Currency Derivatives under a Minimal Market Model with Random Scaling

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Abstract. This paper uses an alternative, parsimonious stochastic volatility model to describe the dynamics of a currency market for the pricing and hedging of derivatives. Time transformed squared Bessel processes are the basic driving factors of the minimal market model. The time transformation is characterized by a random scaling, which provides for realistic exchange rate dynamics. The pricing of standard European options is studied. In particular, it is shown that the model produces implied volatility surfaces that are typically observed in real markets.

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1 Introduction

The well-known Black-Scholes model (BSM) uses geometric Brownian motions to model the dynamics of the underlying securities. This leads to a convenient and compact asset pricing model that is characterized by deterministic volatilities. Furthermore, the BSM admits an equivalent risk neutral martingale measure so that the standard risk neutral pricing methodology can be directly applied. In currency markets the volatilities of exchange rates are not deterministic, as has been observed in a wide range of studies, see, for instance, Malz (1997). This is also reflected in deviations of observed derivative prices from the prices predicted under the BSM. Therefore, it is essential to provide reliable prices and hedging prescriptions for a range of derivative securities.

One major line of research uses one-factor local volatility function models, see, for instance, Derman & Kani (1994) and Dupire (1992), to capture the effect of stochastic volatility in an exchange rate. However, for more advanced applications such a model remains somewhat restricted because it allows only one factor to drive the exchange rate dynamics. A multi-factor model is essential if one aims to model the currency market as a consistent family of interrelated cross currency exchange rates. Using intraday exchange rate data, their dynamics have been analyzed in Breymann, Kelly & Platen (2005). This study suggests that an exchange rate should be modeled by a ratio of two time transformed squared Bessel processes. In Platen (2001) the minimal market model (MMM) models exchange rates as such ratios of squared Bessel processes in a parsimonious and consistent manner.

Since the resulting currency model is a generalization of the MMM, it does not allow for the existence of an equivalent risk neutral martingale measure, as pointed out in Heath & Platen (2002a, 2002b). Therefore, to obtain a consistent arbitrage free pricing framework we apply the fair pricing concept of the benchmark approach developed in Platen (2002). More precisely, we model the dynamics of different denominations of the growth optimal portfolio (GOP), which is the portfolio that maximizes expected logarithmic utility. An exchange rate then equals the ratio of two currency denominations of the GOP. This paper documents the fact that realistic implied volatility surfaces for standard European options on exchange rates can be generated naturally from the proposed MMM with random scaling.

The paper is organized as follows. In Section 2 a general currency market is described. Using the benchmark approach, Section 3 introduces the fair pricing of derivatives. A generalization of the MMM with random scaling is proposed in Section 4. The corresponding pricing functionals for a wide class of contingent claims are derived in Section 5. European call and put options on exchange rates are described in Section 6, where a new moment expansion procedure is employed to compute implied volatility surfaces. Finally, in Section 7 a series of numerical experiments are described. These document the type of realistic implied volatility
surfaces that can be obtained under the MMM.

2 Currency Benchmark Model

Consider the evolution of the savings accounts of \(d+1\) currencies, \(d \in \{1, 2, \ldots\}\). To be mathematically precise, these are modeled on a filtered probability space \((\Omega, \mathcal{A}_T, \mathcal{A}, P)\), where the filtration \(\mathcal{A} = (\mathcal{A}_t)_{t \in [0, T]}\) fulfills the usual conditions, see Karatzas & Shreve (1998), with \(\mathcal{A}_0\) being trivial. Here \(\mathcal{A}_t\) describes the information that is available at time \(t \in [0, T]\). The measure \(P\) is the real world probability measure.

Let \(r^j_t\) denote the \(j\)th short rate at time \(t \in [0, T]\), \(j \in \{0, 1, \ldots, d\}\). This is the shortest forward rate for the \(j\)th currency. In this paper we assume, for simplicity, that \(r^j_t = \{r^j_t, t \in [0, T]\}\) is a nonnegative deterministic function of time. Note however that the short rate can be made stochastic without changing significantly the results of this paper. Using these definitions \(r^0_t\) denotes the domestic short rate at time \(t\).

The \(j\)th savings account process \(B^j = \{B^j_t, t \in [0, T]\}\) of the \(j\)th currency, when denominated in this currency, is given by the differential equation

\[
\frac{d}{dt} B^j_t = B^j_t r^j_t dt
\]

for \(t \in [0, T]\) with \(B^j_0 = 1\). The locally riskless savings account can be interpreted as the limit in probability of a sequence of rollover short term bond accounts with times to maturity converging to zero.

The \((i, j)\)th exchange rate \(X^{i, j}_t\) denotes the price of one unit of the \(j\)th currency at time \(t \in [0, T]\), when measured in units of the \(i\)th currency \(i, j \in \{0, 1, \ldots, d\}\). Viewed from the domestic market, the quantity \(X^{0, j}_t\) denotes the exchange rate for the \(j\)th currency. Without loss of generality, we interpret the 0th currency as the domestic currency. It is assumed that all exchange rate processes are continuous.

We introduce \(d+1\) primary security account processes, which are chosen to be the savings accounts. The \(j\)th savings account value at time \(t\), when denominated in units of the domestic currency, is then given by the expression

\[
S^{0,j}_t = X^{0,j}_t B^j_t,
\]

for \(t \in [0, T]\) and \(j \in \{0, 1, \ldots, d\}\).

We now introduce portfolios of primary security accounts, where \(S = \{S^i_t = (S^{0,0}_t, S^{0,1}_t, \ldots, S^{0,d}_t) \top, t \in [0, T]\}\) denotes the vector process of primary security account prices when expressed in units of the domestic currency. We call a predictable, \(S\)-integrable stochastic process \(\delta = \{\delta^i_t = (\delta^0_t, \ldots, \delta^d_t) \top, t \in [0, T]\}\) a strategy. The quantity \(\delta^j_t \in (-\infty, \infty)\) denotes the number of units of the \(j\)th primary security account that are held at time \(t\) according to the strategy \(\delta\). In
addition, let $S^{(\delta)} = \{S_t^{(\delta)}, t \in [0,T]\}$ be the value process of the corresponding portfolio, where $S^{(\delta)}$ is denominated in units of the domestic currency, that is

$$S_t^{(\delta)} = \sum_{j=0}^{d} \delta_t^j S_t^{0,j},$$

almost surely, for $t \in [0,T]$. We say that a strategy $\delta$ is *self-financing* if

$$dS_t^{(\delta)} = \sum_{j=0}^{d} \delta_t^j dS_t^{0,j}$$

for $t \in [0,T]$. Here the stochastic differentials are defined as Itô differentials, see Protter (2004). In practical terms, the self-financing property means that all changes in the value of the portfolio are due to gains or losses from trade. We assume throughout the paper that all portfolios and strategies are self-financing and omit therefore this attribute from now on.

