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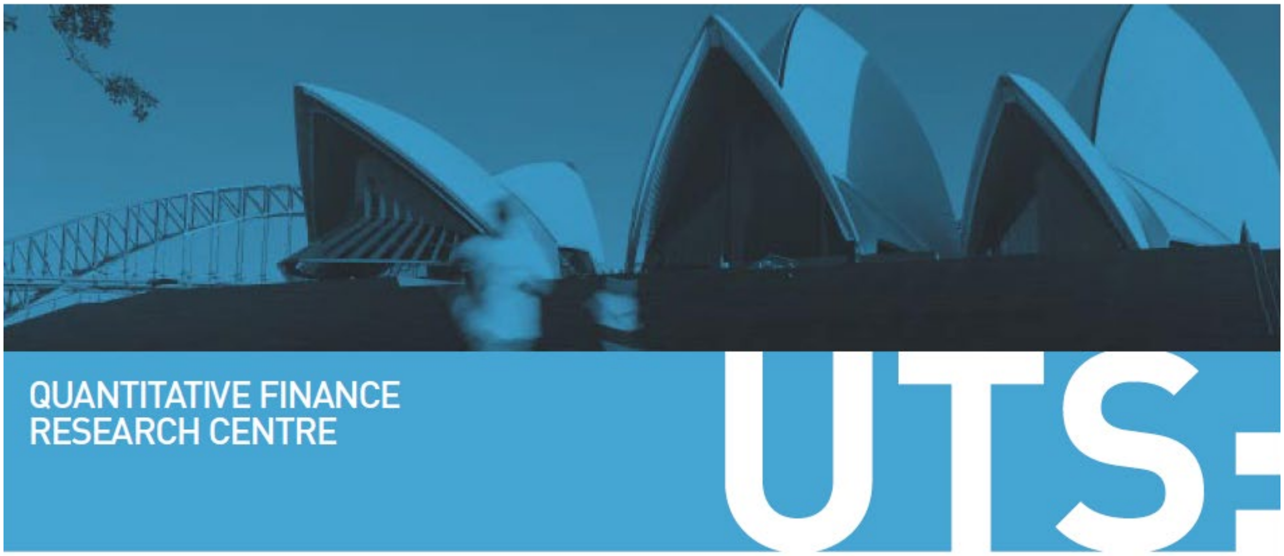

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NO-ARBITRAGE CONCEPTS IN TOPOLOGICAL VECTOR LATTICES

ECKHARD PLATEN AND STEFAN TAPPE

Abstract. We provide a general framework for no-arbitrage concepts in topological vector lattices, which covers many of the well-known no-arbitrage concepts as particular cases. The main structural condition which we impose is that the outcomes of trading strategies with initial wealth zero and those with positive initial wealth have the structure of a convex cone. As one consequence of our approach, the concepts NUPBR, NAA_1 and NA_1 may fail to be equivalent in our general setting. Furthermore, we derive abstract versions of the fundamental theorem of asset pricing. We also consider a financial market with semimartingales which does not need to have a numéraire, and derive results which show the links between the no-arbitrage concepts by only using the theory of topological vector lattices and well-known results from stochastic analysis in a sequence of short proofs.

1. Introduction

Let $(\Omega, \mathscr{G}, \mathbb{P})$ be a probability space, and let $\mathscr{K}_0 \subset L^0(\Omega, \mathscr{G}, \mathbb{P})$ be a set of random variables, where we think of outcomes of trading strategies with initial wealth zero. Then an *arbitrage opportunity* is an element $X \in \mathscr{K}_0$ such that

$$\mathbb{P}(X \geq 0) = 1 \quad \text{and} \quad \mathbb{P}(X > 0) > 0.$$

Therefore, *No Arbitrage* (NA) means that

$$\mathscr{K}_0 \cap L^0_+ = \{0\} \quad \Longleftrightarrow \quad (\mathscr{K}_0 - L^0_+) \cap L^0_+ = \{0\}.$$

It is well-known that for concrete financial models it is easy to find mathematical conditions which are sufficient for NA (like the existence of an equivalent martingale measure), but typically these conditions fail to be necessary for NA. In order to overcome this problem, two approaches have been suggested in the literature:

(1) We choose a subspace of L^0, say L^∞, and define the subset $\mathscr{C} \subset L^\infty$ as

$$\mathscr{C} := (\mathscr{K}_0 - L^0_+) \cap L^\infty.$$

Then NA can equivalently be written as

$$\mathscr{C} \cap L^0_+ = \{0\},$$

and we consider the stronger condition

$$\overline{\mathscr{C}} \cap L^0_+ = \{0\},$$

where the closure is taken with respect to some topology on L^∞. If this topology is the norm topology on L^∞, then we have the well-known concept of NFLVR, see [6]. It is well-known that for suitable semimartingale models in continuous time NFLVR is equivalent to the existence of an equivalent

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local martingale measure; see, for example, the papers [6, 7], the textbook [8], and also the paper [14].

(2) Instead of considering the set $\mathscr{K}_0$ of outcomes of trading strategies with initial wealth zero, we rather consider the outcomes $(\mathscr{K}_\alpha)_{\alpha>0}$ of trading strategies with positive, but arbitrary small initial wealth α. Then the appropriate concepts are NUPBR, NAA_1 and NA_1. It is well-known that for suitable semimartingale models in continuous time these three conditions are equivalent, and that they are satisfied if and only if there exists an equivalent local martingale deflator; see [32], and also the earlier papers [5] and [21].

The goal of this paper is to provide a general mathematical framework for no-arbitrage concepts which goes beyond the settings which have been considered in the literature so far. The idea is as follows. It is known that the space L^0 has rather poor topological properties. It fails to be a locally convex space, and its dual space is typically trivial. However, the space $(L^0, \leq)$ is an example of a topological vector lattice; indeed it is even a so-called Fréchet lattice. The properties of the space L^0 and in particular its positive cone L^0_+ have already been studied in the literature, often with a focus to applications in finance; see, for example [3, 34, 9, 35, 20, 27, 22, 25, 23, 24, 11].

The observation that $(L^0, \leq)$ is a topological vector lattice motivates the general study of no-arbitrage concepts in topological vector lattices. For a topological vector lattice $(V, \leq)$ we consider the positive cone

$$V_+ := \{x \in V : x \geq 0\}.$$

Furthermore, let $\mathscr{K}_0 \subset V$ be a subset and let $(\mathscr{K}_\alpha)_{\alpha>0}$ be a family of subsets such that certain structural conditions are satisfied. In particular, $\mathscr{K}_0$ is supposed to be a convex cone and for each $\alpha > 0$ the set $\mathscr{K}_\alpha$ is convex; we refer to Section 3 for further details. Then NA simply means that

$$\mathscr{K}_0 \cap V_+ = \{0\} \quad \iff \quad (\mathscr{K}_0 - V_+) \cap V_+ = \{0\}.$$

Let us also indicate how the remaining above mentioned no-arbitrage concepts are defined:

(1) Consider the convex cone $\mathscr{K}_0$. Let $U \subset V$ be an ideal which is dense in V. Then $(U, \leq)$ is also a topological vector lattice with positive cone $U_+ = V_+ \cap U$. We define the subset $\mathscr{C} \subset U$ as

$$\mathscr{C} := (\mathscr{K}_0 - V_+) \cap U.$$

Then NA is satisfied if and only if

$$\mathscr{C} \cap U_+ = \{0\}.$$

Let τ be a topology on U. Then we say that NFL_τ holds if the stronger condition

$$\overline{\mathscr{C}}^\tau \cap U_+ = \{0\}$$

is fulfilled. In the particular case $V = L^0$ and $U = L^\infty$, we obtain the well-known concepts NFLVR, NFLBR and NFL; see [8] or [14].

(2) Consider the family $(\mathscr{K}_\alpha)_{\alpha>0}$. We define the family $(\mathscr{B}_\alpha)_{\alpha>0}$ of convex and semi-solid subsets of V_+ as

$$\mathscr{B}_\alpha := (\mathscr{K}_\alpha - V_+) \cap V_+, \quad \alpha > 0.$$

We may think of all nonnegative elements which are equal to or below the outcome of a trading strategy with initial value α. As we will show, then we have $\mathscr{B}_\alpha = \alpha \mathscr{B}$ for each $\alpha > 0$. Therefore, rather than focusing on all outcomes of trading strategies with positive initial wealth, it suffices to concentrate on all outcomes of trading strategies with initial wealth one. Mathematically speaking, we may focus on the set $\mathscr{B} := \mathscr{B}_1$ rather than on the whole family $(\mathscr{B}_\alpha)_{\alpha>0}$. We introduce the no-arbitrage concepts as follows:

- NUPBR holds if $\mathscr{B}$ is topologically bounded.
- NAA_1 holds if $\mathscr{B}$ is sequentially bounded.
- NA_1 holds if $p_{\mathscr{B}}(x) > 0$ for all $x \in V_+ \setminus \{0\}$, where $p_{\mathscr{B}}$ is the Minkowski functional, which can also be interpreted as the minimal superreplication price.

As we will show, in the particular case $V = L^0$ these concepts correspond to the well-known concepts used in the literature.

In this paper we will introduce all these no-arbitrage concepts formally for a topological vector lattice $(V, \leq)$, show the connections between these concepts, and consider the particular situation where the topological vector lattice is the space $(L^0, \leq)$ of all random variables.

In particular, we will show that in a topological vector lattice the concepts NUPBR, NAA_1 and NA_1 are generally not equivalent. More precisely, the concepts NUPBR and NAA_1 are equivalent, and they are satisfied if and only if for every neighborhood of zero the Minkowski functional considered on V_+ is bounded from below by a positive constant outside this neighborhood; see Proposition 3.16. In particular, the Minkowski functional has no zeros on $V_+ \setminus \{0\}$, and therefore NUPBR (or NAA_1) implies NA_1, but we also see that the converse can generally not be true; see Example 2.14 for a counter example. However, in the particular case $V = L^0$ these concepts are known to be equivalent, and in Theorem 5.12 we will present further equivalent conditions, including the von Weizsäcker property and the Banach Saks property of the convex subset $\mathscr{B}$.

In the framework with topological vector lattices we also present versions of the abstract fundamental theorem of asset pricing. Later on this is used for an extension of the well-known no-arbitrage result in discrete time; see Theorem 6.10.

Furthermore, using our general theory we will derive results for no-arbitrage concepts in a market with semimartingales which does not need to have a numéraire, in particular for self-financing portfolios; see Theorem 7.24 and Propositions 7.25–7.28.

The remainder of this paper is organized as follows. In Section 2 we present the required background about topological vector lattices. In Section 3 we introduce no-arbitrage concepts in topological vector lattices. In Section 4 we present versions of the abstract fundamental theorem of asset pricing. In Section 5 we review the no-arbitrage concepts in the particular situation where the topological vector lattice is the space L^0 of random variables. In Section 6 we present a version of the abstract fundamental theorem of asset pricing in L^p-spaces, and derive the mentioned result for financial models in discrete time. In Section 7 we consider a financial market with nonnegative semimartingales which does not need to have a numéraire, and derive consequences for the no-arbitrage concepts; in particular regarding self-financing portfolios.

2. Topological vector lattices

In this section we provide the required background about topological vector lattices and some related results. For further details about topological vector lattices, we refer, for example, to [30, Chap. V].

Let V be a $\mathbb{R}$-vector space. Furthermore, let $\leq$ be a binary relation over V which is reflexive, anti-symmetric and transitive; more precisely:

- We have $x \leq x$ for all $x \in V$.

- If $x \leq y$ and $y \leq x$, then we have $x = y$.
- If $x \leq y$ and $y \leq z$, then we have $x \leq z$.

Then $(V, \leq)$ is called an *ordered vector space* if the following axioms are satisfied:

(1) If $x \leq y$, then we have $x + z \leq y + z$ for all $x, y, z \in V$.
(2) If $x \leq y$, then we have $\alpha x \leq \alpha y$ for all $x, y \in V$ and $\alpha > 0$.

Let V be a topological vector space such that $(V, \leq)$ is an ordered vector space. Then we call $(V, \leq)$ an *ordered topological vector space* if the positive cone

$$V_+ := \{x \in V : x \geq 0\}$$

is closed in V. A *vector lattice* (or a *Riesz space*) is an ordered vector space $(V, \leq)$ such that the supremum $x \vee y$ and the infimum $x \wedge y$ exist for all $x, y \in V$. We introduce further lattice operations. Namely, for $x \in V$ we define the positive part $x^+ := x \vee 0$, the negative part $x^- := -x \vee 0$, and the absolute value $|x| := x \vee (-x)$.

Let $(V, \leq)$ be a vector lattice. A subspace $U \subset V$ is called a *vector sublattice* (or a *Riesz subspace*) of V if $x \vee y \in U$ for all $x, y \in U$. Then $(U, \leq)$ is a vector lattice with positive cone $U_+ = V_+ \cap U$.

A subset $A \subset V$ is called *solid* if for all $x \in A$ and $y \in V$ with $|y| \leq |x|$ we have $y \in A$. A solid subspace $U \subset V$ is called an *ideal*. Every ideal is a vector sublattice of V.

A topological vector space V is called *locally solid* if it has a zero neighborhood basis of solid sets. A vector lattice $(V, \leq)$ is called a *topological vector lattice* if it is a Hausdorff topological vector space which is locally solid.

A topological vector space V is called *completely metrizable* if there is a metric d on V which induces the topology and for which the metric space (V, d) is complete. A *Fréchet lattice* is a completely metrizable topological vector lattice.

For what follows, let $(V, \leq)$ be a topological vector lattice. Recall that we have five lattice operations

$$\begin{aligned}
&V \times V \to V, \quad (x, y) \mapsto x \wedge y, \\
&V \times V \to V, \quad (x, y) \mapsto x \vee y, \\
&V \to V_+, \quad x \mapsto |x|, \\
&V \to V_+, \quad x \mapsto x^+, \\
&V \to V_+, \quad x \mapsto x^-.
\end{aligned}$$

2.1. **Lemma.** *The following statements are true:*

(1) *The lattice operations are continuous.*
(2) *$(V, \leq)$ is an ordered topological vector space.*

Proof. By statement 7.1 on page 234 in [30] the lattice operations are continuous. Furthermore, by statement 7.2 on page 235 in [30] the positive cone V_+ is closed, showing that $(V, \leq)$ is an ordered topological vector space. □

2.2. **Definition.** *Let $\mathscr{B} \subset V$ be a subset.*

(1) *$\mathscr{B}$ is called* topologically bounded *if for every neighborhood $U \subset V$ of zero there is $\alpha > 0$ such that $\mathscr{B} \subset \alpha U$.*
(2) *$\mathscr{B}$ is called* sequentially bounded *if for every sequence $(x_n)_{n \in \mathbb{N}} \subset \mathscr{B}$ and every sequence $(\alpha_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ with $\alpha_n \to 0$ we have $\alpha_n x_n \to 0$.*
(3) *$\mathscr{B}$ is called* circled *(or* balanced*) if*
$$\alpha \mathscr{B} \subset \mathscr{B} \quad \text{for all } \alpha \in [-1, 1].$$
(4) *The* Minkowski functional *$p_{\mathscr{B}} : V \to [0, \infty]$ of $\mathscr{B}$ is defined as*
$$p_{\mathscr{B}}(x) := \inf\{\alpha > 0 : x \in \alpha \mathscr{B}\}, \quad x \in V.$$

2.3. **Lemma.** *If $\mathscr{B} \subset V$ is solid, then it is also circled.*

Proof. Let $x \in \mathscr{B}$ and $\alpha \in [-1,1]$ be arbitrary. Then we have $|\alpha x| = |\alpha| \, |x| \leq |x|$, and hence $\alpha x \in \mathscr{B}$. □

2.4. **Definition.** *Let $\mathscr{B} \subset V_+$ be a subset.*

(1) *$\mathscr{B}$ is called* semi-circled *(or* semi-balanced*) if*
$$\alpha \mathscr{B} \subset \mathscr{B} \quad \text{for all } \alpha \in [0,1].$$
(2) *$\mathscr{B}$ is called* semi-solid *if for all $x \in \mathscr{B}$ and all $y \in V_+$ with $y \leq x$ we have $y \in \mathscr{B}$.*

2.5. **Lemma.** *If $\mathscr{B} \subset V_+$ is semi-solid, then it is also semi-circled.*

Proof. Let $x \in \mathscr{B}$ and $\alpha \in [0,1]$ be arbitrary. Then we have $0 \leq \alpha x \leq x$, and hence $\alpha x \in \mathscr{B}$. □

Recall that a subset $\mathscr{K} \subset V$ is called *convex* if
$$\lambda x + (1-\lambda) y \in \mathscr{K}$$
for all $x, y \in \mathscr{K}$ and all $\lambda \in [0,1]$.

