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Detecting Money Market Bubbles

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Abstract. Using a range of stochastic volatility models well-known in the finance literature, we study the existence of money market bubbles in the US economy. Money market bubbles preclude the existence of a risk-neutral pricing measure. Understanding whether markets exhibit money market bubbles is crucial from the point of view of derivative pricing since their existence implies the existence of a self-financing trading strategy that replicates the savings account's value at a fixed future date at a cheaper cost than the current value of the savings account. The benchmark approach is formulated under the real world probability measure and does not require the existence of a risk neutral probability measure. It hence emerges as the appropriate framework to study the potential existence of money market bubbles. Testing the existence of money market bubbles in the US economy we find that for all models the US market exhibits a money market bubble. This conclusion suggests that for derivative pricing and hedging care should be taken when making assumptions pertaining to the existence of a risk-neutral probability measure. Less expensive hedge portfolios may exist for a wide range of derivatives.

JEL Classification: C6, C63, G1, G13
Key words and phrases: Money market bubbles; strict local martingales; Markov chain Monte Carlo; stochastic volatility models; benchmark approach

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1 Introduction

In this paper, we empirically investigate the potential existence of money market bubbles, as formally introduced in the literature in [29]. Already in [52], the potential existence of this type of classical arbitrage was theoretically pointed out. Money market bubbles form part of a growing literature on bubbles, which are studied in both, the economics and the finance literature, see e.g. [14],[23], [24], [48], [36], [29], [30], [31], [35], [37], [38], and, in particular, the recent article [57] for an overview of the literature. In [29], an asset-price bubble is occurring through an asset with a nonnegative price, which can be replicated at a fixed future time using a self-financing trading strategy whose setup cost is lower than the current value of the asset. As in [29], we place ourselves in a two asset economy. To be specific, we assume the existence of a well-diversified index and a money market account. Then a money market bubble exists if the savings account value can be replicated at a fixed future time using a well-diversified index at a cost that is lower than the value of the savings account. As established in [29], a necessary and sufficient condition for the existence of such a bubble is the failure of the existence of an equivalent risk neutral probability measure or an equivalent martingale measure (EMM), i.e. the Radon-Nikodym derivative of the putative EMM is a strict local martingale. So far though, only [29] and [31] have dealt with money market bubbles. Furthermore, we point out that under the Platen's *benchmark approach*, see e.g. [55], money market bubbles naturally occur, as for example under the minimal market model (MMM), see also [55] and [56]. Should the candidate model rule out money market bubbles, one can investigate if the discounted well-diversified index follows a strict local martingale under a respective EMM. If this is the case, then one has detected an asset-price bubble. The latter have received much attention in the literature, see e.g. [35], [37], [38], and, in particular, the recent article [57]. The conditions for the existence of asset-price bubbles are by now well understood: An asset-price bubble exists if and only if the discounted index follows a strict local martingale under the EMM. The asset-price bubbles exhibit some interesting financial consequences, such as failure of put-call parity, non-convexity of option prices with respect to the underlying for convex

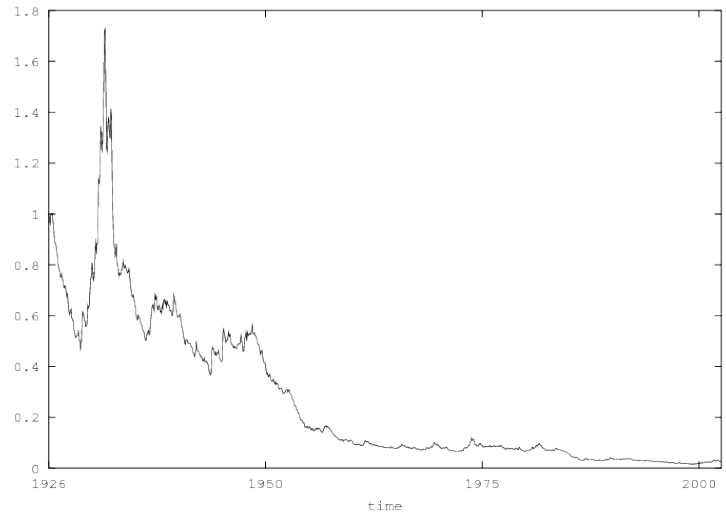

Figure 1.1: Empirical Radon-Nikodym derivative for the US economy.

options with convex payoff functions, to name but a few.

This paper examines the fundamental question of the existence of an EMM when aiming for a candidate model. A motivation for this type of examination is the plot of the US savings account in units of the total S&P 500 return, shown in Figure 1.1, normalised to an initial value of one. It displays an empirical Radon-Nikodym derivative for the putative risk neutral measure for the US market for a complete market model that interprets the S&P 500 as numéraire portfolio, as will be explained later.

By using FX-derivative data, in [5] it has been shown that an equivalent risk neutral probability measure may not exist for certain currency denominations. We aim to employ historical index data to provide alternative evidence in the same direction. The empirical Radon-Nikodym derivative displayed in Figure 1.1 demonstrates a systematic downward trend which is indicative of a nonnegative strict local martingale, which is a strict supermartingale, and hence hints at the presence of a money market bubble. This is opposed to a Radon-Nikodym derivative forming a martingale, where the present value is the best forecast of any future values. We take Figure 1.1 as a motivation for a more thorough investigation of the potential existence of a money market bubble.

To investigate the existence of an EMM, we require an approach more general than the classical no-arbitrage approach with its risk neutral pricing paradigm. In particular, we require an approach which is working under the real world probability measure and more general than the classical approach. The benchmark approach, [55], offers such framework and allows us to rigorously examine the potential existence of an EMM. At the heart of the benchmark approach sits the numéraire portfolio (NP), see [47], which when used as benchmark, makes all nonnegative benchmarked portfolios supermartingales. It is identical to the growth optimal portfolio (GOP), where the latter is mathematically appropriately defined as the portfolio which maximises expected logarithmic utility from terminal wealth, see [43]. The NP plays a fundamental role in derivative pricing, as it serves as numéraire when pricing contingent claims under the real world probability measure. We alert the reader to the fact that when employing the benchmark approach for derivative pricing, one finds the minimal possible price by employing the *real world pricing formula*, [55], which computes the expectation under the real world probability measure using the NP as numéraire. In terms of money market bubbles, we recall that a money market bubble exists if the savings account value at a future time can be replicated using the NP at a setup cost cheaper than the current value of the savings account. More precisely, this paper answers the following questions:

- Does a money market bubble most likely exist in the US market?
- Which is a tractable, parsimonious, yet sufficiently accurate model for the discounted S&P 500?

We answer the first question by connecting two significant streams in the finance literature, namely the literature concerned with the estimation of stochastic volatility models and the literature on the existence of bubbles. Using Markov Chain Monte Carlo (MCMC) methods, see [13], [20] and also the references therein, on parametrised stochastic volatility models describing the S&P 500 index, we approach the second question. We adopt the MCMC approach and fit various stochastic volatility models that are well established in the finance literature to the S&P

500 index, including the Heston, the 3/2, and the continuous time GARCH model. We refer to [45] for an overview of stochastic volatility models, including the latter ones. We remark at this point that there are some criticisms pertaining to some of these models, as for example discussed in [12] and [15]. For all stochastic volatility models considered in this paper, we can present necessary and sufficient conditions for the existence of a money market bubble. We demonstrate that money market bubbles cannot exist under the Heston model, and this model mainly serves as a popular reference model. However, for the remaining models we find that for some parameter regimes, money market bubbles exist. In fact, we find that when fitting these models, they strongly hint at the likely existence of money market bubbles. This suggests that assumptions pertaining to the existence of an EMM should be made with care. Although the main objective of this paper is to investigate the ability to generate bubbles by the selected models, we also compare the model performance based on different selection criteria and find that all alternative models that can generate bubbles also outperform the Heston model, leading to superior fitting of the data.

This paper can be seen as building on [29], [31], [35], and [6]. It is the first paper that tests for the existence of money market bubbles, which have only been discussed in [29], [31], in the same way that [35] was the first paper to investigate empirically the existence of stock price bubbles. Finally, we remark that in [6], local volatility models were fitted to well-diversified indices using techniques that were similar to those extended in [35], used to examine the existence of stock price bubbles. We point out that the results presented in [6] support our findings, which suggest the existence of money market bubbles in the markets under consideration.

