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# Lie Symmetry Methods for Local Volatility Models

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# Lie Symmetry Methods for Local Volatility Models

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**Abstract**

We investigate PDEs of the form $u_t = \frac{1}{2}\sigma^2(t,x)u_{xx} - g(x)u$ which are associated with the calculation of expectations for a large class of local volatility models. We find nontrivial symmetry groups that can be used to obtain standard integral transforms of fundamental solutions of the PDE. We detail explicit computations in the separable volatility case when $\sigma(t,x) = h(t)(\alpha + \beta x + \gamma x^2)$, $g = 0$, corresponding to the so called Quadratic Normal Volatility Model. We also consider choices of $g$ for which we can obtain exact fundamental solutions that are also positive and continuous probability densities.

**Key words:** Lie symmetries, fundamental Solution, PDEs, Local Volatility Models, Normal Quadratic Volatility Model.

## 1 Introduction

A symmetry of a differential equation is a transformation that maps solutions to solutions. Continuous symmetries which have group properties are called Lie symmetries, since the means for computing them were developed in the late nineteenth century by Sophus Lie. A modern account of the theory can be found in Olver's book, Olver (1993) and the books of Bluman, such as Bluman and Kumei (1989).

The applications of Lie symmetry groups are extensive. These range from epidemiology to finance. Examples of these applications in stochastic analysis and finance may be found in references such as Craddock and Lennox (2007), Craddock and Lennox (2009), Craddock and Platen (2004), Lennox (2011), Craddock (2009), Craddock and Lennox (2012), Baldeaux and Platen (2013), Goard (2011), Goard (2000), Goard and Mazur (2013), Andersen (2011), Lipton (2002), Zuhlsdorff (2002), Carr et al. (2006), Carr et al. (2013), Cordoni and Di Persio (2014), Grasselli (2016) and Itkin (2013). This list is by no means exhaustive.

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The aim of this paper is to approach the classical local volatility models from the perspective of Lie symmetry analysis of the associated backward PDEs. The concept of a symmetry group of a PDE was introduced by Sophus Lie in the 1880s, see the collection Lie (1912). Lie's methods have recently been applied to problems in stochastic analysis by, among others, Craddock and Platen (2004), Craddock and Lennox (2007) and Craddock and Lennox (2009), who focused mainly on "generalized square root" models of the form

$$dX_t = f(X_t)dt + X_t^\gamma dW_t,$$

for some constant $\gamma$. These models have obvious financial interpretations.

In this paper, we focus on (local) martingale models of the form

$$dX_t = \sigma(t, X_t)dW_t, \tag{1}$$

together with the associated parabolic backward equation

$$\partial_t u = \frac{1}{2}\sigma^2(t, x)\partial_{xx}u. \tag{2}$$

Here and throughout $\partial_x = \frac{\partial}{\partial x}$ etc. For suitable choices of $\sigma$, this PDE has many symmetries. If we have at hand a Lie Group that leaves the PDE invariant, then we can apply it to any solution to get another solution. The key result for finding the Lie Groups admitted by the PDE is Lie's Theorem. For a PDE of order $n$, we write down the infinitesimal generator $\mathbf{v}$ of the symmetry and obtain its so called $n$-th *prolongation* $\mathrm{pr}^n\mathbf{v}$ (see below). Lie's Theorem says that $\mathbf{v}$ generates a symmetry of the PDE

$$P(x, D^\alpha u) = 0,\ x \in \Omega \subseteq \mathbb{R}^m,\ D^\alpha u = \partial_{x_1^{\alpha_1}\cdots x_n^{\alpha_m}} u,\ \alpha_1 + \cdots + \alpha_m \leq n,$$

if and only if $\mathrm{pr}^n\mathbf{v}[P(x, D^\alpha u)] = 0$ whenever $P(x, D^\alpha u) = 0$. See Olver (1993), Chapter 2.

This condition leads to an equation in a set of independent functions of the derivatives of $u$. As the equation must be true for arbitrary values of these independent functions, their coefficients must vanish, leading to a linear system of equations known as the Determining Equations. Once these equations are solved, one can find the corresponding Lie group admitted by the PDE. For linear parabolic PDEs on the line, Craddock proved that it is always possible to find a symmetry that maps a nonzero solution to a generalised Fourier or Laplace transform of a fundamental solution, see Craddock and Dooley (2010). The question of under what conditions these fundamental solutions yield transition densities for the underlying stochastic process was addressed in Craddock and Lennox (2009) and Craddock (2009).