2.6. **Lemma.** *Let $\mathscr{K} \subset V$ be a subset. We define the subset $\mathscr{B} \subset V_+$ as*
$$\mathscr{B} := (\mathscr{K} - V_+) \cap V_+.$$
Then the following statements are true:

(1) *$\mathscr{B}$ is semi-solid.*
(2) *If $\mathscr{K}$ is convex, then $\mathscr{B}$ is also convex.*

Proof. Let $x \in \mathscr{B}$ and $y \in V$ with $0 \leq y \leq x$ be arbitrary. Then we have $y \in V_+$. Furthermore, we have
$$y = x - (x-y) \in \mathscr{K} - V_+,$$
because $x \in \mathscr{K} - V_+$ and $x - y \in V_+$. Therefore, we have $y \in \mathscr{B}$.

Now, we assume that $\mathscr{K}$ is also convex. Let $x, y \in \mathscr{B}$ and $\lambda \in [0,1]$ be arbitrary. Since V_+ is a convex cone, we have $x + \lambda(y-x) \in V_+$. There exist $c, d \in \mathscr{K}$ and $v, w \in V_+$ such that $x = c - v$ and $y = d - w$. Since $\mathscr{K}$ is convex, we obtain
$$x + \lambda(y-x) = \underbrace{c + \lambda(d-c)}_{\in \mathscr{K}} - \underbrace{(v + \lambda(w-v))}_{\in V_+} \in \mathscr{K} - V_+,$$
showing that $\mathscr{B}$ is also convex. □

2.7. **Lemma.** *Let $U \subset V$ be an ideal which is dense in V. Then for every subset $\mathscr{K} \subset V$ we have*
$$\mathscr{B} \subset \overline{\mathscr{C} \cap U_+},$$
where $\mathscr{B} = (\mathscr{K} - V_+) \cap V_+$, $\mathscr{C} = (\mathscr{K} - V_+) \cap U$ and $U_+ = V_+ \cap U$.

Proof. By Lemma 2.6 the subset $\mathscr{B}$ is semi-solid. Furthermore, $(U, \leq)$ is a topological vector lattice with positive cone U_+ because $U \subset V$ be an ideal. Let $x \in \mathscr{B}$ be arbitrary. Then we have $x \in \mathscr{K} - V_+$ and $x \in V_+$. Since U is dense in V, there is a net $(x_i)_{i \in I} \subset U$ with $x_i \to x$. Since $x \in V_+$ and the lattice operations are continuous, this gives us $x_i^+ \wedge x \to x$. Let $i \in I$ be arbitrary. Then we have $0 \leq x_i^+ \wedge x \leq x$, and hence $x_i^+ \wedge x \in \mathscr{B}$, because $\mathscr{B}$ is semi-solid. In particular, we have $x_i^+ \wedge x \in \mathscr{K} - V_+$. Furthermore, we have $0 \leq x_i^+ \wedge x \leq x_i^+$ and $x_i^+ \in U_+$. Since U_+ is a semi-solid subset of V, we deduce $x_i^+ \wedge x \in U_+$. Consequently, we have $x_i^+ \wedge x \in \mathscr{C} \cap U_+$, and hence $x \in \overline{\mathscr{C} \cap U_+}$. □

2.8. **Lemma.** *For a subset $\mathscr{B} \subset V_+$ the following statements are equivalent:*

(i) *$\mathscr{B}$ is sequentially bounded.*
(ii) *For every sequence $(x_n)_{n \in \mathbb{N}} \subset \mathscr{B}$ and every sequence $(\alpha_n)_{n \in \mathbb{N}} \subset (0, \infty)$ with $\alpha_n \downarrow 0$ we have $\alpha_n x_n \to 0$.*

Proof. (i) $\Rightarrow$ (ii): This implication is obvious.
(ii) $\Rightarrow$ (i): Let $(x_n)_{n \in \mathbb{N}} \subset \mathscr{B}$ and $(\alpha_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ be sequences with $\alpha_n \to 0$. There exist a decreasing sequence $(\beta_n)_{n \in \mathbb{N}} \subset (0, \infty)$ with $\beta_n \downarrow 0$ and an index $n_1 \in \mathbb{N}$ such that

$$|\alpha_n| \leq \beta_n \quad \text{for each } n \geq n_1. \tag{2.1}$$

Indeed, since $\alpha_n \to 0$ there is a subsequence $(n_k)_{k \in \mathbb{N}}$ such that for each $k \in \mathbb{N}$ we have

$$|\alpha_n| \leq k^{-1} \quad \text{for all } n \geq n_k.$$

We define the sequence $(\beta_n)_{n \in \mathbb{N}} \subset (0, \infty)$ as

$$\beta_n := k^{-1} \quad \text{if } n_k \leq n < n_{k+1}.$$

Then we have (2.1). Now, let $U \subset V$ be an arbitrary zero neighborhood. Since V is locally solid, we may assume that U is solid, and hence circled. By assumption there exists an index $N \geq n_1$ such that

$$\beta_n x_n \in U \quad \text{for all } n \geq N.$$

Since U is circled, by (2.1) we also have

$$\alpha_n x_n \in U \quad \text{for all } n \geq N,$$

showing that $\alpha_n x_n \to 0$. □

2.9. **Lemma.** *Let $\mathscr{B} \subset V_+$ be a semi-circled subset. Then the following statements are true:*

(1) *We have $0 \in \mathscr{B}$.*
(2) *For each $\alpha \geq 0$ the set $\alpha \mathscr{B}$ is also semi-circled.*
(3) *We have $\alpha \mathscr{B} \subset \beta \mathscr{B}$ for all $\alpha, \beta \in \mathbb{R}_+$ with $\alpha \leq \beta$.*

Proof. The proof is obvious, and therefore omitted. □

2.10. **Lemma.** *Let $\mathscr{B} \subset V_+$ be a semi-circled subset. Then the following statements are true:*

(1) *We have $p_{\mathscr{B}}(0) = 0$.*
(2) *We have $p_{\mathscr{B}}(x) \leq 1$ for all $x \in \mathscr{B}$.*
(3) *We have $p_{\mathscr{B}}(x) \geq 1$ for all $x \in V_+ \setminus \mathscr{B}$.*
(4) *We have $p_{\mathscr{B}}(\alpha x) = \alpha \cdot p_{\mathscr{B}}(x)$ for all $x \in \mathscr{B}$ and $\alpha \in \mathbb{R}_+$.*
(5) *If $\mathscr{B}$ is semi-solid, then we have $p_{\mathscr{B}}(x) \leq p_{\mathscr{B}}(y)$ for all $x, y \in \mathscr{B}$ with $x \leq y$.*

Proof. The first three statements are obvious. Let $x \in \mathscr{B}$ and $\alpha \in \mathbb{R}_+$ be arbitrary. We may assume that $\alpha > 0$ because otherwise the identity follows from the first statement. Since $\mathscr{B}$ is semi-circled, for each $\beta > 0$ we have $\alpha x \in \beta \mathscr{B}$ if and only if $x \in \frac{\beta}{\alpha} \mathscr{B}$, and for each $\gamma > 0$ we have $x \in \gamma \mathscr{B}$ if and only if $\alpha x \in \alpha \gamma \mathscr{B}$. Therefore, we have

$$\begin{aligned} p_{\mathscr{B}}(\alpha x) &= \inf\{\beta > 0 : \alpha x \in \beta \mathscr{B}\} \\ &= \alpha \cdot \inf\{\gamma > 0 : x \in \gamma \mathscr{B}\} = \alpha \cdot p_{\mathscr{B}}(x). \end{aligned}$$

Now assume that $\mathscr{B}$ is semi-solid, and let $x, y \in \mathscr{B}$ with $x \leq y$ be arbitrary. Then for each $\alpha > 0$ with $y \in \alpha\mathscr{B}$ we have $x \in \alpha\mathscr{B}$, and hence

$$\begin{aligned} p_{\mathscr{B}}(x) &= \inf\{\alpha > 0 : x \in \alpha\mathscr{B}\} \\ &\leq \inf\{\alpha > 0 : y \in \alpha\mathscr{B}\} = p_{\mathscr{B}}(y), \end{aligned}$$

completing the proof. □

For each $\alpha \geq 0$ we agree on the notation

$$\{p_{\mathscr{B}} \leq \alpha\} := \{x \in V : p_{\mathscr{B}}(x) \leq \alpha\}.$$

2.11. **Lemma.** *Let $\mathscr{B} \subset V_+$ be a semi-circled subset. Then we have*

$$\alpha\mathscr{B} = \{p_{\mathscr{B}} \leq \alpha\} \cap V_+ \quad \textit{for each } \alpha \in (0,1).$$

Proof. Let $x \in \mathscr{B}$ be arbitrary. Then we have $p_{\mathscr{B}}(x) \leq \alpha$ if and only if

$$\inf\{\beta > 0 : x \in \beta\mathscr{B}\} \leq \alpha.$$

Since $\mathscr{B}$ is semi-circled, by Lemma 2.9 this is the case if and only if $x \in \alpha\mathscr{B}$. This proves

$$\alpha\mathscr{B} = \{p_{\mathscr{B}} \leq \alpha\} \cap \mathscr{B}.$$

Since $p_{\mathscr{B}}(x) \geq 1$ for all $x \in V_+ \setminus \mathscr{B}$, this completes the proof. □

2.12. **Proposition.** *Let $\mathscr{B} \subset V_+$ be a semi-circled subset. Then the following statements are equivalent:*

(i) *$\mathscr{B}$ is topologically bounded.*
(ii) *$\mathscr{B}$ is sequentially bounded.*
(iii) *For each neighborhood $U \subset V$ of zero there exists $\alpha \in (0,1)$ such that*

$$\{p_{\mathscr{B}} \leq \alpha\} \cap V_+ \subset U \cap V_+.$$

In either case, we have $p_{\mathscr{B}}(x) > 0$ for all $x \in V_+ \setminus \{0\}$.

Proof. (i) ⇔ (ii): See, for example, statement (3) on page 153 in [26] or statement 5.3 on page 26 in [30].
(i) ⇔ (iii): The subset $\mathscr{B}$ is topologically bounded if and only if for each neighborhood U of zero there exists $\alpha \in (0,1)$ such that $\alpha\mathscr{B} \subset U \cap V_+$. Using Lemma 2.11 completes the proof.
The additional statement is obvious. □

Hence, in the situation of Proposition 2.12 for every neighborhood of zero the Minkowski functional $p_{\mathscr{B}}$ considered on V_+ is bounded from below by a positive constant outside this neighborhood. In particular, it has no zeros on $V_+ \setminus \{0\}$.

2.13. **Proposition.** *Let $\mathscr{B} \subset V_+$ be a semi-circled subset. Then the following statements are equivalent:*

(i) *We have $p_{\mathscr{B}}(x) > 0$ for all $x \in V_+ \setminus \{0\}$.*
(ii) *We have $\bigcap_{\alpha > 0} \alpha\mathscr{B} = \{0\}$.*

Proof. (i) ⇒ (ii): Let $x \in V_+ \setminus \{0\}$ be arbitrary. If $x \notin \mathscr{B}$, then $x \notin \bigcap_{\alpha>0} \alpha\mathscr{B}$. Hence, we may assume that $x \in \mathscr{B} \setminus \{0\}$. Then we have $p_{\mathscr{B}}(x) > 0$, and hence there exists $\alpha > 0$ with $\alpha < p_{\mathscr{B}}(x)$. This gives us $x \notin \alpha\mathscr{B}$, and in particular $x \notin \bigcap_{\alpha>0} \alpha\mathscr{B}$.
(ii) ⇒ (i): Let $x \in V_+ \setminus \{0\}$ be arbitrary. By assumption there is $\alpha > 0$ such that $x \notin \alpha\mathscr{B}$. By Lemma 2.9 we deduce that $x \notin \beta\mathscr{B}$ for all $\beta \in [0, \alpha]$. Hence $p_{\mathscr{B}}(x) \geq \alpha > 0$. □

The following example shows that a convex, semi-solid subset $\mathscr{B} \subset V_+$ with $\bigcap_{\alpha>0} \alpha\mathscr{B} = \{0\}$ does not need to be topologically bounded.

2.14. **Example.** *Let* $V = \ell^2(\mathbb{N})$ *be the space of all square-integrable sequences, equipped with the Hilbert space topology induced by the norm*

$$\|x\| = \bigg(\sum_{k=1}^{\infty} |x_k|^2 \bigg)^{1/2}, \quad x \in V. \tag{2.2}$$

We agree to write $x \leq y$ *if* $x_k \leq y_k$ *for all* $k \in \mathbb{N}$*. Then* $(V, \leq)$ *is a vector lattice, and the positive cone is given by*

$$V_+ = \{x \in V : x_k \geq 0 \text{ for each } k \in \mathbb{N}\}.$$

Furthermore, for each $x \in V$ *be have* $x^+ = (x_k^+)_{k \in \mathbb{N}}$*,* $x^+ = (x_k^+)_{k \in \mathbb{N}}$ *and* $|x| = (|x_k|)_{k \in \mathbb{N}}$*. Hence, taking into account (2.2) the system* $(U_\epsilon)_{\epsilon > 0}$ *given by*

$$U_\epsilon = \{x \in V : \|x\| < \epsilon\}, \quad \epsilon > 0$$

is a zero neighborhood basis of V *consisting of solid sets, showing that* $(V, \leq)$ *is a topological vector lattice. We define the sequence* $(f_k)_{k \in \mathbb{N}}$ *as* $f_k := k\, e_k$*, where* e_k *denotes the* k*th unit vector. Furthermore, we define the subset* $\mathscr{B} \subset V_+$ *as the convex hull*

$$\mathscr{B} := \operatorname{co}\big(\{0\} \cup \{f_k : k \in \mathbb{N}\}\big).$$

Then $\mathscr{B}$ *is unbounded, because* $\|f_k\| \to \infty$ *for* $k \to \infty$*, and* $\mathscr{B}$ *consists of all linear combinations*

$$x = \sum_{k=1}^{n} \lambda_k f_k \tag{2.3}$$

for some $n \in \mathbb{N}$*, where* $\lambda_k \geq 0$ *for* $k = 1, \ldots, n$ *and* $\sum_{k=1}^n \lambda_k \leq 1$*. From this representation we see that* $\mathscr{B}$ *is semi-solid. For each* $\alpha > 0$ *the set* $\alpha \mathscr{B}$ *consists of all* $x \in V_+$ *with representation (2.3) such that* $\lambda_k \geq 0$ *for* $k = 1, \ldots, n$ *and* $\sum_{k=1}^n \lambda_k \leq \alpha$*. Let* $x \in \mathscr{B} \setminus \{0\}$ *with representation (2.3) be arbitrary. Since* $x \neq 0$*, we have* $\lambda > 0$*, where* $\lambda := \sum_{k=1}^n \lambda_k$*, and hence* $x \notin \alpha \mathscr{B}$ *for each* $\alpha \in (0, \lambda)$*. Consequently, we have* $\bigcap_{\alpha > 0} \alpha \mathscr{B} = \{0\}$*.*

However, surprisingly there are some examples of topological vector lattices V where every convex, semi-solid subset $\mathscr{B} \subset V_+$ with $\bigcap_{\alpha > 0} \alpha \mathscr{B} = \{0\}$ is topologically bounded. As we will see in Section 5 later on, this is in particular the case if $V = L^0$ is the space of all random variables defined on some probability space.

