Equilibrium Analysis of Conformity and Influence on a Social Network

David Goldbaum†
April 29, 2014

Abstract

Desiring conformity but lacking common labels with which to identify the different options, a population employs reliable social connections to identify paths through which decisions can decimate. A leader serves the population by coordinating adoption and therefore increasing conformity. A premium for adopting the popular option in advance of others in the population ensures that the cooperative solution generates asymmetry in the payoffs. Motivated by an inquiry into the role of individuals and group connectivity in the emergence of social phenomenon, the present paper identified the set of social structures that are the product of equilibrium strategy profiles. While individual best response strategy and reward depend on the player’s position in the social network of the particular equilibrium realized, the features that characterize the equilibrium structure are found to be universal.

Keywords: Leader, Network, Cooperation, Competition
(JEL Codes: D85, D71, C71)

He enjoys being a player in the power game of art, particularly at this level where patronage can have an impact on public consciousness.1

1 Introduction

Consider a game in which high payoffs are achieved through coordination in the absence of a common language, in the nature of Crawford and Haller (1990). If coordination is to be achieved then strategy profiles must develop over time through path dependent play. Now suppose the options, along with the language, change with each decision. In the face of the lack in continuity in the options, the environment developed allows players to build consistency in their relationships in order to achieve the coordinated outcomes. As an additional challenge,

---

1 The project benefited from the University of Technology Sydney Business Faculty Readership program and from Australian Research Council’s Linkage Projects funding (project number LP0990750)
2 Economics Discipline Group, University of Technology Sydney, PO Box 123 Broadway, NSW 2007 Australia, david.goldbaum@uts.edu.au
3 Thornton (2009) discussing the art collector David Teiger.
the coordinated payoffs are asymmetric so that the players have non-cooperative preferences over the different coordinated outcomes.

The game is motivated by an interest in why and how a population might organize to produce social phenomenon. A key feature of the environment includes a reward to conformity when selecting from a set of options and a premium for adopting the popular choice ahead of the larger population. In the world of avant-garde art, Thornton (2009) observes that a collector’s reputation is based on his or her success in being an early collector of an emergent artist’s works. At the same time the successful emergence of aspiring artists is driven, in part, by the reputation of the collectors acquiring their works. The scenario is such that an individual benefits from acting in advance of a socially determined phenomenon, the emergence of which may be influenced by the individual’s own action. The opening quote’s description of an art collector suggests a desire to be influential in determining which new artists become recognized.\(^2\) The same incentive for early adoption of a subsequently popular choice applies to a wide range of economic actors, from designers and retailers, particularly in fashionable popularity-driven consumer products, to real estate developers, foreign direct investors, revolutionaries, and politicians.

In the developed game, coordination is made possible by the endogenous formation of directed links allowing the player who originates the link to imitate the target of the link. Coordination manifests as a leader-follower structure in the form of a network of followers on the periphery and a single core leader. The leading player selects from the available options with this choice then disseminating through the population of followers. The asymmetry in the payoff is a consequence of the premium earned by the earlier adopters. The benefit of a link is greater for the target than it is for the player who forms the link.

Players are \textit{ex ante} homogeneous in that prior to organizing they face the same payment opportunities, there are no explicit costs for either moving or waiting, and they all have equal access to when and how to choose among the options. Thus, while the presence of a leader serves to make the followers better off, the leader’s ability to do so is derived entirely from the followers’ willingness to concede the leadership position to a single individual and adopt optimizing strategies accordingly. In the absence of leadership characteristics, an element of

\(^2\)Thornton (2009) also quotes Teiger in a statement that reflects a desire to preempt popularity, “My goal is to acquire works that great museums letch after.” (p. 100). Thornton observes, “Unlike other industries, where buyers are anonymous and interchangeable, here, artists’ reputations are enhanced or contaminated by the people who own their work.” (p.88).
the model's development is to resolve how the contradictory forces between cooperating with and competing for the leadership position impact the social structure. All Nash equilibrium strategy profiles produce a social structure consisting of a leader-follower structure and for every population member there is a Nash equilibrium with that player in the leader position.

Strategy profiles that produce multiple leader-follower hierarchies are generally unsupported as equilibrium. The decision that sustains a second hierarchy is not in the hands of the leader but in those of the followers who typically are better served by a single large hierarchy than multiple smaller hierarchies. Caveats do apply.

The present paper identifies strategies profiles that constitute equilibrium for a single period of play and resulting social structures generated by the strategy profiles. Section 2 introduces a network structure, strategies, and reward structure. Many characteristics of the large population solution are observed in the examples of two or three players included in this section. Other aspects of the solution are only revealed when considering general solutions for populations of arbitrary size as developed in Sections 3 and 4. Section 3 develops the condition for which the equilibrium social structure consists of a single leader and a population of followers. Section 4 largely eliminates alternative structures with multiple leaders from the set of Nash equilibria. Structures are also evaluated against best response cascades as a mechanism to determine the resiliency of a particular leader's position. The concluding section speculates on mechanisms for leader emergence. Formal development and proofs are provided in the appendices.

1.1 Related literature

The players in Crawford and Haller (1990) overcome the absence of a common language by using experience and recall to identify a strategy profile that produced the desired coordination. In the current investigation, the options and labels are new to the players at each decision. Players cannot rely on their choice history to facilitate coordination but can organize to make use of universally known social connections to identify paths through which decisions can decimate through the population.

The desire to adopt a popular choice means that a component of the reward is much like that which results from the strategic complementarities found in Katz and Shapiro (1985). Coordination in adoption also plays a role in the Brock and Durlauf (2001) model of conformity.

\[^3\]The static game equilibrium can be thought of as the play in the terminal period of a dynamic game.
and the Arthur (1994) El Farol problem. Classic evidence of social influence in individual decision even in the absence of mechanical complementarities can be found in Whyte (1954), Katz and Lazarsfeld (1955) and Arndt (1967). More recent evidence involves modern technology, such as mobile phone networks as in Hill et al. (2006), and online chat rooms as in Dwyer (2007).

Social networks are an important tool for examining social influence, including the early examples of Schelling (1971), Schelling (1973), Katz and Shapiro (1985), and Banerjee (1992), all of which model the bi-directional interaction between individual decisions and global behavior.\footnote{Watts (2001) and Jackson and Watts (2002) offer useful literature reviews of works on social influence.}

The architecture of a social network defines how information and influence disseminate across a population, as emphasized by Ellison (1993). Brock and Durlauf (2001) is an example for which peer effects are global, allowing tractable analysis using mean-field interactions. Cowan and Jonard (2004) document the impact of local and global connectivity on overall knowledge across a population. The leader and follower structure of the present setting emerges endogenously to serve social and individual interests.

Jackson and Wolinsky (1996) provides the setting employed in a number of models examining endogenous network formation, including Bala and Goyal (2000), Watts (2001), Jackson and Watts (2002), and Dutta and Jackson (2000), and with some modification, Kirman et al. (2007). In these models, the individual payoff function is based on connectivity. The individuals form links that maximize connectivity at minimal cost. The socially influenced network structure that emerges differs in the current project since the individual’s objective is to establish a link that serves a particular purpose: to facilitate early adoption of a subsequently popular trend, an objective not necessarily served by maximizing connectivity. The social phenomenon which is a consequence of the social network determines the payoff. In the current setting, an individual’s payoff is not a direct function of connectivity.

Galeotti and Goyal (2010) considers a model of endogenous network formation and information acquisition. The setting shares in the asymmetry of the payoff with the core population paying more to obtain the information that is eventually shared by the entire population. Acemoglu et al. (2010) also considers the utility gain of the information that flows over the network rather than employing network connectivity as the source of utility.

The present paper identifies the set of social structures that are the product of equilibrium strategies for a single period of play. Multiple Nash equilibria exist and, unlike Crawford and
Haller (1990), the asymmetry in the payoff means that the players have conflicting interests with regards to which equilibrium emerges. The two player version of the current environment reflects the endogenous heterogeneity that can emerge in R&D and duopoly games, as in Reinganum (1985), Sadanand (1989), Hamilton and Slutsky (1990), Amir and Wooders (1998), and Tesoriere (2008). Such games may be played over two stages, but the parallel emerges in the decision regarding when the player wishes to act. Amir et al. (2010) generalizes the issue of symmetry breaking, as is the case when a leader and follower emerge. The general n player game retains the issues regarding asymmetry in outcome while introducing new strategy possibilities. It also introduces the possibility of best response cascades as in Dixit (2003) and Heal and Kunreuther (2010) as ways of refining the equilibrium set.

The potentially large population envisioned for the current investigation implies a leader whose decision drives subsequent adoption, as is true in the herding models such as Ellison and Fudenberg (1995) and Banerjee and Fudenberg (2004), but the setting is quite different. Typical in the herding literature, the network is employed to gather exogenously determined intrinsic product quality information. Here, the information being transmitted from the leader is endogenously determined popularity. It may be possible to distinguish, for example, restaurants by the quality of their food, but the value in going to a particular bar or club is defined socially.

The benefits of early adoption have appeared in models such as in the Pesendorfer (1995) early adoption of new fashion and in Challet et al. (2001), where a trading strategy is profitable if adopted before it becomes popular but not if adopted after.

2 Model

Let $N = \{1, \ldots, n\}$ be a set of players and let the $n \times n$ matrix $g$ describe the potential (directed) links between players. If $i$ can form a link to $j$ then $g_{i,j} = 1$ and $g_{i,j} = 0$ otherwise. Let $g_{i,i} = 1$ always. Write $N^d(i; g) = \{j \in N \setminus \{i\} | g_{i,j} = 1\}$ for a set of players to which $i$ can form a link.

Let $O = \{O_1, O_2, \ldots, O_m\}$ be a set of options or alternatives. To implement the absence of a common labeling of the options, let $K = \{\text{“A”}, \text{“B”}, \ldots\}$ be a set of $m$ labels for these options and let the private one-to-one function $f_i$, determined by nature, map from labels to options, $f_i : K \rightarrow O$. Each player thus privately observes a set of labeled options $K_i = \{\text{“A”}, \text{“B”}, \ldots\}$. Each player sees different labels and for every $i, j$ pair there is a one-to-one function
$h_{i,j} : K_i \rightarrow K_j$ that is unknown to the players.

Let $a_i$ denote the action of player $i$. Players act simultaneously with each player choosing (i) one of the $m$ options, or (ii) to link to another player. In the former case, if player $i$ chooses option $O_k$, then assign $a_i = f_i^{-1}(O_k) = k_i$. If player $i$ links to player $j$, then assign $a_i = j$. A player who chooses an option is said to lead while a player who links to another is said to follow. Thus, the set of actions for player $i$ is $A_i = K_i \cup N \setminus \{i\}$. Write $a = (a_1, \ldots, a_n)$ for an action profile, where $a_i \in A_i$. Let $\mathcal{A}(g,m)$ be the set of all possible action profiles given $g$ and $m$.

An action profile $a$ induces an $n \times n$ matrix $s$ describing the actual links between the players as determined by their actions. If $a_i = j$ then let $s_{i,j} = 1$ and if $a_i \in K$, such that $i$ leads, then $s_{i,i} = 1$. Otherwise, $s_{i,j} = 0$. Thus, for the matrix $s$, we have $s \cdot 1 = 1$. Allow that $j$ is a predecessor of $i$ if $s_{i,j} = 1$ or if there is a sequence of players $j_1, \ldots, j_r$ such that $s_{i,j_1} = \ldots = s_{j_r,i} = 1$. Write $N^P(i; s)$ for the predecessors of $i$. Allow that $j$ is a successor of $i$ if $s_{j,i} = 1$ or if there is a sequence of players $j_1, \ldots, j_r$ such that $s_{j,j_1} = \ldots = s_{j_r,i} = 1$. Write $N^S(i; s)$ for the successors of $i$. A leader is an agent who leads and has a non-empty set of successors. It is possible to lead without being a leader.