In this paper we give sufficient conditions directly on $\sigma$ for the existence of a Lie group admitted by the PDE (2). We compute symmetries which lead directly to a Fourier integral transform of the fundamental solution of the backwards PDE.

In the special case where $\sigma(t, x) = h(t)(\alpha + \beta x + \gamma x^2)$ (i.e. with $\sigma$ a separable function with a polynomial of degree two dependence on the state variable $X_t$) we exponentiate the group to explicitly find the symmetries of the equation by considering separately the cases of two distinct real roots, a single real root and two distinct complex roots. This local volatility case corresponds to the so called Quadratic Normal Volatility model that has been investigated, among others, by Zuhlsdorff (2002), Andersen (2011) and Carr et al. (2013). For this model, we provide the explicit expression for the (positive) fundamental solution that are also probability densities, thus giving an analytical counterpart to the probabilistic justification of the tractability of this model presented in Carr et al. (2013).

For pricing, we have the following generic situation. Generally one can show, for example by arbitrage arguments, that the price $u(S,t)$ for a derivative security on an underlying $S \in \Omega \subseteq \mathbb{R}^n$, is given by a solution of the terminal value problem for the parabolic PDE

$$u_t + Lu = 0,\ u(S,T) = f(S), S \in \Omega \subseteq \mathbb{R}^n, t \in [0,T]. \tag{3}$$

One transforms this to the initial value problem

$$u_t = Lu = 0,\ u(S,0) = f(S), S \in \Omega \subseteq \mathbb{R}^n, t \in [0,T], \tag{4}$$

by letting $t \to T - t$. Clearly we require a unique, continuous price. For hedging we also require the existence of a certain number of derivatives. This amounts to asking under which circumstances the Cauchy problem (4) has a unique, smooth solution. There is extensive literature on the question of existence and uniqueness of solutions. We cannot survey it all here, but Theorem 16 of Friedman (1964) tells us that if $u_t = Lu$ is uniformly parabolic, with smooth coefficients having bounded first and second derivatives in $\Omega \times [0,T]$ then (4) has a unique solution satisfying $\int_0^T \int_{\mathbb{R}^n} |u(x,t)| \exp(-k|x|^2) dxdt < \infty$ for some constant $k > 0$. This solution will also be smooth if $f$ is not too pathological. Generally speaking, we get non-uniqueness for smooth solutions only when the initial data grows extremely rapidly. This is precisely the situation in the famous example of non-uniqueness for the heat equation that was constructed by Tychonoff Tychonoff (1935). Financial considerations typically rule out such pathological situations.

Thus for pricing, we require fundamental solutions that return smooth continuous solutions of the pricing PDE, given reasonable payoffs. In this paper we exhibit such fundamental solutions. It is also sometimes possible to obtain fundamental solutions that are not continuous and which return solutions that are discontinuous at some point, typically the origin. We rule out such solutions on the grounds that the price should depend continuously on the underlying.

## 2 Lie Symmetries for PDEs

In this Section, we will give a short introduction to Lie Groups of transformations with a particular focus on their applications to the solution of PDE's. In this introduction we follow closely the presentation given in Bluman and Kumei (1989). Another standard reference for this topic is Olver (2000). Then we will focus on the relations between symmetry analysis of PDE's and transform of solutions of PDE's as studied in Craddock and Platen (2004), Craddock and Lennox (2007) and Craddock and Lennox (2009).

## 2.1 Invariance of a Differential Equation

Let us consider now the second order PDE in two independent variables $(x,t)$ of the form

$$F(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt}) = 0. \tag{5}$$

We work exclusively on a second order PDE in two independent variables since this is the setting which interests us here. Of course the results we are about to discuss are valid for ODE's as well and for differential equations of any order in general.

We now give a definition that will be crucial in the following. The PDE (5) is said to admit the Lie Group of transformations $(X_\epsilon, T_\epsilon, U_\epsilon)_\epsilon$, for $\epsilon > 0$, if the family of solutions of (5) is an invariant family of surfaces for $(X_\epsilon, T_\epsilon, U_\epsilon)_\epsilon$.

In other words, this implies that if we have at hand a Lie Group $G$ that leaves the PDE invariant, then we can apply it to any solution to get another solution. If the solution we start from is itself an invariant surface, then by applying the Lie Group to it, we will end up with the solution itself. Here the crucial concept is that the Lie Group must act on the space where the solutions of the PDE lives, in our case the $(x,t,u)$-space.