Recall that a subset $\mathscr{B} \subset V_+$ is unbounded if and only if there exist sequences $(x_n)_{n \in \mathbb{N}} \subset \mathscr{B}$ and $(\alpha_n)_{n \in \mathbb{N}} \subset (0, \infty)$ with $\alpha_n \downarrow 0$ such that $\alpha_n x_n \not\to 0$. In the upcoming definition, we make a stronger assumption for unbounded subsets, which are convex and semi-solid.

2.15. **Definition.** *The topological vector lattice* $(V, \leq)$ *admits nontrivial minimal elements for unbounded, convex and semi-solid subsets of* V_+ *if for each unbounded, convex and semi-solid subset* $\mathscr{B} \subset V_+$ *there are* $x \in \mathscr{B} \setminus \{0\}$*, and sequences* $(x_n)_{n \in \mathbb{N}} \subset \mathscr{B}$ *and* $(\alpha_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+$ *with* $\alpha_n \downarrow 0$ *such that* $x \leq \alpha_n x_n$ *for each* $n \in \mathbb{N}$*.*

2.16. **Theorem.** *Suppose that* $(V, \leq)$ *admits nontrivial minimal elements for unbounded, convex and semi-solid subsets of* V_+*. Then for every convex, semi-solid subset* $\mathscr{B} \subset V_+$ *the following statements are equivalent:*

(i) $\mathscr{B}$ *is topologically bounded.*
(ii) $\mathscr{B}$ *is sequentially bounded.*
(iii) *We have* $p_{\mathscr{B}}(x) > 0$ *for all* $x \in V_+ \setminus \{0\}$*.*
(iv) *We have* $\bigcap_{\alpha > 0} \alpha \mathscr{B} = \{0\}$*.*

Proof. By virtue of Propositions 2.12 and 2.13, we only need to prove the implication (iii) $\Rightarrow$ (ii). Suppose that $\mathscr{B}$ is not sequentially bounded. Then there exist sequences $(x_n)_{n\in\mathbb{N}} \subset \mathscr{B}$ and $(\alpha_n)_{n\in\mathbb{N}} \subset (0,\infty)$ with $\alpha_n \downarrow 0$ and an element $x \in \mathscr{B} \setminus \{0\}$ such that $x \leq \alpha_n x_n$ for each $n \in \mathbb{N}$. By Lemma 2.10 we have

$$p_{\mathscr{B}}(x) \leq p_{\mathscr{B}}(\alpha_n x_n) = \alpha_n \cdot p_{\mathscr{B}}(x_n) \to 0 \quad \text{for } n \to \infty,$$

and hence the contradiction $p_{\mathscr{B}}(x) = 0$. □

2.17. **Proposition.** *Suppose that the topological vector lattice $(V, \leq)$ is locally convex with a family $(\rho_i)_{i\in I}$ of seminorms satisfying the following two conditions:*

(1) *For all $x, y \in V_+$ we have $x \leq y$ if and only if $\rho_i(x) \leq \rho_i(y)$ for all $i \in I$.*
(2) *For each $f : I \to \mathbb{R}_+$ there exists $x \in V_+$ with $\rho_i(x) = f(i)$ for all $i \in I$.*

Then $(V, \leq)$ admits nontrivial minimal elements for unbounded, convex and semi-solid subsets of V_+.

Proof. Let $\mathscr{B} \subset V_+$ be an unbounded, convex and semi-solid subset. Then there exist sequences $(x_n)_{n\in\mathbb{N}} \subset \mathscr{B}$ and $(\alpha_n)_{n\in\mathbb{N}} \subset (0,\infty)$ with $\alpha_n \downarrow 0$ such that $\alpha_n x_n \not\to 0$. Hence, there exists $i \in I$ such that $\rho_i(\alpha_n x_n) \not\to 0$. Therefore, there exist $\epsilon > 0$ and a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $\rho_i(\alpha_{n_k} x_{n_k}) \geq \epsilon$ for each $k \in \mathbb{N}$. Let $f : I \to \mathbb{R}_+$ be the function given by $f(i) := \epsilon$ and $f(j) := 0$ for all $j \in I \setminus \{i\}$. By assumption there exists $x \in V_+$ such that $\rho_j(x) = f(j)$ for all $j \in I$. This gives us $\rho_i(x) = \epsilon$ and $\rho_j(x) = 0$ for all $j \in I \setminus \{i\}$. Therefore, we have $\rho_j(x) \leq \rho_j(\alpha_{n_k} x_{n_k})$ for all $k \in \mathbb{N}$ and all $j \in J$, and hence $x \leq \alpha_{n_k} x_{n_k}$ for all $k \in \mathbb{N}$. Note that $x \in \mathscr{B} \setminus \{0\}$, because $\rho_i(x) > 0$ and $\mathscr{B}$ is semi-solid. □

2.18. **Remark.** *According to Proposition 2.17 the following examples of topological vector lattices $(V, \leq)$ admit nontrivial minimal elements for unbounded, convex and semi-solid subsets of V_+, which means that Theorem 2.16 applies:*

- *The Euclidean space $V = \mathbb{R}^n$, equipped with the usual Euclidean topology.*
- *The space $V = \ell^0(\mathbb{N})$ of all sequences, equipped with the topology of pointwise convergence.*
- *The space V consisting of all mappings $f : D \to \mathbb{R}$ on some domain D, equipped with the topology of pointwise convergence.*

As we will see later on, the space $V = L^0$ is also such an example; see Proposition 5.5 below.

3. No-arbitrage concepts in topological vector lattices

In this section we introduce no-arbitrage concepts in topological vector lattices. Let $(V, \leq)$ be a topological vector lattice. Furthermore, let $\mathscr{K}_0 \subset V$ be a subset. We may think of outcomes of trading strategies with initial value zero. Throughout this section, we make the following assumption.

3.1. **Assumption.** *We assume that $\mathscr{K}_0$ is a convex cone.*

3.2. **Definition.** *$\mathscr{K}_0$ satisfies* NA (No Arbitrage) *if $\mathscr{K}_0 \cap V_+ = \{0\}$.*

We define the subset $\mathscr{B}_0 \subset V_+$ as

$$\mathscr{B}_0 := (\mathscr{K}_0 - V_+) \cap V_+.$$

The following auxiliary result is obvious.

3.3. **Lemma.** *The following statements are equivalent:*

(i) *$\mathscr{K}_0$ satisfies NA.*
(ii) *We have $(\mathscr{K}_0 - V_+) \cap V_+ = \{0\}$.*
(iii) *We have $\mathscr{B}_0 = \{0\}$.*

Let $U \subset V$ be an ideal which is dense in V. We define the convex cone $\mathscr{C} \subset U$ as

$$\mathscr{C} := (\mathscr{K}_0 - V_+) \cap U.$$

3.4. **Lemma.** *The following statements are equivalent:*

(i) *$\mathscr{K}_0$ satisfies NA.*
(ii) *We have $(\mathscr{K}_0 - V_+) \cap U_+ = \{0\}$.*
(iii) *We have $\mathscr{C} \cap U_+ = \{0\}$*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii): Taking into account Lemma 3.3, these implications are obvious.
(iii) $\Rightarrow$ (i): Let $x \in \mathscr{B}_0$ be arbitrary. By Lemma 2.7 there is a net $(x_i)_{i \in I} \subset \mathscr{C} \cap U_+$ such that $x_i \to x$. By assumption we have $x_i = 0$ for each $i \in I$, and hence $x = 0$. □

3.5. **Definition.** *Let τ be a topology on U. We say that $\mathscr{K}_0$ satisfies* NFL_τ (No Free Lunch with respect to τ) *if*

$$\overline{\mathscr{C}}^{\tau} \cap U_+ = \{0\}.$$

3.6. **Proposition.** *Let τ_1 and τ_2 be two topologies on U such that $\tau_1 \subset \tau_2$. If $\mathscr{K}_0$ satisfies NFL$_{\tau_1}$, then it also satisfies NFL$_{\tau_2}$.*

Proof. By assumption we have $\overline{\mathscr{C}}^{\tau_2} \subset \overline{\mathscr{C}}^{\tau_1}$, whence the statement follows. □

Now, let τ be a topology on U.

3.7. **Proposition.** *If $\mathscr{K}_0$ satisfies NFL$_\tau$, then $\mathscr{K}_0$ also satisfies NA.*

Proof. This is an immediate consequence of Lemma 3.4. □

3.8. **Corollary.** *Suppose that $\mathscr{C}$ is closed in U with respect to τ. Then the following statements are equivalent:*

(i) *$\mathscr{K}_0$ satisfies NFL$_\tau$.*
(ii) *$\mathscr{K}_0$ satisfies NA.*

Proof. This is an immediate consequence of Lemma 3.4. □

3.9. **Corollary.** *Suppose that $\mathscr{K}_0 - V_+$ is closed in V, and that $\sigma \cap U \subset \tau$, where σ denotes the topology on V. Then the following statements are equivalent:*

(i) *$\mathscr{K}_0$ satisfies NFL$_\tau$.*
(ii) *$\mathscr{K}_0$ satisfies NA.*

Proof. The convex cone $\mathscr{C}$ is closed in U with respect to τ. Indeed, let $(x_i)_{i \in I} \subset \mathscr{C}$ be a net and $x \in U$ be an element such that $x_i \xrightarrow{\tau} x$. Since $\sigma \cap U \subset \tau$, we also have $x_i \xrightarrow{\sigma} x$. Since $\mathscr{K}_0 - V_+$ is closed in V, we deduce that $x \in \mathscr{K}_0 - V_+$. Consequently, the statement follows from Corollary 3.8. □

Now, let $(\mathscr{K}_\alpha)_{\alpha > 0}$ be a family of subsets of V. We may think of outcomes of trading strategies with initial value α. Throughout this section, we make the following assumption.

3.10. **Assumption.** *We assume that*

$$ax + by \in \mathscr{K}_{a\alpha + b\beta} \tag{3.1}$$

for all $a, b \in \mathbb{R}_+$, $\alpha, \beta > 0$ with $a\alpha + b\beta > 0$ and $x \in \mathscr{K}_\alpha$, $y \in \mathscr{K}_\beta$.

Then for each $\alpha > 0$ the set $\mathscr{K}_\alpha$ is convex, and the union

$$\mathscr{K}_{>0} := \left(\bigcup_{\alpha>0} \mathscr{K}_\alpha \right) \cup \{0\}$$

is a convex cone. We define the family $(\mathscr{B}_\alpha)_{\alpha>0}$ of subsets of V_+ as

$$\mathscr{B}_\alpha := (\mathscr{K}_\alpha - V_+) \cap V_+, \quad \alpha > 0.$$

We may think of all nonnegative elements which are equal to or below the outcome of a trading strategy with initial value α. By Lemma 2.6 for each $\alpha > 0$ the set $\mathscr{B}_\alpha$ is convex and semi-solid. We set $\mathscr{B} := \mathscr{B}_1$.

3.11. **Lemma.** *We have $\mathscr{B}_\alpha = \alpha\mathscr{B}$ for each $\alpha > 0$.*

Proof. Let $\alpha > 0$ be arbitrary. Furthermore, let $x \in \mathscr{B}$ be arbitrary. Then we have $x \in V_+$ and $x \leq y$ for some $y \in \mathscr{K}_1$. Note that $\alpha x \in V_+$ and $\alpha x \leq \alpha y$. Moreover, by (3.1) we have $\alpha y \in \mathscr{K}_\alpha$. Therefore, we have $\alpha x \in \mathscr{B}_\alpha$, showing that $\alpha\mathscr{B} \subset \mathscr{B}_\alpha$.

Now, let $x \in \mathscr{B}_\alpha$ be arbitrary. Then we have $x \in V_+$ and $x \leq y$ for some $x \in \mathscr{K}_\alpha$. Note that $\alpha^{-1}x \in V_+$ and $\alpha^{-1}x \leq \alpha^{-1}y$. Moreover, by (3.1) we have $\alpha^{-1}y \in \mathscr{K}_1$. Therefore, we have $\alpha^{-1}x \in \mathscr{B}$, and hence $x \in \alpha\mathscr{B}$, showing that $\mathscr{B}_\alpha \subset \alpha\mathscr{B}$. □

Consequently, it suffices to concentrate on all outcomes of trading strategies with initial wealth one rather than focusing on all outcomes of trading strategies with positive initial wealth, and for our upcoming no-arbitrage concepts it is enough to focus on the convex subset $\mathscr{B}$.

3.12. **Definition.** *We introduce the following concepts:*

(1) *$\mathscr{K}_1$ satisfies* NUPBR (No Unbounded Profit with Bounded Risk) *if $\mathscr{B}$ is topologically bounded.*
(2) *$\mathscr{K}_1$ satisfies* NAA_1 (No Asymptotic Arbitrage of the 1st Kind) *if $\mathscr{B}$ is sequentially bounded.*
(3) *$\mathscr{K}_1$ satisfies* NA_1 (No Arbitrage of the 1st Kind) *if $p_{\mathscr{B}}(x) > 0$ for all $x \in V_+ \setminus \{0\}$.*

3.13. **Remark.** *By Lemma 3.11 the following statements are equivalent:*

(i) *$\mathscr{K}_1$ satisfies NUPBR.*
(ii) *$\mathscr{B}_\alpha$ is topologically bounded for all $\alpha > 0$.*
(iii) *$\mathscr{B}_\alpha$ is topologically bounded for some $\alpha > 0$.*

3.14. **Remark.** *By Lemma 2.8 the subset $\mathscr{K}_1$ satisfies NAA_1 if and only if for each sequence $(\alpha_n)_{n\in\mathbb{N}} \subset (0,\infty)$ with $\alpha_n \downarrow 0$ and every sequence $(x_n)_{n\in\mathbb{N}} \subset V_+$ with $x_n \in \mathscr{B}_{\alpha_n}$ for each $n \in \mathbb{N}$ we have $x_n \to 0$.*

3.15. **Remark.** *By virtue of Lemma 3.11, the Minkowski functional $p_{\mathscr{B}} : V \to [0,\infty]$ can be written as*

$$p_{\mathscr{B}}(x) = \inf\{\alpha > 0 : x \in \mathscr{B}_\alpha\}, \quad x \in V.$$

Hence $p_{\mathscr{B}}(x)$ has the interpretation of the minimal superreplication price *of x. Thus $\mathscr{K}_1$ satisfies NA_1 if and only if the superreplication price $p_{\mathscr{B}}(x)$ is strictly positive for every strictly positive element $x \in V_+ \setminus \{0\}$.*

3.16. **Proposition.** *The following statements are equivalent:*

(i) *$\mathscr{K}_1$ satisfies NUPBR.*
(ii) *$\mathscr{K}_1$ satisfies NAA_1.*
(iii) *For each neighborhood U of zero there exists $\alpha \in (0,1)$ such that*

$$\{p_{\mathscr{B}} \leq \alpha\} \cap V_+ \subset U \cap V_+.$$

If the previous conditions are fulfilled, then $\mathscr{K}_1$ satisfies NA_1.

Proof. This is a direct consequence of Proposition 2.12. □

3.17. **Theorem.** *Suppose that $(V, \leq)$ admits nontrivial minimal elements for unbounded, convex and semi-solid subsets of V_+. Then the following statements are equivalent:*

(i) *$\mathscr{K}_1$ satisfies NUPBR.*
(ii) *$\mathscr{K}_1$ satisfies NAA_1.*
(iii) *$\mathscr{K}_1$ satisfies NA_1.*
(iv) *We have $\bigcap_{\alpha>0} \mathscr{B}_\alpha = \{0\}$.*

Proof. This is a consequence of Theorem 2.16. □

Now, we consider $\mathscr{K}_0$ and $(\mathscr{K}_\alpha)_{\alpha>0}$ together. The following remark provides a sufficient condition ensuring that Assumptions 3.1 and 3.10 are fulfilled.