Let $N^L(s)$ denote the set of players who lead. If player $i$ leads and player $j$ is a successor of $i$, this makes player $i$ player $j$’s leader. Note that each player $i$ has at most one player who leads as a predecessor, that is $|N^L(s) \cap N^P(i; s)| \in \{0, 1\}$ for each $i$. It is possible for a successor to be without a leader. Additionally, let $L_i$ identify the predecessor of $i$ who is a leader.

**Example 1.** Consider a population of twelve players arranged in a ring with each player able to link to her nearest neighbor on either side. For $m = 2$, the set of feasible action profiles includes, as an illustrative example, the action

$$a = (f_1^{-1}(O_1), f_2^{-1}(O_1), 2, 5, 6, f_6^{-1}(O_1), 6, 7, f_5^{-1}(O_2), 9, 12, 11).$$

Figure 1 includes graphical representations of $g$ and $s$. Here, $N^L(s) = \{1, 2, 6, 9\}$ and for $i \in \{4, 5, 7, 8\} L_i = 6$. In Figure 1c, the leaders all occupy the root node of a tree capturing the hierarchical social structure determined by $s$ with predecessors above successors. Player 1, without successors, occupies a trivial tree. Player 9 is the only leader to have selected $O_2$. Player 5 has 6 as a predecessor and 4 as a successor. Players 11 and 12 fail to adopt one of the
Figure 1: A $g$ and feasible $s$ for a population of $n = 12$ players with $m = 2$ options. The structure of $g$ is a ring with $N^d(1; g) = \{12, 2\}$, $N^d(12; g) = \{11, 1\}$ and $N^d(i; g) = \{i - 1, i + 1\}$ otherwise. Frame (1a) is a graphical depiction of the $g$ matrix. Frame (1b) is a graphical depiction of the $s$ matrix resulting from the actions $a = (f_1^{-1}(O_1), f_2^{-1}(O_1), 2, 5, 6, f_6^{-1}(O_1), 6, 7, f_9^{-1}(O_2), 9, 12, 11)$. A leader’s choice is included in parenthesis. Frame (1c) depicts the groupings implied by $s$ as trees (or “hierarchies”) with option selected above the trees, dashed arrows indicating a leader’s option choice, and predecessors positioned above successors. Followers 11 and 12, lacking a path to one of the options, are placed at the bottom.
options as they are successors to each other and thus without a leader.

A notion of distance is needed in order to specify payoffs. Figure 1c includes a route from each player to their option through their predecessors and their leader’s option selection. Define the distance from a player \( i \) to her adopted option as the number of players between \( i \) and the option. Using \( d_i \) to denote player \( i \)’s distance,

\[
d_i = \begin{cases} 
0 & i \in N^L(s) \\
1 & s_{i,j} = 1, j \in N^L(s) \\
r + 1 & s_{i,j} = \ldots = s_{j,r} = 1, j \in N^L(s) \\
\infty & \text{otherwise.}
\end{cases}
\]

Use \( D(s) \) to denote the matrix for which each row contains the ordered pair \((d_i, L_i)\). A second useful distance measure is the distance from a successor to a predecessor. Use \( d_{i,j} \) to denote the distance from successor \( i \) to predecessor \( j \) measured in the number of links connecting \( i \) to \( j \). Observe that when \( L_i = j \), \( d_{i,j} = d_i \).

Let \( N^J(i; a) \) denote the set of players who adopt the same option as does player \( i \) (inclusive of \( i \)). Let \( N^T(i; a) \) denote the subset of \( N^J(i; a) \) who are of greater distance from the option than is \( i \). Formally, \( N^J(i; a) = \{ j \in N| f_i(k_i) = f_j(k_j) \} \) and \( N^T(i; a) = \{ j \in N^J(i; a)|d_j > d_i \} \). Let \( \mu^J_i = |N^J(i; a)| \) and \( \mu^T_i = |N^T(i; a)| \).

The payoff for player \( i \) is

\[
\pi(i; s) = a_J(\mu^J_i - 1) + a_T\mu^T_i 
\]

with coefficients \( a_T \geq 0 \) and \( a_J \geq 0 \). The first element of the payoff is the “conformity” component, much like the community effect of Blume and Durlauf (2001). The second element in (1) reflects the distance advantage a player has over other players.\(^7\) Let \( \Pi(a) = (\pi_1, \ldots, \pi_n)' \).

In Figure 1c of Example 1, for player \( i \in \{1, \ldots, 8\} \), \( N^J(i; a) = \{1, \ldots, 8\} \) so that \( \mu^J_i = 8 \). In

---

5 The notion of time and distance are interchangeable when adoption disseminates at a rate of one unit of time per link.

6 The linearity employed in (1) is later relaxed for limited exploration of alternatives.

7 The model generalizes to one in which instead of rewarding early adoption there is a cost or penalty to late adoption. Similar to the examples used in Brindisi et al. (2011) in which late adopters pay higher costs, payoffs become \( \pi(i; s) = b_J(\mu^J_i - 1) - b_T(\mu^J_i - \mu^T_i - 1) \) where \( b_J \) is the per member conformity payoff and \( b_T \) is the cost associated with each player who acts concurrent or in advance of player \( i \) on the same option. With \( b_J = a_J + a_T \) and \( b_T = a_T \), the two scenarios are isomorphic.
addition, for $i \in \{3, 5, 7\}$, $N_T(i; a) = \{4, 8\}$, reflecting that all players of equal distance from $O_1$ benefit equally from the players who are of greater distance. Relative distance from the adopted option determines payoff, not succession. Player 9, having chosen differently than the other leading players, benefits only from her successor, player 10. For players $i \in \{4, 8, 10, 11, 12\}$, $N_T(i; a) = \emptyset$. Players 11 and 12, failing to adopt a choice receive no payoff, nor do they contribute to the payoff of any other player. Table 1 reports the payoff to each player based on the action $a$ from example 1.

2.1 Strategic behavior

For any two leading players $i$ and $j$, players believe that $\Pr(f_i(k_i) = f_j(k_j)) = 1/m$, a belief that is consistent with uncertainty about $h_{i,j}$. The uncertainty in whether two leaders will match options means that there can be a random element to pure strategy payoffs. A couple of small-$n$ examples illustrate the issues and outcomes inherent to the setting.

Example 2. $n = 2$, $m = 2$, $a_J = 1$, $a_T = x > 0$ and $g = \frac{1}{2 \times 2}$. The payoff matrix associated with each possible outcome is listed in Table 2a. Table 2b is the proper normal-form game. The payoff table includes all possible actions by each player and the expected payoff associated with the uncertain outcome produced when both players lead. The game can be reduced, as in Table 2c, to eliminate the inconsequential distinction between leading with “A” and leading with “B”.

The two Nash equilibrium strategy profiles both produce one leader and one follower. For the equilibrium with player 2 leading player 1, player 2 receives the higher payoff for being the leader. Player 1’s lower payoff remains higher than the expected payoff that can be obtained from also leading. Player 2’s selection of $a_2 \in \{“A”, “B”\}$ has no impact on the realized payoffs in equilibrium nor does the choice effect expected payoffs in non-equilibrium play. The

<table>
<thead>
<tr>
<th>Player</th>
<th>$\mu_i^J$</th>
<th>$\mu_i^T$</th>
<th>$\pi_i$</th>
<th>Player</th>
<th>$\mu_i^J$</th>
<th>$\mu_i^T$</th>
<th>$\pi_i$</th>
<th>Player</th>
<th>$\mu_i^J$</th>
<th>$\mu_i^T$</th>
<th>$\pi_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
<td>5</td>
<td>22</td>
<td>5</td>
<td>8</td>
<td>2</td>
<td>13</td>
<td>9</td>
<td>2</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>5</td>
<td>22</td>
<td>6</td>
<td>8</td>
<td>5</td>
<td>22</td>
<td>10</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>2</td>
<td>13</td>
<td>7</td>
<td>8</td>
<td>2</td>
<td>13</td>
<td>11</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>0</td>
<td>7</td>
<td>8</td>
<td>8</td>
<td>0</td>
<td>7</td>
<td>12</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1: Payoff calculations for $a = (f_1^{-1}(O_1), f_2^{-1}(O_1), 2, 5, 6, f_6^{-1}(O_1), 6, 7, f_9^{-1}(O_2), 9, 12, 11)$ using payoff parameters $a_J = 1$ and $a_T = 3$. 


symmetry of the game means that there is also an equilibrium with player 1 leading player 2. The players want to avoid the strategy profile in which both lead. They also want to avoid the outcome produced when both imitate the other.\(^8\)

**Example 3.** \(n = 3, m = 2, a_J = 1, a_T = x > 0\) and \(g = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} \times \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \). A larger population introduces the possibility of acting as a minority player. Table 3 reports the expected payoff matrix for the actions for players 1 and 2 based on player 3 leading. If player 2 also chooses to lead, then player 1 should imitate but is indifferent with regards to whom to imitate. Either way, the social structure has one leader-successor pairing and one player leading but without a successor. Player 2’s choice from the option set, \(a_2 = "A"\) or \(a_2 = "B"\), is inconsequential to the expected payoffs.

If player 2 chooses to imitate player 1, then player 1’s best response is to imitate player 3. With player 2 as a successor, by relinquishing the lead to player 3, player 1 benefits from the enlarged group, increasing the conformity payoff, without sacrificing the distance advantage.

---

\(^8\)The mixed strategy solution for this example has \(Pr_i(\text{lead}) = (1 + x)/(3 + x)\). The value of the game in the mixed strategy solution is \(v = (2 + 2x)/(3 + 2x)\). Since, \(v < 1\) for all \(x \geq 0\), the value of the mixed strategy solution is always less than the follower’s payoff in the pure strategy game.
Table 3: Example 3 relevant expected payoff table for $n = 3$, $k = 2$, $a_J = 1$, $a_T = x > 0$, $s_{3,3} = 1$

If player 2 imitates player 3, player 1’s optimal strategy depends on the value of $x$. For $x < 2$, then player 1 again chooses to succeed 3, either directly or indirectly by succeeding 2. Player 1 is indifferent between direct or indirect succession because, whether alone or alongside player 2, she is the most distant player.

The structure is again symmetric. For $x < 2$, the equilibrium is a social structure in which one player leads and the other two imitate. It would be self-serving for each successor to imitate the leader directly, but successor $i$ can costlessly improve the payoff of successor $j$ by routing her imitation of the leader through player $j$. All three players want to avoid the presence of a self-referencing loop which could be seen as the unfortunate outcome of the two imitators both looking to generously improve the other successor’s payoff.

For $x > 2$, if player 2 imitates player 3, then player 1’s optimal strategy is to lead. This strategy is motivated by the $1/2$ probability that $f_1(k_1) = f_3(k_3)$ when both 1 and 3 lead. In this case player 1 gains the distance advantage over player 2. This gamble becomes worthwhile for sufficiently large $x$. Notice that player 1 only takes this gamble when player 2 follows player 3. Without an imitator of player 3 there is no inducement for player 1 to gamble with leading. Regardless of $x$, if both 2 and 3 lead, player 1 will want to imitate.

With the symmetry of the game, there are six equilibria for $x > 2$, each consisting of one player as leader, one player following the leader, and the third player leading but without a follower. All are equivalent subject to a relabeling of the players.