We introduce a vector field

$$\mathbf{v} = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \phi(x,t,u)\partial_u, \tag{6}$$

which is the infinitesimal generator of $G$. We can extend the action of $G$ in a natural way to act also on the derivatives of $u$ up to second order, by essentially requiring that the chain rule holds. We call this the second prolongation of $G$ and denote it by $G^{(2)}$. The generator of $G^{(2)}$ is the second prolongation of $\mathbf{v}$. This is given by (see e.g. Bluman and Kumei (1989) and Olver (1993)):

$$\mathrm{pr}^2\mathbf{v} = \mathbf{v} + \phi^x\frac{\partial}{\partial u_x} + \phi^t\frac{\partial}{\partial u_t} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{xt}\frac{\partial}{\partial u_{xt}} + \phi^{tt}\frac{\partial}{\partial u_{tt}}.$$

Expressions for $\phi^t$ etc are given by the prolongation formula given below.

The main result due to Lie (see e.g. Bluman and Kumei (1989)) is that the PDE (5) admits the Lie Group of transformations $(X_\epsilon, T_\epsilon, U_\epsilon)$ if and only if

$$F(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt}) = 0 \to (G^{(2)}F)(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt}) = 0,$$

that is the surface $\{F(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt}) = 0\}$ is invariant for the second prolongation $G^{(2)}$ of the group action $G$. We refer e.g. to Bluman and Kumei (1989) for the construction of the second prolongation for a group. At the level of vector fields, this leads to Lie's Theorem which states that $\mathbf{v}$ generates a one parameter group of symmetries if and only if

$$\mathrm{pr}^2\mathbf{v}[F(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt})] = 0$$

whenever $F(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt}) = 0$.

The coefficients $\phi^x, \phi^t$ etc are given by the prolongation formula, first published by Olver (1978).

$$\phi_j^{\mathbf{J}}(x,u^{(n)}) = D_{\mathbf{J}}\big(\phi_j - \sum_{i=1}^{p}\xi^i u_i^j\big) + \sum_{i=1}^{p}\xi^i u_{\mathbf{J},i}^j, \tag{7}$$

where $u^j_i = \dfrac{\partial u^j}{\partial x^i}$, and $u^j_{\mathbf{J},i} = \dfrac{\partial u^j_{\mathbf{J}}}{\partial x^i}$, and $D_{\mathbf{J}}$ is the total differentiation operator.

In practice, we use the variable names rather than the multi-indices in the exponents. So we write $\phi^{xx}$ rather than $\phi^{1,1}$. The coefficient functions $\phi^x$ and $\phi^t$ are

$$\begin{aligned}
\phi^x &= D_x\left(\phi - \xi u_x - \tau u_t\right) + \xi u_{xx} + \tau u_{xt}\\
&= \big(\phi_x + \phi_u u_x - \xi_x u_x - \xi_u u_x^2 - \xi u_{xx} - \tau_x u_t - \tau_u u_x u_t - \tau u_{xt}\big)\\
&\quad + \xi u_{xx} + \tau u_{xt}\\
&= \phi_x + (\phi_u - \xi_x)u_x - \xi_u u_x^2 - \tau_x u_t - \tau_u u_x u_t.
\end{aligned}$$

Similarly, $\phi^t = D_t\left(\phi - \xi u_x - \tau u_t\right) + \xi u_{xt} + \tau u_{tt}$, which leads to

$$\phi^t = \phi_t - \xi_t u_x + (\phi_u - \tau_t)u_t - \xi_u u_x u_t - \tau_u u_t^2,$$

and

$$\begin{aligned}
\phi^{xx} &= D_{xx}\left(\phi - \xi u_x - \tau u_t\right) + \xi u_{xxx} + \tau u_{xxt}\\
&= \phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (\phi_{uu} - 2\xi_{xu})u_x^2 - 2\tau_{xu}u_x u_t\\
&\quad - \xi_{uu}u_x^3 - \tau_{uu}u_x^2 u_t + (\phi_u - 2\xi_x)u_{xx} - 2\tau_x u_{xt} - 3\xi_u u_x u_{xx}\\
&\quad - \tau_u u_t u_{xx} - 2\tau_u u_x u_{xt}.
\end{aligned}$$

The coefficients $\phi^{xt}$ and $\phi^{tt}$ can be obtained in the same manner, but we do not need them for our calculations.