The presence of money market bubbles violates the "no free lunch with vanishing risk" (NFLVR) condition, see [16], [17], which is the no-arbitrage condition that underpins classical risk neutral pricing. The models we present still satisfy the more general "no unbounded profit with bounded risk" (NUPBR) condition, see [41], which is sufficient for pricing, hedging, risk management and utility maximisation in a general semimartingale framework. They also fit under the benchmark approach, which only assumes the existence of the numéraire portfolio and automatically excludes, so called, strong arbitrage, which would create strictly positive wealth from a nonnegative portfolio with zero initial capital, see [55]. From the point of view of classical economic intuition though, a reader acquainted with derivative pricing, based on the absence of classical arbitrage, might challenge the existence of money market bubbles: Shorting the asset and replicating it using the index would then produce a certain positive cash-flow either at the time the arbitrage is set up or, equivalently, at the time when the arbitrage strategy matures. One could argue that arbitrageurs would employ the arbitrage strategy by investing accordingly. The key problem with the above argument is that it is essentially a "one-period model" argument: One only considers the portfolio value at two points in time, when the arbitrage is set up, and when the arbitrage matures. However, the argument ignores the portfolio values between these two time points, which could become negative with strictly positive probability. This has important (corporate finance) implications, as liabilities must be secured by posting collateral. This is explored in [46], where a model is presented for which an EMM does not exist, and under which arbitrages are considered as risky investments. It is shown that it can be optimal for the investor to underinvest in the arbitrage opportunity and not to take the largest arbitrage position allowed by the collateral constraints. Clearly, if the investor only takes a limited position in the arbitrage position or avoids taking a position in the arbitrage opportunity, then there is no reason for the bubble to not to exist.

The remainder of the paper is structured as follows: in Section 2, we introduce the modelling framework which will be used to detect money market bubbles, in Section 3 we introduce the statistical methodology used to detect bubbles, and in Section 4 we present the empirical results. Section 5 interprets the results and Section 6 concludes the paper.

2 Money Market Bubbles

In this section, we introduce a framework that will allow us to empirically test the existence of money market bubbles. We place ourselves in the setup presented in [29], Section 2, and formulate our model on a filtered probability space $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P)$, where P is the real world probability measure and $\underline{\mathcal{A}} = (A_t)_{t\geq 0}$ is a filtration modelling the evolution of information and satisfying the usual conditions. We assume that the squared volatility V_t of the NP, which is a state variable, is the unique strong solution of the stochastic differential equation (SDE)

$$dV_t = a(V_t)dt + b(V_t)dW_t^1\,, \tag{2.1}$$

where V_0 is strictly positive. The process W_t^1 is a Brownian motion under the real world probability measure P. In the following subsections we introduce specific models for V_t, specifying $a(\cdot)$ and $b(\cdot)$. In particular, we assume that the functions $a(\cdot)$ and $b(\cdot)$ employed in this paper are continuous functions on $(0,\infty)$ with $b^2(\cdot) > 0$. Secondly, we present a model for the discounted numéraire portfolio (NP). In particular, we assume that the discounted NP satisfies the SDE

$$dS_t = S_t\left(V_t dt + \sqrt{V_t}dW_t^2\right)\,, \tag{2.2}$$

where $\sqrt{V_t}$ denotes the volatility and also the market price of risk, which we specify further below. W_t^2 is another Brownian motion and we assume that the Brownian motions W_t^1 and W_t^2 can be correlated with constant correlation ρ. Finally, we introduce the money market account,

$$dB_t = r_t B_t dt\,, \tag{2.3}$$

$t \geq 0$, where $B_0 = 1$. We now provide a definition for a money market bubble, which is adopted from [29], see Definition 2.1.

Definition 2.1. *The money market account has a bubble, if there is a self-financing portfolio with path-wise nonnegative wealth that costs less than the money market account and replicates the money market account at a fixed future date. The value of the money market bubble is the difference between the money market's price and the lowest cost replicating strategy.*

Under the benchmark approach the NP is employed as benchmark and all securities denominated in units of the benchmark are called benchmarked securities. Next, we introduce the benchmarked money market account $\hat{B} = \left\{\hat{B}_t = \frac{1}{S_t}\,,\; t \geq 0\right\}$, which is simply the inverse of the discounted NP and plays a central role in the detection of bubbles. The dynamics of $\hat{B}$ are given by the SDE

$$d\hat{B}_t = -\sqrt{V_t}\hat{B}_t dW_t^2\,, \tag{2.4}$$

to obtain

$$\hat{B}_t = \hat{B}_0 \exp\left(-\int_0^t \sqrt{V_s}dW_s^2 - \frac{1}{2}\int_0^t V_s ds\right)\,.$$

We now introduce the Radon-Nikodym derivative for the putative equivalent martingale measure (EMM), which is given by $\xi_t = \frac{\hat{B}_t}{\hat{B}_0}$, $t \geq 0$. We now recall Proposition 2.1 from [29].

Proposition 2.1. *The process $\xi = \{\xi_t\,,\, t \geq 0\}$ is a strict local martingale if and only if the money market account has a bubble.*

As shown in [29], the process ξ is a strict local martingale, if and only if the process $\tilde{V}$ does not reach zero or infinity in finite time, where $\tilde{V}$ is given by the SDE

$$d\tilde{V}_t = (a(\tilde{V}_t) - \rho\sqrt{\tilde{V}_t}b(\tilde{V}_t))dt + b(\tilde{V}_t)dW_t^1\,, \tag{2.5}$$

with $\tilde{V}_0 = V_0$. We formalise this insight in the following corollary, which reflects the Condition 2' from [29].

Corollary 2.1. *The money market account exhibits a bubble if and only if the solution of* (2.5) *explodes or hits* 0 *in finite time.*

There are necessary and sufficient conditions which allow one to check if a particular one-dimensional diffusion, such as the one in equation (2.5), reaches 0 or infinity. This includes the Feller test, see e.g. Theorem 5.29 (Section 5.5) in [42]. This argument appears in [59], but has subsequently been used to study whether local martingales are strict or not, see e.g. [55] and the references therein; and also to study moment explosions, see e.g. [3]. Important is in this context the work by [32], which formulates in a semimartingale setting the necessary and sufficient conditions when a local martingale is a true martingale.

We now discuss the four stochastic volatility models that will be examined in the paper. All fall into the above framework.

2.1 Heston Model

For the Heston model, see [27], we set

$$a(x) = \kappa(\theta - x)\,,\; b(x) = \sigma\sqrt{x}\,, \tag{2.6}$$

in equation (2.1), where $\kappa > 0$, $\theta > 0$, and $\sigma > 0$. We assume that the Feller condition is in force, i.e. that $2\kappa\theta > \sigma^2$. We then have the following result, which follows immediately from Corollary 2.1.

Theorem 2.1. *Assume that the dynamics of $\hat{B}$ are given by the Heston model. Then $\hat{B}$ is a martingale and the money market account does not exhibit a bubble.*

Proof. The result can be directly established using the argument from [3], which is given in its Lemma 2.3. □

We remark that calibration of the Heston model in the FX-market usually leads to violations of the Feller condition, see e.g. [25]. Also the trajectory in Figure 1.1 would not be typical for the Heston model, since $\hat{B}_t$ evolves more like a strict local martingale than a martingale. Nevertheless, the Heston model will serve us for comparisons. Additionally, we find it interesting to investigate whether the Feller condition is empirically satisfied or not.

2.2 3/2 Nonlinearly Mean Reverting Model

To formulate the nonlinearly mean reverting 3/2 model (3/2N), introduced in [54], [28], [1] and [52], and further developed in [45], we set

$$a(x) = \kappa x(\theta - x)\,,\; b(x) = \sigma x^{3/2}\,, \tag{2.7}$$

where $\kappa > 0$, $\theta > 0$, and $\sigma > 0$. Under the 3/2N model, for certain parameter combinations the process $\hat{B}$ is a martingale, but otherwise a strict local martingale.

Theorem 2.2. *Assume that the dynamics of $\hat{B}$ are given by the $3/2N$ model. Then $\hat{B}$ is a martingale, if and only if*

$$\frac{\kappa}{\sigma} + \frac{\sigma}{2} \geq -\rho\,.$$

Proof. This result was first proven in [5] to which we refer. □

In Section 3, we will estimate the parameters of the 3/2N model, to determine whether $\hat{B}$ follows potentially a strict local martingale.