A “strongly connected” network is one for which every player pair \{i, j\} has either $g_{i,j} = 1$ or there exists $j_1, \ldots, j_m$ such that $g_{i,j_1} = \ldots = g_{j_m,j} = 1$. As a consequence, for every \{i, j\} pair there is a directed path from $i$ to $j$. Let $G(n)$ be the universe of strongly connected networks based on a population size $n$. Both examples 2 and 3 are based on a $g$ that is a complete graph.
(all players are able to link to any other player directly). For \( n = 2 \) the only strongly connected graph is the complete graph. There are 18 possible \( g \in G(3) \) with five that are unique to a relabeling of the players. For any \( g \in G(3) \), the benefits to coordinating on a choice through imitation is unchanged. The following can be demonstrated to be true for all \( g \in G(3) \).

1. For \( x < 2 \), an action profile is a Nash equilibrium if and only if it produces one leader and two successors.

2. For \( x > 2 \), an action profile is a Nash equilibrium if and only if it produces two players who lead (only one is a leader) and one successor.

3. The set of Nash equilibria include actions profiles that produce \( i \) as the unique leader for all \( i \in \{1, 2, 3\} \).

From 1 and 2 above, every equilibrium action produces one and only one non-trivial tree. From 3 it is always possible that any one of the players can hold the favorable position of leader of the non-trivial tree.

To be developed in the following sections, the features found in the \( n = 3 \) population generalize for any size population occupying a strongly connected network. These features are

- Pure strategy Nash equilibria exist
- A unique leader, possibly in the presence of other leading players, is the equilibrium social structure (caveats apply).
- All players \( i \in N \) can be the equilibrium leader of the non-trivial tree.
- There is a single equilibrium leader. Whether the leader is the only agent who leads in equilibrium depends on a formula expressed primarily in terms of \( \frac{a_J}{a_T} \) and \( m \). Above a threshold, the entire population joins to form a single tree. For \( \frac{a_J}{a_T} \) below the threshold, the sized of the single non-trivial tree declines in \( \frac{a_J}{a_T} \).

3 **Equilibrium**

This section formally develops the behavior observed in the two examples of Section 2.1 while generalizing to a population of size \( n \) and a network of potential links \( g \in G(n) \). A population
of \( n > 3 \) makes feasible coexisting multiple non-trivial trees. Some additional aspects of the equilibrium actions only come to light when considering a larger \( n \). There is, for example, a special case social structure that requires \( n \geq 8 \).

The equilibrium concept employed is that of a pure strategy Nash equilibrium. The following establishes two components of a player’s decision. First it is established that if a player chooses to follow rather than to lead, then it is best to imitate a player offering the shortest distance, measured in number of links, to the player’s leader. Conditions under which a player prefers to follow rather than lead are then developed. The conditions governing individual player decisions are then employed to develop a general condition ensuring a single leader as equilibrium.

3.1 Hierarchies

The term hierarchy refers to a non-trivial tree. Let \( h(i; g) \) be the set of \( s \) given \( g \) such that \( i \in N^L(s) \) with a non-trivial tree of successors. Let \( H(i; g) \) represent the set of \( s \) given \( g \) such that player \( i \) is the exclusive member of \( N^L(s) \) with a successor population \( N^S(i; s) = N \setminus \{i\} \).

**Lemma 1.** For every \( g \in G(n) \) there exists, for all \( i \), at least one \( a \) such that \( s \in H(i; g) \).

Note that the \( a \) generating \( s \in H(i; g) \) is generally not unique. In particular, \( s \in H(i; g) \) is independent of \( a_i \in K \). In addition, the set \( H(i; g) \) will generally contain more than one element as there generally exists more than one path through which any follower \( j \) can link to leader \( i \).

**Lemma 2.** For \( j \in N^S(i; s) \) and \( j \in N^S(i; s') \), where \( s \) has \( s_{j,ja} = 1 \) and \( s' \) has \( s_{j,jb} = 1 \), then for \( d_{ja,i} \leq d_{jb,i} \),

\[
\pi_j(s) = \begin{cases} 
\pi_j(s') & \text{if } d_{ja,i} = d_{jb,i} \\
\geq \pi_j(s') & \text{if } d_{ja,i} < d_{jb,i}.
\end{cases}
\]

Lemma 2 follows from the fact that \( \mu_{j}^{T} \) is non-decreasing as \( j \in N^S(j; s) \) reduces her distance to \( i \) for all \( s' \in H(i; g) \) given \( s_{-j} = s_{-j} \).

Let \( H^*(i; g) \) be the set of \( s \in H(i; g) \) such that each \( j \neq i \) employs an imitation strategy offering a distance-minimizing sequence of links to the leader. Note that if \( s' \) exists such that \( \{s, s'\} \subseteq H^*(i; g) \), then \( d_j(s) = d_j(s') \) for all \( j \in N \). As a result, all \( s \in H^*(i; g) \) offer exactly the same payoffs across individuals. These features are illustrated in Figure 2. Observe that
Figure 2: Possible $s \in H(1; g)$ for $g$ with $N^d(1) = \{3, 4\}$, $N^d(2) = \{1, 3\}$, $N^d(3) = \{1, 4\}$, and $N^d(4) = \{2, 3\}$, in frame (a) player 2 fails to use a distance-minimizing chain of links to connect to player 1. In frames (b), and (c) all followers employ distance-minimizing links demonstrating that player 4 has two distance minimizing routes by which to link to player 1. For $s \in H^*(1; g)$, let $h^*(i; g)$ be the set of strategies for which each successor of $i$ imitates the player offering the shortest distance from $i$.

Consider $s \in H(i; g)$, and $j \in N^S(i; s)$. Let $N^x(j; s)$ be the set of successors of $i$ who are of distance no greater than $d_{j,i}$. Let $N^y(i; s)$ be the set of successors of $i$ with a distance greater than $d_{j,i}$ who are not successors of $j$. Formally, $N^x(j; s) = \{j_x \in N^S(i; s) \setminus \{i\} | d_{x,i} \leq d_{j,i}\}$, $N^y(j; s) = \{j_y \in N^S(i; s) \setminus N^S(j; s) | d_{y,i} > d_{j,i}\}$. Let $N^z(j; s)$ be the successors of $i$ who are also successors of $j$, so that $N^z(j; s) = N^S(j; s)$. Let $\mu^x(j) = |N^x(j; s)|$, $\mu^y(j) = |N^y(j; s)|$ and, $\mu^z(j) = |N^z(j; s)|$. In Figure 3, the node labels identify the agent’s position relative to player $j$ with, $\mu^x(j) = 1$, $\mu^y(j) = 3$ and $\mu^z(j) = 2$. Observe the identities for any $s \in H(i; g)$

$$\mu^I_j = 2 + \mu^x(j) + \mu^y(j) + \mu^z(j) = n$$  \hspace{1cm} (2)

$$\mu^T_j = \mu^y(j) + \mu^z(j).$$  \hspace{1cm} (3)

As seen in Example 3, a follower can possibly increase her distance to the leader without reducing her own payoff. The condition for this to be feasible is developed formally in the Appendix. It requires that the follower $j$ has $\mu^y(j) = 0$ and at least one member of $N^x(j; s)$ in her contact list who is at the same distance from $i$. Let $H'(i; g)$ be the set of $s \in H(i; g)$ such that each $j \neq i$ employs an imitation strategy offering a payoff equal to that earned by $s^*_j$, where $s^* \in H^*(i; g)$, and without sacrificing any payoff gain from another player’s deviation
Proposition 1. If $s^* \in H(i;g)$ is a Nash Equilibrium, then either $s^* \in H^*(i;g)$ or $s^* \in H'(i;g)$ and there exists an $s^{**} \in H^*(i;g)$ that is also a Nash Equilibrium.

Proof. See Appendix

The proof of Proposition 1 demonstrates that for $s \in H(i;g) \backslash H'(i;g)$, there exists at least one successor of $i$ with the option to shorten her distance to $i$ and increase her payoff by imitating someone else in her $N_d(i; s)$ set. For $s^* \in H'(i;g)$, no opportunity exists for a player to improve her payoff while continuing as a successor of $i$.

The analysis to follow considers whether $s \in H^*(i;g)$ constitutes an equilibrium and explores the conditions under which $s \notin H(i;g)$ can be an equilibrium. If $s \in H^*(i;g)$ is a Nash equilibrium, so are all $s' \in H'(i;g)$.9

Consider a deviation from $s \in H^*(i;g)$ by some $j \in N^S(i; s)$ to lead rather than follow so that the resulting $s' \notin H(i;g)$. Let

$$A(j; s) := (m - 1)(a_J + (a_T + a_J)\mu_y(j)) + ((m - 1)a_J - a_T)\mu_x(j).$$

9Risk aversion based arguments can also be employed to eliminate the need to consider $s \in H'(i;g) \backslash H^*(i;g)$ as possible Nash equilibrium. An agent who unilaterally increases her distance from the leader without loss in her expected payoff exposes herself to lost opportunity to a higher payoff or to membership in a self-referencing loop if another player engages in the same activity.
and let
\[ B(m, n, a_J, a_T) := (m - 1) \frac{a_J}{a_T} - \left( 1 - \frac{1}{n - 1} \right). \] (5)

**Proposition 2.** Consider \( s \in H(i; g) \), \( B \geq 0 \) is a necessary and sufficient condition that no \( j \in N^S(i; s) \) can improve her payoff by switching strategy to lead.

**Proof.** See Appendix  

The proof of Proposition 2 establishes that \( A(j; s) \geq 0 \) ensures that player \( j \in N^S(i; s) \) prefers her current imitation strategy over leading. Since the first term of (4) is weakly positive, a sufficient condition for \( A(j; s) \geq 0 \) for all \( j \in N^S(i; s) \), \( s \in H(i; g) \), is that \( a_T \leq (m - 1)a_J \) or
\[(m - 1)\frac{a_J}{a_T} - 1 \geq 0. \] (6)

From (6), \( a_J > a_T \) ensures that for \( m > 1 \) only \( i \) leads. A setting with \( a_J < a_T \) introduces the possibility that there exists “too few” \( m \) or “too little” \( a_J \) to support \( s \in H^*(i; g) \) as equilibrium. Too few \( m \) entices potential successors to instead lead, taking advantage of the relatively high probability that \( f_i(k_i) = f_j(k_j) \). Too little \( a_J \) fails to adequately reward those most distant from \( i \) for following.

Were player \( j \) in Figure 3 to lead, she retains the population \( N^z(j; s) \) players as successors regardless of the outcome. With \( \{i, j\} = N^L(s) \), the probability \( f_i(k_i) = f_j(k_j) \) is \( \frac{1}{m} \). If \( f_i(k_i) = f_j(k_j) \), then \( N^T(j; a) = N^x(j; s) \cup N^y(j; s) \cup N^z(j; s) \). Player \( j \) thus retains her distance advantage over the \( N^y \) and \( N^z \) populations and gains a distance advantage over the \( N^x \) population. With probability \( \frac{m - 1}{m} \), \( f_i(k_i) \neq f_j(k_j) \) in which case player \( j \) retains successors \( N^z \) but loses her distance advantage over the \( N^y \) population. Player \( j \) also loses her conformity in choice with \( \{i\} \cup N^x(j; s) \cup N^y(j; s) \). A player \( j_n \) who is most distant from \( i \) has the most to gain from leading and the least to lose from giving up her position as a successor of \( i \) as captured by \( A(j_n; s) \leq A(j; s) \) for all \( j \in N^S(i; s) \). Since \( A(j_n; s) = B \), \( A(j; s) \geq 0 \) for all \( j \in N \{i\} \) if and only if \( B \geq 0 \).

**Proposition 3.** \( B \geq 0 \) is a necessary and sufficient condition for \( \{H^*(i; g)\}_{i \in N} \) to be a set of equilibrium strategies.