3.18. **Remark.** *Suppose that*

$$ax + by \in \mathscr{K}_{a\alpha+b\beta} \tag{3.2}$$

for all $a, b \in \mathbb{R}_+$, $\alpha, \beta \in \mathbb{R}_+$ and $x \in \mathscr{K}_\alpha$, $y \in \mathscr{K}_\beta$. Then $\mathscr{K}_0$ is a convex cone, and we have (3.2) for all $a, b \in \mathbb{R}_+$, $\alpha, \beta > 0$ with $a\alpha + b\beta > 0$ and $x \in \mathscr{K}_\alpha$, $y \in \mathscr{K}_\beta$.

3.19. **Proposition.** *Suppose that $\mathscr{B}_0 \subset \mathscr{B}_\alpha$ for each $\alpha > 0$. If $\mathscr{K}_1$ satisfies NA_1, then $\mathscr{K}_0$ satisfies NA.*

Proof. This is a consequence of Theorem 3.17 and Lemma 3.3. □

3.20. **Proposition.** *Let τ be a topology on U such that*

$$\bigg(\bigcap_{\alpha>0} \mathscr{B}_\alpha \bigg) \cap U \subset \overline{\mathscr{C}}^\tau.$$

If $\mathscr{K}_0$ satisfies NFL_τ, then $\mathscr{K}_1$ satisfies NA_1.

Proof. By assumption we have

$$\bigg(\bigcap_{\alpha>0} \mathscr{B}_\alpha \bigg) \cap U_+ = \{0\}.$$

Let $x \in \bigcap_{\alpha>0} \mathscr{B}_\alpha$ be arbitrary. Since U is dense in V, there exists a net $(x_i)_{i\in I} \subset U$ such that $x_i \to x$. Since the lattice operations are continuous, we obtain $x_i^+ \wedge x \to x$. Let $i \in I$ be arbitrary. Then we have $0 \leq x_i^+ \wedge x \leq x$. Since $\bigcap_{\alpha>0} \mathscr{B}_\alpha$ is semi-solid, we have $x_i^+ \wedge x \in \bigcap_{\alpha>0} \mathscr{B}_\alpha$. Furthermore, we have $0 \leq x_i^+ \wedge x \leq x_i^+$ and $x_i^+ \in U_+$. Since U_+ is a semi-solid subset of V, we deduce $x_i^+ \wedge x \in U_+$. Consequently, we have $x = 0$. □

4. Versions of the abstract fundamental theorem of asset pricing

In this section, we present versions of the abstract fundamental theorem of asset pricing in our present framework with topological vector lattices. Our results are similar to those in [13, 28, 4, 29], where also further refinements can be found. In this section, we provide a comparatively simple framework which will enable us to prove Theorem 6.10 concerning no-arbitrage in discrete time later on.

As in Section 3, let $(V, \leq)$ be a topological vector lattice, and let $\mathscr{K}_0 \subset V$ be a convex cone. As already mentioned, we may think of the outcomes of trading strategies with initial value zero. Furthermore, let $U \subset V$ be an ideal, and let τ be a topology on U. We assume that the topological vector lattice $(U, \leq)$ is locally convex. Recall that the convex cone $\mathscr{C} \subset U$ is defined as

$$\mathscr{C} := (\mathscr{K}_0 - V_+) \cap U,$$

and that NFL$_\tau$ means $\overline{\mathscr{C}}^\tau \cap U_+ = \{0\}$.

We denote by U' the space of all continuous linear functionals with respect to τ. A functional $x' \in U'$ is called *positive* if $x'(U_+) \subset \mathbb{R}_+$. We denote by U'_+ the set of all positive linear functionals. Note that U'_+ is a convex cone in U'. Furthermore, we denote by U'_{++} the set of all positive functionals $x' \in U'_+$ such that $x'(x) > 0$ for all $x \in U_+ \setminus \{0\}$.

4.1. **Definition.** *A positive functional $x' \in U'_+$ is called* separating *for $\mathscr{C}$ if $x'(y) \leq 0$ for all $y \in \mathscr{C}$.*

4.2. **Definition.** *A functional $x' \in U'_+$ which is separating for $\mathscr{C}$ is called* strictly separating *for $\mathscr{C}$ if $x' \in U'_{++}$.*

Let $x' \in U'_+$ be a functional which is separating for $\mathscr{C}$. Then we have

$$x'(y) \leq 0 \leq x'(z) \quad \text{for all } y \in \mathscr{C} \text{ and } z \in U_+,$$

showing that x' separates the sets $\mathscr{C}$ and U_+. If x' is even strictly separating for $\mathscr{C}$, then we have

$$x'(y) \leq 0 < x'(z) \quad \text{for all } y \in \mathscr{C} \text{ and } z \in U_+ \setminus \{0\}.$$

4.3. **Lemma.** *Let $\mathscr{C} \subset U$ be a closed convex cone such that*

$$-U_+ \subset \mathscr{C} \quad \text{and} \quad \mathscr{C} \cap U_+ = \{0\}. \tag{4.1}$$

Then for each $x \in U_+ \setminus \{0\}$ there exists a separating functional $x' \in U'_+$ for $\mathscr{C}$ such that $x'(x) > 0$.

Proof. Let $x \in U_+ \setminus \{0\}$ be arbitrary. By (4.1) we have $x \notin \mathscr{C}$. Hence, by [1, Cor. 5.84] there exists a continuous linear functional $x' \in U'$ such that $x'(x) > 0$ and $x'(y) \leq 0$ for all $y \in \mathscr{C}$. Let $z \in U_+$ be arbitrary. By (4.1) we have $-z \in \mathscr{C}$, and hence $x'(z) \geq 0$. □

4.4. **Theorem** (Abstract Fundamental Theorem of Asset Pricing, Version 1)**.** *The following statements are equivalent:*

(i) *$\mathscr{K}_0$ satisfies NFL$_\tau$.*
(ii) *For each $x \in U_+ \setminus \{0\}$ there exists a separating functional $x' \in U'_+$ for $\mathscr{C}$ such that $x'(x) > 0$.*

Proof. (i) ⇒ (ii): Since $\mathscr{K}_0$ satisfies NFL$_\tau$, we have

$$-U_+ \subset \overline{\mathscr{C}}^\tau \quad \text{and} \quad \overline{\mathscr{C}}^\tau \cap U_+ = \{0\}.$$

Noting that $\overline{\mathscr{C}}^\tau$ is a closed convex cone, by Lemma 4.3 there exists a separating functional $x' \in U'_+$ for $\overline{\mathscr{C}}^\tau$ such that $x'(x) > 0$. Of course, x' is also a separating functional for $\mathscr{C}$.

(ii) ⇒ (i): Let $x \in U_+ \setminus \{0\}$ be arbitrary. Then we have $x \notin \overline{\mathscr{C}}^\tau$. Indeed, otherwise there is a net $(x_i)_{i \in I} \subset \mathscr{C}$ such that $x_i \to x$. Then we have $x'(x_i) \leq 0$ for all $i \in I$, and hence the contradiction $x'(x) \leq 0$. □

4.5. **Definition.** *Let $\mathscr{X} \subset U_+ \setminus \{0\}$ and $\mathscr{X}' \subset U'_+ \setminus \{0\}$ be subsets. Then $\mathscr{X}'$ is called* strictly positive separating *for $\mathscr{X}$ if for each $x \in \mathscr{X}$ there exists $x' \in \mathscr{X}'$ such that $x'(x) > 0$.*

The upcoming notion is inspired by the Halmos-Savage theorem; see, for example [10, Thm. 1.61]. In [13] this condition is called *Lindelöf condition*.

4.6. **Definition.** *The locally convex space (U, τ) has the* Halmos-Savage property *if for every subset $\mathscr{X}' \subset U'_+ \setminus \{0\}$ which is strictly positive separating for $U_+ \setminus \{0\}$ there is a countable subset $\mathscr{Y}' \subset \mathscr{X}'$ which is strictly positive separating for $U_+ \setminus \{0\}$.*

The upcoming definition is inspired by [29].

4.7. **Definition.** *The locally convex space* (U,τ) *has the* Kreps-Yan property *if for every closed convex cone* $\mathscr{C} \subset U$ *satisfying (4.1) there exists a strictly separating functional* $x' \in U'_{++}$ *for* $\mathscr{C}$*.*

4.8. **Remark.** *In* [29] *it was shown that every Banach ideal space* U *on a* σ*-finite measure space* $(\Omega, \mathscr{F}, \mu)$ *has the Kreps-Yan property.*

4.9. **Proposition.** *If a normed space* U *has the Halmos-Savage property, then it also has the Kreps-Yan property.*

Proof. Let $\mathscr{C} \subset U$ be a closed convex cone such that (4.1) holds true. By Lemma 4.3 for each $x \in U_+ \setminus \{0\}$ there exists a separating functional $x' \in U'_+$ for $\mathscr{C}$ such that $x'(x) > 0$. Let $\mathscr{X}' \subset U'_+ \setminus \{0\}$ be the collection of all these functionals. Then $\mathscr{X}'$ is strictly positive separating for $U_+ \setminus \{0\}$. Since U has the Halmos-Savage property, there exists a countable family $(x'_n)_{n\in\mathbb{N}} \subset \mathscr{X}'$ such that $\{x'_n : n \in \mathbb{N}\}$ is strictly positive separating for $U_+ \setminus \{0\}$. Without loss of generality, we may assume that $\|x'_n\| = 1$ for each $n \in \mathbb{N}$. Thus, by the completeness of U' we can define

$$x' := \sum_{n=1}^{\infty} \frac{x'_n}{2^n} \in U'.$$

Then we have $x'(y) \leq 0 < x'(z)$ for all $y \in \mathscr{C}$ and $z \in U_+ \setminus \{0\}$. □

4.10. **Theorem** (Abstract Fundamental Theorem of Asset Pricing, Version 2)**.** *Suppose that the locally convex space* (U,τ) *has the Kreps-Yan property. Then the following statements are equivalent:*

(i) $\mathscr{K}_0$ *satisfies NFL*$_\tau$*.*
(ii) *There exists a strictly separating functional* $x' \in U'_{++}$ *for* $\mathscr{C}$*.*

Proof. (i) ⇒ (ii): Since $\mathscr{K}_0$ satisfies NFL$_\tau$, we have

$$-U_+ \subset \overline{\mathscr{C}}^\tau \quad \text{and} \quad \overline{\mathscr{C}}^\tau \cap U_+ = \{0\}.$$

Noting that $\overline{\mathscr{C}}^\tau$ is a closed convex cone, there exists a strictly separating functional $x' \in U'_{++}$ for $\overline{\mathscr{C}}^\tau$. Of course, x' is also a strictly separating functional for $\mathscr{C}$.
(ii) ⇒ (i): This is an immediate consequence of Theorem 4.4. □

4.11. **Corollary.** *Suppose that the locally convex space* (U,τ) *has the Kreps-Yan property. If* $\mathscr{C}$ *is closed in* U *with respect to* τ*, then the following statements are equivalent:*

(i) $\mathscr{K}_0$ *satisfies NA.*
(ii) *There exists a strictly separating functional* $x' \in U'_{++}$ *for* $\mathscr{C}$*.*

Proof. This is a consequence of Theorem 4.10 and Corollary 3.8. □

4.12. **Corollary.** *Suppose that the locally convex space* (U,τ) *has the Kreps-Yan property, and that* $\sigma \cap U \subset \tau$*, where* σ *denotes the topology on* V*. If* $\mathscr{K}_0 - V_+$ *is closed in* V*, then the following statements are equivalent:*

(i) $\mathscr{K}_0$ *satisfies NA.*
(ii) *There exists a strictly separating functional* $x' \in U'_{++}$ *for* $\mathscr{C}$*.*

Proof. This is a consequence of Theorem 4.10 and Corollary 3.9. □

5. The space of random variables

In this section we will consider our abstract no-arbitrage concepts on the space of random variables, and review the concepts which are known in the literature. Let $(\Omega, \mathscr{G}, \mathbb{P})$ be a probability space. We denote by $V = L^0(\Omega, \mathscr{G}, \mathbb{P})$ the space of all equivalence classes of real-valued random variables, in short $V = L^0$. Here two random variables X and Y are identified if $\mathbb{P}(X = Y) = 1$. Furthermore, we write $X \leq Y$ if $\mathbb{P}(X \leq Y) = 1$. The space L^0 equipped with the metric

$$d(X, Y) = \mathbb{E}[|X - Y| \wedge 1], \quad X, Y \in L^0 \tag{5.1}$$

is a topological vector space, and convergence with respect to this metric is convergence in probability; that is, we have $d(X_n, X) \to 0$ if and only if $X_n \xrightarrow{\mathbb{P}} X$. For this statement see, for example Exercise A.8.9 on page 450 in [2]. The positive cone of L^0 is denoted by L^0_+.

5.1. **Proposition.** *The space $(L^0, \leq)$ is a Fréchet lattice.*

Proof. See [1, Thm. 13.41]. □

5.2. **Remark.** *Let us list some further properties of the topological vector lattice $(L^0, \leq)$.*

- *If $(\Omega, \mathscr{G}, \mathbb{P})$ is non-atomic, then the dual space of L^0 is trivial, and hence in L^0 is not locally convex; see* [1, Thm. 13.41].
- *L^0 is a so-called F-space. This follows from Proposition 5.1 and the definition (5.1) of the metric d.*
- *If $\mathbb{Q} \approx \mathbb{P}$ is an equivalent probability measure, then the new metric*

$$d_{\mathbb{Q}}(X, Y) = \mathbb{E}_{\mathbb{Q}}[|X - Y| \wedge 1], \quad X, Y \in L^0$$

  induces the same topology on L^0; see [18].

5.3. **Definition.** *A subset $\mathscr{B} \subset V$ is called* bounded in probability *if for each $\epsilon > 0$ there exists $c > 0$ such that*

$$\sup_{X \in \mathscr{B}} \mathbb{P}(|X| \geq c) < \epsilon.$$

5.4. **Lemma.** *For a subset $\mathscr{B} \subset V$ the following statements are equivalent:*

(i) *$\mathscr{B}$ is topologically bounded.*
(ii) *$\mathscr{B}$ is bounded in probability.*

Proof. See, for example Exercise A.8.18 on page 451 in [2]. □

5.5. **Proposition.** *$(L^0, \leq)$ admits nontrivial minimal elements for unbounded, convex and semi-solid subsets of L^0_+.*

Proof. We follow the proof of [19, Prop. 1.2] rather closely. Let $\mathscr{B} \subset L^0_+$ be an unbounded, convex and semi-solid subset. By [3, Lemma 2.3] there exists an event $\Omega_u \in \mathscr{G}$ with $\mathbb{P}(\Omega_u) > 0$ such that for each $\epsilon > 0$ there exists $X \in \mathscr{B}$ with

$$\mathbb{P}(\Omega_u \cap \{X < \epsilon^{-1}\}) < \epsilon.$$

We define the sequence $(\alpha_n)_{n \in \mathbb{N}} \subset (0, \infty)$ as

$$\alpha_n := \frac{\mathbb{P}(\Omega_u)}{2^{n+1}} \quad \text{for each } n \in \mathbb{N}.$$

Then we have $\alpha_n \downarrow 0$, and there is a sequence $(X_n)_{n \in \mathbb{N}} \subset \mathscr{B}$ such that

$$\mathbb{P}(\Omega_u \cap \{X_n < \alpha_n^{-1}\}) < \alpha_n \quad \text{for each } n \in \mathbb{N}.$$

We define the sequence $(A_n)_{n\in\mathbb{N}} \subset \mathscr{G}$ as $A_n := \Omega_u \cap \{X_n \geq \alpha_n^{-1}\}$. We set $A := \bigcap_{n\in\mathbb{N}} A_n \in \mathscr{G}$ and define $X \in L^0_+$ as $X := \mathbb{1}_A$. Then we have

$$0 \leq X = \mathbb{1}_A \leq \mathbb{1}_{A_n} \leq \alpha_n X_n \quad \text{for each } n \in \mathbb{N}.$$

In particular, since $\mathscr{B}$ is semi-solid, we have $X \in \mathscr{B}$. Furthermore, we have

$$\begin{aligned}\mathbb{P}(\Omega_u \setminus A) &= \mathbb{P}\bigg(\bigcup_{n\in\mathbb{N}} (\Omega_u \setminus A_n)\bigg) \leq \sum_{n\in\mathbb{N}} \mathbb{P}(\Omega_u \setminus A_n) = \sum_{n\in\mathbb{N}} \mathbb{P}(\Omega_u \cap \{X_n < \alpha_n^{-1}\}) \\ &< \sum_{n\in\mathbb{N}} \alpha_n = \sum_{n\in\mathbb{N}} \frac{\mathbb{P}(\Omega_u)}{2^{n+1}} = \frac{\mathbb{P}(\Omega_u)}{2},\end{aligned}$$

and hence $\mathbb{P}(A) > 0$, showing that $X \in \mathscr{B} \setminus \{0\}$. □

As in Section 3, let $\mathscr{K}_0 \subset L^0$ be a subset such that Assumption 3.1 is fulfilled; that is $\mathscr{K}_0$ is a convex cone. As already mentioned, we may think of outcomes of trading strategies with initial value zero. Then we can define the concept NA as in Section 3. In order to introduce further concepts in the present setting, we fix some $p \in [1, \infty]$. Note that the space L^p is an ideal which is dense in L^0.