Of particular importance in our analysis will be the *growth optimal portfolio* (GOP). This is the portfolio that maximizes expected logarithmic utility from terminal wealth, see Kelly (1956), Long (1990) or Karatzas & Shreve (1998). It has been shown in Platen (2002) that the given continuous complete market has a unique GOP. According to the above mentioned literature, the value $D_t^0$ at time $t$ of the GOP, when denominated in units of the domestic currency, satisfies the *stochastic differential equation* (SDE)

$$dD_t^0 = D_t^0 \left( r_t^0 dt + \sum_{k=1}^{d} \theta_t^{0,k} \left( \theta_t^{0,k} dt + dW_t^k \right) \right)$$

for $t \in [0,T]$ with $D_0^0 > 0$. Here the processes $W^k = \{W_t^k, t \in [0,T]\}$, $k \in \{1,2,\ldots,d\}$, denote independent standard Wiener processes. Note that the volatilities $\theta_t^{0,k}$, $k \in \{1,2,\ldots,d\}$, of the GOP are the corresponding domestic *market prices of risk* and the risk premium of the GOP is the sum of the squares of these market prices of risk.

It has been shown in Platen (2004a, 2004b, 2005) that the GOP $D^0$ can be approximated by a well diversified accumulation world stock index and can therefore be interpreted as a market index. Also Breymann, Kelly & Platen (2004) have found in a comparison of intraday world stock indices that the market capitalization weighted world stock index is a good proxy for the GOP. We assume that a total return world stock index (WSI) can be considered as a good approximation for the GOP.

To illustrate these ideas we show in Figure 1 the values of a WSI in British Pound and US dollar over the period from 1970 until 2004. This index has been constructed for illustrative purposes by using market capitalization and stock index data as available from Thomson Financial. Note that for these currency denominations the index is on average growing over time.
Let us denote by $D_i^t$ the denomination of the GOP in units of the $i$th currency, $i \in \{0, 1, \ldots, d\}$. It is straightforward to conclude by symmetry that as in (2.5) the GOP $D_i^t$, when denominated in units of the $i$th currency, must satisfy the SDE

$$dD_i^t = D_i^t \left( r_i^t dt + \sum_{k=1}^{d} \theta_{i,k}^t \left( \theta_{i,k}^t dt + dW_k^t \right) \right)$$  \hspace{1cm} (2.6)$$

for $t \in [0, T]$ and $i \in \{0, 1, \ldots, d\}$. Here $\theta_{i,k}^t$ is the market price of risk of the $i$th currency denomination with respect to the $k$th Wiener process $W^k$ at time $t$. We assume that the $i$th denomination of the GOP remains strictly positive, that is, $D_i^t > 0$ almost surely for all $t \in [0, T]$ and $i \in \{0, 1, \ldots, d\}$. Furthermore, the volatility processes $\theta_{i,k}^t$ are assumed to be predictable and such that

$$\int_0^T \sum_{k=1}^{d} \sum_{j=0}^{d} \left( \theta_{i,k}^s \right)^2 ds < \infty$$

almost surely.

Obviously, since the GOP is unique, the $(i, j)$th exchange rate at time $t$ can be expressed by the ratio

$$X_{i,j}^t = \frac{D_i^t}{D_j^t}$$  \hspace{1cm} (2.7)$$

for $t \in [0, T]$ and $i, j \in \{0, 1, \ldots, d\}$. Using the Itô formula together with (2.6) and (2.7) it can be seen that

$$dX_{i,j}^t = X_{i,j}^t \left( \left( r_i^t - r_j^t + \sum_{k=1}^{d} \theta_{i,k}^t \left( \theta_{i,k}^t - \theta_{j,k}^t \right) \right) dt + \sum_{k=1}^{d} \left( \theta_{i,k}^t - \theta_{j,k}^t \right) dW_k^t \right)$$  \hspace{1cm} (2.8)$$
For $t \in [0, T]$ with initial value $X_{0,i,j}^t = \frac{D_i^t}{D_0^t} > 0$ and $i, j \in \{0, 1, \ldots, d\}$.

Furthermore, by (2.2), (2.1) and (2.7), the value of the $j$th savings account, when denominated in units of the domestic currency, is given by the formula

$$S_{t}^{0,j} = \frac{D_t^{0}}{D_t^{j}} B_t^{j}$$

for $t \in [0, T]$ and $j \in \{0, 1, \ldots, d\}$. By application of the Itô formula we obtain from (2.9), (2.1) and (2.6) for $S_{t}^{0,j}$ the SDE

$$dS_{t}^{0,j} = S_{t}^{0,j} \left( r_t^0 dt + \sum_{k=1}^{d} \left( \theta_{t}^{0,k} - \theta_{t}^{j,k} \right) \left( \theta_{t}^{0,k} dt + dW_t^{k} \right) \right)$$

(2.10)

for $t \in [0, T]$ with initial value $S_0^{0,j} > 0$ and $j \in \{0, 1, \ldots, d\}$.

To avoid redundant primary security accounts we assume that the domestic volatility matrix $b_t = [b_{t,k}^{j,k}]_{j,k=1}^d$ with $j, k$th domestic volatility

$$b_{t,k}^{j,k} = \theta_{t}^{0,k} - \theta_{t}^{j,k}$$

(2.11)

is invertible, see Platen (2002). We call the above model a currency benchmark model, where the GOP plays the role of the benchmark.

### 3 Benchmarked Securities and Fair Pricing

Under the benchmark approach, we call any price that is expressed in units of the GOP a benchmarked price. Thus, the benchmarked $j$th savings account value $\hat{S}_{t}^{(j)}$ at time $t$ is obtained by the ratio

$$\hat{S}_{t}^{(j)} = S_{t}^{0,j} \frac{D_t^{0}}{D_t^{j}}$$

(3.1)

for $t \in [0, T]$ and $j \in \{0, 1, \ldots, d\}$.