2.3 Linearly Mean Reverting 3/2 Model

To formulate the linearly mean reverting 3/2L model, see [40], we set

$$a(x) = \kappa(\theta - x)\,,\; b(x) = \sigma x^{3/2}\,, \tag{2.8}$$

where $\kappa > 0$, $\theta > 0$, and $\sigma > 0$. For this model we obtain that for certain parameter combinations the process $\hat{B}$ is a martingale, but otherwise a strict local martingale.

Theorem 2.3. *Assume that the dynamics of $\hat{B}$ are given by the linearly mean reverting 3/2 model. Then $\hat{B}$ is a martingale, if and only if*

$$\frac{\sigma}{2} \geq -\rho\,. \tag{2.9}$$

Proof. The proof follows from Proposition 2.5 in [3]. Since we consider the benchmarked savings account, we replace ρ in Proposition 2.5 by $-\rho$. □

2.4 GARCH Model

Similar to the continuous time limit of the GARCH model, see [51], we consider also in continuous time what we call here a GARCH model, which was e.g. studied in [26]. It is characterised by the choice

$$a(x) = \kappa(\theta - x)\,,\; b(x) = \sigma x\,, \tag{2.10}$$

where $\kappa > 0$, $\theta > 0$, and $\sigma > 0$. We obtain that for certain parameter combinations of this model that the process $\hat{B}$ is a martingale but otherwise a strict local martingale.

Theorem 2.4. *Assume that the dynamics of $\hat{B}$ are given by the GARCH model. Then $\hat{B}$ is a martingale, if and only if*

$$\rho > 0\,. \tag{2.11}$$

Proof. The proof follows from Proposition 2.5 in [3]. Since we consider the benchmarked savings account, we replace ρ in Proposition 2.5 by $-\rho$. $\square$

Note that the modelling framework introduced above is flexible enough to be extended to *multi-factor* stochastic volatility models and models which include jumps. However, for the purpose of this paper we concentrate on the one-factor stochastic volatility models. In the next section, we discuss the statistical methodology that will be used to estimate the models presented in this section. Having estimated the models, we can use Theorems 2.2, 2.3, and 2.4, respectively, to determine if a money market bubble is present or not assuming the respective model.

3 Markov-Chain Monte-Carlo Estimation

We now discuss the statistical methodology used in this paper, which is based on [13] and [20]. Firstly, we define the logarithm of the benchmarked savings account $Y_t = \log(\hat{B}_t)$ to obtain the SDE

$$dY_t = -\frac{1}{2}V_t dt - \sqrt{V_t}dW_t^2\,. \tag{3.12}$$

To estimate the model, we use an Euler discretization scheme and use the time step size $\Delta = 1$, which corresponds to one day. We define the log-returns of the benchmarked savings account $R_t = Y_t - Y_{t-1}$, so that

$$\begin{aligned} R_t &= -\frac{1}{2}\int_{t-1}^{t} V_s ds - \int_{t-1}^{t} \sqrt{V_s}dW_s^2 \\ &\approx -\frac{1}{2}V_{t-1} - \sqrt{V_{t-1}}(W_t^2 - W_{t-1}^2) \end{aligned}$$

$$= -\frac{1}{2}V_{t-1} - \sqrt{V_{t-1}}\varepsilon_t^y\,, \tag{3.13}$$

where $\varepsilon_t^y = W_t^2 - W_{t-1}^2$, $t \in \{1, 2, ...\}$. For the variance processes we assume a general specification,

$$\begin{aligned} V_t &= V_{t-1} + \int_{t-1}^{t} a(V_s)ds + \int_{t-1}^{t} b(V_s)dW_s^1 \\ &\approx V_{t-1} + a(V_{t-1}) + b(V_{t-1})(W_t^1 - W_{t-1}^1)\,. \end{aligned} \tag{3.14}$$

We note that all models discussed in this paper can be represented approximately via the recursive relationship

$$V_t \approx V_{t-1} + \kappa(V_{t-1})^a(\theta - V_{t-1}) + \sigma(V_{t-1})^b\varepsilon_t^v \tag{3.15}$$

for $a \in \{0, 1\}$, $b \in \left\{\frac{1}{2}, 1, \frac{3}{2}\right\}$ and $\varepsilon_t^v = W_t^1 - W_{t-1}^1$. In particular, we have

$$V_t \approx V_{t-1} + \kappa(\theta - V_{t-1}) + \sigma\sqrt{V_{t-1}}\varepsilon_t^v \quad (a = 0, b = 1/2\text{: Heston}) \tag{3.16}$$

$$V_t \approx V_{t-1} + \kappa V_{t-1}(\theta - V_{t-1}) + \sigma(V_{t-1})^{3/2}\varepsilon_t^v \quad (a = 1, b = 3/2\text{: 3/2 model}) \tag{3.17}$$

$$V_t \approx V_{t-1} + \kappa(\theta - V_{t-1}) + \sigma(V_{t-1})^{3/2}\varepsilon_t^v \quad (a = 1, b = 3/2\text{: 3/2 linear drift model}) \tag{3.18}$$

$$V_t \approx V_{t-1} + \kappa(\theta - V_{t-1}) + \sigma V_{t-1}\varepsilon_t^v \quad (a = 0, b = 1\text{: GARCH}) \tag{3.19}$$

The underlying problem setup involves the estimation of the parameter vector $\mathbf{\Theta} = (\theta, \kappa, \sigma, \rho)^\top$, as well as, the volatility $\mathbf{V}$, which is a state variable. In a Bayesian context each of these unobserved variables is treated as a parameter to be estimated. This leads to a high-dimensional posterior distribution that is not known. In order to compute the moments of the posterior, we would have to compute a high dimensional integral, which is not feasible. Therefore, we rely on the Markov-Chain Monte-Carlo (MCMC) method to compute the moments of the parameter values and latent variables conditional on the observed data. These moments are used as a point estimator for the parameters and the latent variables. Bayesian MCMC methods for stochastic volatility models have been developed in [34]. MCMC generates samples from a given target distribution, in our case $p(\mathbf{\Theta}, \mathbf{V}|\mathbf{R})$ - the joint distribution of the parameter vector $\mathbf{\Theta}$ and the state variable $\mathbf{V}$, given the observed returns. The basis for MCMC estimation is provided by the Hammersley-Clifford theorem, which states that under certain regularity conditions the joint posterior distribution can be completely characterised by the complete conditional distributions. In order to compute the posterior distribution note that the joint distribution of (R_t, V_t) follows a bivariate normal distribution with the following parameters:

$$
\begin{aligned}
\boldsymbol{\mu} &= \begin{pmatrix} -\frac{1}{2}V_{t-1} \\ V_{t-1} + \kappa V_{t-1}^{a}(\theta - V_{t-1}) \end{pmatrix} \\
\boldsymbol{\Sigma} &= \begin{pmatrix} V_{t-1} & -\rho\sigma V_{t-1}^{b+0.5} \\ -\rho\sigma V_{t-1}^{b+0.5} & \sigma^2 V_{t-1}^{2b} \end{pmatrix}.
\end{aligned} \tag{3.20}
$$

Given the normality of the joint distribution we can determine the conditional distribution of returns $p(R_t|V_t, V_{t-1}, \boldsymbol{\Theta})$. It follows a normal distribution with parameters

$$
\begin{aligned}
\mu_{R_t|V_t} &= -\frac{1}{2}V_{t-1} - \rho\sigma^{-1}V_{t-1}^{0.5-b}(V_t - V_{t-1} - \kappa V_{t-1}^{a}(\theta - V_{t-1})) \\
\sigma^2_{R_t|V_t} &= V_{t-1}(1-\rho^2).
\end{aligned} \tag{3.21}
$$

On the other hand, for the conditional distribution for $p(V_t|R_t, V_{t-1}, \boldsymbol{\Theta})$ we obtain a normal distribution with

$$
\begin{aligned}
\mu_{V_t|R_t} &= V_{t-1} + \kappa(V_{t-1})^{a}(\theta - V_{t-1}) - \rho\sigma V_{t-1}^{b-0.5}(R_t + \frac{1}{2}V_{t-1}) \\
\sigma^2_{V_t|R_t} &= \sigma^2 V_{t-1}^{2b}(1-\rho^2).
\end{aligned} \tag{3.22}
$$

In general, the posterior is given by the following expression:

$$
p(\boldsymbol{\Theta}, \mathbf{V}|\mathbf{R}) \propto p(\mathbf{R}|\boldsymbol{\Theta}, \mathbf{V})p(\boldsymbol{\Theta}, \mathbf{V}) = p(\mathbf{R}|\boldsymbol{\Theta}, \mathbf{V})p(\mathbf{V}|\boldsymbol{\Theta})p(\boldsymbol{\Theta}) \tag{3.23}
$$

where $p(\mathbf{R}|\boldsymbol{\Theta}, \mathbf{V})$ denotes the *likelihood* and $p(\boldsymbol{\Theta}, \mathbf{V})$ is the *prior distribution.*