**Proof.** See Appendix
Proposition 3 follows naturally from Propositions 1 and 2. The condition \( B \geq 0 \) is determined by universal parameters not specific to the potential network \( g \) or the particular strategy profile \( a \) that produces \( s \in H^*(i; g) \), a reflection of the fact that the payoffs at both the top and the bottom of the hierarchy depend only on the size of the hierarchy and not its organization. If \( B \geq 0 \), then the equilibrium structure consists of a single leader with the entire remaining population as successors. Each successor imitates in order to minimize the distance to the leader. The equilibrium condition does not establish or limit who in the population holds the single leader position and it is available to everyone in the population.

4 Other structures

The decision by player \( j \) regarding whether to imitate and thus join \( N^S(i; s) \) or to lead has payoff consequences for the entire population. If \( s \in H^*(i; g) \) is an equilibrium, it is appropriate to consider just how fragile or stable the optimizing decisions supporting the tree structure rely on the coordination of the other players. After developing the equilibrium structure for \( B < 0 \), this section considers the best response to a single player deviation under \( B \geq 0 \).

4.1 Multiple agents lead

From Proposition 2, \( B < 0 \) does not support the single-leader/one-tree structure as an equilibrium. Given an \( s \in H^*(i; g) \), for \( B < 0 \), at least one of the most distant followers prefers to also lead. This section develops an equilibrium structure given that \( s \in H(i; g) \) is not an equilibrium.

Let \( h_L(i, \mu^*_i; g) \) represent the set of actions for which \( i \)'s successor population is of size \( \mu^*_i < n - 1 \) and the \( n - \mu^*_i - 1 \) most distant players from \( i \) lead rather than follow. For \( s \in h_L(i, \mu^*_i; g) \) then \( s \) is the analog based on each follower choosing a predecessor who minimizes her distance to \( i \). To develop \( h_L(i, \mu^*_i; g) \) formally, for any \( s \in H(i; g) \), let \( r_{j,i}(s) \) be \( j \)'s rank when ordering players based on their distance from \( i \). The closest successor of \( i \) has \( r_{j,i}(s) = 1 \). Players of equal distance receive consecutive ranks, the order of which does not impact the findings. Let \( N^S(i; s) \) be the set of successors to \( i \) with rank \( r_{j,i} \leq r \). Then, for \( s' \in H(i; g) \), \( h_L(i, \mu^*_i; g) = \{ s \in h(i; g) \} \) such that for \( j \in N^S(i; s') \), \( s_j = s'_j \) and for \( j \in N \setminus N^S(i; s') \), \( s_{j,j} = 1 \).
Let
\[ D(\mu_i^z; m, n, a_J, a_T) = (m - 1) \frac{a_J}{a_T} - \left(1 - \frac{1}{\mu_i^z}\right). \] (7)

Allow \( \mu^* \) to represent the value of \( \mu_i^z \) that solves \( D = 0 \),
\[ \mu^* = \frac{1}{1 - (m - 1) \frac{a_J}{a_T}}. \] (8)

**Proposition 4.** For \( B < 0 \), if \( s^* \in h_L(i, \mu_i^z; g) \) is a Nash equilibrium, then \( s^* \in h^*_L(i, \tilde{n}; g) \) where \( \tilde{n} \) is an integer value with \( |\tilde{n} - \mu^*| < 1 \).

**Proof.** See Appendix

The proof of Proposition 4 establishes that for \( s \in h_L(i, \mu_i^z; g) \), the same \( A(j; s) \geq 0 \) condition developed in Proposition 2 ensures that player \( j \in N^S(i; s) \) prefers her current imitation strategy over leading. The difference is that \( \mu^z(j) \) and \( \mu^y(j) \) vary not only with \( j \)'s relative distance from \( i \), but also with \( \mu^z(i) \), the size of the tree under leader \( i \). Identify the most distant successor of \( i \) given \( s \) as \( j_{\mu_i^z} \in N^S(i; s) \). The proof demonstrates that \( A(j_1, s) > 0 \), \( A(j_{\mu_i^z}; s) \) is decreasing in \( \mu_i^z \) and since \( B < 0 \), \( A(j_{\tilde{n} - 1}; s) \) is assured to be negative. At size \( \mu_i^z = \tilde{n} \), the most distant \( j \in N^S(i; s) \) prefers following to leading while at size \( \mu_i^z = \tilde{n} + 1 \), the most distant \( j \in N^S(i; s) \) prefers to lead.

As was the case in Proposition 3, the equilibrium size is determined by universal parameters, independent of the particular \( i \in N \) leading the single non-trivial tree and of the particular structure of the tree other than its size. In order to be able to claim that \( \{h^*_L(i, \tilde{n}; g)\}_{i \in N} \) is the set of equilibrium, other \( h^*(i; g) \) structures consisting of multiple non-trivial trees must be eliminated as possible equilibrium.

### 4.2 Single and multiple leaders

The possibility that multiple hierarchies may exist in equilibrium must be considered for both the \( B \geq 0 \) and \( B < 0 \) cases. In general, actions that combine to produce multiple hierarchies are not consistent with equilibrium play. Two exceptional scenarios do exist for \( B \geq 0 \) as developed in Proposition 5 below. There are no exceptions when \( B < 0 \).
4.2.1 \( B \geq 0 \)

Let \( H(i_A, i_B; g) \) represent the set of structures in which every player in the population is a successor to one of two leaders, producing trees under leaders \( i_A \) and \( i_B \). In \( H^*(i_A, i_B; g) \), each successor employs the shortest path to the chosen leader. In \( H^{**}(i_A, i_B; g) \) each successor employs the shortest path to a leader. Let \( \mu^z_h = |N_S(i_h; s)| \) indicating the number of successors in the \( i_h \)-led tree \( h \in \{A, B\} \). Without loss in generality, assume \( \mu^z_A \geq \mu^z_B \).

For \( h \in \{A, B\} \), let \( j_h \) represent \( j \in N^S(i_h; s) \). Let \( d\mu = \mu^z_A - \mu^z_B \) so that \( d\mu \) captures the population size differential between the two trees. For \( s \in H(i_A, i_B; g) \), let \( N^{AB}(i_A, i_B; s) \) represent the set of players possessing potential links to predecessors in both trees. The population \( N^{AB}(i_A, i_B; s) \) also potentially includes \( i_h \) if \( i_h \) has contacts in the \( i_{-h} \)-led tree. A strongly connected \( g \) ensures that the two sets \( N^{AB}(i_A, i_B; s) \cap \{N^S(i_h; s), i_h\} \) are both non-empty.

With two non-trivial trees, there is a need to identify and label populations in the \( i_{-h} \)-led tree based on their position relative to \( j_h \). Let \( N^{i_{-h}}(j; s) \) represent the subset of \( N^S(i_{-h}; s) \) who are more distant from \( i_{-h} \) than is player \( j \) from \( i_h \). For example, \( N^{i_{-h}}_B(j_B; s) = \{ j \in N^S(i_A; s) | d_{j, i_A} > d_{j_B, i_B} \} \). Let \( \mu^{i_{-h}}(j) = |N^{i_{-h}}(j; s)| \). Let \( s' = s_{-j_h} \times s'_{j_h} \) be the structure produced by \( j_h \) switching predecessors in order to become a member of the \( i_{-h} \)-led tree. The alternative structure identifies populations \( N^{i_{-h}}(j_h; s') \) and \( N^{i_{-h}}(j_h) = N^{i_{-h}}(j_h; s') \). The former is the population of players in \( j_h \)'s current tree who are more distant from \( i_h \) than is \( j_h \) from \( i_{-h} \) in \( s' \). The latter is the population in the \( i_{-h} \) led tree more distant from \( i_{-h} \) than \( j_h \) in
s'. Let $\mu^h_{h}(j_h) = |N^h_{h}(j_h; s')|$ and $\mu^v_{h}(j_h) = |N^v(j_h; s')|$. The example presented in Figure 4 illustrates these relative values.

Let

$$E_A(j_A; m, a_J, a_T) := (m - 1)\frac{a_J}{a_T}(d\mu - 1 - \mu^z(j_A))$$

(9)

$$-(\mu^A_B(j_A) - \mu^A_B(j_A) - m(\mu^y(j_A) - \mu^y_B(j_A)))$$

$$E_B(j_B; m, a_J, a_T) := -(m - 1)\frac{a_J}{a_T}(d\mu + 1 + \mu^z(j_B))$$

(10)

$$+(\mu^A_B(j_B) - \mu^A_B(j_B) + m(\mu^y(j_B) - \mu^y_B(j_B))).$$

**Proposition 5.** Given $B \geq 0$, a necessary and sufficient condition for $s \in H^*(i_A, i_B; g)$ to be a Nash equilibrium is for $N^d(i_h; g) \cap N^S(i_h; g) = \emptyset$ and $E_h(j) \geq 0$ for all $j \in N^{AB}(i_A, i_B; s) \backslash \{i_A, i_B\}$, $h \in \{A, B\}$. For those players occupying a terminal node of a follower tree branch,

1. the most distant successors have,

   (a) for every $j_B \in \{N^{AB}(i_A, i_B; s) \mid \mu^z(j) = \mu^y(j) = 0\}$, $(m - 1)\frac{a_J}{a_T} < \frac{\mu^A_B(j_B) - \mu^A_B(j_B)}{d\mu + 1}$,

   (b) for every $j_A \in \{N^{AB}(i_A, i_B; s) \mid \mu^z(j) = \mu^y(j) = 0\}$, $(m - 1)\frac{a_J}{a_T} \geq \frac{\mu^A_B(j_A)}{d\mu - 1}$,

2. for all $j_B \in \{N^S(i_B, s) \cap N^{AB}(i_A, i_B; s) \mid \mu^z(j_B) = 0, \mu^y(j_B) > 0\}$,

   $$(m - 1)\frac{a_J}{a_T} < \frac{m(\mu^y(j_B) - \mu^y_B(j_B)) + (\mu^A_B(j_B) - \mu^A_B(j_B))}{(d\mu + 1)}.$$ 

**Proof.** See Appendix.

The expressions $E_A$ and $E_B$ are derived for $j \in N^{AB}(i_A, i_B; s)$ from the expected payoff premium offered by the player’s current position in the $i_h$-led tree over the expected payoff available from changing predecessors in order to join the $i_{-h}$-led tree, $E_h = \mathbb{E}(\pi_{h}(j_h; s)) - \mathbb{E}(\pi_{-h}(j_h; s'))$. Expectations are taken over the possible maps of $f$. The conditions that produce $E_h \geq 0$ can potentially hold for any follower.

In both $E_A$ and $E_B$, the first term captures the payoff gain attributable to the size advantage of the $i_A$-led tree. The second term captures the position-specific advantage for player $j_h$. Reflected in the $E_h$ equations is the fact that were $j_h$ to switch affiliation to the $i_{-h}$-led tree,
she retains the $N^S(j_h, s)$ population as successors. Also reflected in the equations is that were $j_h$ to switch affiliation, she loses her distance advantage over the $N^Y(j_h; s)$ population.

The individual circumstance under which $E_h(i) \geq 0$ depends on player $i$’s position in $s$. For the leaders, $E_h(i_h) < 0$ for all $s \in H^*(i_A, i_B; g)$. The opportunity to retain their relative position over their successors while enlarging the size of the tree to which they are affiliated makes following the other leader always preferred to remaining a leader. That $E_h(i_h) < 0$ is a reflection of Proposition 2. For $s \in H^*(i_A, i_B; g)$ to be an equilibrium, Proposition 5 excludes structures in which either leader’s contact list includes a member of the other tree.