By application of the Itô formula we obtain from (3.1), (2.10) and (2.5) for the benchmarked $j$th savings account $\hat{S}_{t}^{(j)}$ the SDE

$$d\hat{S}_{t}^{(j)} = -\hat{S}_{t}^{(j)} \sum_{k=1}^{d} \theta_{t}^{j,k} dW_t^{k}$$

(3.2)

for $t \in [0, T], j \in \{0, 1, \ldots, d\}$. Note that the SDE (3.2) for $\hat{S}_{t}^{(j)}$ is driftless and $\hat{S}_{t}^{(j)}$ is therefore a nonnegative ($\mathbb{A}, P$)-local martingale, see Protter (2004),
\( j \in \{0, 1, \ldots, d\} \). Similarly, by application of the Itô formula it follows from (2.4), (2.9) and (2.6) that any benchmarked portfolio

\[
\hat{S}_t^{(d)} = \frac{S_t^{(d)}}{D_t^0}
\]

is also driftless, that is

\[
d\hat{S}_t^{(d)} = \sum_{j=0}^{d} \delta_t^{(j)} d\hat{S}_t^{(j)}
\]

for \( t \in [0, T] \). This means that benchmarked portfolio processes are \((\mathcal{A}, P)\)-local martingales. Consequently, any nonnegative benchmarked portfolio is an \((\mathcal{A}, P)\)-supermartingale, see Protter (2004). By using this property it is shown in Platen (2002) that from zero initial capital one cannot generate, by using any nonnegative self-financing portfolio, some strictly positive wealth with strictly positive probability. In this sense the above benchmark model is arbitrage free.

The Radon-Nikodym derivative \( \Lambda^{(j)} = \{ \Lambda_t^{(j)}, t \in [0, T] \} \), with

\[
\Lambda_t^{(j)} = \frac{\hat{S}_t^{(j)}}{\hat{S}_0^{(j)}}
\]

for \( t \in [0, T] \) and \( j \in \{0, 1, \ldots, d\} \), for the candidate risk neutral measure of the \( j \)th currency denomination equals up to a constant factor the \( j \)th benchmarked savings account. Obviously, by (3.2) the process \( \Lambda^{(j)} \) is an \((\mathcal{A}, P)\)-local martingale, \( j \in \{0, 1, \ldots, d\} \). Most of the literature on currency derivative pricing makes the assumption that there exist equivalent risk neutral martingale measures for all currency denominations. This means that one assumes that the Radon-Nikodym derivatives \( \Lambda^{(0)}, \Lambda^{(1)}, \ldots, \Lambda^{(d)} \) of the corresponding candidate risk neutral measures are for all currency denominations martingales.

To see whether this assumption makes practical sense we plot in Figure 2 the Radon-Nikodym derivative for the US dollar denomination, where we interpret again the WSI as GOP. Here the US short rate has been set to about 5%, which is above its average of 4.1%, see Dimson, Marsh & Staunton (2002). It is obvious that the Radon-Nikodym derivative process shows a systematic decline over time. This can be observed for all major currencies. In addition, it is widely recognized that the world stock portfolio will outperform in the long run the savings account of any currency, see Dimson, Marsh & Staunton (2002). This provides further evidence that the Radon-Nikodym derivative, given in (3.5), should systematically decline in the long run. Such a systematic decline is not typical for trajectories of martingales, however, it is typical for trajectories of nonnegative strict supermartingales, as is suggested by the MMM that we will introduce later. The systematic decline of the graphs in Figure 2 can be taken as a warning that one should avoid the standard risk neutral pricing methodology for realistic currency market models.
Since we do not rely on the standard risk neutral pricing methodology we need a more general pricing concept. For the pricing of derivatives we apply the concept of fair pricing, as introduced in Platen (2002).

A price process is called fair if, when expressed in units of the GOP, forms an \((\mathcal{A}, P)\)-martingale. This means, benchmarked fair prices are martingales. However note that savings accounts do not need to be fair. Also in a risk neutral world, where an equivalent risk neutral martingale measure exists, one can show that risk neutral prices of derivatives are fair.

For a maturity date \( T \in (0, \infty) \) let \( H_T \) denote an \( \mathcal{A}_T \)-measurable, nonnegative contingent claim, which is expressed in units of the domestic currency and has finite expected benchmarked value, that is

\[
E \left( \frac{H_T}{D_T^0} \bigg| \mathcal{A}_t \right) < \infty \tag{3.6}
\]

for all \( t \in [0, T] \). The benchmarked, fair price \( \hat{u}_{H_T}(t) \) at time \( t \) of this contingent claim is given by the conditional expectation

\[
\hat{u}_{H_T}(t) = E \left( \frac{H_T}{D_T^0} \bigg| \mathcal{A}_t \right) \tag{3.7}
\]

for \( t \in [0, T] \). This specification makes the benchmarked price process \( \hat{u}_{H_T} = \{\hat{u}_{H_T}(t), t \in [0, T]\} \) an \((\mathcal{A}, P)\)-martingale that matches the contingent claim \( H_T \) at the maturity date \( T \). Any nonnegative portfolio that replicates the contingent claim is according to (3.4) and (3.3) an \((\mathcal{A}, P)\)-supermartingale. It has therefore an initial value greater or equal to the fair value. The fair value describes in the given currency benchmark model the minimal price that allows one to replicate the contingent claim by a nonnegative portfolio, see Platen (2002).
The fair price \( u_{HT}(t) \) of the contingent claim \( H_T \) at time \( t \), when expressed in units of the domestic currency, is then according to (3.3) obtained by the fair pricing formula

\[
u_{HT}(t) = D_0^t \hat{u}_{HT}(t)
\] (3.8)

for \( t \in [0,T] \). An analogous formula holds for each currency denomination.

