymbol{x}_{\boldsymbol{\rho} t}^{0\prime}, \\
\boldsymbol{M}_3 &= T^{-1}\sum_{t=1}^{T}\boldsymbol{x}_{\boldsymbol{\rho} t}^0\left(\boldsymbol{P}_t^{-1} \otimes \boldsymbol{P}_t^{-1} + \left(\boldsymbol{P}_t^{-1} \otimes \boldsymbol{I}\right)\boldsymbol{K}\left(\boldsymbol{P}_t^{-1} \otimes \boldsymbol{I}\right)\right)\boldsymbol{x}_{\boldsymbol{\rho} t}^{0\prime}, \\
\boldsymbol{M}_4 &= T^{-1}\sum_{t=1}^{T}\begin{bmatrix}\boldsymbol{x}_{1t}^0 & & 0 \\ & \ddots & \\ 0 & & \boldsymbol{x}_{Nt}^0\end{bmatrix}\begin{bmatrix}1_1'\boldsymbol{P}_t^{-1} \otimes 1_1' + 1_1' \otimes 1_1'\boldsymbol{P}_t^{-1} \\ \vdots \\ 1_N'\boldsymbol{P}_t^{-1} \otimes 1_N' + 1_N' \otimes 1_N'\boldsymbol{P}_t^{-1}\end{bmatrix}\boldsymbol{x}_{\boldsymbol{\varphi} t}^{0\prime}, \\
\boldsymbol{M}_5 &= T^{-1}\sum_{t=1}^{T}\boldsymbol{x}_{\boldsymbol{\rho} t}^0\left(\boldsymbol{P}_t^{-1} \otimes \boldsymbol{P}_t^{-1} + \left(\boldsymbol{P}_t^{-1} \otimes \boldsymbol{I}\right)\boldsymbol{K}\left(\boldsymbol{P}_t^{-1} \otimes \boldsymbol{I}\right)\right)\boldsymbol{x}_{\boldsymbol{\varphi} t}^{0\prime}, \\
\boldsymbol{M}_6 &= T^{-1}\sum_{t=1}^{T}\boldsymbol{x}_{\boldsymbol{\varphi} t}^0\left(\boldsymbol{P}_t^{-1} \otimes \boldsymbol{P}_t^{-1} + \left(\boldsymbol{P}_t^{-1} \otimes \boldsymbol{I}\right)\boldsymbol{K}\left(\boldsymbol{P}_t^{-1} \otimes \boldsymbol{I}\right)\right)\boldsymbol{x}_{\boldsymbol{\varphi} t}^{0\prime},
\end{aligned}$$

the information matrix $\mathfrak{I}(\boldsymbol{\theta}_0)$ is approximated by

$$\begin{aligned}
\hat{\mathfrak{I}}_T(\boldsymbol{\theta}_0) &= T^{-1}\sum_{t=1}^{T}E\left[\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\theta}}\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\theta}'} \mid \mathcal{F}_{t-1}\right] \\
&= \begin{bmatrix}\boldsymbol{M}_1 & \boldsymbol{M}_2 & \boldsymbol{M}_4 \\ \boldsymbol{M}_2' & \boldsymbol{M}_3 & \boldsymbol{M}_5 \\ \boldsymbol{M}_4' & \boldsymbol{M}_5' & \boldsymbol{M}_6\end{bmatrix}.
\end{aligned}$$

The block of the inverse of $\hat{\Im}_T(\boldsymbol{\theta}_0)$ corresponding to the correlation and transition parameters is given by

$$\left(\begin{bmatrix} \boldsymbol{M}_3 & \boldsymbol{M}_5 \\ \boldsymbol{M}_5' & \boldsymbol{M}_6 \end{bmatrix} - \begin{bmatrix} \boldsymbol{M}_2' \\ \boldsymbol{M}_4' \end{bmatrix} \boldsymbol{M}_1^{-1} \begin{bmatrix} \boldsymbol{M}_2 & \boldsymbol{M}_4 \end{bmatrix} \right)^{-1}$$