In (3.23) we can write $p(\mathbf{R}|\boldsymbol{\Theta}, \mathbf{V}) = \prod_{t=1}^{T} p(R_t|V_t, V_{t-1}, \boldsymbol{\Theta})$ by conditional independence and $p(\mathbf{V}|\boldsymbol{\Theta}) \propto \prod_{t=1}^{T} p(V_t|V_{t-1}, \boldsymbol{\Theta})$ by the Markov property. Using the fact that

$$
\begin{aligned}
p(\mathbf{R}|\boldsymbol{\Theta}, \mathbf{V})p(\mathbf{V}|\boldsymbol{\Theta}) &\propto \prod_{t=1}^{T} p(R_t|V_t, V_{t-1}, \boldsymbol{\Theta})p(V_t|V_{t-1}, \boldsymbol{\Theta}) \\
&= \prod_{t=1}^{T} p(R_t, V_t|V_{t-1}, \boldsymbol{\Theta}),
\end{aligned} \tag{3.24}
$$

we can rewrite (3.23) as follows:

$$p(\mathbf{\Theta}, \mathbf{V}|\mathbf{R}) \;\; \propto \;\; \prod_{t=1}^{T} p(R_t, V_t|V_{t-1}, \mathbf{\Theta})p(\mathbf{\Theta}). \tag{3.25}$$

To update estimated parameter values in each iteration, the MCMC algorithm draws from its posterior distribution conditional on the current values of all other parameters and state variables. Therefore, in order to reduce the influence of the starting point and to assure that stationarity is achieved, the general approach is to discard a burn-in period of the first h iterations. The iterations after the burn-in period provide a representative sample from the joint posterior, and averaging over the non-discarded iterations provides an estimate for posterior means of parameters and latent variables.

Sampling from the conditional posterior distribution can be implemented by either using the *Gibbs sampler* introduced by [21], or the *Metropolis-Hasting algorithm*, see [49].

The Gibbs sampler is applied if the complete conditional distribution to sample from is known. The MCMC algorithm with the Gibbs step samples iteratively drawing from the following conditional posteriors, see [22]:

$$\begin{array}{ll} \text{Parameters}: & p(\mathbf{\Theta}_i|\mathbf{\Theta}_{-i}, \mathbf{V}, \mathbf{R}), \; i = 1, \ldots, N \\ \text{Volatility}: & p(V_t|\mathbf{\Theta}, V_{t+1}, V_{t-1}, \mathbf{R}), \; t = 1, \ldots, T, \end{array}$$

where N denotes the number of draws, T is the number of observations and $\mathbf{\Theta}_{-i}$ is the parameter vector $\mathbf{\Theta}$ without the $i-$th element.

To start the procedure, we have to specify the prior distributions. When possible we assume the so-called conjugate priors, which after multiplying with the likelihood lead to a posterior distribution belonging to the same family of distributions as the prior itself. Wherever possible, we choose standard conjugate priors, which allow us to draw from the conditional posteriors directly. For the model parameters we specify conjugate priors and provide further details on parameter draws in the appendix.

If some conditional distributions cannot be sampled directly, as in the case with variance, where the complete conditional distribution is not recognizable, we apply the Metropolis-Hasting algorithm. For details on the Metropolis-Hasting algorithm we refer to [39]. We can use the following expression for the complete conditional posterior:

$$\begin{aligned} p(V_t|\mathbf{\Theta},\mathbf{R}) &\propto \prod_{t=1}^{T} p(R_t, V_t|V_{t-1},\mathbf{\Theta})p(\mathbf{\Theta}) \\ &= \prod_{t=1}^{T} p(V_t|R_t, V_{t-1},\mathbf{\Theta})p(R_t|V_{t-1},\mathbf{\Theta})p(\mathbf{\Theta}) \\ &\propto \prod_{t=1}^{T} p(V_t|R_t, V_{t-1},\mathbf{\Theta})p(R_t|V_{t-1},\mathbf{\Theta}). \end{aligned} \tag{3.26}$$

Thus, the complete conditional distribution for V_t, after removing all terms that do not depend on V_t, reduces to

$$\begin{aligned} p(V_t|\mathbf{\Theta},\mathbf{R}) &\propto p(V_{t+1}|R_{t+1}, V_t,\mathbf{\Theta})p(R_{t+1}|V_t,\mathbf{\Theta})p(V_t|R_t, V_{t-1},\mathbf{\Theta}) \\ &= p(R_{t+1}, V_{t+1}|V_t,\mathbf{\Theta})p(V_t|R_t, V_{t-1},\mathbf{\Theta}). \end{aligned} \tag{3.27}$$

In the last expression, $p(R_{t+1}, V_{t+1}|V_t,\mathbf{\Theta})$ follows a bivariate normal distribution and $p(V_t|R_t, V_{t-1},\mathbf{\Theta})$ follows a univariate normal distribution. The Metropolis-Hasting step proposes a new variance $V_t^{(i)}$ in the i^{th} draw by drawing from $p(V_t|R_t, V_{t-1},\mathbf{\Theta})$ and accepting that draw with probability

$$\min\left\{\frac{p(R_{t+1}, V_{t+1}|V_t^{(i)},\mathbf{\Theta})}{p(R_{t+1}, V_{t+1}|V_t^{(i-1)},\mathbf{\Theta})}, 1\right\}. \tag{3.28}$$

4 Model Testing

In addition to detecting money market bubbles, this paper also aims to determine which model among the considered ones fits best the discounted S&P 500 data. This section aims to present several decision criteria that quantify the model performance or the fit to the data. The first procedure to investigate the model performance is the visual inspection, achieved by comparing quantile-to-quantile (QQ) plots. Since residuals from the return and the variance equations are assumed to follow normal distributions, one can contrast the quantiles of these error distributions against normal quantiles in the QQ plot. The degree of model misspecification (non-normality of residuals) is defined by how far the estimated residuals in the log-return and the variance equation deviate from the 45 degrees line. Secondly, we will use the deviance information criterion (DIC) proposed in [60], which allows us to compare non-nested models. Finally, we compare the estimated volatility path with the path of the Chicago Board Options Exchange Market Volatility Index (VIX)[4], as it has been the best measure capturing the market perspective on the implied volatility of the S&P 500 index options. In order to allow the comparison with the VIX, we will use the S&P 500 index as our proxy for the NP in Section 5. The following subsection deals with the DIC in more details.

4.1 Deviance Information Criterion

In a non-Bayesian setting deviance is used as a characteristic that estimates the number of the parameters in the underlying model: It refers to the difference in log-likelihoods between the fitted and the saturated model, that is, the one which yields a perfect fit of the data. Model comparison can be performed based on the fit, which is measured by the deviance statistic and the complexity, which is the number of free parameters in the model. In analogy to the well-known criteria like the AIC introduced in [2] or BIC constructed in [58], the authors of [18], [60] and [8] have developed the deviance information criterion (DIC) as a Bayesian model choice criterion. The DIC solves the problem of comparing complex hierarchical models (among others Bayesian models) when the number of parameters is not clearly defined. The DIC consists of two components: a term $\bar{D}$ that measures the goodness of fit and a penalty term p_D accounting for the model complexity:

$$\mathrm{DIC} = \bar{D} + p_D. \tag{4.29}$$

The first term can be calculated as follows:

$$\bar{D} = E_{\boldsymbol{\Theta}|\mathbf{R}}\{D(\boldsymbol{\Theta})\} = E_{\boldsymbol{\Theta}|\mathbf{R}}\{-2\log f(\mathbf{R}|\boldsymbol{\Theta})\}, \tag{4.30}$$

[4]Available from http://www.cboe.com/micro/vix/historical.aspx

where $\mathbf{R}$ denotes returns and $\mathbf{\Theta}$ is a vector of parameters. The better the model fits the data, the larger is the likelihood; i.e. smaller values of $\bar{D}$ indicate a better model fit. In fact, since the DIC already includes a penalty term p_D, it could be better thought of as a measure of 'model adequacy' rather than a measure of fit, although these terms can be used interchangeably.