If there exists an equivalent risk neutral martingale measure \( \hat{P} \) with Radon-Nikodym derivative \( \Lambda_T^0 = \frac{d\hat{P}}{dP} \bigg|_{A_T} = \frac{B_T^0}{B_T^0} D_0^t \) at time \( t \), then the fair pricing formula (3.8) coincides with the classical risk neutral pricing formula, as can be seen by the relations

\[
u_{HT}(t) = E \left( \frac{D_0^t B_0^t B_0^t \Lambda_T^0 H_T}{D_T^0} \bigg| A_t \right) = E \left( \frac{\Lambda_T^0 B_T^0 B_T^0 H_T}{\Lambda_T^0} \bigg| A_t \right) = \hat{E} \left( \frac{B_T^0}{B_T^0} H_T \bigg| A_t \right).
\]

Here \( \hat{E} \) denotes expectation under the domestic equivalent risk neutral martingale measure, see Platen (2002). However, as indicated previously, a realistic currency benchmark model is unlikely to have an equivalent risk neutral martingale measure. This is why we will use the fair pricing formula (3.8).

### 4 Minimal Market Model with Random Scaling

The model that we consider now is a generalization of the minimal market model (MMM) suggested in Platen (2001, 2002). The MMM generates stochastic volatilities that involve transformations of squared Bessel processes. As discussed in Platen (2001) and Breymann, Kelly & Platen (2004), the MMM captures a number of important stylized empirical facts observed for indices and exchange rates.

Under the generalized version of the MMM that we are going to consider, the \( j \)th denomination \( D_T^j \) of the GOP at time \( t \in [0,T] \) is modeled by the equation

\[
u_{ij} = (Z_T^j)^{\nu_j/2} B_t^j,
\] (4.1)

where \( Z_T^j = \{Z_T^j, t \in [0,T]\} \) is a time transformed squared Bessel process of dimension \( \nu_j > 2 \), which satisfies the SDE

\[
u_{ij} = \frac{\nu_j}{4} \sum_{k=1}^d (q_{i,k}^j)^2 \gamma_t dt + \sqrt{Z_T^j \gamma_t} \sum_{k=1}^d q_{i,k}^j dW_t^k
\] (4.2)

for \( t \in [0,T] \) with \( Z_0^j = (D_0^j)^{\nu_j/2} > 0, j \in \{0,1,\ldots,d\} \). We know, see Revuz & Yor (1999), that the above time transformed squared Bessel process stays almost surely strictly positive. The \( (j,k) \)th scaling level \( q_{i,k}^j = \{q_{i,k}^j, t \in [0,T]\} \) will be specified below, as well as the scaling process \( \gamma = \{\gamma_t, t \in [0,T]\} \). In general, one could introduce a separate scaling for each squared Bessel process. However, for simplicity we consider here only a single common scaling process \( \gamma \). The scaling
levels $q^{j,k}$ can, for instance, be modeled as continuous time Markov chains. For simplicity, we assume here that these are constants.

By application of the Itô formula it follows from (4.1) and (4.2) that the $j$th denomination $D^j_t$ of the GOP at time $t$ satisfies the SDE (2.5) with $(j,k)$th volatility, that is $j,k$th market price of risk, $\theta^{j,k}_t$, that is

$$\theta^{j,k}_t = \left(\nu_j^2 - 1\right) q^{j,k} \sqrt{\frac{\gamma_t}{Z_j^t}}$$

for $t \in [0,T], \ j \in \{0,1,\ldots,d\}, \ k \in \{1,2,\ldots,d\}$. This volatility is stochastic since it is proportional to the inverse of the square root of the $j$th squared Bessel process $Z^j$. We allow the scaling $\gamma_t$ to be random to generate globally for the currency market random time transformations for the squared Bessel processes involved. One can say that the random scaling models the trading activity in the currency market. To model this in detail, we assume that the random scaling $\gamma_t$ satisfies the SDE

$$d\gamma_t = a(t, \gamma_t) dt + b(t, \gamma_t) \left( \sum_{k=1}^{d} \varrho^k dW^k_t + \sqrt{1 - \sum_{k=1}^{d} (\varrho^k)^2} d\tilde{W}_t \right)$$

for $t \in [0,T]$ with $\gamma_0 \geq 0$. Here $\tilde{W} = \{\tilde{W}_t, \ t \in [0,T]\}$ is an independent standard Wiener process that does not drive the trading noise of the market. Furthermore, the $k$th scaling correlation $\varrho^k$ is assumed to be a constant. The drift coefficient function $a(\cdot, \cdot)$ and the diffusion coefficient function $b(\cdot, \cdot)$ are given functions to be chosen such that the SDE (4.4) has a unique strong solution and the scaling $\gamma_t$ remains almost surely nonnegative for all $t \in [0,T]$.

The above formulation of the random scaling is still rather general. Below we will specify further details based on empirical evidence. We then obtain a Markovian multi-factor currency benchmark model. Current numerical methods applied to derivative pricing problems work reasonably well only up to two or three dimensions. This is also the dimensionality of the model considered here. Numerically, we are therefore on the borderline of what is achievable with current algorithms and techniques.

As we will see below, when pricing currency options, the derivative prices are strongly influenced by the randomness in the scaling. To be more specific, we introduce the market activity process $m = \{m_t, \ t \in [0,T]\}$, see Breymann, Kelly & Platen (2004, 2005) and Heath & Platen (2004), given by

$$m_t = \frac{\gamma_t}{\xi_t}$$

with the exponential function

$$\xi_t = \xi_0 \exp \left\{ \int_0^t \eta_s ds \right\}$$
for $t \in [0, T]$. Here $\xi_0 > 0$ is a constant and $\eta_t$ is the piecewise constant deterministic long term net growth rate of the market. If one thinks of the market activity $m_t$ as fluctuating around one, then the net growth rate process $\eta = \{\eta_t, t \in [0, T]\}$ needs to offset the average growth in the random scaling process $\gamma$ over the long term. This is why the ratio $\frac{\eta}{\xi}$ appears in (4.5).

The market activity $m_t$ is designed to model the normalized trading activity at time $t$. By using intraday data the market activity process for the US dollar denomination has been analyzed in Breymann, Kelly & Platen (2004). According to these results, a realistic market activity process is obtained when the diffusion coefficient function in (4.4) is chosen to be multiplicative with

$$b(t, \gamma) = \beta_t \gamma. \quad (4.7)$$

for $(t, \gamma) \in [0, T] \times (0, \infty)$. In addition, these results suggest that the drift coefficient function has the form

$$a(t, \gamma_t) = \xi_t \beta_t^2 A\left(\frac{\gamma_t}{\xi_t}\right) + \gamma_t \eta_t \quad (4.8)$$

for $t \in [0, T]$ and $\gamma \geq 0$, where the activity volatility $\beta = \{\beta_t, t \in [0, T]\}$ is some deterministic function of time. Using Itô’s formula together with (4.4), (4.5) and (4.6) the SDE for $m_t$ is given by

$$dm_t = \beta_t^2 A(m_t) dt + \beta_t m_t \left( \sum_{k=1}^{d} \rho^k dW^k_t + \sqrt{1 - \sum_{k=1}^{d} (\rho^k)^2} d\tilde{W}_t \right)$$

for $t \in [0, T]$.