5.6. **Definition.** *We introduce the following concepts:*

(1) $\mathscr{K}_0$ *satisfies* NFL_p (No Free Lunch with respect to L^p) *if it satisfies* NFL_{τ_1}*, where* τ_1 *is the weak-** *topology on* L^p *with respect to* L^q *and* $q \in [1, \infty]$ *is such that* $\frac{1}{p} + \frac{1}{q} = 1$.
(2) $\mathscr{K}_0$ *satisfies* NFLBR_p (No Free Lunch with Bounded Risk with respect to L^p) *if it satisfies* NFL_{τ_2}*, where* τ_2 *is the sequential weak-** *topology on* L^p *with respect to* L^q *and* $q \in [1, \infty]$ *is such that* $\frac{1}{p} + \frac{1}{q} = 1$.
(3) $\mathscr{K}_0$ *satisfies* NFLVR_p (No Free Lunch with Vanishing Risk with respect to L^p) *if it satisfies* NFL_{τ_3}*, where* τ_3 *is the norm topology on* L^p.

In case $p = \infty$ we agree to write NFLVR, NFLBR and NFL rather than NFLVR_∞, NFLBR_∞ and NFL_∞. These are the well-known no-arbitrage concepts which are widely used in the literature; see for example [8] or [14].

5.7. **Proposition.** *We have the implications (i) ⇒ (ii) ⇒ (iii) ⇒ (iv), where:*

(i) $\mathscr{K}_0$ *satisfies* NFL_p.
(ii) $\mathscr{K}_0$ *satisfies* $NFLBR_p$.
(iii) $\mathscr{K}_0$ *satisfies* $NFLVR_p$.
(iv) $\mathscr{K}_0$ *satisfies NA.*

Proof. Since $\tau_1 \subset \tau_2 \subset \tau_3$, this is an immediate consequence of Propositions 3.6 and 3.7. □

Recall that the convex cone $\mathscr{C} \subset L^p$ is given by

$$\mathscr{C} = (\mathscr{K}_0 - L^0_+) \cap L^p.$$

5.8. **Corollary.** *Suppose that* $\mathscr{C}$ *is closed in* L^p *with respect to* $\| \cdot \|_{L^p}$*. Then the following statements are equivalent:*

(i) $\mathscr{K}_0$ *satisfies* $NFLVR_p$.
(ii) $\mathscr{K}_0$ *satisfies NA.*

Proof. This is an immediate consequence of Corollary 3.8. □

5.9. **Corollary.** *Suppose that* $\mathscr{K}_0 - L^0_+$ *is closed in* L^0*. Then the following statements are equivalent:*

(i) $\mathscr{K}_0$ *satisfies* $NFLVR_p$.
(ii) $\mathscr{K}_0$ *satisfies NA.*

Proof. Since $\|X_n - X\|_{L^p} \to 0$ implies $X_n \xrightarrow{\mathbb{P}} X$, this is a consequence of Corollary 3.9. □

Now, let $(\mathscr{K}_\alpha)_{\alpha>0}$ be a family of subsets of L^0_+ such that Assumption 3.10 is fulfilled. As already mentioned, we may think of the outcomes of trading strategies with initial value α. Recall that we had defined the family $(\mathscr{B}_\alpha)_{\alpha>0}$ of convex, semi-solid subsets of L^0_+ as

$$\mathscr{B}_\alpha := (\mathscr{K}_\alpha - L^0_+) \cap L^0_+, \quad \alpha > 0,$$

and that we have set $\mathscr{B} := \mathscr{B}_1$. In Definition 3.12 we had defined the concepts NUPBR, NAA_1 and NA_1. Lemma 5.4 shows that NUPBR corresponds to the well-known respective concept that is usually used in the finance literature. By Remark 3.15 the concept NA_1 corresponds to the respective concept that is usually used in the finance literature. The following result shows that also NAA_1 corresponds to the well-known respective concept that is usually used in the finance literature; cf. for example [15].

5.10. **Lemma.** *The following statements are equivalent:*

(i) *$\mathscr{K}_1$ satisfies NAA_1.*

(ii) *For each sequence $(\alpha_n)_{n\in\mathbb{N}} \subset (0,\infty)$ with $\alpha_n \downarrow 0$ and every sequence $(X_n)_{n\in\mathbb{N}} \subset L^0_+$ with $X_n \in \mathscr{B}_{\alpha_n}$ for each $n \in \mathbb{N}$ we have*

$$X_n \xrightarrow{\mathbb{P}} 0.$$

(iii) *For each sequence $(\alpha_n)_{n\in\mathbb{N}} \subset (0,\infty)$ with $\alpha_n \downarrow 0$ and every sequence $(X_n)_{n\in\mathbb{N}} \subset L^0_+$ with $X_n \in \mathscr{B}_{\alpha_n}$ for each $n \in \mathbb{N}$ we have*

$$\lim_{n\to\infty} \mathbb{P}(X_n \geq 1) = 0.$$

Proof. (i) ⇔ (ii): See Remark 3.14.
(ii) ⇒ (iii): This implication is obvious.
(iii) ⇒ (ii): Let $\epsilon > 0$ be arbitrary. We set $Y_n := X_n/\epsilon$ and $\beta_n := \alpha_n/\epsilon$ for each $n \in \mathbb{N}$. Then we have $\beta_n \downarrow 0$ and $Y_n \in \mathscr{B}_{\beta_n}$ for each $n \in \mathbb{N}$ as well as

$$\mathbb{P}(X_n \geq \epsilon) = \mathbb{P}(Y_n \geq 1) \to 0.$$

Since $\epsilon > 0$ was arbitrary, this shows $X_n \xrightarrow{\mathbb{P}} 0$. □

For the proof of the upcoming Theorem 5.12 we require the following auxiliary result.

5.11. **Lemma.** *Let $(X_n)_{n\in\mathbb{N}} \subset L^0_+$ be a sequence such that for some $\epsilon > 0$ we have*

$$\mathbb{P}(X_n \geq n) \geq \epsilon \quad \textit{for each } n \in \mathbb{N}.$$

Then for each subsequence $(X_{n_k})_{k\in\mathbb{N}}$ there exists a sequence $(a_k)_{k\in\mathbb{N}} \subset (0,\infty)$ with $a_k \to \infty$ such that

$$\mathbb{P}\bigg(\frac{1}{k}\sum_{l=1}^k X_{n_l} \geq a_k\bigg) \geq \frac{\epsilon}{2} \quad \textit{for each } k \in \mathbb{N}. \tag{5.2}$$

Proof. Let $k \in \mathbb{N}$ be arbitrary. Then we have

$$\mathbb{P}\bigg(\frac{X_{n_l}}{k} \geq \frac{n_l}{k}\bigg) \geq \epsilon \quad \text{for all } l = 1,\dots,k.$$

Therefore, by [8, Lemma 9.8.6] for each $\delta \in (0,1)$ we have

$$\mathbb{P}\bigg(\frac{1}{k}\sum_{l=1}^k X_{n_l} \geq \frac{\delta\epsilon}{k}\sum_{l=1}^k n_l\bigg) \geq (1-\delta)\epsilon.$$

Now, we set $\delta := \frac{1}{2}$ and define the sequence $(a_k)_{k \in \mathbb{N}} \subset (0,\infty)$ as

$$a_k := \frac{\delta \epsilon}{k} \sum_{l=1}^{k} n_l \quad \text{for each } k \in \mathbb{N}.$$

Then we have (5.2) and

$$a_k \geq \frac{\delta \epsilon}{k} \sum_{l=1}^{k} l = \frac{\delta \epsilon}{k} \frac{k(k+1)}{2} = \frac{\delta \epsilon (k+1)}{2} \to \infty \quad \text{for } k \to \infty,$$

completing the proof. □

We say that the subset $\mathscr{B}$ is $L^1(\mathbb{Q})$-*bounded* for some equivalent probability measure $\mathbb{Q} \approx \mathbb{P}$ on $(\Omega, \mathscr{G})$ if

$$\sup_{X \in \mathscr{B}} \mathbb{E}_{\mathbb{Q}}[X] < \infty.$$

Furthermore, we say that the convex subset $\mathscr{B}$ has the *Banach Saks property* with respect to almost sure convergence (convergence in probability) if every sequence $(X_n)_{n \in \mathbb{N}} \subset \mathscr{B}$ has a subsequence $(X_{n_k})_{k \in \mathbb{N}}$ which is almost surely Cesàro convergent (Cesàro convergent in probability) to a finite nonnegative random variable $X \in L^0_+$. Similarly, we say that the convex subset $\mathscr{B}$ has the *von Weizsäcker property* with respect to almost sure convergence (convergence in probability) if for every sequence $(X_n)_{n \in \mathbb{N}} \subset \mathscr{B}$ there exist a subsequence $(X_{n_k})_{k \in \mathbb{N}}$ and a finite nonnegative random variable $X \in L^0_+$ such that for every further subsequence $(n_{k_l})_{l \in \mathbb{N}}$ and every permutation $\pi : \mathbb{N} \to \mathbb{N}$ the sequence $(X_{n_{k_{\pi(l)}}})_{l \in \mathbb{N}}$ is almost surely Cesàro convergent (Cesàro convergent in probability) to X.

5.12. **Theorem.** *The following statements are equivalent:*

- (i) $\mathscr{K}_1$ *satisfies NUPBR.*
- (ii) $\mathscr{K}_1$ *satisfies* NAA_1.
- (iii) $\mathscr{K}_1$ *satisfies* NA_1.
- (iv) *We have* $\bigcap_{\alpha > 0} \mathscr{B}_\alpha = \{0\}$.
- (v) *There exists an equivalent probability measure* $\mathbb{Q} \approx \mathbb{P}$ *such that* $\mathscr{B}$ *is* $L^1(\mathbb{Q})$-*bounded.*
- (vi) *There exists an equivalent probability measure* $\mathbb{Q} \approx \mathbb{P}$ *with bounded Radon-Nikodym derivative* $\frac{d\mathbb{Q}}{d\mathbb{P}}$ *such that* $\mathscr{B}$ *is* $L^1(\mathbb{Q})$-*bounded.*
- (vii) $\mathscr{B}$ *has the von Weizsäcker property with respect to almost sure convergence.*
- (viii) $\mathscr{B}$ *has the von Weizsäcker property with respect to convergence in probability.*
- (ix) $\mathscr{B}$ *has the Banach Saks property with respect to almost sure convergence.*
- (x) $\mathscr{B}$ *has the Banach Saks property with respect to convergence in probability.*
- (xi) *For every sequence* $(X_n)_{n \in \mathbb{N}} \subset \mathscr{B}$ *there exist a subsequence* $(X_{n_k})_{k \in \mathbb{N}}$ *and a probability measure* μ *on* $(\mathbb{R}_+, \mathscr{B}(\mathbb{R}_+))$ *such that* $\mathbb{P} \circ X_{n_k} \stackrel{w}{\to} \mu$ *for* $k \to \infty$.

Proof. (i) ⇔ (ii) ⇔ (iii) ⇔ (iv): These equivalences are a consequence of Theorem 3.17 and Proposition 5.5.
(i) ⇒ (vi): Since $\mathscr{B}$ is convex, this implication follows from [3, Lemma 2.3(3)].
(vi) ⇒ (v): This implication is obvious.
(v) ⇒ (i): Since $\mathscr{B}$ is convex, this implication follows from [18, Prop. 1.16].
(i) ⇒ (vii): Let $(X_n)_{n \in \mathbb{N}} \subset \mathscr{B}$ be an arbitrary sequence. By the von Weizsäcker theorem there exist a subsequence $(X_{n_k})_{k \in \mathbb{N}}$ and a nonnegative random variable $X : \Omega \to [0,\infty]$ such that for every further subsequence $(n_{k_l})_{l \in \mathbb{N}}$ and every permutation $\pi : \mathbb{N} \to \mathbb{N}$ the sequence $(X_{n_{k_{\pi(l)}}})_{l \in \mathbb{N}}$ is almost surely Cesàro convergent (Cesàro convergent in probability) to X; see [34] and [16, Thm. 5.2.3]. Since $\mathscr{B}$ is bounded

in probability, we have $X < \infty$ almost surely, that is $X \in L^0_+$; see, for example [33, Cor. 2.12].
The implications (vii) $\Rightarrow$ (viii) $\Rightarrow$ (x) and (vii) $\Rightarrow$ (ix) $\Rightarrow$ (x) are obvious.
(x) $\Rightarrow$ (i): Suppose that $\mathscr{B}$ is not bounded in probability. Then there are $\epsilon > 0$ and a sequence $(X_n)_{n\in\mathbb{N}} \subset \mathscr{B}$ such that

$$\mathbb{P}(X_n \geq n) \geq \epsilon \quad \text{for each } n \in \mathbb{N}.$$

By assumption there exist a subsequence $(X_{n_k})_{k\in\mathbb{N}}$ and a nonnegative random variable $X \in L^0_+$ such that $X_{n_k} \xrightarrow{\mathbb{P}} X$, where

$$X_{n_k} := \frac{1}{k}\sum_{l=1}^{k} X_{n_l} \quad \text{for each } k \in \mathbb{N}.$$

By Lemma 5.11 there exists a sequence $(a_k)_{k\in\mathbb{N}} \subset (0,\infty)$ with $a_k \to \infty$ such that

$$\mathbb{P}(\bar{X}_{n_k} \geq a_k) \geq \frac{\epsilon}{2} \quad \text{for each } k \in \mathbb{N}.$$

Since $\bar{X}_{n_k} \xrightarrow{\mathbb{P}} X$, there exists an index $k_0 \in \mathbb{N}$ such that

$$\mathbb{P}(|\bar{X}_{n_k} - X|) \leq \frac{\epsilon}{4} \quad \text{for each } k \geq k_0.$$

Note that for each $k \in \mathbb{N}$ we have

$$\begin{aligned}\{\bar{X}_{n_k} \geq a_k\} &\subset \left\{X \geq \frac{a_k}{2}\right\} \cup \left\{\bar{X}_{n_k} - X \geq \frac{a_k}{2}\right\}\\ &\subset \left\{X \geq \frac{a_k}{2}\right\} \cup \left\{|\bar{X}_{n_k} - X| \geq \frac{a_k}{2}\right\}.\end{aligned}$$

Therefore, for all $k \geq k_0$ we have

$$\mathbb{P}\bigg(X \geq \frac{a_k}{2}\bigg) \geq \mathbb{P}(\bar{X}_{n_k} \geq a_k) - \mathbb{P}\bigg(|\bar{X}_{n_k} - X| \geq \frac{a_k}{2}\bigg) \geq \frac{\epsilon}{2} - \frac{\epsilon}{4} = \frac{\epsilon}{4}.$$

Since $a_k \to \infty$, we obtain $\mathbb{P}(X = \infty) > 0$, which contradicts $X \in L^0_+$.
(i) $\Leftrightarrow$ (xi): This equivalence is a consequence of Prohorov's theorem. □

5.13. **Remark.** *If the convex subset $\mathscr{B}$ is closed, then it is bounded in probability (which means that $\mathscr{K}_1$ satisfies NUPBR) if and only if it is convexly compact; see* [35] *for further details.*

Now, we consider $\mathscr{K}_0$ and $(\mathscr{K}_\alpha)_{\alpha>0}$ together.