The second component measures the complexity of the model by the effective number of parameters:

$$
\begin{aligned}
p_D &= \bar{D} - D(\bar{\mathbf{\Theta}}) = E_{\mathbf{\Theta}|\mathbf{R}}\{D(\mathbf{\Theta})\} - D\{E_{\mathbf{\Theta}|\mathbf{R}}(\mathbf{\Theta})\} \\
&= E_{\mathbf{\Theta}|\mathbf{R}}\{-2\log f(\mathbf{R}|\mathbf{\Theta})\} + 2\log f(\mathbf{R}|\bar{\mathbf{\Theta}}).
\end{aligned} \tag{4.31}
$$

Since p_D is considered to be the posterior mean of the deviance (average of log-likelihood ratios) minus the deviance evaluated at the posterior mean (likelihood evaluated at average), it can be used to quantify the number of free parameters in the model. Furthermore, defining $-2\log f(\mathbf{R}|\mathbf{\Theta})$ to be the residual information in the data $\mathbf{R}$ conditional on $\mathbf{\Theta}$, and interpreting it as a logarithmic penalty, see [44], [9], p_D can be regarded as the expected excess value of the true over the estimated residual information in data $\mathbf{R}$ conditional on $\mathbf{\Theta}$, and thus, can be thought of as the expected reduction in penalty or uncertainty.

From (4.31) we obtain: $\bar{D} = D(\bar{\mathbf{\Theta}}) + p_D$, and thus, DIC can be rewritten as the estimate of the fit plus twice the number of effective parameters (measure of complexity):

$$
\mathrm{DIC} = D(\bar{\mathbf{\Theta}}) + 2p_D. \tag{4.32}
$$

5 Empirical Analysis

In this section we analyse the performance of the different models using time series of daily log-returns for the benchmarked savings account $\hat{B} = \frac{B}{S}$, where the S&P 500 index is used as benchmark, or numéraire. The time period analysed here covers data from 1 January 1980 to 31 December 2011.[5] Note, the considered time period covers the most prominent crises discussed in

[5]In the following, we discuss the results only for this time period. However, alternative time frames (with or without inclusion of crisis periods) have been used as a robustness check, and the results appear to be quantitatively similar. The results are available from the authors upon request.

the literature, which include the 1987 market crash, the LTCM and the Russian crisis of 1998, the market decline in 2000, the dot-com bubble of 2001, and the subprime and financial crisis of 2007-2009.

In the following, we present the findings on parameter estimates and hence the implications for the presence of money market bubbles discussed in Section 2. Furthermore, we discuss model selection using various selection criteria presented in Section 4. Note however, model comparison is not the primary goal of this study (which is to determine whether a money market bubble is present or not), but merely serves as an indication for the best model under consideration. Given the information about the presence of money market bubbles, it could obviously be used for pricing of contingent claims.

Table 1 summarises parameter estimates for different stochastic volatility model specifications. Note, the reported results are obtained using daily returns in percentages defined as $100 \times (\log(\hat{B}_t) - \log(\hat{B}_{t-1}))$. First of all, we find a large negative correlation ρ between fluctuations in the return and the variance, ranging from -0.53 to -0.24, indicating that the leverage effect is the strongest (weakest) for the Heston (3/2L) model specifications. The long-run mean of the variance θ ranges between 0.36 and 0.46 indicating long-run yearly volatility $\sqrt{252 \times \theta}$ to fall in a range between 9.5% and 10.8%. Note that κ is not directly comparable between linear ($a = 0$) and nonlinear ($a = 1$) models, see [13]. Among linear model specifications κ tends to be larger for the Heston and GARCH models, indicating greater speed of adjustment to the long-run variance level. For the non-linear case (3/2N model) the speed of mean reversion is given by κV_t.

Finally, σ is comparable for a given diffusion specification, i.e., for a given b, but not across different diffusion specifications, see [13]. For the 3/2N and 3/2L models one observes similar values for σ corresponding to about 0.12. Note that our results are in line with those reported in the literature, see e.g. [20] and [13][6], even if we work with the benchmarked savings account

[6] [13] use time frame from 1996 to 2004, which does not incorporate the most pronounced market crashes as in 1987 as well as the recent financial crisis. We have also used this time period to estimate the parameters, and the results are consistent with [13].

rather than with the S&P 500.[7]

We now discuss the possible existence of money market bubbles: Note, under the Heston model the dynamics of $\hat{B}$ is always a martingale. We observe that for the 3/2N, 3/2L, as well as, GARCH models the inequalities formulated in Section 2 for the presence of martingales, are violated, indicating that the price processes for all these models resemble strict local martingales. As an implication of these results, we are now interested to learn which of the considered models provides the best fit to the data.

To decide on model performance, to start with we first use visual inspection by looking at Figures 6.3 and 6.4 that show the QQ probability plot of residuals in the return and the variance equations, respectively.[8] The QQ plot contrasts the quantiles of the estimated residuals with the quantiles from the standard normal distribution. A deviation of the residual data from the 45 degrees line indicates strong non-normality of residuals and thus, the evidence of misspecification. From the upper left panel of the two figures, which correspond to the Heston model, we observe that the fat low tail for the return residuals (corresponding to negative returns), as well as, the fat upper tail for the variance residuals (corresponding to high volatility) could not be captured appropriately. All alternative model specifications seem to perform similarly (and outperform the Heston model) based on the QQ plot for return innovations (Figure 6.3), whereas the GARCH model tends to slightly outperform the competing models in capturing well high negative returns. The latter result becomes more pronounced in Figure 6.4 when examining variance residuals: Here, the QQ plot for the GARCH model shows no severe deviation from the 45 degrees line.

Secondly, we use the DIC statistic to rank the models. Comparing DIC values across different model specifications, we observe that the 3/2N model is ranked first, whereas the Heston model is ranked last.[9] Note that this study considers stochastic volatility models without incorporating

[7] Remember, in order to make the results comparable, the parameter estimates have to be converted into the units used, as e.g., yearly values in [13].

[8] This is done by first, estimating the model parameters, and then solving equations (3.13) and (3.14) for ε_t^y and ε_t^v, respectively. If the model would fit data well, the residuals should be Gaussian.

[9] Note, for the applications we have in mind, that is, pricing under the benchmark approach using the real world probability measure, we are interested in those models that allow strict local martingales. Thus, the Heston

jump components. Although several studies treat jumps as secondary effect, if at all (see e.g. [4] and [20]), others find an inclusion of the jump components essential (see e.g. [7], [20] and [11]). Naturally, including jumps (in returns or returns and volatility) might result in different ranking, and as documented e.g. in [33], the best performing stochastic volatility specification does not necessarily lead to the best performing overall specification when a jump component is added to it. Since the objective of this study is to determine whether different model specifications allow strict local martingales (as supported by empirical evidence), we stay within the most parsimonious model specifications, that is, stochastic volatility models (without jumps). However, our methodology can be extended to include jumps.

Finally, we compare model performance by their ability to capture observed volatility provided by the VIX. Figure 6.2 shows the annualised daily spot volatility path $\sqrt{V_t}$[10] using the dotted line for each model specification, whereas the solid green line represents the VIX volatility index. All volatility paths are shown on the same scale, going from 0% to 100% in annual terms. Although the overall pattern in volatility is similar across models, when volatility increases it tends to do so more sharply for the 3/2N and the 3/2L models, whereas the Heston model exhibits the least number of spikes in volatility compared to the remaining models.[11] Thus, although all models appear to resemble the VIX movements, the Heston model seems to underestimate spikes in the VIX, which could be noticed already visually from the figure. This observation is confirmed when computing model errors reported in Table 2. Here, we present summary statistics for the relative error δ_t defined as the percentage deviation from VIX, $\delta_t = (Vola_t - VIX_t)/VIX_t$, its absolute value $|\delta_t|$ as well as their sum of errors $\sum_{t=1}^{T} \delta_t$ and $\sum_{t=1}^{T} |\delta_t|$, respectively. We observe small negative values for the relative errors δ_t, ranging from -0.46% to -0.18%, indicating that all models tend to slightly underestimate the VIX. While the Heston model under-performs all other models, the 3/2L and the GARCH model perform equally well (based on the mean and the median statistics for δ_t). This result also holds when $\sum_{t=1}^{T} \delta_t$ is used as a decision criteria. Note,

model is considered here only as a reference model for performance.

[10]The spot volatility path $\sqrt{V_t}$ is estimated via the Metropolis-Hasting algorithm as described in Section 3.

[11]These observations are in line with [13] and [33].