The particular choice of the function $A(\cdot)$ controls the type of feedback which characterizes the random scaling. In Breymann, Kelly & Platen (2004) it was pointed out that a good choice for the function $A(\cdot)$ is of the form

$$A(m) = \left(\frac{p-1}{2} - \frac{g}{2} m\right) m \quad (4.9)$$

for $m \in [0, \infty)$, with some speed of adjustment parameter $g$ and some reference level $p$. The market activity process can be shown to have the gamma density as its stationary density with mean $\frac{p-1}{g}$ and variance $\frac{1}{g}$ for parameters $g > 0$ and $p > 1$, see Heath & Platen (2004).

5 Pricing Functions for Contingent Claims

Let us now check, whether the above MMM with random scaling can provide currency derivative prices that are consistent with those observed in the market.
For the study of currency derivatives let us, for simplicity, consider the case of two independent squared Bessel processes $Z_0$ and $Z_1$, satisfying according to (4.2) the SDE
\[ dZ_i^t = \nu_i \frac{(q_{i,i+1})^2 \gamma_t}{2} dt + \sqrt{Z_i^t \gamma_t q_{i,i+1}} dW_i^{i+1} \] (5.1)
for $t \in [0, T]$ and $i \in \{0, 1\}$ with the constant scaling levels $q_{0,1}$ and $q_{1,2}$. By (4.3) and (2.5) the corresponding $i$th denomination $D_i^t$ of the GOP then satisfies the SDE
\[ dD_i^t = \left( r_i^t + \left( \frac{\nu_i - 2}{2} \right)^2 (q_{i,i+1})^2 \right) \gamma_t dW_i^{i+1} + \frac{\nu_i - 2}{2} q_{i,i+1} \sqrt{\gamma_t Z_i^t} dt \] (5.2)
for $t \in [0, T], i \in \{0, 1\}$. Now, consider the case where the contingent claim $H_T$, when expressed in units of the domestic currency, has the form
\[ H_T = H_T(D_0^T, D_1^T, \gamma_T) \] (5.3)
which is that of a European currency option with maturity $T$. Due to the given Markovian structure, the corresponding benchmarked fair price process $\hat{u}_{H_T}$, given by (3.7), is such that
\[ \hat{u}_{H_T}(t) = \hat{u}(t, D_0^t, D_1^t, \gamma_t) \] (5.4)
Here the function $\hat{u} : [0, T] \times (0, \infty)^3 \rightarrow [0, \infty)$ is differentiable with respect to time $t$ on $(0, T)$ and twice differentiable with respect to the components $(D_0^t, D_1^t, \gamma_t)$ on $(0, \infty)^3$. This allows us to apply the Itô formula to obtain the martingale representation
\[
\frac{H_T(D_0^t, D_1^t, \gamma_T)}{D_0^T} = \hat{u}(t, D_0^t, D_1^t, \gamma_t) \\
+ \int_t^T \frac{\nu_0 - 2}{2} q_{0,1} \sqrt{\frac{\gamma_s}{Z_0^s}} D_0^s \frac{\partial \hat{u}(t, D_0^s, D_1^s, \gamma_s)}{\partial D_0^s} dW_1^s \\
+ \int_t^T \frac{\nu_1 - 2}{2} q_{1,2} \sqrt{\frac{\gamma_s}{Z_1^s}} D_1^s \frac{\partial \hat{u}(t, D_0^s, D_1^s, \gamma_s)}{\partial D_1^s} dW_1^s \\
+ \int_t^T b(s, \gamma_s) \frac{\partial \hat{u}(t, D_0^s, D_1^s, \gamma_s)}{\partial \gamma} \left( \sum_{k=1}^{2} \rho^k dW_1^k + \left( 1 - \sum_{k=1}^{2} \rho^k \right) d\tilde{W}_s \right) (5.5)
\]
for $t \in [0, T)$. Here the function $\hat{u}$ satisfies by (5.2) and (4.4) the Kolmogorov backward equation
\[ L^0 \hat{u}(t, D_0^t, D_1^t, \gamma) = 0 \] (5.6)
with operator
\[ L^0 \hat{u}(t, D^0, D^1, \gamma) = \frac{\partial \hat{u}(t, D^0, D^1, \gamma)}{\partial t} + D^0 \left( r_t^0 + \frac{(\nu_0 - 2)(q_0^0)^2}{2} \frac{\partial \hat{u}(t, D^0, D^1, \gamma)}{\partial D^0} \right) \frac{\partial \hat{u}(t, D^0, D^1, \gamma)}{\partial D^0} + D^1 \left( r_t^1 + \frac{(\nu_1 - 2)(q_1^1)^2}{2} \frac{\partial \hat{u}(t, D^0, D^1, \gamma)}{\partial D^1} \right) \frac{\partial \hat{u}(t, D^0, D^1, \gamma)}{\partial D^1} \]

\[ + \frac{1}{2} \frac{(\nu_0 - 2)(q_0^0)^2}{(D^0_{B^0})^2} \frac{\partial^2 \hat{u}(t, D^0, D^1, \gamma)}{\partial (D^0)^2} + \frac{1}{2} \frac{(\nu_1 - 2)(q_1^1)^2}{(D^1_{B^1})^2} \frac{\partial^2 \hat{u}(t, D^0, D^1, \gamma)}{\partial (D^1)^2} + a(t, \gamma) \frac{\partial \hat{u}(t, D^0, D^1, \gamma)}{\partial \gamma} + \frac{1}{2} (b(t, \gamma))^2 \frac{\partial^2 \hat{u}(t, D^0, D^1, \gamma)}{\partial \gamma^2} + b(t, \gamma) g^1 D^0 \frac{\nu_0 - 2}{2} q_0^0,1 \sqrt{\gamma} \left( B_t^0 \right)^{-\frac{1}{\nu_0 - 2}} \frac{\partial^2 \hat{u}(t, D^0, D^1, \gamma)}{\partial \gamma \partial D^0} \]