from which the south-east $N(N-1) \times N(N-1)$ block $[\hat{\Im}_T(\boldsymbol{\theta}_0)]^{-1}_{(\boldsymbol{\rho},\boldsymbol{\rho})}$ corresponding to the correlation parameters can be extracted. The test statistic evaluated at the restricted maximum likelihood estimates is then

$$LM_{PCCC} = T^{-1}\boldsymbol{q}(\hat{\boldsymbol{\theta}})' \left[\hat{\Im}_T(\hat{\boldsymbol{\theta}})\right]^{-1}_{(\boldsymbol{\rho},\boldsymbol{\rho})} \boldsymbol{q}(\hat{\boldsymbol{\theta}}). \tag{22}$$

In (22), $\boldsymbol{q}(\hat{\boldsymbol{\theta}})$ is the $N(N-1) \times 1$ block of the score vector corresponding to the correlation parameters whose elements equal $\sum_{t=1}^{T} \frac{\partial l_t(\hat{\boldsymbol{\theta}})}{\partial \rho_{ij}}$ if the correlation between assets i and j is constrained to be constant and zero otherwise. Under the null hypothesis, the LM-statistic has an asymptotic χ^2_R distribution where R is the number of restrictions to be tested.

The Wald statistic is

$$W_{PCCC} = T\boldsymbol{a}(\hat{\boldsymbol{\theta}})' \left(\boldsymbol{A} \left[\hat{\Im}_T(\hat{\boldsymbol{\theta}})\right]^{-1}_{(\boldsymbol{\rho},\boldsymbol{\rho})} \boldsymbol{A}' \right)^{-1} \boldsymbol{a}(\hat{\boldsymbol{\theta}}) \tag{23}$$

where $\hat{\boldsymbol{\theta}}$ is the maximum likelihood estimates of the full STCC–GARCH model, $\boldsymbol{a}$ is the $\frac{N(N-1)}{2} \times 1$ vector of constraints, more specifically

$$\boldsymbol{a} = vecl\left((\boldsymbol{P}_1 - \boldsymbol{P}_2) \odot \mathbb{I}_{restr}\right),$$

where $\mathbb{I}_{restr}$ is an $\frac{N(N-1)}{2} \times \frac{N(N-1)}{2}$ indicator matrx with the (i,j) element equal to one if the correlation between assets i and j is constrained equal and zero otherwise, $\boldsymbol{A} = \frac{\partial \boldsymbol{a}}{\partial \boldsymbol{\rho}'}$, and $[\hat{\Im}_T(\hat{\boldsymbol{\theta}})]^{-1}_{(\boldsymbol{\rho},\boldsymbol{\rho})}$ is the $N(N-1) \times N(N-1)$ block corresponding the correlation parameters of the inverse of $\hat{\Im}_T$. Under the null hypothesis, the Wald statistic is also asymptotically χ^2_R distributed where R is the number of restrictions to be tested.

Test of partially constant correlations against a less restricted STCC–GARCH model

In this final case, the alternative model is partially constant STCC–GARCH model. Let there be k pairs of variables with constant correlations in the alternative model. The test is as above but with the following changes to definitions and notations. The vector of correlation parameters is $\boldsymbol{\rho} = (\boldsymbol{\rho}_1', \boldsymbol{\rho}_2^{P\prime})'$, where the $\frac{N(N-1)}{2} - k \times 1$ vector $\boldsymbol{\rho}_2^P$ is $vecl\boldsymbol{P}_2$ without the constant elements. Then the partial derivatives are as above, with the modification that

$$\frac{\partial\left(vec\boldsymbol{P}_t\right)'}{\partial \boldsymbol{\rho}}$$

is as above, but with k rows deleted so that the remaining rows are corresponding to the elements in $\boldsymbol{\rho} = (\boldsymbol{\rho}_1', \boldsymbol{\rho}_2^{P\prime})'$, and of the first $\frac{N(N-1)}{2}$ rows, the k rows corresponding to the constant correlations in the alternative model are multiplied with 1 instead of $1 - G_t$. With this modification the calculation of the statistics is straightforward, and again the asymptotic distribution under the null hypothesis is χ^2_R where R is the number of restrictions to be tested.

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