Case 5.1 identifies the situation in which, as the most distant successors of $i_A$ and $i_B$ respectively, neither $\{j_A, j_B\} \in N^{AB}(i_A, i_B; s)$ finds it advantageous to switch predecessors. As the most distant followers of their respective leaders, they have two sources of expected reward; (i) the conformity payoff gained by following the chosen leader and (ii) for $d_{j_A, i_A} \neq d_{j_B, i_B}$, the player with the shorter distance gains, in expectation, a benefit from the non-trivial $N^h_{-h}(j_h, s)$ population. The former reward favors the larger tree so that, in general, $j_B$ would want to switch affiliation to the $i_A$-led tree. For $d_{j_B, i_B} < d_{j_A, i_A}$ the latter favors the smaller tree. When the inequality in 5.1a holds, then latter reward can be sufficiently large to generate in $j_B$ a preference for the smaller $i_B$-led tree. In this case, $s \in H^*(i_A, i_B; g)$ can be an equilibrium if there is not $j_A$ able to take advantage of the same distance advantage enjoyed by $j_B$ by switching to the $i_B$-led tree, as is the case if the inequality in 5.1b holds.

To jointly satisfy the two inequalities in 1a and 1b requires the $i_A$ tree structure to be sparsely populated at distances less than or equal to the distance $d_{j_B, i_B}$, that there is a large population of successors to $i_A$ at the distance of $d_{j_B, i_B} + 1$, and that at distances greater than $d_{j_B, i_B} + 1$ the $i_A$-led tree be sparsely populated or unoccupied. It also imposes that the potential link for $j_B$ in the $i_A$-led tree place $j_B$ at the distance $d_{j_B, i_A} = d_{j_B, i_B} + 1$ and that the potential link for $j_A$ in the $i_B$-led tree place $j_A$ at distance $d_{j_A, i_B} = d_{j_B, i_B} + 1$. The exact threshold relationships are discussed in more detail in the Appendix. These features ensure that there is a population over which $j_B$ enjoys a distance advantage in expectation only as a member of the $i_B$-led tree but that $j_A$ cannot also benefit from by switching affiliation. The structure depicted in Figure 5 satisfies the inequalities of 5.1 with $m = 2$ and $(m - 1) \frac{\alpha_j}{\alpha_T} = 1$.

The inequality in 5.2 reflects $E_B \geq 0$ for a player $j_B$ with $\mu^z(j_B) = 0$ and $\mu^y(j_B) > 0$. In
Figure 5: $s \in H^*(i_A, i_B; g)$ for which neither $j_A$ nor $j_B$ wishes to switch trees despite having the option to do so. The dashed link is the position available to $j_h$ in the $i_{-h}$ tree. Here, $d\mu = 3$, $\mu^A_B(j_A) = 1$, $\mu^A_B(j_B) = 9$, and $\mu^y_A(j_B) = 2$. Let $m = 2$ and $(m - 1)\frac{a_T}{a_T} = 1$, then $\frac{1}{2} \leq \frac{5}{3} < \frac{5}{4}$ satisfies the two condition in 1a and 1b of Proposition 5.

this case, the preference for affiliating with the $i_B$-led tree despite its smaller size originates from distance advantage $j_B$ holds over the $N^y(j_B, s)$ population. If $j_B$ is the only member of $N^B(i_A, i_B; s) \cap (N^S(i_B; s) \cup \{i_B\})$, then her preference for the status quo preserves $s \in H^*(i_A, i_B; g)$ as a Nash equilibrium. With $m = 2$ and $(m - 1)\frac{a_T}{a_T} = 1$, the inequality in 5.2 is satisfied for $s \in H^*(i_A, i_B; g)$ depicted in Figure 4, with $1 < 5/3$. Player 3’s distance advantage over player 4 in the tree under player 1 offers greater benefit to player 3 than does switching affiliation to the more populous tree under player 5. This particular multiple leader equilibrium scenario will be revisited in section 4.3.2 to consider the resilience of this type of Nash equilibrium structure.

4.2.2 $B < 0$

Proposition 4 was established based on the presumption that the only alternative action to following the leader is to lead. Consider the broader set of actions such that for $B < 0$, some or all of the $n - n_i$ individuals not in $N^S(i_A; s)$ form an second hierarchy to that led by $i_A$. For $\{j_1, j_2\} \notin N^S(i_A; s)$, the ability to form such a hierarchy is not assured by strong connectivity since it is possible that the chain of links connecting $j_1$ to $j_2$ passes through a member of $N^S(i_A; s)$. To consider an environment friendly to the development of an alternate hierarchy, presume a strongly connected sub-population consisting of those individuals not in the $i_A$-led tree. Let

$$F(i_A, i_B, j; s) := (m - 1)\frac{a_T}{a_T} - \left(1 + \frac{\mu^z_A - 1 - \mu^z_B(j)}{\mu^z_B}\right).$$

(11)
Proposition 6. Given $B < 0$, for $s$ to be a Nash equilibrium requires $s \in h_i^*(i, \pi; g)$.

Proof. See Appendix

The condition $F \geq 0$ is necessary for $j$, as the most distant member of the $i_B$-tree, to prefer following $i_B$ to leading. The condition is the same whether $s \in H^*(i_A, i_B; g)$ or $s \in h^*(i_A, i_B; g)$ with $\mu_L > 2$. The proof of Proposition 6 demonstrates that $B < 0$ ensures $F < 0$ for all $j$ and all $s' \in h(i_A, i_B; g)$. If a player is not going to be a successor of $i_A$, leading is a better option than being a successor to some $i_B$.

For $B < 0$, the attraction that induces the most distant follower of $i_A$ to abandon the certain formity payoff is the possibility of earning the leader’s payoff. To abandon the $i_A$-led tree to join a $i_B$-led tree undermines the purpose since, from a position following $i_B$, some of the $N_S(i_A, s)$ population would have already adopted before $j$.

4.2.3 Alternate environments

Consider a violation of the assumption of a strongly connected population. Let $N_A$ and $N_B$ be two non-overlapping populations with $N = \{N_A, N_B\}$. Let the matrices $g_A$ and $g_B$ each capture the potential links within the respective populations, each of which is strongly connected. The matrix $g$ has the structure

$$
g = \begin{bmatrix} g_A & 0 \\
0 & g_B \end{bmatrix}.
$$

The off-diagonal zero sub-matrices indicate the absence of potential links between members of $N_A$ and $N_B$. Without loss in generality allow $N_A$ to be the population for which, given $s \in H^*(i_A, i_B; g)$, $\mu_B^j(j_{n_B}) \geq 0$ indicating that the most distant successor of $i_A$ is of a distance greater or equal to the most distant successor of $i_B$.

Proposition 7. Given $\mu_B^j \geq 1$, $F(i_A, i_B, j_{n_B}^*; s) \geq 0$ is a necessary and sufficient condition for $s \in H^*(i_A, i_B; g)$ to be a Nash equilibrium.

Proof. See Appendix

From Proposition 6, $F(i_A, i_B, j_{n_B}^*; s) \geq 0$ is the condition for the most distant potential successor of $i_B$ to prefer following over leading. For $B \geq 0$, a second leader with followers is possible when linking to an existing leader is not possible. The threshold minimum value for
\[(m - 1)a_J/a_T \text{ to satisfy } F \geq 0 \text{ is specific to the pair } \{i_A, i_B\} \text{ due to the pair specific } \mu_B^A(j_{\mu_B^i}) \text{ value. The value } \mu_A^i - \mu_B^A(j_{\mu_B^i}) \text{ is the size of the population in the } i_A\text{-led tree of a distance from } i_A \text{ equal to or less than the distance from } j_{\mu_B^i} \text{ to } i_B. \text{ This population has to contain at least one member so that in (11), } \mu_A^i - 1 - \mu_B^A(j_{\mu_B^i}) \geq 0 \text{ and thus}
\[
\left(1 + \frac{\mu_A^i - 1 - \mu_B^A(j_{\mu_B^i})}{\mu_B^i}\right) \geq 1 > \left(1 - \frac{1}{n - 1}\right). \tag{12}
\]
The minimum threshold on \((m - 1)a_J/a_T\) to maintain } s \in H^*(i_A, i_B; g) \text{ as equilibrium in the presence of unlinked subpopulations is greater than the threshold necessary to maintain } s \in H^*(i_A; g) \text{ as equilibrium when the entire population is connected to } i_A.

Another alternative to consider is the incorporation of a minority game component, similar to Arthur (1994), as a penalty for excessive popularity. Let } J(x) \text{ represent the possibly non-linear payoff function for the conformity component. Consider}
\[
J(\mu^J) = \begin{cases} a_J(\mu^J - 1) & \text{for } \mu^J \leq n^a \\ 0 & \text{otherwise} \end{cases} \tag{13}
\]
with } 2 \leq n^a < n - 2. \text{ The result is an environment that supports a hierarchy of size } \mu_A^i + 1 \leq n^a. \text{ Again, those not in the main hierarchy may lead or join to form an alternate hierarchy. Such a model imposes a penalty for excessive conformity should } f_{i_A}(k_{i_A}) = f_{i_B}(k_{i_B}), \text{ though the reward to early adoption remains for those with followers. For } n - \mu_A^i - 1 \leq n^a, \text{ the condition ensuring } s \in \{H^*(i_A, i_B; g) | \mu_A^i = n^a - 1\} \text{ as an equilibrium is}
\[
(m - 1)a_J/a_T \geq \frac{\mu_A^i + \mu_B^i - 1 - \mu_B^A(j)}{(\mu_B^i + 1)(\mu^l - 3) + (m - 1)\mu^l - 2 \mu_B^i}, \tag{14}
\]
a higher threshold than the condition } F \geq 0. \text{ Also, } \mu_A^i \text{ takes the value } n^a - 1. \text{ When this condition does not hold, the interior solution for } \mu_B^i \text{ can be computed.}^{10}

\[^{10}\text{When the condition in (14) does not hold, players otherwise at the bottom of the } i_B\text{-led tree choose instead to lead. The condition for } j, \text{ as the most distant successor of } i_B, \text{ to prefer following to leading is}
\[
(m - 1)a_J/a_T \geq \frac{\mu_A^i + \mu_B^i - 1 - \mu_B^A(j)}{(\mu_B^i + 1)(\mu^l - 3) + (m - 1)\mu^l - 2 \mu_B^i},
\]
where } \mu^l - 2 \text{ is the number of leaders excluding } i_A \text{ and } i_B. \text{ The } \mu_B^i \text{ that sets this equal to zero is the equilibrium size of the } i_B\text{-led hierarchy.}
More generally, a concave conformity function with \( J(1) = 0, J'(1) > 0, \) and \( J''(x) < 0 \) might seem a reasonable alternative to the presumed linear function. This can be accomplished, for example, by replacing the constant \( a_J \) with a function \( a_J = a_J(\mu) \) for which \( a''_J(\mu) < 0 \). Since it is the decision of the player at the greatest distance from the leader who drives the \( B \geq 0 \) condition, substituting an increasing concave \( a_J(n) \) for \( a_J \) alters the value of \( B \) but leaves Propositions 3 and 4 largely unchanged.

Without linearity, computing expected payoffs for fractional participation becomes considerably more laborious. Nonetheless, inference can be made based on the linear analysis. Concavity in \( J \) decreases the incentive for conformity with large populations while maintaining an incentive to conform in small groups. Overcoming the single hierarchy equilibrium becomes possible if \( J(\mu^x_A) < J(\mu^x_B) \) for \( \mu^x_A > \mu^x_B \). A monotonically increasing concave function that retains \( J'(x) > 0 \) produces a single-hierarchy equilibrium. Ensuring multiple hierarchies in equilibrium requires \( J'(x) \leq 0 \) for large \( x \). A function with \( J(1) = 0, J'(1) > 0 \) and \( J(x) \to 0 \) as \( x \to n \) would offer higher conformity payoff for smaller hierarchies, possibly inducing those players who otherwise are most distant from \( i_A \) to abandon a large \( N^S(i_A; s) \) in favor of forming alternative hierarchies.