\[ + b(t, \gamma) g^2 D^1 \frac{\nu_1 - 2}{2} q_1^1,2 \sqrt{\gamma} \left( B_t^1 \right)^{-\frac{1}{\nu_1 - 2}} \frac{\partial^2 \hat{u}(t, D^0, D^1, \gamma)}{\partial \gamma \partial D^1} \] (5.7)

for \((t, D^0, D^1, \gamma) \in (0, T) \times (0, \infty)^3\) with terminal condition
\[ \hat{u}(T, D^0, D^1, \gamma) = \frac{H_T(D^0, D^1, \gamma)}{D^0} \] (5.8)

for \((D^0, D^1, \gamma) \in (0, \infty)^3\). The above set of equations describes the pricing functions for a range of currency derivatives. Some of these will be studied below.

This PDE formulation can be extended to include the case of many path dependent derivatives, such as barrier and binary options, by appropriate modification of the boundary conditions, see Heath & Platen (1996). Note that in these cases the PDE operator appearing in (5.7) does not change.

### 6 European Options on Exchange Rates

Let us now apply the benchmarked pricing formula (3.7) for computing a benchmarked European call option price on the exchange rate process \(X_t^{0,1} = \{X_t^{0,1} = \)
\( \frac{D_0^T}{D_T^1}, t \in [0,T] \) \) with strike \( K \) and maturity date \( T \). Then the benchmarked fair call option price \( \hat{c}_{T,K}(t, D_0^T, D_1^T, \gamma_t) \) at time \( t \in [0,T] \) is given by

\[
\hat{c}_{T,K}(t, D_0^T, D_1^T, \gamma_t) = E\left( \left( \frac{X_T^{0,1}}{D_T^0} - K \right)^+ \bigg| \mathcal{A}_t \right)
\]

\[
= E\left( \left( \frac{1}{D_T^1} - \frac{K}{D_T^0} \right)^+ \bigg| \mathcal{A}_t \right) \quad (6.1)
\]

for \( t \in [0,T] \). The corresponding fair call option price \( c_{T,K}(t, D_0^T, D_1^T, \gamma_t) \), expressed in units of the domestic currency, see (3.8), takes the form

\[
c_{T,K}(t, D_0^T, D_1^T, \gamma_t) = D_0^T E\left( \left( \frac{1}{D_T^1} - \frac{K}{D_T^0} \right)^+ \bigg| \mathcal{A}_t \right) \quad (6.2)
\]

for \( t \in [0,T] \). The benchmarked pricing function \( \hat{c}_{T,K} \) satisfies the PDE (5.6) with terminal condition

\[
\hat{c}_{T,K}(T, D_0^0, D_1^1, \gamma) = \left( \frac{1}{D_T^1} - \frac{K}{D_T^0} \right)^+ \quad (6.3)
\]

for \( (D_0^0, D_1^1, \gamma) \in (0, \infty)^3 \).

Note that the PDE (5.7) and (5.8) is a three-dimensional PDE that is difficult to approximate using standard numerical methods. A valuation procedure that is numerically tractable and can be widely applied for European style currency derivatives is as follows:

Using (3.7) the fair prices \( P_0^T(t, D_0^0, \gamma_t) \) and \( P_1^T(t, D_0^0, D_1^1, \gamma_t) \) for the domestic and foreign zero coupon bonds with a maturity date \( T \), when denominated in domestic currency, are given by

\[
P_0^T(t, D_0^0, \gamma_t) = D_0^T E\left( \left( \frac{1}{D_T^0} \right)^+ \bigg| \mathcal{A}_t \right) \quad (6.4)
\]

and

\[
P_1^T(t, D_0^0, D_1^1, \gamma_t) = D_0^T E\left( \left( \frac{1}{D_T^1} \right)^+ \bigg| \mathcal{A}_t \right) \quad (6.5)
\]

for \( t \in [0,T] \), respectively.

Again using (3.7) the fair price \( p_{T,K}(t, D_0^0, D_1^1, \gamma_t) \) for a European put option with strike \( K \) and maturity date \( T \) is given by

\[
p_{T,K}(t, D_0^0, D_1^1, \gamma_t) = D_0^T E\left( \left( \frac{K}{D_T^0} - \frac{1}{D_T^1} \right)^+ \bigg| \mathcal{A}_t \right) \quad (6.6)
\]

for \( t \in [0,T] \). The corresponding put-call parity relation between European put and call options is expressed via the equation

\[
p_{T,K}(t, D_0^0, D_1^1, \gamma_t) = c_{T,K}(t, D_0^0, D_1^1, \gamma_t) + K P_0^T(t, D_0^0, \gamma_t) - P_1^T(t, D_0^0, D_1^1, \gamma_t) \quad (6.7)
\]
for \( t \in [0, T] \).

Let us now define the time transformation \( \varphi \) by the differential

\[
d\varphi(t) = \gamma_t \, dt
\]

for \( t \in [0, T] \) with \( \varphi(0) = 0 \). Consider the time transformed stochastic process

\[
V^i = \{V^i_{\varphi(t)}, \varphi \in [0, \varphi(T)]\}
\]

given by

\[
V^i_{\varphi(t)} = Z^i_t
\]

for \( t \in [0, T] \) and \( i \in \{0, 1\} \). Using (5.1), (6.8) and (6.9) it can be seen that

\[
dV^i_{\varphi(t)} = \frac{\nu_i}{4} (q^{i,i+1})^2 \gamma_t \, dt + q^{i,i+1} \sqrt{V^i_{\varphi(t)}} \, dW^i_{\varphi(t)},
\]

where

\[
dU^{i+1}_{\varphi(t)} = \sqrt{\gamma_t} \, dW^i_{\varphi(t)}
\]

for \( t \in [0, T] \) and \( i \in \{0, 1\} \). Using (6.8) the quadratic variation of \( U^{i+1} \) is given

\[
\langle U^{i+1} \rangle_t = \int_0^t \gamma_s \, ds = \varphi(t)
\]

for \( t \in [0, T] \) and thus \( U^{i+1} \) is a Wiener process in \( \varphi \)-time, \( i \in \{0, 1\} \). Consequently, in the new \( \varphi \)-time scale \( V^i \) is a squared Bessel process of dimension \( \nu_i \) that does not depend on the random scaling process \( \gamma \).