5.14. **Proposition.** *Suppose that $\mathscr{B}_0 \subset \mathscr{B}_\alpha$ for each $\alpha > 0$. If $\mathscr{K}_1$ satisfies NA$_1$, then $\mathscr{K}_0$ satisfies NA.*

Proof. This is an immediate consequence of Proposition 3.19. □

Recall that the convex cone $\mathscr{C} \subset L^p$ is given by

$$\mathscr{C} = (\mathscr{K}_0 - L^0_+) \cap L^p$$

for some $p \in [1,\infty]$.

5.15. **Proposition.** *Suppose that*

$$\bigg(\bigcap_{\alpha>0} \mathscr{B}_\alpha\bigg) \cap L^p \subset \overline{\mathscr{C}}^{\|\cdot\|_{L^p}}.$$

If $\mathscr{K}_0$ satisfies NFLVR$_p$, then $\mathscr{K}_1$ satisfies NA$_1$.

Proof. This is an immediate consequence of Proposition 3.20. □

6. Abstract fundamental theorem of asset pricing in L^p-spaces

In this section we present a version of the abstract fundamental theorem of asset pricing in L^p-spaces. Let $(\Omega, \mathscr{G}, \mathbb{P})$ be a probability space.

6.1. **Proposition.** *Let Φ be an arbitrary set of random variables. Then there exists a numeric random variable $X^* : \Omega \to \mathbb{R} \cup \{\infty\}$ with the following properties:*

1. *For each $X \in \Phi$ we have $\mathbb{P}(X \leq X^*) = 1$.*
2. *For every numeric random variable $Y : \Omega \to \mathbb{R} \cup \{\infty\}$ such that $\mathbb{P}(X \leq Y) = 1$ for all $X \in \Phi$ we have $\mathbb{P}(X^* \leq Y) = 1$.*

Furthermore, the random variable X^* is $\mathbb{P}$-almost surely unique, and there exists a countable subset $\Psi^* \subset \Phi$ such that

$$\mathbb{P}\bigg(X^* = \sup_{X \in \Psi^*} X \bigg) = 1.$$

Proof. This follows from [10, Thm. A.37] and its proof. □

The random variable X^* from Proposition 6.1 is called the *essential supremum* of Φ with respect to $\mathbb{P}$, and we write

$$\operatorname*{ess\,sup}_{X \in \Phi} \Phi := X^*.$$

6.2. **Lemma.** *Let $p, q \in [1, \infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $\mathscr{X} \subset L_+^p \setminus \{0\}$ and $\mathscr{Y} \subset L_+^q \setminus \{0\}$ be subsets such that the following conditions are fulfilled:*

1. *We have $\{\mathbb{1}_A : A \in \mathscr{F} \text{ with } \mathbb{P}(A) > 0\} \subset \mathscr{X}$.*
2. *For each $X \in \mathscr{X}$ there exists $Y \in \mathscr{Y}$ such that $\mathbb{E}[XY] > 0$.*

Then there exists a countable subset $\mathscr{Z} \subset \mathscr{Y}$ such that for each $X \in \mathscr{X}$ there exists $Z \in \mathscr{Z}$ such that $\mathbb{E}[XZ] > 0$.

Proof. We define the family of random variables

$$\Phi := \{\mathbb{1}_{\{Y > 0\}} : Y \in \mathscr{Y}\},$$

and the essential supremum

$$Z^* := \operatorname*{ess\,sup}_{Z \in \Phi} Z.$$

By Proposition 6.1 we have $Z^* : \Omega \to \{0, 1\}$. We claim that $\mathbb{P}(Z^* = 1) = 1$. Indeed, set $A := \{Z^* = 0\}$ and suppose that $\mathbb{P}(A) > 0$. Then for each $Y \in \mathscr{Y}$ we have $Y = 0$ almost surely on A, which implies the contradiction $\mathbb{E}[Y \mathbb{1}_A] = 0$ for all $Y \in \mathscr{Y}$. Furthermore, by Proposition 6.1 there exists a sequence $(Y_n)_{n \in \mathbb{N}} \subset \mathscr{Y}$ such that

$$\mathbb{P}\bigg(Z^* = \sup_{n \in \mathbb{N}} \mathbb{1}_{A_n} \bigg) = 1,$$

where $A_n := \{Y_n > 0\}$ for each $n \in \mathbb{N}$. Since $\mathbb{P}(Z^* = 1) = 1$, we have $\mathbb{P}(\bigcup_{n \in \mathbb{N}} A_n) = 1$. Let $X \in \mathscr{X}$ be arbitrary, and set $B := \{X > 0\}$. Then we have $\mathbb{P}(B) > 0$, and hence there is an index $n \in \mathbb{N}$ such that $\mathbb{P}(B \cap A_n) > 0$. Therefore, we have $\mathbb{E}[XY_n] > 0$, completing the proof. □

For each $q \in [1, \infty]$ the set L_{++}^q consists of all $X \in L_+^q$ such that $\mathbb{P}(X > 0) = 1$.

6.3. **Lemma.** *Let $x' \in (L^p)'$ for some $p \in [1, \infty)$ be arbitrary, and let $q \in (1, \infty]$ be the unique number such that $\frac{1}{p} + \frac{1}{q} = 1$. Then the following statements are true:*

1. *There exists a unique random variable $Y \in L^q$ such that*
$$x'(X) = \mathbb{E}[XY] \quad \text{for all } X \in L^p.$$
2. *If $x' \in (L^p)'_+$, then we have $Y \in L_+^q$.*
3. *If $x' \neq 0$, then we have $Y \neq 0$.*

(4) *If* $x' \in (L^p)'_{++}$*, then we have* $Y \in L^q_{++}$.

Proof. The first statement follows from the Riesz representation theorem. Now assume that $x' \in (L^p)'_+$. Then we have

$$\mathbb{E}[Y \mathbb{1}_A] \geq 0 \quad \text{for all } A \in \mathscr{F},$$

and hence $Y \in L^q_+$. The third statement is obvious because $Y = 0$ implies $x' = 0$. Next, assume that $x' \in (L^p)'_{++}$. Then we have

$$\mathbb{E}[Y \mathbb{1}_A] > 0 \quad \text{for all } A \in \mathscr{F} \text{ with } \mathbb{P}(A) > 0.$$

Suppose that $Y \notin L^q_{++}$. Then there exists $A \in \mathscr{F}$ such that $Y = 0$ almost surely on A, and we obtain the contradiction $\mathbb{E}[Y \mathbb{1}_A] = 0$. Therefore, we have $Y \in L^q_{++}$, completing the proof. □

6.4. **Proposition.** *For each* $p \in [1, \infty)$ *the Banach space* L^p *has the Halmos-Savage property.*

Proof. Let $\mathscr{X}' \subset (L^p)'_+ \setminus \{0\}$ be strictly positive separating for $L^p_+ \setminus \{0\}$. By Lemma 6.3 there exists $Y \in L^q_+ \setminus \{0\}$ such that

$$x'(X) = \mathbb{E}[XY] \quad \text{for all } X \in L^p.$$

We denote by $\mathscr{Y} \subset L^q_+ \setminus \{0\}$ the collection of all these random variables. By Lemma 6.2 there exists a countable subset $\mathscr{Z} \subset \mathscr{Y}$ such that for each $X \in \mathscr{X}$ there exists $Z \in \mathscr{Z}$ with $\mathbb{E}[XZ] > 0$. Consequently, there exists a countable subset $\mathscr{Z}' \subset \mathscr{Y}'$ which is strictly positive separating for $L^p_+ \setminus \{0\}$. □

6.5. **Definition.** *Let* $\mathscr{C} \subset L^1(\mathbb{P})$ *be a subset. An equivalent probability measure* $\mathbb{Q} \approx \mathbb{P}$ *on* $(\Omega, \mathscr{G})$ *is called a* separating measure *for* $\mathscr{C}$ *if we have* $\mathscr{C} \subset L^1(\mathbb{Q})$ *and*

$$\mathbb{E}_{\mathbb{Q}}[X] \leq 0 \quad \textit{for all } X \in \mathscr{C}.$$

6.6. **Remark.** *If* $\mathscr{C} \subset L^1(\mathbb{P})$ *and* $\mathbb{Q} \approx \mathbb{P}$ *is an equivalent probability measure with bounded Radon-Nikodym derivative* $\frac{d\mathbb{Q}}{d\mathbb{P}}$*, then we automatically have* $\mathscr{C} \subset L^1(\mathbb{Q})$.

Now, let $\mathscr{K}_0 \subset L^0$ be a subset such that Assumption 3.1 is fulfilled; that is $\mathscr{K}_0$ is a convex cone. As already mentioned, we may think of outcomes of trading strategies with initial value zero. We recall that the convex cone $\mathscr{C} \subset L^p$ is given by

$$\mathscr{C} = (\mathscr{K}_0 - L^0_+) \cap L^p.$$

6.7. **Theorem** (Abstract Fundamental Theorem of Asset Pricing, Version in L^p-spaces)**.** *For each* $p \in [1, \infty)$ *the following statements are equivalent:*

(i) $\mathscr{K}_0$ *satisfies NFLVR*$_p$.
(ii) *There exists a separating measure* $\mathbb{Q} \approx \mathbb{P}$ *for* $\mathscr{C}$ *with Radon-Nikodym derivative* $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^q$*, where* $q \in (1, \infty]$ *is such that* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. (i) ⇒ (ii): By Proposition 6.4 and Proposition 4.9 the Banach space L^p has the Kreps-Yan property. Therefore, by Theorem 4.10 there exists a strictly separating functional $x' \in (L^p)'_{++}$. By Lemma 6.3 there exists $Y \in L^q_{++}$ such that

$$x'(X) = \mathbb{E}[XY] \quad \text{for all } X \in L^p.$$

Without loss of generality, we may assume that $\mathbb{E}[Y] = 1$. Let $\mathbb{Q} \approx \mathbb{P}$ be the equivalent probability measure on $(\Omega, \mathscr{G})$ with Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}} = Y$. Since x' is a separating functional, the probability measure $\mathbb{Q}$ is a separating measure for $\mathscr{C}$.

(ii) $\Rightarrow$ (i): We set $Y := \frac{d\mathbb{Q}}{d\mathbb{P}} \in L^q_{++}$, and define the continuous linear functional $x' \in (L^p)'$ as

$$x'(X) := \mathbb{E}[XY] = \mathbb{E}_{\mathbb{Q}}[X] \quad \text{for all } X \in L^p.$$

Then x' is a strictly separating functional for $\mathscr{C}$, and the implication follows from Theorem 4.10. □

6.8. **Corollary.** *Suppose that $\mathscr{C}$ is closed in L^p with respect to $\| \cdot \|_{L^p}$ for some $p \in [1, \infty)$. Then the following statements are equivalent:*

(i) *$\mathscr{K}_0$ satisfies NA.*
(ii) *There exists a separating measure $\mathbb{Q} \approx \mathbb{P}$ for $\mathscr{C}$.*
(iii) *There exists a separating measure $\mathbb{Q} \approx \mathbb{P}$ for $\mathscr{C}$ with Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^q$, where $q \in (1, \infty]$ is such that $\frac{1}{p} + \frac{1}{q} = 1$.*

Proof. (i) $\Rightarrow$ (iii): By Corollary 5.8 the convex cone $\mathscr{K}_0$ satisfies NFLVR$_p$. Hence this implication is a consequence of Theorem 6.7.
(iii) $\Rightarrow$ (ii): This implication is obvious.
(ii) $\Rightarrow$ (i): Let $X \in \mathscr{C} \cap L^p_+$ be arbitrary. Since $\mathbb{Q}$ is a separating measure for $\mathscr{C}$, we have $\mathbb{E}_{\mathbb{Q}}[X] \leq 0$, and hence $X = 0$. Consequently, we have $\mathscr{C} \cap L^p_+ = \{0\}$, and by Lemma 3.4 it follows that $\mathscr{K}_0$ satisfies NA. □

6.9. **Corollary.** *Suppose that $\mathscr{K}_0 - L^0_+$ is closed in L^0. Then the following statements are equivalent:*

(i) *$\mathscr{K}_0$ satisfies NA.*
(ii) *There exists a separating measure $\mathbb{Q} \approx \mathbb{P}$ for $\mathscr{C}$.*
(iii) *There exists a separating measure $\mathbb{Q} \approx \mathbb{P}$ for $\mathscr{C}$ with bounded Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^\infty$.*

Proof. Since $\|X_n - X\|_{L^1} \to 0$ implies $X_n \xrightarrow{\mathbb{P}} X$, this is a consequence of Corollary 6.8 with $p = 1$ and $q = \infty$. □

Now, we consider the discrete time setting. Let $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space with discrete filtration $\mathbb{F} = (\mathscr{F}_t)_{t=0,\ldots,T}$ for some $T \in \mathbb{N}$. Let $\mathbb{X} = \{X^1, \ldots, X^d\}$ be a discounted market consisting of $d \in \mathbb{N}$ assets $X^i = (X^i_t)_{t=0,\ldots,T}$ for $i = 1, \ldots, d$. We assume that $X^i \geq 0$ for each $i = 1, \ldots, d$. Consider the convex cone

$$\mathscr{K}_0 := \left\{ \sum_{t=1}^T \xi_t \cdot (X_t - X_{t-1}) : \xi \text{ is a strategy} \right\},$$

where every predictable process ξ (that is ξ_t is $\mathscr{F}_{t-1}$-measurable for each $t = 1, \ldots, T$) is called a strategy. As usual, we say that an equivalent probability measure $\mathbb{Q} \approx \mathbb{P}$ is an *equivalent martingale measure (EMM)* for $\mathbb{X}$ if $X^1, \ldots, X^d$ are $\mathbb{Q}$-martingales. The following result extends the well-known no-arbitrage result in discrete time (see, for example [10] or [17]) by additionally providing a characterization in terms of separating measures. Now the convex cone $\mathscr{C} \subset L^1$ is given by

$$\mathscr{C} = (\mathscr{K}_0 - L^0_+) \cap L^1.$$

6.10. **Theorem.** *The following statements are equivalent:*

(i) *$\mathscr{K}_0$ satisfies NA.*
(ii) *There exists an EMM $\mathbb{Q} \approx \mathbb{P}$ for X.*
(iii) *There exists an EMM $\mathbb{Q} \approx \mathbb{P}$ for X with bounded Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^\infty$.*
(iv) *There exists a separating measure $\mathbb{Q} \approx \mathbb{P}$ for $\mathscr{C}$.*

(v) *There exists a separating measure $\mathbb{Q} \approx \mathbb{P}$ for $\mathscr{C}$ with bounded Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^\infty$.*

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii): See [10, Thm. 5.16] or [17, Thm. 1].
(i) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v): This follows by combining Corollary 6.9 and [17, Thm. 1]. □

7. Financial market with semimartingales

In this section we consider a financial market with nonnegative semimartingales which does not need to have a numéraire. We will derive consequences for the no-arbitrage concepts considered so far; in particular regarding self-financing portfolios.