$|\delta_t|$ ignores the direction of the deviation and does not cancel out under- and over-estimation effects. It leads to comparable values for all models allowing for money market bubbles (with 3/2L being slightly superior to the remaining models), whereas the Heston model, which assumes $\hat{B}$ to be a martingale, underperforms significantly all models under consideration.

6 Conclusion

Bubbles in financial market models have received a lot of attention in the recent literature, though most of it has been focused on asset pricing bubbles, which assume the existence of an EMM. Money market bubbles emerge if and only if an EMM fails to exist, which means that the widely practiced risk neutral pricing is not theoretically justified. However, the benchmark approach, which is formulated under the real world probability measure, is still applicable. Using stochastic volatility models well established in the literature, namely the Heston, the 3/2 with linear (L) and non-linear (NL) drift, and the continuous GARCH limit, we test for the existence of money market bubbles in the US market. For this purpose, we use daily log-returns of the benchmarked savings account from 1 January 1980 to 31 December 2011. We find that all models that can produce money market bubbles do so when we fit these.

Although the main objective of this study was to investigate whether the selected models generate bubbles when fitted, we also compared the model performance based on different selection criteria. We observed that the Heston model performs worst out of all models under consideration. This model does not allow the presence of a money market bubble. All alternative model specifications (3/2N, 3/2L and GARCH) outperform the Heston model, when using as selection criteria the deviance information criterion, QQ-plots, as well as, model errors computed as relative deviations from the observed volatility (VIX). Comparing model performance across those which allow for money market bubbles, we observe mixed results: While we tend to favour the 3/2L and the GARCH model based on the smallest model errors and QQ-plots, respectively; the 3/2N is superior to other models when using DIC as a decision criteria.

Model	Heston	3/2N	3/2L	GARCH
(a,b)	(0.0;0.5)	(1.0;1.5)	(0.0;1.5)	(0.0;1.0)
κ	0.0609 (0.0053)	0.0347 (0.0037)	0.0117 (0.0040)	0.0531 (0.0059)
θ	0.3604 (0.0237)	0.3696 (0.0426)	0.4660 (0.1100)	0.3860 (0.0202)
σ_v	0.1378 (0.0079)	0.1285 (0.0070)	0.1243 (0.0070)	0.1432 (0.0093)
ρ	-0.5356 (0.0325)	-0.4726 (0.0381)	-0.2462 (0.0349)	-0.4172 (0.0344)
DIC	22,276 (4)	22,047 (1)	22,145 (2)	22,173 (3)

Table 1: Parameter estimates and their standard errors (in parentheses) for different stochastic volatility model specifications. The reported parameter estimates are obtained using data on the S&P 500 benchmarked savings account daily returns during the time period from 1 January 1980 to 31 December 2011.

The results presented in this study lead to several questions, which motivate further research. In particular, its implications can be used when pricing and hedging derivative contracts under the benchmark approach like, put and call options, as well as, long-dated insurance claims, exploiting the presence of a money market bubble.

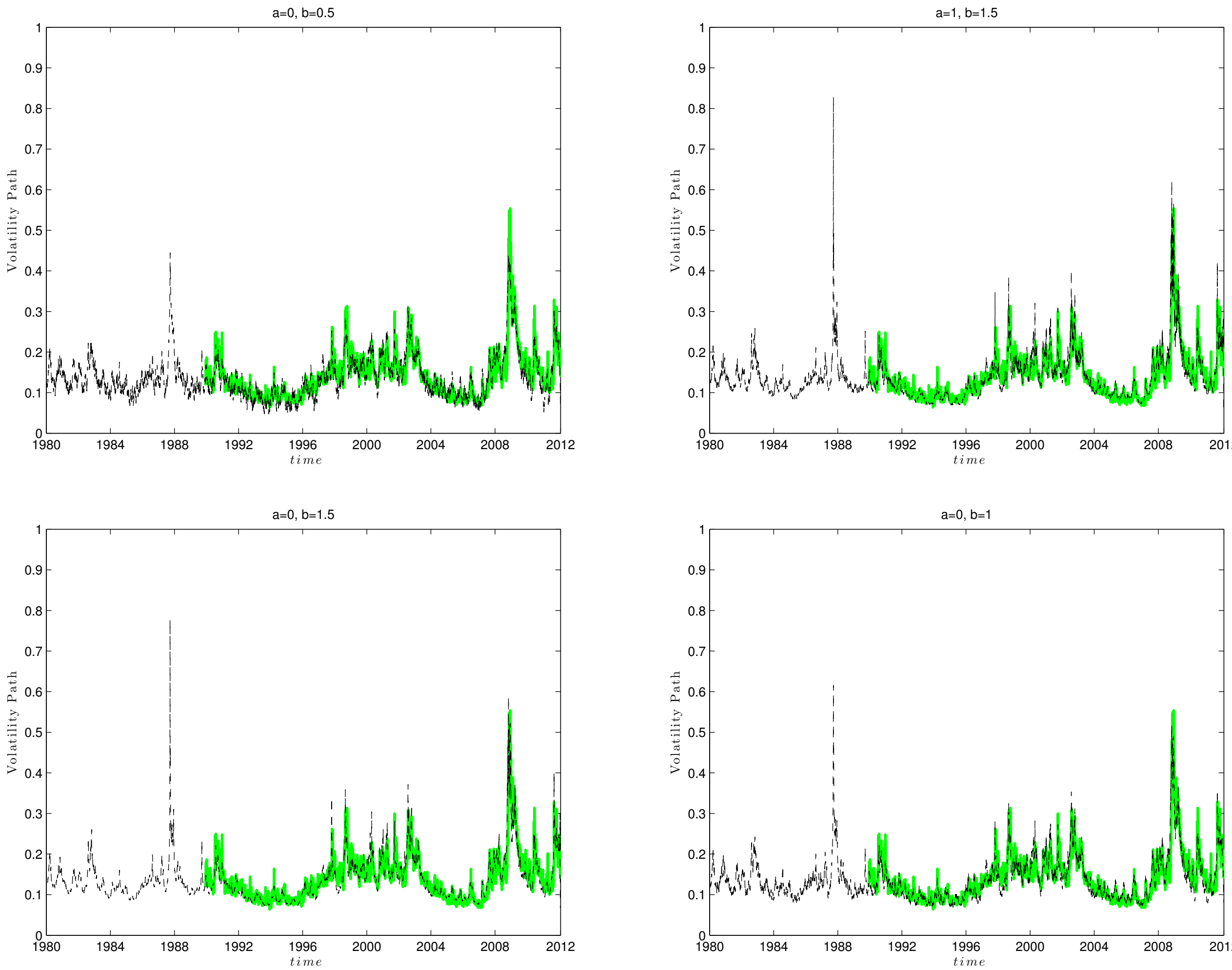

Figure 6.2: Annualised daily spot volatility path $\sqrt{V_t}$ (dotted line) during time period from 1 January 1980 to 31 December 2011 for each stochastic volatility model specification: Heston (upper left), 3/2 (upper right), 3/2 with linear drift (lower left), GARCH (lower right); the VIX volatility index (solid green line).