Using (4.1), (6.1), (6.8) and (6.9) the benchmarked fair price

\[
\tilde{c}_{T,K}(t, \varphi(t), V^0_{\varphi(t)}, V^1_{\varphi(t)}, \gamma_t) = \tilde{c}_{T,K}(t, D^0_t, D^1_t, \gamma_t)
\]

of a European call can also be expressed in the form

\[
\tilde{c}_{T,K}(t, \varphi(t), V^0_{\varphi(t)}, V^1_{\varphi(t)}, \gamma_t) = E\left( \left( \frac{1}{B^1_T(V^1_{\varphi(T)})^{\frac{1}{2}} - \frac{K}{B^2_T(V^2_{\varphi(T)})^{\frac{1}{2}}} \right)^+ \mid \mathcal{A}_t \right)
\]

for \( t \in [0, T] \).

Consider the filtrations \( \mathcal{A}^1_{t \geq 0} \) and \( \mathcal{A}^2_{t \geq 0} \) generated by the independent Wiener processes \( W^1 \) and \( W^2 \), respectively, together with the filtration \( \tilde{\mathcal{A}}_{t \geq 0} \) generated by the Wiener process \( \tilde{W} \). The filtration \( \mathcal{A}_{t \geq 0} \) appearing in (6.12) can be defined as the joint filtration of \( \mathcal{A}^1_{t \geq 0} \), \( \mathcal{A}^2_{t \geq 0} \) and \( \tilde{\mathcal{A}}_{t \geq 0} \). That is,

\[
\mathcal{A}_t = \mathcal{A}^1_t \otimes \mathcal{A}^2_t \otimes \tilde{\mathcal{A}}_t
\]

for \( t \in [0, T] \). Let \( \mathcal{A}^*_t \) be the joint filtration given by

\[
\mathcal{A}^*_t = \mathcal{A}^1_t \otimes \mathcal{A}^2_t \otimes \tilde{\mathcal{A}}_T
\]
for \( t \in [0, T] \). Therefore, \( \mathcal{A}^\gamma_t \) contains all of the information regarding the full evolution of the random scaling process \( \gamma \) over the time interval \([0, T]\).

The conditional expectation (6.12) for the benchmarked fair price of a European call can now be rewritten in the form

\[
\tilde{c}_{T,K}(t, \varphi(t), V_{\varphi(t)}^0, V_{\varphi(t)}^1, \gamma_t) = E \left( c_{\varphi(T),T,K}(t, \varphi(t), V_{\varphi(t)}^1, V_{\varphi(t)}^2) \bigg| \mathcal{A}_t \right),
\]

(6.13)

where

\[
c_{\varphi(T),T,K}(t, \varphi(t), V_{\varphi(t)}^1, V_{\varphi(t)}^2) = E \left( \left( \frac{1}{B_T^1(V_{\varphi(T)}^1)^{\frac{\nu}{2} - 1}} - \frac{K}{B_T^2(V_{\varphi(T)}^2)^{\frac{\nu}{2} - 1}} \right)^+ \bigg| \mathcal{A}_t^\gamma \right)
\]

for \( t \in [0, T] \).

For simplicity, we will use the model formulation with \( \varrho^1 = \varrho^2 = 0 \). Then for fixed values \( \varphi(t) \) and \( \varphi(T) \) the function \( c_{\varphi(T),T,K}(t, \varphi(t), V_{\varphi(t)}^1, V_{\varphi(t)}^2) \) can be calculated using the known transition densities of the squared Bessel processes \( V^1 \) and \( V^2 \). To approximate (6.13) we will use the following useful moment expansion

\[
E \left( c_{\varphi(T),T,K}(t, \varphi(t), V_{\varphi(t)}^1, V_{\varphi(t)}^2) \bigg| \mathcal{A}_t \right) \approx c_{\varphi(T),T,K}(t, \varphi(t), V_{\varphi(t)}^1, V_{\varphi(t)}^2) \bigg|_{\tau = E(\varphi(T)|\mathcal{A}_t)} + \frac{1}{2} \frac{\partial^2}{\partial \tau^2} c_{\varphi(T),T,K}(t, \varphi(t), V_{\varphi(t)}^1, V_{\varphi(t)}^2) \bigg|_{\tau = E(\varphi(T)|\mathcal{A}_t)} E \left( (\varphi(T) - E(\varphi(T)|\mathcal{A}_t))^2 \bigg| \mathcal{A}_t \right)
\]

(6.15)

for \( t \in [0, T] \). This expression requires an estimate of the conditional variance \( E((\varphi(T) - E(\varphi(T)|\mathcal{A}_t))^2|\mathcal{A}_t) \). Fortunately, this can be conveniently approximated using a one-dimensional coupled system of PDEs. For this purpose we define a function \( \tau_1 : [0, T] \times (0, \infty)^2 \rightarrow (0, \infty) \) by

\[
\tau_1(t, \gamma, \varphi) = E \left( \varphi(T) \bigg| \mathcal{A}_t \right)
\]

(6.16)

for \( t \in [0, T] \). Using the Kolmogorov backward equation, (4.4) and (6.8), this function satisfies the PDE

\[
\frac{\partial \tau_1(t, \gamma, \varphi)}{\partial t} + a(t, \gamma) \frac{\partial \tau_1(t, \gamma, \varphi)}{\partial \gamma} + \frac{1}{2} b^2(t, \gamma) \frac{\partial^2 \tau_1(t, \gamma, \varphi)}{\partial \gamma^2} + \gamma \frac{\partial \tau_1(t, \gamma, \varphi)}{\partial \varphi} = 0
\]

(6.17)

for \( (t, \gamma, \varphi) \in (0, T) \times (0, \infty)^2 \) with terminal condition

\[
\tau_1(t, \gamma, \varphi) = \varphi
\]

(6.18)

for \( (\gamma, \varphi) \in (0, \infty)^2 \).