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$ be a stochastic basis satisfying the usual conditions, see [12, Def. I.1.3]. Furthermore, we assume that $\mathscr{F}_0 = \{\Omega, \emptyset\}$. Then every $\mathscr{F}_0$-measurable random variable is $\mathbb{P}$-almost surely constant. Let $\mathbb{L}$ be the space of all equivalence classes of adapted, càdlàg processes $X : \Omega \times \mathbb{R}_+ \to \mathbb{R}$, where two processes X and Y are identified if X and Y are indistinguishable, that is if almost all paths of X and Y coincide; see [12, I.1.10]. Let $(\mathbb{K}_\alpha)_{\alpha \geq 0}$ be a family of subsets of $\mathbb{L}$ such that for each $\alpha \geq 0$ and each $X \in \mathbb{K}_\alpha$ we have $X_0 = \alpha$. Throughout this section, we make the following assumptions.

7.1. **Assumption.** *We assume that $\mathbb{K}_0$ is a convex cone.*

7.2. **Assumption.** *We assume that*

$$aX + bY \in \mathbb{K}_{a\alpha + b\beta}. \tag{7.1}$$

for all $a, b \in \mathbb{R}_+$, $\alpha, \beta > 0$ with $a\alpha + b\beta > 0$ and $X \in \mathbb{K}_\alpha$, $Y \in \mathbb{K}_\beta$.

The following remark provides a sufficient condition ensuring that Assumptions 7.1 and 7.2 are fulfilled.

7.3. **Remark.** *Suppose that*

$$aX + bY \in \mathbb{K}_{a\alpha + b\beta} \tag{7.2}$$

for all $a, b \in \mathbb{R}_+$, $\alpha, \beta \in \mathbb{R}_+$ and $X \in \mathbb{K}_\alpha$, $Y \in \mathbb{K}_\beta$. Then $\mathbb{K}_0$ is a convex cone, and we have (7.2) for all $a, b \in \mathbb{R}_+$, $\alpha, \beta > 0$ with $a\alpha + b\beta > 0$ and $X \in \mathscr{K}_\alpha$, $Y \in \mathscr{K}_\beta$.

The following example shows that the framework considered in [15, Appendix A] is contained in our present setting.

7.4. **Example.** *Let $\mathbb{X} \subset \mathbb{L}$ be a convex set of processes such that $X_0 = 0$ and $X \geq -1$ for each $X \in \mathbb{X}$. We define the family $(\mathbb{K}_\alpha)_{\alpha \geq 0}$ as*

$$\mathbb{K}_0 := \mathbb{R}_+ \mathbb{X},$$
$$\mathbb{K}_\alpha := \alpha(1 + \mathbb{X}), \quad \alpha > 0.$$

Then Assumptions 7.1 and 7.2 are fulfilled. Indeed, the set $\mathbb{K}_0$ is a convex cone. Let $a, b \in \mathbb{R}_+$, $\alpha, \beta > 0$ with $a\alpha + b\beta > 0$ and $X \in \mathbb{K}_\alpha$, $Y \in \mathbb{K}_\beta$ be arbitrary. Then there are $Z, W \in \mathbb{X}$ such that $X = \alpha(1 + Z)$ and $Y = \beta(1 + W)$. Since $\mathbb{X}$ is convex, we obtain

$$\begin{aligned} aX + bY &= a\alpha(1 + Z) + b\beta(1 + W) \\ &= a\alpha + b\beta + (a\alpha + b\beta)\left(\frac{a\alpha}{a\alpha + b\beta} Z + \frac{b\beta}{a\alpha + b\beta} W\right) \in \mathbb{K}_{a\alpha + b\beta}, \end{aligned}$$

showing that (7.1) is fulfilled.

As we will see, in all examples which we consider in this section later on, relation (7.2) from Remark 7.3 will be satisfied. Now, let $T \in (0, \infty)$ be a fixed terminal time. We define the family $(\mathscr{K}_\alpha)_{\alpha \geq 0}$ of subsets of $L^0 = L^0(\Omega, \mathscr{F}_T, \mathbb{P})$ as

$$\mathscr{K}_\alpha := \{X_T : X \in \mathbb{K}_\alpha\}. \tag{7.3}$$

Then we are in the framework of Section 5. The next result shows that Assumptions 3.1 and 3.10 are fulfilled.

7.5. **Lemma.** *The following statements are true:*

(1) $\mathscr{K}_0$ *is a convex cone.*
(2) *We have*

$$a\xi + b\eta \in \mathscr{K}_{a\alpha+b\beta}$$

for all $a, b \in \mathbb{R}_+$, $\alpha, \beta > 0$ *with* $a\alpha + b\beta > 0$ *and* $\xi \in \mathscr{K}_\alpha$, $\eta \in \mathscr{K}_\beta$.

Proof. Note that $\varphi : \mathbb{L} \to L^0$ given by $\varphi(X) := X_T$ is a linear operator such that $\varphi(\mathbb{K}_\alpha) = \mathscr{K}_\alpha$ for each $\alpha \geq 0$. Therefore, $\mathscr{K}_0$ is also a convex cone. Let $a, b \in \mathbb{R}_+$, $\alpha, \beta > 0$ with $a\alpha + b\beta > 0$ and $\xi \in \mathscr{K}_\alpha$, $\eta \in \mathscr{K}_\beta$ be arbitrary. Then there exist $X \in \mathbb{K}_\alpha$ and $Y \in \mathbb{K}_\beta$ such that $\xi = \varphi(X)$ and $\eta = \varphi(Y)$. Therefore, by the linearity of φ we obtain

$$a\xi + b\eta = a\varphi(X) + b\varphi(Y) = \varphi(aX + bY) \in \varphi(\mathbb{K}_{a\alpha+b\beta}) = \mathscr{K}_{a\alpha+b\beta},$$

completing the proof. □

As in Section 3, we define the family $(\mathscr{B}_\alpha)_{\alpha\geq 0}$ of convex, semi-solid subsets of L^0_+ as

$$\mathscr{B}_\alpha := (\mathscr{K}_\alpha - L^0_+) \cap L^0_+, \quad \alpha \geq 0,$$

and we set $\mathscr{B} := \mathscr{B}_1$. Furthermore, we define the convex cone $\mathscr{C} \subset L^\infty$ as

$$\mathscr{C} := (\mathscr{K}_0 - L^0_+) \cap L^\infty.$$

Now, we consider particular examples for the family of processes $(\mathbb{K}_\alpha)_{\alpha\geq 0}$. Let $I \neq \emptyset$ be an arbitrary index set, and let $(S^i)_{i\in I}$ be a family of semimartingales. We assume that $S^i \geq 0$ for each $i \in I$. We define the *market* $\mathbb{S} := \{S^i : i \in I\}$. For an $\mathbb{R}^d$-valued semimartingale X we denote by $L(X)$ the set of all X-integrable processes in the sense of vector integration; see [31] or [12, Sec. III.6]. For $\delta \in L(X)$ we denote by $\delta \cdot X$ the stochastic integral according to [31]. For a finite set $F \subset I$ we define the multi-dimensional semimartingale $S^F := (S^i)_{i\in F}$.

7.6. **Definition.** *We call a process* $\delta = (\delta^i)_{i\in I}$ *a* strategy *for* $\mathbb{S}$ *if there is a finite set* $F \subset I$ *such that* $\delta^i = 0$ *for all* $i \in I \setminus F$ *and we have* $\delta^F \in L(S^F)$.

7.7. **Definition.** *We denote by* $\Delta(\mathbb{S})$ *the set of all strategies* δ *for* $\mathbb{S}$.

7.8. **Definition.** *For a strategy* $\delta \in \Delta(\mathbb{S})$ *we set*

$$\delta \cdot S := \delta^F \cdot S^F,$$

where $F \subset I$ *denotes the finite set from Definition 7.6.*

7.9. **Theorem.** [31, Thm. 4.3] *Let* $\delta_1, \delta_2 \in \Delta(\mathbb{S})$ *and* $\alpha_1, \alpha_2 \in \mathbb{R}$ *be arbitrary. Then we have*

$$\alpha_1\delta_1 + \alpha_2\delta_2 \in \Delta(\mathbb{S})$$

and

$$(\alpha_1\delta_1 + \alpha_2\delta_2) \cdot S = \alpha_1(\delta_1 \cdot S) + \alpha_2(\delta_2 \cdot S).$$

7.10. **Definition.** *For* $\alpha \in \mathbb{R}$ *and strategy* $\delta \in \Delta(\mathbb{S})$ *we define the* integral process $I^{\alpha,\delta} := \alpha + \delta \cdot S$.

7.11. **Definition.** *For a strategy* $\delta \in \Delta(\mathbb{S})$ *we define the* portfolio $S^\delta := \delta \cdot S$, *where we use the short-hand notation*

$$\delta \cdot S := \sum_{i\in F} \delta^i S^i$$

with $F \subset I$ *denoting the finite set from Definition 7.6.*

7.12. **Definition.** *A strategy $\delta \in \Delta(\mathbb{S})$ and the corresponding portfolio S^δ are called* self-financing *for $\mathbb{S}$ if $S^\delta = S_0^\delta + \delta \cdot S$.*

7.13. **Definition.** *We denote by $\Delta_{\rm sf}(\mathbb{S})$ the set of all self-financing strategies for $\mathbb{S}$.*

The following auxiliary result is obvious.

7.14. **Lemma.** *For a strategy $\delta \in \Delta(\mathbb{S})$ the following statements are equivalent:*
 (i) *We have $\delta \in \Delta_{\rm sf}(\mathbb{S})$.*
 (ii) *We have $S^\delta = I^{\alpha,\delta}$, where $\alpha = S_0^\delta$.*

Recall that a process X is called admissible if $X \geq -a$ for some constant $a \in \mathbb{R}_+$.

7.15. **Definition.** *We introduce the following families:*
 (1) *We define the family of all* integral processes $(\mathbb{I}_\alpha(\mathbb{S}))_{\alpha \geq 0}$ *as*
$$\mathbb{I}_\alpha(\mathbb{S}) := \{I^{\alpha,\delta} : \delta \in \Delta(\mathbb{S})\}, \quad \alpha \geq 0.$$
 (2) *We define the family of all* admissible integral processes $(\mathbb{I}_\alpha^{\rm adm}(\mathbb{S}))_{\alpha \geq 0}$ *as*
$$\mathbb{I}_\alpha^{\rm adm}(\mathbb{S}) := \{X \in \mathbb{I}_\alpha(\mathbb{S}) : X \textit{ is admissible}\}, \quad \alpha \geq 0.$$
 (3) *We define the family of all* nonnegative integral processes $(\mathbb{I}_\alpha^+(\mathbb{S}))_{\alpha \geq 0}$ *as*
$$\mathbb{I}_\alpha^+(\mathbb{S}) := \{X \in \mathbb{I}_\alpha(\mathbb{S}) : X \geq 0\}, \quad \alpha \geq 0.$$
 (4) *We denote by $(\mathscr{I}_\alpha(\mathbb{S}))_{\alpha \geq 0}$, $(\mathscr{I}_\alpha^{\rm adm}(\mathbb{S}))_{\alpha \geq 0}$ and $(\mathscr{I}_\alpha^{\rm adm}(\mathbb{S}))_{\alpha \geq 0}$ the respective families of random variables defined according to (7.3).*

7.16. **Remark.** *Consider the particular case where $X^i \equiv 1$ for some $i \in I$. Then the market $\mathbb{S}$ can be interpreted as discounted price processes of risky assets with respect to some savings account, and the families $(\mathbb{I}_\alpha(\mathbb{S}))_{\alpha \geq 0}$, $(\mathbb{I}_\alpha^{\rm adm}(\mathbb{S}))_{\alpha \geq 0}$, $(\mathbb{I}_\alpha^+(\mathbb{S}))_{\alpha \geq 0}$ can be regarded as wealth processes in this case.*

7.17. **Lemma.** *Let $a, b \in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$ and $\delta, \vartheta \in \Delta(\mathbb{S})$ be arbitrary. Then we have*
$$I^{a\alpha + b\beta, a\delta + b\vartheta} = aI^{\alpha,\delta} + bI^{\beta,\vartheta}.$$

Proof. Using Theorem 7.9 we have
$$\begin{aligned} aI^{\alpha,\delta} + bI^{\beta,\vartheta} &= a(\alpha + \delta \cdot S) + b(\beta + \vartheta \cdot S) \\ &= (a\alpha + b\beta) + a(\delta \cdot S) + b(\vartheta \cdot S) \\ &= (a\alpha + b\beta) + (a\delta + b\vartheta) \cdot S \\ &= I^{a\alpha + b\beta, a\delta + b\vartheta}, \end{aligned}$$
completing the proof. □

Recall that we had defined the family $(\mathscr{B}_\alpha)_{\alpha \geq 0}$ of convex, semi-solid subsets of L_+^0 as
$$\mathscr{B}_\alpha := (\mathscr{K}_\alpha - L_+^0) \cap L_+^0, \quad \alpha \geq 0.$$

7.18. **Lemma.** *Suppose that $(\mathscr{K}_\alpha)_{\alpha \geq 0}$ is one of the families*
$$(\mathscr{I}_\alpha(\mathbb{S}))_{\alpha \geq 0}, (\mathscr{I}_\alpha^{\rm adm}(\mathbb{S}))_{\alpha \geq 0}, (\mathscr{I}_\alpha^+(\mathbb{S}))_{\alpha \geq 0}.$$
Then we have $\mathscr{B}_0 \subset \mathscr{B}_\alpha$ for each $\alpha > 0$.

Proof. Let $\alpha > 0$ and $\xi \in \mathscr{B}_0$ be arbitrary. Then we have $\xi \in L_+^0$ and there exists $\delta \in \Delta(\mathbb{S})$ such that $\delta \cdot S \in \mathbb{K}_0$ and $\xi \leq (\delta \cdot S)_T$. Therefore, we have
$$\xi \leq \alpha + (\delta \cdot S)_T.$$
Note that $\alpha + \delta \cdot S \in \mathbb{I}_\alpha(\mathbb{S})$. Furthermore, if $\delta \cdot S \in \mathbb{I}_\alpha^{\rm adm}(\mathbb{S})$, then $\alpha + \delta \cdot S \in \mathbb{I}_\alpha^{\rm adm}(\mathbb{S})$, and if $\delta \cdot S \in \mathbb{I}_\alpha^+(\mathbb{S})$, then $\alpha + \delta \cdot S \in \mathbb{I}_\alpha^+(\mathbb{S})$. We conclude that $\xi \in \mathscr{B}_\alpha$. □

Recall that we had defined the convex cone $\mathscr{C} \subset L^\infty$ as

$$\mathscr{C} = (\mathscr{K}_0 - L^0_+) \cap L^\infty.$$

7.19. **Lemma.** *Suppose that $(\mathscr{K}_\alpha)_{\alpha \geq 0}$ is one of the families*

$$(\mathscr{I}_\alpha(\mathbb{S}))_{\alpha \geq 0}, (\mathscr{I}^{\mathrm{adm}}_\alpha(\mathbb{S}))_{\alpha \geq 0}.$$

Then we have

$$\bigg(\bigcap_{\alpha > 0} \mathscr{B}_\alpha \bigg) \cap L^\infty \subset \overline{\mathscr{C}}^{\| \cdot \|_{L^\infty}}.$$

Proof. Let

$$\xi \in \bigg(\bigcap_{\alpha > 0} \mathscr{B}_\alpha \bigg) \cap L^\infty$$

be arbitrary. Furthermore, let $\alpha > 0$ be arbitrary. Then there is a strategy $\delta^\alpha \in \Delta(\mathbb{S})$ such that

$$\xi \leq \alpha + (\delta^\alpha \cdot S)_T.$$

We have $\alpha + \delta^\alpha \cdot S \in \mathbb{I}_\alpha(\mathbb{S})$ and $\delta^\alpha \cdot S \in \mathbb{I}_0(\mathbb{S})$. If $\alpha + \delta^\alpha \cdot S \in \mathbb{I}^{\mathrm{adm}}_\alpha(\mathbb{S})$, then we have $\delta^\alpha \cdot S \in \mathbb{I}^{\mathrm{adm}}_0(\mathbb{S})$. We set

$$\xi_\alpha := \xi \wedge (\delta^\alpha \cdot S)_T.$$

Since $(\delta^\alpha \cdot S)_T \in \mathscr{K}_0$, we have $\xi_\alpha \in \mathscr{K}_0 - L^0_+$. Furthermore, we have

$$|\xi - (\delta^\alpha \cdot S)_T| \leq \alpha.$$

Since $\xi \in L^\infty$, we deduce that $(\delta^\alpha \cdot S)_T \in L^\infty$, and hence $\xi_\alpha \in L^\infty$, showing that $\xi_\alpha \in \mathscr{C}$. Moreover, we have

$$|\xi_\alpha - (\delta^\alpha \cdot S)_T| \leq \alpha.$$

Therefore, we obtain $\|\xi_\alpha - \xi\|_{L^\infty} \to 0$ for $\alpha \downarrow 0$, showing that $\xi \in \overline{\mathscr{C}}^{\| \cdot \|_{L^\infty}}$. □