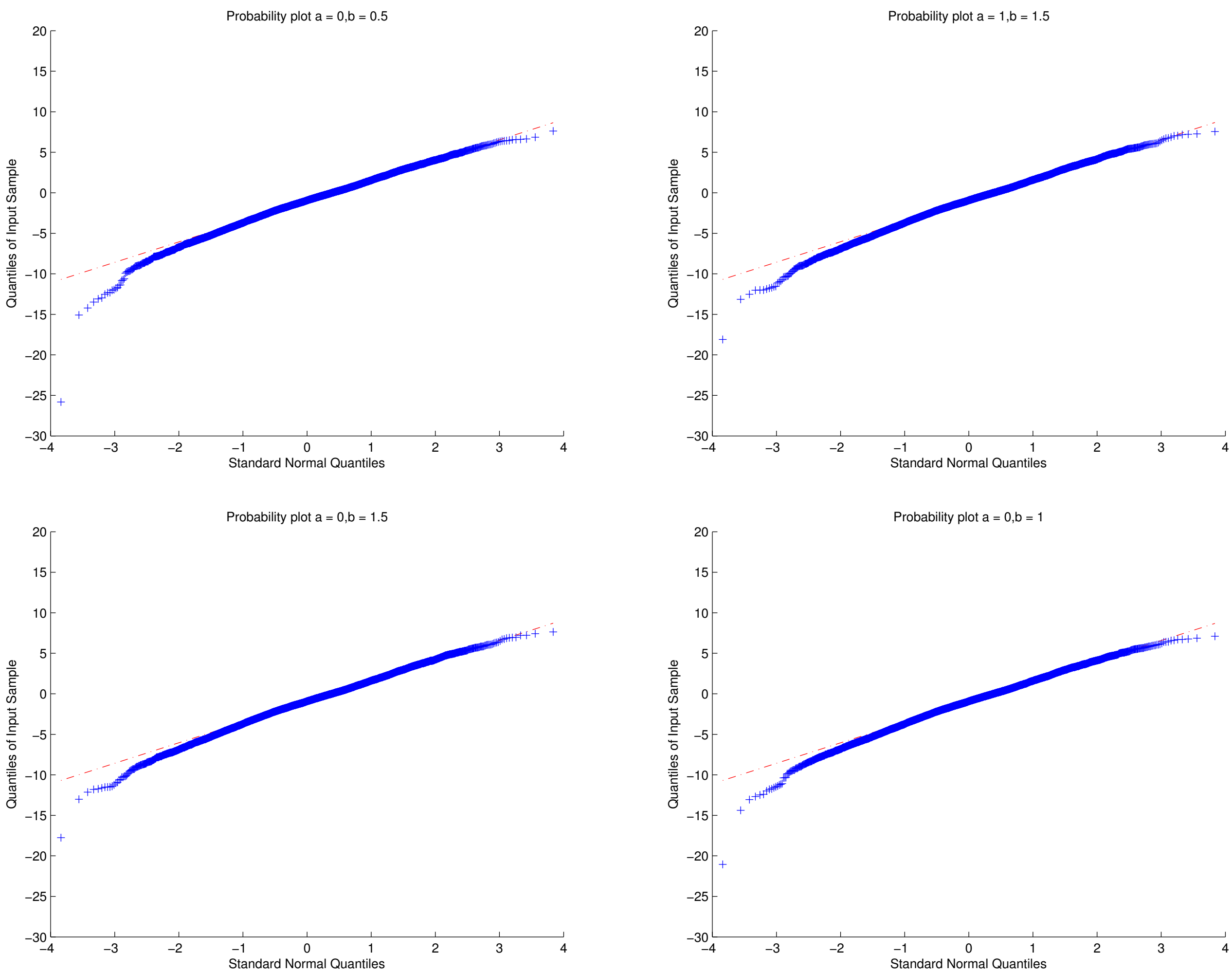

Figure 6.3: The QQ probability plot: quantiles from the return equation against quantiles from the normal distribution for each stochastic volatility model specification: Heston (upper left), 3/2 (upper right), 3/2 with linear drift (lower left), GARCH (lower right).

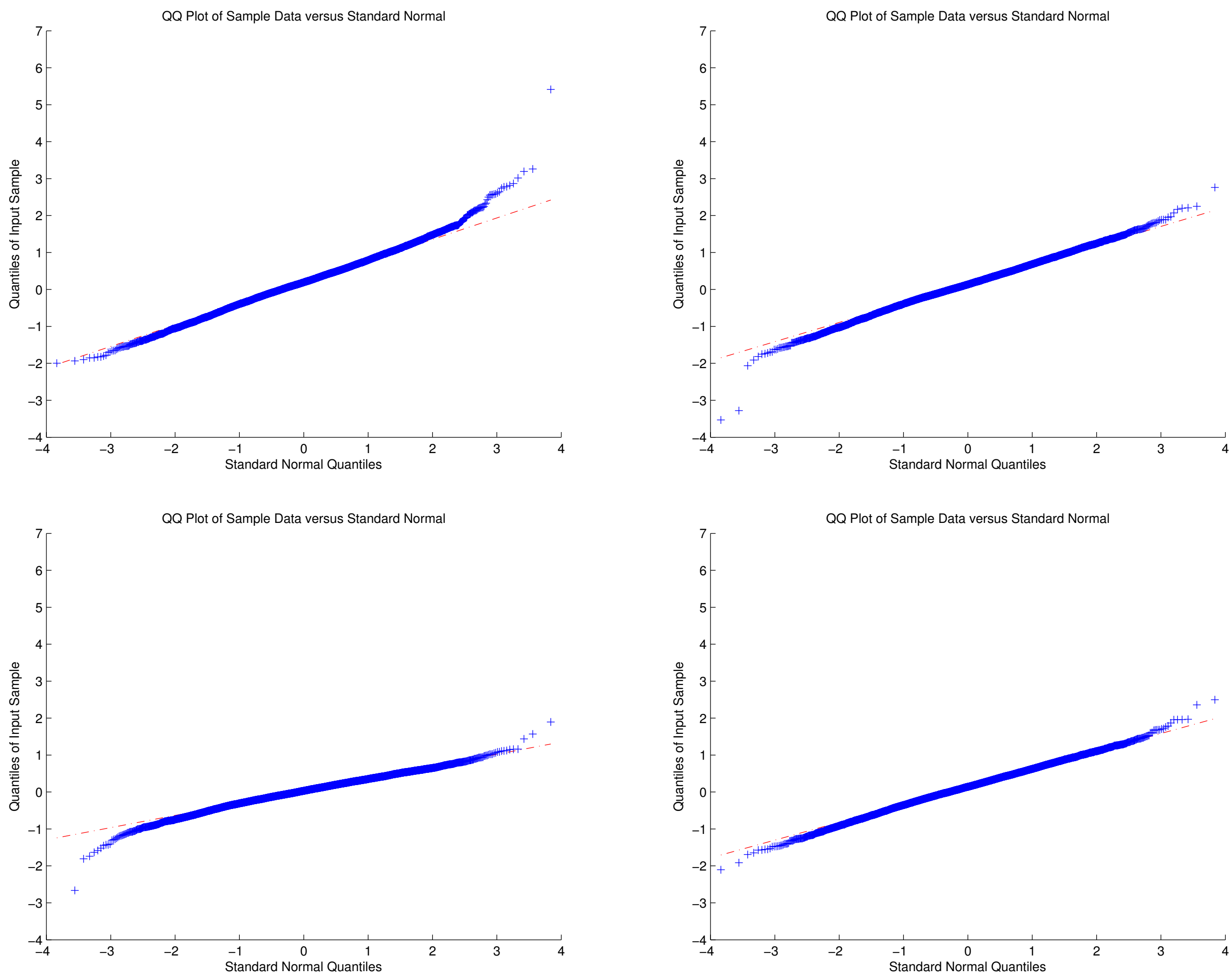

Figure 6.4: The QQ probability plot: quantiles from the variance equation against quantiles from the normal distribution for each stochastic volatility model specification: Heston (upper left), 3/2 (upper right), 3/2 with linear drift (lower left), GARCH (lower right).

Model	**Heston**	**3/2N**	**3/2L**	**GARCH**
(a,b)	(0.0;0.5)	(1.0;1.5)	(0.0;1.5)	(0.0;1.0)
Summary statistics for $\delta_t = (Vola_t - VIX_t)/VIX_t$				
mean	-0.0046 (4)	-0.0043 (3)	-0.0018 (1)	-0.0020 (2)
std.dev	0.1800	0.1586	0.1572	0.1621
median	-0.0207 (3)	-0.0216 (4)	-0.0169 (2)	-0.0165 (1)
min	-0.5428	-0.6954	-0.5782	-0.4978
max	1.0486	1.1853	1.1367	0.9200
Summary statistics for $\lvert\delta_t\rvert = \lvert Vola_t - VIX_t\rvert/VIX_t$				
mean	0.1403 (4)	0.1198 (2)	0.1186 (1)	0.1253 (3)
std.dev	0.1128	0.1040	0.1032	0.1028
median	0.1153 (4)	0.0986 (2)	0.0968 (1)	0.1035 (3)
min	0.0000	0.0000	0.0000	0.0000
max	1.0486	1.1853	1.1367	0.9200
Sum of errors				
$\sum_{t=1}^{T} \delta_t$	-25.4774 (4)	-23.5729 (3)	-9.8642 (1)	-11.2436 (2)
$\sum_{t=1}^{T} \lvert\delta_t\rvert$	777.3140 (4)	663.5747 (2)	656.9741 (1)	694.2380 (3)

Table 2: Model errors with ranking (in parentheses) for different stochastic volatility model specifications. The reported models are obtained using data on the benchmarked (with S&P 500) savings account daily log-returns during the time period from 1 January 1980 to 31 December 2011.