The solution to this two-dimensional PDE is given by

\[
\tau_1(t, \gamma, \varphi) = \tau_{1,0}(t, \gamma) + \varphi \tau_{1,1}(t, \gamma),
\]
where \( \tau_{1,0} : [0, T] \times (0, \infty) \to (0, \infty) \) and \( \tau_{1,1} : [0, T] \times (0, \infty) \to (0, \infty) \) are functions that satisfy the coupled system of PDEs

\[
\frac{\partial \tau_{1,0}(t, \gamma)}{\partial t} + a(t, \gamma) \frac{\partial \tau_{1,0}(t, \gamma)}{\partial \gamma} + \frac{1}{2} b^2(t, \gamma) \frac{\partial^2 \tau_{1,0}(t, \gamma)}{\partial \gamma^2} + \gamma \tau_{1,1}(t, \gamma) = 0 \quad (6.19)
\]

and

\[
\frac{\partial \tau_{1,1}(t, \gamma)}{\partial t} + a(t, \gamma) \frac{\partial \tau_{1,1}(t, \gamma)}{\partial \gamma} + \frac{1}{2} b^2(t, \gamma) \frac{\partial^2 \tau_{1,1}(t, \gamma)}{\partial \gamma^2} = 0 \quad (6.20)
\]

for \( (t, \gamma) \in (0, T) \times (0, \infty) \) with terminal condition

\[
\tau_{1,0}(T, \gamma) = 0 \quad (6.21)
\]

and

\[
\tau_{1,1}(T, \gamma) = 1 \quad (6.22)
\]

for \( \gamma \in (0, \infty) \). A similar valuation procedure can be used to approximate the function \( \tau_2 : [0, T] \times (0, \infty)^2 \to (0, \infty) \) given by

\[
\tau_2(t, \gamma_t, \varphi(t)) = E \left( \varphi^2(T) \mid \mathcal{A}_t \right) \quad (6.23)
\]

for \( t \in [0, T] \). The functions \( \tau_1 \) and \( \tau_2 \) given in (6.16) and (6.23), respectively, are needed to approximate the conditional variance appearing in the expansion (6.15). For all of the numerical results described in this section the above expansion method was successfully used.

### 7 Numerical Results

To illustrate the type of pricing effects that can be obtained, Figure 3 shows a term structure of implied volatilities as a function of the strike \( K \) and maturity date \( T \) generated for the above MMM. The default parameters used were \( \nu_0 = \nu_1 = 4.0 \), scaling levels \( q^{0,1} = q^{1,2} = 0.2 \), short rates \( r_0^t = r_1^t = 0.05 \) initial values \( D_0^0 = D_0^1 = 1.0 \) and maturity date \( T = 1.0 \). For the random scaling process \( \gamma \) the default parameters used were \( \varrho = 3 \), \( g = 2 \), \( \eta = 0.048 \) and \( \beta = 0.6 \) with initial values \( \xi_0 = 1.0 \) and \( \gamma_0 = 1.0 \).

For the numerical results shown here, implied volatilities were calculated by using both the domestic and foreign zero coupon bonds to infer the discount factor used in the Black-Scholes formula for exchange rate call and put options. This ensures that the implied volatilities calculated for a call and put option with the same strike and maturity date are the same.

Inspection of Figure 3 shows that for the default parameters used the at-the-money implied volatilities increase gradually as the maturity date increases. The implied volatility smile curvature, evident in Figure 3, is due to the scaling process \( \gamma \). This type of shape for the term structure of implied volatilities is very typical for observed currency options.
Figure 3: Implied call volatilities as function of strike $K$ and time $t$.

Figure 4: Implied call volatilities as function of strike $K$ and scaling level $q^{1,2}$ with $q^{0,1} + q^{1,2} = 0.4$.

For a fixed maturity date $T = 0.25$, Figure 4 shows implied volatilities for European call options as a function of the scaling level $q^{1,2}$. Here the normalization condition $q^{0,1} + q^{1,2} = 0.4$ is imposed to ensure that the at-the-money implied volatilities are approximately the same for different values of $q^{1,2}$. Note that by changing the magnitude of $q^{1,2}$ relative to $q^{0,1}$ the implied volatility curve can be changed from a predominately negatively skewed smile to a predominantly positively skewed smile. For a range of stochastic volatility models this is generally only obtained by using different values for the degree of correlation between the driving Wiener processes for the models. Here it is obtained more naturally and directly by using the scaling levels $q^{0,1}$ and $q^{1,2}$.

For a fixed maturity date $T = 0.25$, Figure 5 displays implied volatilities as a function of the strike $K$ and the dimension $\nu_2$ with $\nu_1 = \nu_2$. It can be seen
Figure 5: Implied call volatilities as function of strike $K$ and dimension $\nu_2$ with $\nu_1 = \nu_2$.

from Figure 5 that a lower dimension produces more curvature for the implied volatility smile. For higher dimensions some degree of curvature is still present. This is due to the natural curvature produced by the random scaling process $\gamma$.

Figure 6: Implied call volatilities as function of strike $K$ and scaling volatility $\beta$.

Finally, Figure 6 shows implied volatilities as a function of the strike $K$ and scaling volatility $\beta$. Note the dramatic increase in curvature if the scaling process $\gamma$ is made more volatile. This figure shows the impact of using a random scaling process and its effect on the curvature of implied volatilities. The above figures illustrate how the curvature and skewness of the implied volatilities of exchange rate options change depending on the parameter choice for the MMM. From a practical point of view it is clear that with realistic parameter choices one can capture using the above parsimonious model the typical implied volatility surfaces of exchange rate options. As shown in Heath & Platen (2004) the corresponding...
index version of this model easily generates the negatively skewed implied volatility surfaces commonly observed for index options. In addition this reduced index model produces realistic fair bonds and corresponding forward rate curves.

Conclusion

This paper proposes a minimal market model with random scaling to describe the dynamics of a currency market. This parsimonious and realistic model is based on specifying suitable representations for the growth optimal portfolio and its different denominations in the domestic and foreign currencies. Time transformed squared Bessel processes arise for each of the associated factors. In addition, the time transformation is itself modeled using a random scaling process. The particular time transformation has been chosen to represent the effect of having random market activity. The fair pricing concept is applied to price European style derivative securities because the model does not admit an equivalent risk neutral martingale measure.

Numerical results for the proposed three-factor model have been obtained, which show realistic implied volatility surfaces for European currency options. A powerful numerical expansion method is described that can be used to efficiently price European style derivatives.

References