### 4.3 Sequential play

#### 4.3.1 Subgame perfect hierarchies

The single leader and follower tree structure can also be supported in a game with sequential play by a subgame perfect equilibrium (SPE). Typically, the first mover can establish herself as the leader and the remaining population adopts the following strategy to best accommodate this reality. There are instances, though, in which the first mover does not end up as the leader in the equilibrium structure, as illustrated in Example 4 below. The inability of player \( i \) to lead in a SPE indicates a susceptibility of the Nash equilibrium \( s \in H^*(i; g) \) to disruption by a player whose deviation from Nash equilibrium play, in a cascade of best responses, would leave to a different Nash equilibrium \( s' \notin H^*(i; g) \) favored by the original deviant player.

**Example 4.** Table 4 lists the potential links of network \( g \). Each member of the population has two contacts. Figure 6 depicts \( \{s^1\} = H^*(i; g) \) and \( s^3 \in H^*(j; g) \), both Nash equilibria. Given \( i, j, x, z_1, \) and \( z_2 \) as the order of play \( s^3 \), rather than \( s^1 \), is the SPE despite \( i \)'s first mover
<table>
<thead>
<tr>
<th>Players</th>
<th>Contacts</th>
</tr>
</thead>
<tbody>
<tr>
<td>i</td>
<td>x</td>
</tr>
<tr>
<td>j</td>
<td>i</td>
</tr>
<tr>
<td>x</td>
<td>i</td>
</tr>
<tr>
<td>z₁</td>
<td>j</td>
</tr>
<tr>
<td>z₂</td>
<td>j</td>
</tr>
</tbody>
</table>

Table 4: Links of \(g\) in an example for which not all \(s \in H^*(i; g)\) are subgame perfect equilibrium.

![Diagram](image)

Figure 6: The tree structures of \(s^1\) and \(s^3\) are both Nash equilibria structures based on \(g\) as defined in Table 4. Only \(s^3\) is a SPE given moves in the order \(i, j, x, z_1\), then \(z_2\). The structure \(s^2\) reflects the best response by players \(x, z_1,\) and \(z_2\) if faced with both \(i\) and \(j\) leading. The fact that \(j\) prefers \(s^2\) to \(s^1\) is what undermines player \(i\) leadership when considering a cascade of best responses.

option to establish herself as a leader. The full extensive-form game is included in Appendix B.1. Best response to player \(i\) leading produces \(s^2 \in h^*_L(j, \mu^*(j); g)\) in which both \(i\) and \(j\) lead but \(j\) attracting all of the followers. Two critical features present in \(s^1\) are essential to exclude it from the set of SPE. First, player \(j\)’s has an advantage in attracting followers despite \(i\) moving first. Because \(z_1\) and \(z_2\) have no choice but to follow \(j\), \(j\) has the larger population of followers regardless of \(x\)’s decision regarding whom to follow. This compels \(x\) to follow \(j\). Second, the potential defector, \(j\), must be motivated to defect despite \(i\)’s lead. Here, \(\pi(j; s^2) > \pi(j; s^1)\).

The motivation in this example comes from player \(x\). Agent \(x\) best responds by following \(j\) when both \(i\) and \(j\) lead. In \(s^1\), \(x \notin \mu^T(j; s^1)\) but \(x \in \mu^T(j; s^2)\).

Similar to the best response cascade discussed in Heal and Kunreuther (2010), were the population to start from \(s^1\), the cascade of best responses to the single deviation by player \(j\) transitions the population from one equilibrium to another. Trivially, in the two player game in Example 2 both Nash equilibria are susceptible to a defection-induced cascade. Such an outcome
is not particularly enlightening for refining the possible list of Nash equilibrium structures that might be observed. As illustrated in Example 4, the SPE sets a higher standard to induce player \( j \) into disrupting a Nash equilibrium.

### 4.3.2 Multiple hierarchies as subgame perfect

Appendix B.2 offers an analysis of an \( s \in H^*(i_A, i_B; g) \) structure that is a Nash equilibrium because it satisfies condition 2 of Proposition 5 recall that \( j_B \) is characterized by \( \mu^z(j_B) = 0, \mu^x(j_B) > 0, \) and \( N_{AB} \cap N^S(i_B) = \{j_B\} \). Such an \( s \) is not a subgame perfect equilibrium regardless of the order of play within the players of the \( i_B \)-tree. The reason is that as the only conduit through which \( i_B \) and the \( N^S(i_B, s) \) population can join \( N^S(i_A, s) \), player \( j_B \)'s switch to join the \( i_A \)-led tree enables the remainder of \( N^S(i_B, s) \) to also join in following \( N^S(i_A, s) \) in a cascade of best responses. The end result is to player \( j_B \)'s advantage as she now leads \( i_B \) and the former \( N^S(i_B, s) \) population.

### 5 Conclusion

Strategic behavior in an environment that rewards early adoption of a popular choice results in an equilibrium structure consisting of a leader with a hierarchy of followers. The desire to conform assures the presences of a hierarchy in the equilibrium structure. The relative magnitude of the premium for early adoption determines the size of the equilibrium hierarchy. A sufficiently small premium extends the benefit of cooperation to everyone in the hierarchy so that the entire population follows a single leader. The competition inspired by a large premium induces those who benefit least from the hierarchical structure to abandon their position in the hierarchy in favor of an independent gamble, diminishing in size the hierarchy though never fully eliminating it.

Relaxing some of the assumptions that support the single hierarchy formation outcome could allow multiple coexisting hierarchies in equilibrium. Already discussed is concavity in the conformity payoff. Introducing innate preferences would be another mechanism for generating multiple hierarchies. An individual will not join the hierarchy if she has a strong preference different from that of the leader.

Innate preferences invite exploration of other strategic behaviors as well. Once the consumers
have preference over the set of options they face, there is a need on the part of the consumer
to balance the two social aspects to consumption with their own innate preference. Once
consumers are defined in a preference space, there is considerable opportunity to examine how
these populations organize themselves and how leaders emerge to serve a population defined by
their preferences. Another modification that could act as a catalyst for change is to introduce
cost for information gathering. Stability of a leader’s position, the sustainable size of a hierarchy,
and how broad a population a leader can influence are all rich areas for investigation.

Left unresolved is the process by which the equilibrium structure can emerge. The substan-
tial coordination involved, confounded by the asymmetry of the equilibrium payoff, makes the
realization of an equilibrium structure in a single round of play highly unlikely. Like Craw-
ford and Haller (1990), coordination actions will have to emerge over time as the product of
path dependent play. While the solution cannot depend on consistency of the unknown map
between player’s strategies, as this changes with each new decision, the analysis developed here
rests on the possibility that coordination can emerge as the consequence of building consistency
in player relationships. The computation model in Goldbaum (2013) points to a mechanism
by which an endogenous social network can emerge around a leader based on path dependent
play. Developing the equilibrium strategy to the dynamic game in support of the computational
model would be a natural extension.
A Appendix: Formal development of Section 3

Formally, \( H(i; g) = \{ s | N^L(s) = \{ i \}, N^S(i; s) = N \backslash \{ i \} \} \). The sets of structures consisting of distance minimizing linkage paths from follow to leader are

\[
H^*(i; g) = \{ s \in H(i; g) | \forall j \in N \backslash \{ i \}, s_j = \arg \min(d_{j,i}(g)) \}
\]

and

\[
h^*(i; g) = \{ s \in h(i; g) | \forall j \in N^S(i; s), s_j = \arg \min(d_{j,i}(g)) \}.
\]

A.1 Conditions for a non-empty \( H'(i; g) \backslash H^*(i; g) \)

For \( s \in H^*(i; g) \), the payoff to player \( j \neq i \) is determined by the population sizes \( \mu^L = n \) and \( \mu^T(j) = \mu^y(j) + \mu^z(j) \). The value of \( \mu^L \) is independent of \( j \)'s position in the tree. If \( \mu^y(j) = 0 \) and there exists a non-empty set \( \{ j_1 \in N^d(j; g) | d_{j_1,i} = d_{j,i} \} \) then were \( j \) to switch to imitate \( j_1 \), her distance (and that of all her successors) to \( i \) increases by one. In the resulting \( s' \in H'(i; g) \backslash H^*(i; g) \), \( \mu^y(j; s') = \mu^y(j; s) = 0 \) and \( \mu^z(j; s') = \mu^z(j; s) \) so that \( \mu^T(j) \) is unchanged.

For every player \( j_2 \in N^S(j; s) \), \( \pi(j; s') = \pi(j; s) \) so that none of the successors of \( j \) are harmed by the increased distance to \( i \). An additional restriction for \( s' \) to be an equilibrium is that no \( j_2 \in N^S(j; s) \) has \( j_1 \in N^d(j_2; g) \). That is, no immediate successor of \( j \) has an alternate contact also at distance \( d_{j,i} \) from \( i \). Were this option to exist, when \( j \) increases her distance, then \( j_2 \) can maintain her original distance from \( i \), receive \( \pi(j_2; s'') > \pi(j_2; s) \) by gaining a timing advantage over the other immediate successor of \( j \) while causing \( \pi(j; s''') < \pi(j; s) \).

A.2 Proof of Proposition 1

By Lemma 2, \( \pi_j \) generally increases as \( j \in N^S(i; s) \) decreases her distance to \( i \). By construction, for \( s \in H(i; g) \backslash H^*(i; g) \) there exists \( j \in N^S(i; s) \) for whom \( s_{j,j_a} = 1 \) and a non-empty \( \{ j_b \in N^d(j; g) | d_{j_b,i} < d_{j_a,i} \} \) for which the \( s' \) from setting \( s_{j,j_b} = 1 \) results in \( \pi_j(s') > \pi_j(s) \). Thus, \( s \) cannot be an equilibrium. For \( s^* \in H'(i; g) \), there is no opportunity for \( j \in N^S(i; s) \) to improve her payoff while remaining a successor to \( i \). If any \( s \in H(i; g) \) is a Nash equilibrium, \( s^* \) is a Nash Equilibrium. If an equilibrium \( s^* \in H'(i; g) \backslash H^*(i; g) \) exists then, but construction, there exists \( j \in N^S(i; s) \) for whom \( s_{j,j_a} = 1 \) and for which the \( s^{**} \in H^*(i; g) \) from setting \( s_{j,j_b} = 1 \)
generates a self-referencing loop and π \geq \pi \text{ in the hierarchy over leading.}

The condition A \in \{1, \ldots, n\}, following is inferior to leading as it

\[ \pi(j; s^*) = \pi(j; s^*) \text{ for all } j_b \in N^d(j; g) \mid d_{j_b,i} < d_{j_a,i} \text{ is also an equilibrium.} \]

A.3 Proof of Proposition 2

**Proof.** Let \( s_{-j} \) indicate the strategies of all agents in \( N \setminus \{j\} \). For \( s \in H(i; g) \), let \( \sigma' = s_j \times s_{-j} \)
and \( s_{j,j} = 1 \) producing \( \sigma' \in h(i; g) \). The expected payoff to leading for player \( j \) is

\[ \mathbb{E}(\pi(j; \sigma')) = \frac{1}{m}(a_j \mu_j^I + a_T(\mu_j^T - 1)) + \frac{m-1}{m}(a_j(\mu_j^I) + a_T(\mu_j^T)) \] (15)

where \( \mu_j^I, \mu_j^T, \mu_j^I, \) and \( \mu_j^T \) are all based on \( s \). Let \( f \) be the vector function of each player’s
map from labels to objects, \( \{f_1, \ldots, f_n\} \). Expectations are taken over the possible maps \( f \). The necessary condition to keep player \( j \) from leading is

\[ a_j(\mu_j^I) + a_T(\mu_j^T) \geq \frac{1}{m}(a_j(\mu_j^I) + a_T(\mu_j^T - 1)) + \frac{m-1}{m}(a_j(\mu_j^I) + a_T(\mu_j^T)) \] (16)

Using (2) and (3), this becomes

\[ a_Jn + a_T(\mu^y(j) + \mu^z(j)) \geq \left( \frac{1}{m}(a_Jn + a_T(n - 2)) + \frac{m-1}{m}(a_J(\mu^n(j) + 1) + a_T\mu^z(j)) \right) \] (17)

The condition \( A(j; s) \geq 0 \), derived from (17), ensures that player \( j \in N^S(i; s) \) prefers her
position in the hierarchy over leading.