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7 Appendix: Description of the MCMC Algorithm

7.1 Model

The discretised model for the logarithm of the benchmarked savings account $Y_t = \log \hat{B}_t$ takes the form for the time step size $\Delta = 1$:

$$\begin{aligned} Y_t - Y_{t-1} = R_t &\approx -\frac{1}{2}V_{t-1} - \sqrt{V_{t-1}}\varepsilon_t^y \\ V_t - V_{t-1} &\approx \kappa(V_{t-1})^a(\theta - V_{t-1}) + \sigma(V_{t-1})^b\varepsilon_t^v \end{aligned} \tag{7.33}$$

where $\varepsilon_t^y = W^2(t) - W^2(t-1)$ and $\varepsilon_t^v = W^1(t) - W^1(t-1)$ are standard normal random variables with correlation ρ.

7.2 Priors

In order to simplify the MCMC algorithm, we rely on conjugate priors for the model parameters. These are chosen in line with the previous literature, see e.g. [20], [34] and [33] to guarantee a low level of information. Denoting by $\mathcal{N}(\mu, \sigma^2)$ a normal distribution with mean μ and variance σ^2 and by $\mathcal{IG}(\alpha, \beta)$ an inverse Gamma distribution with a shape parameter α and a scale parameter β, we have

- $\theta \sim \mathcal{N}(0, 1)$
- $\kappa \sim \mathcal{N}(0, 1)$
- $\omega^2 = \sigma^2(1 - \rho^2)$; $\omega \sim \mathcal{IG}(2, 200)$
- $\psi = \sigma\rho$; $\psi|\omega^2 \sim \mathcal{N}(0, 1/2\omega^2)$

7.3 Posteriors

Whenever possible, we use conjugate priors for the parameters in order to derive the complete conditionals. The only parameter (latent variable) that requires a Metropolis-Hastings step is the variance. The posterior is given by the following expression

$$p(\mathbf{\Theta}, \mathbf{V}|\mathbf{R}) \propto p(\mathbf{R}|\mathbf{\Theta}, \mathbf{V})p(\mathbf{\Theta}, \mathbf{V}) = p(\mathbf{R}|\mathbf{\Theta}, \mathbf{V})p(\mathbf{V}|\mathbf{\Theta})p(\mathbf{\Theta}).$$

Note the following about the last expression

$$\begin{aligned} p(\mathbf{R}|\mathbf{\Theta}, \mathbf{V})p(\mathbf{V}|\mathbf{\Theta}) &\propto \prod_{t=1}^{T} p(R_t|V_t, V_{t-1}, \mathbf{\Theta})p(V_t|V_{t-1}, \mathbf{\Theta}) \\ &= \prod_{t=1}^{T} p(R_t, V_t|V_{t-1}, \mathbf{\Theta}) \end{aligned}$$

since by conditional independence we have $p(\mathbf{R}|\mathbf{\Theta}, \mathbf{V}) = \prod_{t=1}^{T} p(R_t|V_t, V_{t-1}, \mathbf{\Theta})$, and by the Markov property we have $p(\mathbf{V}|\mathbf{\Theta}) \propto \prod_{t=1}^{T} p(V_t|V_{t-1}, \mathbf{\Theta})$.

From equation (7.33) we see that the joint distribution is bivariate normal with the following parameters

$$\begin{aligned} \boldsymbol{\mu} &= \begin{pmatrix} -\frac{1}{2}V_{t-1} \\ V_{t-1} + \kappa V_{t-1}^{a}(\theta - V_{t-1}) \end{pmatrix} \\ \mathbf{\Sigma} &= \begin{pmatrix} V_{t-1} & -\rho\sigma V_{t-1}^{b+0.5} \\ -\rho\sigma V_{t-1}^{b+0.5} & \sigma^2 V_{t-1}^{2b} \end{pmatrix}. \end{aligned} \tag{7.34}$$

Given normality of the joint distribution we can determine $p(R_t|V_t, V_{t-1}, \mathbf{\Theta})$. It is also normal with parameters given by

$$\begin{aligned} \mu_{R_t|V_t} &= -\frac{1}{2}V_{t-1} - \rho\sigma^{-1}V_{t-1}^{0.5-b}(V_t - V_{t-1} - \kappa V_{t-1}^{a}(\theta - V_{t-1})) \\ \sigma^2_{R_t|V_t} &= V_{t-1}(1-\rho^2). \end{aligned} \tag{7.35}$$

On the other hand, for $p(V_t|R_t, V_{t-1}, \mathbf{\Theta})$ we obtain a normal distribution with

$$\begin{aligned} \mu_{V_t|R_t} &= V_{t-1} + \kappa(V_{t-1})^{a}(\theta - V_{t-1}) - \rho\sigma V_{t-1}^{b-0.5}(R_t + \frac{1}{2}V_{t-1}) \\ \sigma^2_{V_t|R_t} &= \sigma^2 V_{t-1}^{2b}(1-\rho^2). \end{aligned}$$

We can rewrite the posterior as follows:

$$p(\mathbf{\Theta}, \mathbf{V}|\mathbf{R}) \propto \prod_{t=1}^{T} p(R_t, V_t|V_{t-1}, \mathbf{\Theta})p(\mathbf{\Theta}).$$

Since we cannot sample from the posterior we have to construct the complete conditionals, in order to use MCMC. The algorithms to draw from the complete conditional in each iteration is outlined below. Most of the complete conditional distributions can be derived using standard results in, e.g., [10].

- **Drawing** θ: The complete conditional distribution has the form of a regression with

  $$y_t = \left[V_t - V_{t-1} + \kappa V_{t-1}^{a+1} + \rho\sigma V_{t-1}^{b-0.5}(R_t + \frac{1}{2}V_{t-1})\right] / \left[\sigma V_{t-1}^{b}\sqrt{1-\rho^2}\right]$$

  as a dependent variable and

  $$x_t = \left[\kappa V_{t-1}^{a}\right] / \left[\sigma V_{t-1}^{b}\sqrt{1-\rho^2}\right]$$

  as an explanatory variable. The regression has a known error variance of one.

- **Drawing** κ: The complete conditional distribution has the form of a regression with

$$y_t = \left[V_t - V_{t-1} + \rho\sigma V_{t-1}^{b-0.5}(R_t + \frac{1}{2}V_{t-1})\right] / \left[\sigma V_{t-1}^{b}\sqrt{1-\rho^2}\right]$$

as a dependent variable and

$$x_t = \left[V_{t-1}^{a}(\theta - V_{t-1})\right] / \left[\sigma V_{t-1}^{b}\sqrt{1-\rho^2}\right]$$

as an explanatory variable. The regression has a known error variance of one.

- **Drawing** ρ **and** σ: We follow [34] and use reparametrisation $\psi = \sigma\rho$ and $\omega^2 = \sigma^2(1-\rho^2)$. This yields to the regression with

$$y_t = \left[V_t - V_{t-1} - \kappa V_{t-1}^{a}(\theta - V_{t-1})\right] / \left[V_{t-1}^{b}\right]$$

and

$$x_t = -V_t^{-1/2}(R_t + \frac{1}{2}V_{t-1}).$$

The unknown homoscedastic error variance is given by $\omega^2 = \sigma^2(1-\rho^2)$. Using this setup, we are able to draw ψ and ω^2. The parameters of interest are then given by $\sigma^2 = \omega^2 + \psi^2$ and $\rho = \psi/\sigma$.

- **Drawing** V_t: Variance V_t is a latent variable that does not have a recognizable complete conditional and therefore, we rely on the Metropolis-Hastings algorithm. As outlined in Section 3, the complete conditional distribution for V_t can be written as

$$p(V_t|\mathbf{\Theta}, \mathbf{R}) \propto p(R_{t+1}, V_{t+1}|V_t, \mathbf{\Theta})p(V_t|R_t, V_{t-1}, \mathbf{\Theta}),$$

where $p(R_{t+1}, V_{t+1}|V_t, \mathbf{\Theta})$ follows the bivariate normal distribution and $p(V_t|R_t, V_{t-1}, \mathbf{\Theta})$ follows a univariate normal distribution. The Metropolis-Hasting step proposes a new variance $V_t^{(i)}$ in the i^{th} draw by drawing from $p(V_t|R_t, V_{t-1}, \mathbf{\Theta})$ and accepting that draw with probability

$$\min\left\{\frac{p(R_{t+1}, V_{t+1}|V_t^{(i)}, \mathbf{\Theta})}{p(R_{t+1}, V_{t+1}|V_t^{(i-1)}, \mathbf{\Theta})}, 1\right\}.$$