7.20. **Definition.** *We introduce the following families:*

1. *We define the family of* self-financing portfolios $(\mathbb{P}_{\mathrm{sf},\alpha}(\mathbb{S}))_{\alpha \geq 0}$ *as*
$$\mathbb{P}_{\mathrm{sf},\alpha}(\mathbb{S}) := \{ S^\delta : \delta \in \Delta_{\mathrm{sf}}(\mathbb{S}) \text{ and } S^\delta_0 = \alpha \}, \quad \alpha \geq 0.$$
2. *We define the family of* admissible self-financing portfolios $(\mathbb{P}^{\mathrm{adm}}_{\mathrm{sf},\alpha}(\mathbb{S}))_{\alpha \geq 0}$ *as*
$$\mathbb{P}^{\mathrm{adm}}_{\mathrm{sf},\alpha}(\mathbb{S}) := \{ X \in \mathbb{P}_{\mathrm{sf},\alpha}(\mathbb{S}) : X \text{ is admissible} \}, \quad \alpha \geq 0.$$
3. *We define the family of* nonnegative self-financing portfolios $(\mathbb{P}^{+}_{\mathrm{sf},\alpha}(\mathbb{S}))_{\alpha \geq 0}$ *as*
$$\mathbb{P}^{+}_{\mathrm{sf},\alpha}(\mathbb{S}) := \{ X \in \mathbb{P}_{\mathrm{sf},\alpha}(\mathbb{S}) : X \geq 0 \}, \quad \alpha \geq 0.$$
4. *We denote by $(\mathscr{P}_{\mathrm{sf},\alpha}(\mathbb{S}))_{\alpha \geq 0}$, $(\mathscr{P}^{\mathrm{adm}}_{\mathrm{sf},\alpha}(\mathbb{S}))_{\alpha \geq 0}$ and $(\mathscr{P}^{\mathrm{adm}}_{\mathrm{sf},\alpha}(\mathbb{S}))_{\alpha \geq 0}$ the respective families of random variables defined according to (7.3).*

For each $i \in I$ we denote by $e_i \in \Delta(\mathbb{S})$ the strategy with components

$$e^j_i = \begin{cases} 1, & \text{if } j = i, \\ 0, & \text{otherwise.} \end{cases}$$

7.21. **Lemma.** *The following statements are true:*

1. *For each $i \in I$ we have $e_i \in \Delta_{\mathrm{sf}}(\mathbb{S})$.*

(2) *Let* $\delta, \vartheta \in \Delta_{\mathrm{sf}}(\mathbb{S})$ *and* $a, b \in \mathbb{R}$ *be arbitrary. Then we have* $a\delta + b\vartheta \in \Delta_{\mathrm{sf}}(\mathbb{S})$ *and*

$$S^{a\delta+b\vartheta} = aS^{\delta} + bS^{\vartheta}.$$

Proof. We have

$$S^{e_i} = e_i \cdot S = e_i \cdot S_0 + e_i \cdot (S - S_0) = S_0^{e_i} + e_i \bullet S,$$

proving the first statement. Now, let $\delta, \vartheta \in \Delta_{\mathrm{sf}}(\mathbb{S})$ and $a, b \in \mathbb{R}$ be arbitrary. Then we have

$$S^{a\delta+b\vartheta} = (a\delta + b\vartheta) \cdot S = a(\delta \cdot S) + b(\vartheta \cdot S) = aS^{\delta} + bS^{\vartheta}.$$

Since δ and ϑ are self-financing, we have

$$\begin{aligned} S^{\delta} &= S_0^{\delta} + \delta \bullet S, \\ S^{\vartheta} &= S_0^{\vartheta} + \vartheta \bullet S. \end{aligned}$$

Therefore, using Theorem 7.9 we obtain

$$\begin{aligned} S^{a\delta+b\vartheta} &= aS^{\delta} + bS^{\vartheta} \\ &= a(S_0^{\delta} + \delta \bullet S) + b(S_0^{\vartheta} + \vartheta \bullet S) \\ &= a(\delta_0 \cdot S_0) + b(\vartheta_0 \cdot S_0) + a(\delta \bullet S) + b(\vartheta \bullet S) \\ &= (a\delta_0 + b\vartheta_0) \cdot S_0 + (a\delta + b\vartheta) \bullet S \\ &= S_0^{a\delta+b\vartheta} + (a\delta + b\vartheta) \bullet S, \end{aligned}$$

showing that $a\delta + b\vartheta \in \Delta_{\mathrm{sf}}(\mathbb{S})$. □

Recall that we had defined the family $(\mathscr{B}_\alpha)_{\alpha \geq 0}$ of convex, semi-solid subsets of L_+^0 as

$$\mathscr{B}_\alpha := (\mathscr{K}_\alpha - L_+^0) \cap L_+^0, \quad \alpha \geq 0.$$

7.22. **Lemma.** *Suppose we have* $S_0^i > 0$ *for some* $i \in I$, *and let* $(\mathscr{K}_\alpha)_{\alpha \geq 0}$ *be one of the families*

$$(\mathscr{P}_{\mathrm{sf},\alpha}(\mathbb{S}))_{\alpha \geq 0}, (\mathscr{P}_{\mathrm{sf},\alpha}^{\mathrm{adm}}(\mathbb{S}))_{\alpha \geq 0}, (\mathscr{P}_{\mathrm{sf},\alpha}^{+}(\mathbb{S}))_{\alpha \geq 0}.$$

Then we have $\mathscr{B}_0 \subset \mathscr{B}_\alpha$ *for each* $\alpha > 0$.

Proof. Let $\alpha > 0$ and $\xi \in \mathscr{B}_0$ be arbitrary. Then we have $\xi \in L_+^0$ and there exists $\delta \in \Delta_{\mathrm{sf}}(\mathbb{S})$ such that $S^{\delta} \in \mathbb{K}_0$ and $\xi \leq S_T^{\delta}$. We define

$$\theta := \frac{\alpha e_i}{S_0^i} \quad \text{and} \quad \vartheta := \delta + \theta.$$

By Lemma 7.21 we have $\theta, \vartheta \in \Delta_{\mathrm{sf}}(\mathbb{S})$ and

$$S^{\vartheta} = S^{\delta} + S^{\theta}.$$

We have $S^{\theta} = S \cdot \theta \geq 0$, because $S^i \geq 0$. Therefore, we obtain

$$\xi \leq S_T^{\delta} \leq S_T^{\delta} + S_T^{\theta} \leq S_T^{\vartheta}.$$

Note that $S_0^{\delta} = 0$ and $S_0^{\theta} = \alpha$. Therefore, we have $S_0^{\vartheta} = \alpha$, and hence $S^{\vartheta} \in \mathbb{P}_{\mathrm{sf},\alpha}(\mathbb{S})$. Furthermore, if $S^{\delta} \in \mathbb{P}_{\mathrm{sf},0}^{\mathrm{adm}}(\mathbb{S})$, then $S^{\vartheta} \in \mathbb{P}_{\mathrm{sf},\alpha}^{\mathrm{adm}}(\mathbb{S})$, and if $S^{\delta} \in \mathbb{P}_{\mathrm{sf},0}^{+}(\mathbb{S})$, then $S^{\vartheta} \in \mathbb{P}_{\alpha,\mathrm{sf}}^{+}(\mathbb{S})$. We conclude that $\xi \in \mathscr{B}_\alpha$. □

Recall that we had defined the convex cone $\mathscr{C} \subset L^{\infty}$ as

$$\mathscr{C} = (\mathscr{K}_0 - L_+^0) \cap L^{\infty}.$$

7.23. **Lemma.** *Suppose that* $S_0^i > 0$ *and* $S_T^i \in L^\infty$ *for some* $i \in I$, *and let* $(\mathscr{K}_\alpha)_{\alpha \geq 0}$ *be one of the families*

$$(\mathscr{P}_{\mathrm{sf},\alpha}(\mathbb{S}))_{\alpha \geq 0}, (\mathscr{P}_{\mathrm{sf},\alpha}^{\mathrm{adm}}(\mathbb{S}))_{\alpha \geq 0}.$$

Then we have

$$\bigg(\bigcap_{\alpha > 0} \mathscr{B}_\alpha \bigg) \cap L^\infty \subset \overline{\mathscr{C}}^{\| \cdot \|_{L^\infty}}.$$

Proof. Let

$$\xi \in \bigg(\bigcap_{\alpha > 0} \mathscr{B}_\alpha \bigg) \cap L^\infty$$

be arbitrary. Furthermore, let $\alpha > 0$ be arbitrary. Then there is a self-financing strategy $\delta^\alpha \in \Delta_{\mathrm{sf}}(\mathbb{S})$ such that $S^{\delta^\alpha} \in \mathbb{K}_\alpha$ and $\xi \leq S_T^{\delta^\alpha}$. We define

$$\theta^\alpha := \frac{\alpha e_i}{S_0^i} \quad \text{and} \quad \vartheta^\alpha := \delta^\alpha - \theta^\alpha.$$

By Lemma 7.21 we have $\theta^\alpha, \vartheta^\alpha \in \Delta_{\mathrm{sf}}(\mathbb{S})$ and

$$S^{\vartheta^\alpha} = S^{\delta^\alpha} - S^{\theta^\alpha}.$$

Furthermore, we set

$$\xi_\alpha := \xi - S_T^{\theta^\alpha}.$$

Note that

$$S_T^{\theta^\alpha} = \frac{\alpha S_T^i}{S_0^i} \in L_+^\infty,$$

and hence $\xi_\alpha \in L^\infty$. Furthermore, we have $S^{\theta^\alpha} \in \mathbb{K}_\alpha$, and hence $S^{\vartheta^\alpha} \in \mathbb{K}_0$. Therefore, we obtain

$$\xi_\alpha = \xi - S_T^{\theta^\alpha} \leq S_T^{\delta^\alpha} - S_T^{\theta^\alpha} = S_T^{\vartheta^\alpha} \in \mathscr{K}_0,$$

and hence $\xi_\alpha \in \mathscr{C}$. Moreover, we have

$$\| \xi - \xi_\alpha \|_{L^\infty} = \| S_T^{\theta^\alpha} \| = \frac{\alpha}{S_0^i} \| S_T^i \|_{L^\infty} \to 0$$

for $\alpha \downarrow 0$, showing that $\xi \in \overline{\mathscr{C}}^{\| \cdot \|_{L^\infty}}$. □

Now, we are ready to state our main results of this section. Once again, we point out that the market $\mathbb{S}$ does not need to have a numéraire, and that the upcoming results in particular concern self-financing portfolios.

7.24. **Theorem.** *Let* $(\mathscr{K}_\alpha)_{\alpha \geq 0}$ *be one of the families*

$$(\mathscr{I}_\alpha(\mathbb{S}))_{\alpha \geq 0}, (\mathscr{I}_\alpha^{\mathrm{adm}}(\mathbb{S}))_{\alpha \geq 0}, (\mathscr{I}_\alpha^{+}(\mathbb{S}))_{\alpha \geq 0},$$
$$(\mathscr{P}_{\mathrm{sf},\alpha}(\mathbb{S}))_{\alpha \geq 0}, (\mathscr{P}_{\mathrm{sf},\alpha}^{\mathrm{adm}}(\mathbb{S}))_{\alpha \geq 0}, (\mathscr{P}_{\mathrm{sf},\alpha}^{+}(\mathbb{S}))_{\alpha \geq 0}.$$

Then the following statements are equivalent:

(i) $\mathscr{K}_1$ *satisfies NUPBR.*
(ii) $\mathscr{K}_1$ *satisfies* NAA_1.
(iii) $\mathscr{K}_1$ *satisfies* NA_1.
(iv) *We have* $\bigcap_{\alpha > 0} \mathscr{B}_\alpha = \{0\}$.

Proof. By Lemmas 7.17 and 7.21 we have

$$aX + bY \in \mathbb{K}_{a\alpha + b\beta}.$$

for all $a, b \in \mathbb{R}_+$, $\alpha, \beta \in \mathbb{R}_+$ and $X \in \mathbb{K}_\alpha$, $Y \in \mathbb{K}_\beta$, showing that Assumptions 7.1 and 7.2 are fulfilled. Therefore, the stated equivalences are a consequence of Theorem 5.12. □

7.25. **Proposition.** *Let* $(\mathscr{K}_\alpha)_{\alpha \geq 0}$ *be one of the families*

$$(\mathscr{I}_\alpha(\mathbb{S}))_{\alpha \geq 0}, (\mathscr{I}_\alpha^{\mathrm{adm}}(\mathbb{S}))_{\alpha \geq 0}, (\mathscr{I}_\alpha^{+}(\mathbb{S}))_{\alpha \geq 0}.$$

If $\mathscr{K}_1$ *satisfies* NA_1*, then* $\mathscr{K}_0$ *satisfies NA.*

Proof. This is a consequence of Proposition 5.14 and Lemma 7.18. □

7.26. **Proposition.** *Let* $(\mathscr{K}_\alpha)_{\alpha \geq 0}$ *be one of the families*

$$(\mathscr{I}_\alpha(\mathbb{S}))_{\alpha \geq 0}, (\mathscr{I}_\alpha^{\mathrm{adm}}(\mathbb{S}))_{\alpha \geq 0}.$$

If $\mathscr{K}_0$ *satisfies NFLVR, then* $\mathscr{K}_1$ *satisfies* NA_1*.*

Proof. This is a consequence of Proposition 5.15 and Lemma 7.19. □

7.27. **Proposition.** *Suppose we have* $S_0^i > 0$ *for some* $i \in I$*, and let* $(\mathscr{K}_\alpha)_{\alpha \geq 0}$ *be one of the families*

$$(\mathscr{P}_{\mathrm{sf},\alpha}(\mathbb{S}))_{\alpha \geq 0}, (\mathscr{P}_{\mathrm{sf},\alpha}^{\mathrm{adm}}(\mathbb{S}))_{\alpha \geq 0}, (\mathscr{P}_{\mathrm{sf},\alpha}^{+}(\mathbb{S}))_{\alpha \geq 0}.$$

If $\mathscr{K}_1$ *satisfies* NA_1*, then* $\mathscr{K}_0$ *satisfies NA.*

Proof. This is a consequence of Proposition 5.14 and Lemma 7.22. □

7.28. **Proposition.** *Suppose that* $S_0^i > 0$ *and* $S_T^i \in L^\infty$ *for some* $i \in I$*, and let* $(\mathscr{K}_\alpha)_{\alpha \geq 0}$ *be one of the families*

$$(\mathscr{P}_{\mathrm{sf},\alpha}(\mathbb{S}))_{\alpha \geq 0}, (\mathscr{P}_{\mathrm{sf},\alpha}^{\mathrm{adm}}(\mathbb{S}))_{\alpha \geq 0}.$$

If $\mathscr{K}_0$ *satisfies NFLVR, then* $\mathscr{K}_1$ *satisfies* NA_1*.*

Proof. This is a consequence of Proposition 5.15 and Lemma 7.23. □

We emphasize that the previous results are proven in a rather straightforward manner, only relying on results about topological vector lattices and well-known results from stochastic analysis.

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