Rearrange \( A(j; s) \geq 0 \) as expressed in (4) to get

\[ (\mu^z(j) + 1)(a_T - (m - 1)a_J) - \mu^y(j)(a_T + a_J) \leq a_T. \] (18)

With \( \mu^y(j) = 0 \) and \( \mu^z(j) = n - 2 \), the player(s) most distant from the leader has the most
to gain and the least to lose from leading. For the most distant player(s) from \( i \) in \( s \), so that
\( j_n = \arg\max_{j \in N^S(i; s)} A(j_n; s) = B \). For all \( j \in \{N^S(i; s) \mid d_{j,i} > d_{j_n,i} \}, A(j; s) > A(j_n; s) \) so that
\( B \geq 0 \) is necessary and sufficient to ensure \( A(j; s) \geq 0 \) holds \( \forall j \in N \setminus \{i\} \).

A.4 Proof of Proposition 3

**Proof.** Consider \( s \in H'(i; g) \) and \( j \in N \setminus \{i\} \). For player \( i \), following is inferior to leading as it
generates a self-referencing loop and \( \pi(i, s_{-i}) = 0 \). Let \( s^1 = s^1_j \times s_{-j} \) where \( s^1_j \in H(i; g) \setminus H'(i; g) \).
By the construction of $H'(i; g)$, for all $j \in N \{i\}$, $\pi(j; s) > \pi(j; s')$. For $B \geq 0$, for all $j \in N \{i\}$, $\pi(j; s) > \mathbb{E}(\pi(j; s_{j,j} = 1 \times s_{-j}))$. Thus, for all $j$, $s \in H'(i; g)$ is a Nash equilibrium. \qed

### A.5 Proof of Proposition 4

**Proof.** For $s \in h_L(i, \mu^*_i; g)$, let $C(j; s)$ capture the expected payoff premium earned by $j$ for being a successor of $i$ over leading. For $s' = s_{-j} \times s'_{j}$, with $s'_{j,j} = 1$,

$$C(j; s, s') := \mathbb{E}(a_T(\mu^T(j; s)) + a_J(\mu^J(j; s))) - \mathbb{E}(a_T(\mu^T(j; s')) + a_J(\mu^J(j; s'))) \quad (19)$$

Let $\mu^l = |N^L(s)|$. For agent $j \in N^S(i; s)$

$$\mu^x(j) + \mu^y(j) + \mu^z(j) + \mu^l + 1 = n$$

Expectations are taken over the possible maps $f$. Substituting for the position dependent expected values of $\mu^J$ and $\mu^T$ in $C(j; s, s')$ produces (before the canceling of terms)

$$C = m(a_J(1 + \mu^x(j) + \mu^y(j) + \mu^z(j)) + a_T(\mu^y(j) + \mu^z(j))) + (\mu^l - 1)a_J$$
$$- (m(a_J + a_T)\mu^x(j) + a_J(1 + \mu^x(j) + \mu^y(j) + a_T(\mu^x(j) + \mu^y(j))) + (\mu^l - 1)a_J)$$

What remains after canceling out the two $(\mu^l - 1)a_J$ terms reduces to $A(j; s)$ as in (4).

For $j \in N^L(s) \{i\}$, $A(j; s) \geq 0$ indicates that the player would prefer imitating $i$ over leading. With $B < 0$, for $s \in H(i; g)$ then $A(j_n; s) < 0$. The most distant successor of $i$ content with the follow strategy has

$$A(j_n^*; s) = (m - 1)a_J + ((m - 1)a_J - a_T)(\mu^*_z - 1) \quad (20)$$

and $D = A(j_{\mu^*_z}; s)/a_T\mu^*_z$. By $B \leq 0$, $(a_T - (m - 1)a_J) > 0$ so that $A(j_{\mu^*_z}; s)$ decreases as the size of the tree increases. For $\mu^*_z = 1$, $A(j_1, s) = (m - 1)a_J > 0$ while $B \leq 0$ means that for $\mu^*_z = n - 1$, $A(j_n; s) < 0$.

The value of $\mu^*_z$ that sets $D = 0$ need not be an integer. Player $j$ with rank $r_{j,i}(s) = \text{ceil}(\mu^*)$ so that $n \in \{\text{floor}(\mu^*), \text{ceil}(\mu^*)\}$ depending on whether the marginal $j$’s payoff is better following or leading. An $s \in h_L(i, n; g)\backslash h^*_L(i, n; g)$ cannot be an equilibrium because either there are
members of $N^S(i; s)$ able to improve their payoff by choosing a different predecessor offering a shorter distance to $i$ or there is a member of $N^L(s)$ able to improve her payoff by choosing to follow a predecessor offering a shorter distance to $i$ than the current most distance follower. For $s \in h^*_L(i, \bar{n}; g)$, no player is able to improve her payoff through unilateral deviation. 

A.6 Proof of Proposition 5

Let

$$H(i_A, i_B; g) = \{ s \in h(i_A; g) \cap h(i_B; g) \mid N^L(s) = \{i_A, i_B\} \}$$

so that $H(i_A, i_B; g)$ is the set of strategies in which everyone in the population imitates one of two leader. In addition, let

$$H^*(i_A, i_B; g) = \{ s \in h^*(i_A; g) \cap h^*(i_B; g) \mid N^L(s) = \{i_A, i_B\} \}$$

and

$$H^{**}(i_A, i_B; g) = \{ s \in H(i_A, i_B; g) \forall j \in N \setminus \{i_A, i_B\}, s_j = \arg \min \{d_{j,i_A}, d_{j,i_B}\} \}.$$

for $s \in H^*(i_A, i_B; g)$ every follower $j \in N \setminus \{i_A, i_B\}$ employs the strategy $s_j$ that minimizes $j$’s distance to the selected leader. In $H^{**}(i_A, i_B; g)$ every $j \in N \setminus \{i_A, i_B\}$ chooses the strategy $s_j$ that minimizes $j$’s distance to a leader. The latter’s resulting social structure can be thought of as jointly minimal distance.

Let $\mu^*_A \geq \mu^*_B$ and let

$$N^{AB}(i_A, i_B; s) = \{ j \mid N^d(j; g) \cap \{i_A \cup N^S(i_A; s)\} \neq \emptyset, N^d(j; g) \cap \{i_B \cup N^S(i_B; s)\} \neq \emptyset \}$$

so that player $j \in N^{AB}(i_A, i_B; s)$ has potential links to members of both of the trees. With strongly connected graph, $N^{AB}(i_A, i_B; s) \cap \{i_h \cup N^S(i_h; s)\}, h = A, B$ are both nonempty sets. The set $\{i_h \cup N^S(i_h; s)\} \setminus N^{AB}(i_A, i_B; s) h = A, B$ can be nonempty as well, indicating that possibly $i_h$ and some $j \in N^S(i_h; s)$ have no direct potential link to $N^S(i_{-h}; s)$ with the current $s$. 

32
In a two tree setting, the expected payoff is

$$\mathbb{E}(\pi_h(j; s)) = \frac{m - 1}{m} \left( a_T(\mu^y(j) + \mu^z(j)) + a_j \mu^h_h \right)$$

$$+ \frac{1}{m} \left( a_T(\mu^y(j) + \mu^z(j) + \mu^h_h(j)) + a_j(n-1) \right)$$

where $h \in \{A, B\}$ and $\mu^h_h(j) = |\{j' \in N^S(i_h; s) | d_{j',i_h} > d_{j,i_h} \}|$. The first term of (21) captures the payoff in the probability $(m-1)/m$ event that $f_{i_A}(k_{i_A}) \neq f_{i_B}(k_{i_B})$ and the second term captures the probability $1/m$ event that $f_{i_A}(k_{i_A}) = f_{i_B}(k_{i_B})$. From (21),

$$\mathbb{E}(\pi_h(j_h; s)) - \mathbb{E}(\pi_{-h}(j_h; s')) = \frac{1}{m}((m-1)a_j(\mu^h_h - \mu^z_h - 1 - \mu^z(j_h)).$$

$$+a_T(\mu^h_h(j_h) - \mu^z_h(j_h) + m(\mu^y(j_h) - \mu^z_h(j_h))))$$

The $s' \in H^*(i_A, i_B; g)$ alternative to $s$ has $s_j = s_j$ for all $j \neq j_h$ with $j_h \in N^S(i_h; s)$ and has $j_h \in N^S(i_{-h}; s')$. Here, $\mathbb{E}(\pi_{-h}(j_h; s'))$ represents the expected payoff for $j_h$ linking to a predecessor in the $i_{-h}$-led tree. For $d\mu = \mu^z_h - \mu^z_B$ let $E_h = \frac{m}{a_T} \left( \mathbb{E}(\pi_h(j_h; s)) - \mathbb{E}(\pi_{-h}(j_h; s')) \right)$ so that $E_A \geq 0$ corresponds to $\mathbb{E}(\pi_A(j; s)) - \mathbb{E}(\pi_A(j; s')) \geq 0$ and $E_B \geq 0$ corresponds to $\mathbb{E}(\pi_B(j; s)) - \mathbb{E}(\pi_A(j; s')) \geq 0$.

An $s \in H(i_A, i_B; g)$ cannot be an equilibrium unless $s \in H^*(i_A, i_B; g)$. An $s \in H^*(i_A, i_B; g)$ cannot be an equilibrium if there exists a player who can improve her payoff by shortening her distance to one of the leaders. An $s$ such that either $s \in H^{**}(i_A, i_B; g)$ or $s \in H^*(i_A, i_B; g)$ in which there is no improvement to be had by switching trees in order to shorten one’s distance to a leader can be an equilibrium if neither leader has the option to follow a predecessor from the other tree. The constraint on the leader’s contact list follows from the fact that for $i \in N^L(g)$, the condition $E_h(i_h)$ is equivalent to

$$-\left( (m-1)^2 - \left( 1 - \frac{1}{\mu^z_h + 1} \right) \right) \geq 0.$$

Since $\mu^z_h \leq (n-2)$, $B \geq 0$ ensures that $E_h(i_h) \leq 0$ for both leaders. The condition holds at equality only if $B = 0$ and $\mu^z_h = (n-2)$, which cannot hold for both leaders simultaneously.

For $j \in N^{AB}(i_A, i_B; s) \{i_A, i_B\}$, $E_h(j) > 0$ for all $j \in N^{AB}(i_A, i_B; s)$ is possible. The conditions under which this might hold for all $j$ decrease with the number of members of
$N^{AB}(i_A, i_B; s)$ and at the individual level, $E_h(j)$ is decreasing in $\mu^z(j)$, all else equal. Considering those players with $\mu^z(j) = 0$, the two possible positions face different incentives for preferring and preserving the multiple leader structure. The two inequalities of 5.1 follow from simply imposing $\mu^y(j_B) = \mu^z(j_B) = 0$ on $E_B$ and $E_A$ respectively. Jointly, the two inequalities imply

$$\frac{\mu_A^y(j_A)}{(m-1)a_T} + 1 \leq d\mu < \frac{\mu_B^y(j_B) - m\mu_A^y(j_B)}{(m-1)a_T} - 1. \quad (22)$$

The key features needed of $s$ to satisfy (22) are:

1. $\mu_B^y \geq m\mu_A^y(j_B) + \mu_A^z(j_B) + 1$ so that $\mu_B^y$ is larger than the components making up the near and far components of the $i_A$-led tree,

2. a concentration of the $i_A$-led population at the distance $d_{j_B,i_B} + 1$ sufficiently large to have $\mu_A^y \geq \mu_B^y$ despite feature 1,

3. $d_{j_B,i_A} \geq d_{j_B,i_B} + 1$, and

4. $d_{j_A,i_B} = d_{j_B,i_B} + 1$.

Figure 5 is an equilibrium structure satisfying the conditions of case 5.1. Feature 1 requires an “x” population based on the size of the “a” and $\mu_A^y(j_B)$ populations. The “b” population is sufficiently large to produce $\mu_A^y \geq \mu_B^y$ in accordance with Feature 2. So that $j_B$ prefers the $i_B$-led tree, she cannot benefit from the “b” population were she to switch, which is captured by Feature 3. So that $j_A$ does not want to switch to the $i_B$-led tree despite its attractiveness to $j_B$, $j_A$ cannot share in all of the benefits. Feature 4 puts $j_A$ in a position where she fails to share in $j_B$’s distance advantage over the “b” population from the $i_B$-led tree. By feature 3, the “b” population exists within the distance range $d_{j_B,i_B} + 1$ and $d_{j_B,i_A}$ (inclusive) but feature 4 constrains the population to have a distance of $d_{j_B,i_B} + 1$.

The inequality in 5.2 is based on the value of $E_B(j_B)$ for a follower $j_B \in \{N^S(i_B, s) \cap N^{AB}(i_A, i_B; s)|\mu^z(j) = 0, \mu^y(j) > 0\}$. An additional imposition of $\mu_A^y(j_B) = 0$ would imply that player $j_B$ has to join the $i_A$-led tree as the greatest distance, making the $i_A$-led tree less attractive.

For any player in the interior of the tree, if $E_h(j_B) > 0$, it is a reflection of one or both of
the incentives developed under the two scenario to keep \( E_h(j_B) \) high while overwhelming the incentive to join the larger \( i_A \)-led tree with successors in tow.

### A.7 Proof of Proposition 6

**Proof.** Consider the alternative. Let \( s' \in h^*(i_A, i_B; g) \) with \( \mu^z(i_A; s') = \pi \). The population \( j \notin N^S(i_A, s') \) either follows \( i_B \) or leads.

Let player \( j \) be the most distant successor of \( i_B \). Player \( j \) is unable to profitably join \( N^S(i_A; s) \). Her two options are to lead or to follow \( i_B \). Let \( s'' \) be the structure if \( j \) leads with all other player strategies unchanged. The condition \( E(\pi(j; s'')) \geq E(\pi(j; s')) \), indicating a preference for her position in the \( i_B \)-led tree, reduces to \( F(i_A, i_B, j, s) \geq 0 \). Since the closest \( i_B \) has \( d_{j,i_B} = 1 \) and the number of players at a distance of one from \( i_A \) is at least one, \( \mu^A_B(j) \leq \mu^z(i_A; s') - 1 \) for all \( j \in N^S(i_B, s') \) and for all \( s' \in H^*(i_A, i_B; g) \). Since the second term of \( F \) is weakly greater than one, \( F \geq 0 \) violates the assumption \( B < 0 \). For \( s' \in h^*(i_A, i_B; g) \), the value of \( \mu^z_B \) that sets \( F(i_A, i_B, s) = 0 \) has \( \mu^z_B < 0 \) indicating that there is no condition under which a structure with even just one follower of \( i_B \) is an equilibrium. \( \square \)

### B Appendix: Sequential play games

#### B.1 Unsupported leaders

The \( g \) network is as listed in Table 4 of the main paper. For \( B \geq 0 \), followers \( z_1 \) and \( z_2 \) always choose to follow \( j \). The extensive-form game depicted in Table 5 only includes the decisions of \( i, j, \) and \( x \) in that order of play. Each player has the option to lead, labeled “L”, or to imitate the first or second contact. Under “Strategies” are the actions employed to achieve each state, identified by number in the top row of the table. The payoff to each player in each state is listed in the “payoff” section of Table 5. Those payoff areas labeled “loop” are strategies that produce self-referencing imitation loops with no leader within the loop. Since this generates a zero payoff for those in the loop, a structure that includes a loop is never an equilibrium strategy.

State 5 in the decision tree generates \( s^1 \in H^*(i; g) \). The two hierarchies that make up \( H^*(j; g) \) are produced in states 12 and 20. The subgame perfect state dependent strategy of each player is shaded (if viewed in color, they are colored blue for player \( n \), orange for player \( j \),
Table 5: The Nash equilibrium supported-state $5 \in H^*(i, s)$ is not supported by a subgame perfect analysis with $i$, $j$, $x$, $z_1$, $z_2$ as the order of play. The preferred choice of the each player given the downstream choice of the other players is highlighted according to player $i =$ yellow, player $j =$ orange, and player $x =$ light blue. The hierarchy produced in states 12 and 20 are equivalent and are subgame perfect equilibria with player $j$ leading the entire population, including player $i$. 

<table>
<thead>
<tr>
<th>State</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>23</th>
<th>24</th>
<th>25</th>
<th>26</th>
<th>27</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player</td>
<td>i</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>z_1</td>
<td>z_1</td>
<td>z_1</td>
<td>z_1</td>
<td>z_1</td>
<td>z_1</td>
<td>z_1</td>
<td>z_1</td>
<td>z_1</td>
<td>z_1</td>
<td>z_1</td>
</tr>
<tr>
<td>j</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>i</td>
<td>i</td>
<td>i</td>
<td>z_2</td>
<td>z_2</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>i</td>
<td>i</td>
<td>i</td>
<td>z_2</td>
<td>z_2</td>
<td>z_2</td>
<td>L</td>
<td>L</td>
<td>i</td>
<td>i</td>
<td>i</td>
<td>z_2</td>
<td>z_2</td>
<td>z_2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>x</td>
<td>L</td>
<td>i</td>
<td>j</td>
<td>L</td>
<td>i</td>
<td>j</td>
<td>L</td>
<td>i</td>
<td>j</td>
<td>L</td>
<td>i</td>
<td>j</td>
<td>L</td>
<td>i</td>
<td>j</td>
<td>L</td>
<td>i</td>
<td>j</td>
<td>L</td>
<td>i</td>
<td>j</td>
<td>L</td>
<td>i</td>
<td>j</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| Player | i | 0 | 1.1 | 0 | 3.3 | 4.4 | 4.4 | Loop | 0.1 | Loop | 0.4 | 3.4 | Loop | Loop | Loop | 0.3 | 0.4 | 1.4 | Loop | Loop |
| j | 2.2 | 2.2 | 3.3 | 2.3 | 2.4 | 3.4 | 2.2 | Loop | 4.4 | 2.4 | 3.3 | 4.4 | 4.4 |
| x | 0 | 0.1 | 0.3 | 0 | 2.4 | 0.4 | 1.1 | Loop | 1.4 | 4.4 | 0 | 1.4 | 0.4 |
and yellow for player i).

Both states 12 and 20 are subgame perfect solutions to this sequential play game while 5 is excluded from the equilibrium set.

B.2 Multiple hierarchies

A structure \( s \in H^*(i_A, i_B; g) \) like that depicted in Figure 4 conforms to Case 2 of Proposition 5 if \( N^d(1; g) = \{3, 4\} \), \( N^d(2; g) = \{1, 4\} \), \( N^d(3; g) = \{1, 8\} \), \( N^d(4; g) = \{2, 3\} \). The structure \( s \) is an equilibrium if \( E_h(3) > 0 \). For parameter values \( a_J = 0.1 \), \( a_T = 1 \), and with the structure producing population values \( \mu^b(3) = 1 \) and \( d\mu = 2 \), then,

\[
(m - 1) \frac{a_J}{a_T} < \frac{m\mu^b(3)}{d\mu + 1}
\]

satisfying the necessary condition 2 of Proposition 6 for any value of \( m \). In order to have \( B \geq 0 \) requires \( m \geq 8 \). The SPE analysis of a sequential game is based on this environment with the expected payoffs reported in Table 6 computed based on \( m \to \infty \).\(^{11}\)

The two lower sections of Table 6 include the state-dependent payoffs to each player in the player 1-led tree depending on whether player 3 imitates player 1 or joins \( N^S(i_A; s) \), as indicated in the “strategies” section of the table. Excluded from the table are the payoffs associated player 3’s option to lead which is never a preferred action. This leaves 54 possible states as show in Table 6. These are spread out over two sets of columns of 27 payoffs each; the first set is based on player 3 imitating player 1 and the second set in the lower portion is based on player 3 joining \( N^S(i_A; s) \). The table contains all of the information of the remaining extensive-form tree which is large but straight forward to construct. Each state requires the construction of the resulting hierarchy to determine individual payoff. State 5 with player 3 imitating player 1 is a Nash equilibrium reflected in Figure 4.

In a sequential play game, allow the decision proceeds in the order 5 through 10 and then 1, 2, 4, 3. Eliminate \( A \) from player 3’s action set and only strategy profiles of the the upper payoff section of Table 6 can be reached. State 5 is also the SPE of this limited action set. Consistent with the Nash equilibrium, given state 5, player 3 prefers to imitate player 1 to switching to

\(^{11}\)This is for convenience of display only. The relative payoffs reported in Table 6 are unaffected by the value of \( m \) at any parameters that satisfy \( B \geq 0 \).
Table 6: Subgame perfect analysis of a multiple leader Nash equilibrium structure. The Nash equilibrium-supported state 5 is not supported in a subgame analysis (in any order). Play here is in order the 1, 2, 4, 3. Player 3 chooses between imitating player 1 and switching to join the $i_A$-led tree. The preferred choice of each player given the downstream choice of the other players is highlighted according to player 1 = yellow, player 2 = orange, player 4 = light blue, and player 3 = tan. Strategy sets 15 and 18 are equivalent and are subgame perfect equilibria with player 3 following $i_A$ and players 1, 2, and 4 following 3.
Figure 7: Player 3 with $i_A$ as a predecessor and with player 1 and the former $N^S(1; s)$ population as successors following a best response cascade that transitions from a Nash equilibrium $s \in H^*(i_A, 1; g)$ with two leaders to a Nash equilibrium $s' \in H^*(i_A; g)$ with a single leader.

Allow player 3 to freely choose from $a_3 \in \{L, 1, A\}$, then state 5 is not a SPE. Proper sequential play analysis reveals that states 15 and 18 with $a_3 = A$ are both SPE. Despite the early mover advantage to players 1 and 2, players 3 and 4 are both better off in the $i_A$-led tree. If player 4 imitates 2, then player 3 imitates 1 but this is not in player 4’s interest. Player 4 enables player 3’s choice of $A$ by following 3 rather than 2. Without the support of players 3 and 4 as followers, players 1 and 2 are forced to follow 3 as well. The final $s' \in H^*(i_A; g)$ benefits player 3 because she has a distance advantage over 1, 2, and 4 rather than just player 4, as depicted in Figure 7.

References


41