## 2.2 Symmetries and Fundamental Solutions

In a series of articles of increasing generality, Craddock and his coauthors studied the symmetries of some Kolmogorov backward equations associated to a real diffusion. Namely, Craddock and Platen (2004) studied the equation

$$\partial_t u = x\partial_{xx}u + f(x)\partial_x u, \quad x \in \mathbb{R}_+$$

whereas Craddock and Lennox (2007) studied the equation

$$\partial_t u = \sigma x^\gamma \partial_{xx}u + f(x)\partial_x u - \mu x^r u, \quad x \in \mathbb{R}$$

and Craddock and Lennox (2009) studied the equation

$$\partial_t u = \sigma x^\gamma \partial_{xx}u + f(x)\partial_x u - g(x)u, \quad x \in \mathbb{R}. \tag{8}$$

In Craddock and Dooley (2010), it was shown that if the PDE

$$u_t = A(x,t)u_{xx} + B(x,t)u_x + C(x,t)u, \tag{9}$$

has a four dimensional Lie algebra of symmetries, then there always exists a symmetry mapping a nontrivial solution $u$ to a generalised Laplace transform of a product of a fundamental solution and $u$. Dividing out by $u$ yields the desired fundamental solution. If the symmetry algebra is six dimensional, then we obtain a generalised Fourier type transform.

Lie proved that if a PDE of the form (9) has a six dimensional group of symmetries, then it may be transformed to the heat equation by an invertible change of variables. If the symmetry group is four dimensional, then it can be reduced to the form

$$u_t = u_{xx} - \frac{A}{x^2}u, \ A \neq 0, \tag{10}$$

which has a known fundamental solution. Thus if one is interested only in obtaining a fundamental solution for a PDE, we can attempt to reduce it to (10) or the heat equation as appropriate. This procedure has been followed by e.g. Cordoni and Di Persio (2014), Carr et al. (2006), Goard (2000) and Itkin (2013). However it was shown in Craddock (2009), that this will not necessarily produce a fundamental solution which is also a probability density. There are other issues. A simple boundary value problem on, say $[0, 1]$ may be mapped to a more difficult problem for the heat equation (or (10)).

To illustrate this, let $\sigma(x) = b\left(1 + \left(\frac{x-a}{b}\right)^2\right)$ and consider the problem

$$u_t = \frac{1}{2}\sigma^2(x)u_{xx}, \ x \in \mathbb{R}$$

with $u(x, 0) = f(x)$, where $f$ lies in some appropriate function space, say $L^1(\mathbb{R})$. The change of variables $y = \tan^{-1}\left(\frac{x-a}{b}\right)$, $u(x, t) = U\left(\tan^{-1}\left(\frac{x-a}{b}\right), t\right)$ converts this to the PDE

$$\begin{aligned} U_t &= \frac{1}{2}U_{yy} - \tan y U_y, \ y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ U(y, 0) &= f(a + b\tan y). \end{aligned} \tag{11}$$

If we suppose $u$ is zero at $\pm\infty$, then we also require $U(\frac{\pi}{2}, t) = U(-\frac{\pi}{2}, t) = 0$. Using the methodology of Craddock (2009), it is easy to show that the PDE (11) has a fundamental solution $p(t, y, z) = \frac{1}{\sqrt{2\pi t}}e^{t/2 - \frac{(y-z)^2}{2t}}\frac{\cos z}{\cos y}$, defined on all of $\mathbb{R}$ except odd multiples of $\pi/2$. So this does not help us. The further change of variables $U(y, t) = e^{t/2}\sec(y)v(y, t)$ produces the following problem for the one dimensional heat equation:

$$\begin{aligned} v_t &= \frac{1}{2}v_{yy}, \ y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ v(y, 0) &= \cos y f(a + b\tan y), \\ v\left(-\frac{\pi}{2}, t\right) &= v\left(\frac{\pi}{2}, t\right) = 0. \end{aligned}$$

This problem can be solved by Fourier series methods, but it is more complicated than the original problem. In fact as we shall see, it is easier to solve the original problem directly.

A further example is the initial and boundary value problem

$$u_t = u_{xx} + au_x, \ u(x, 0) = f(x), u_x(0, t) = 0. \tag{12}$$

Solution of this problem yields the transition density for a reflected Brownian motion with drift. This is a deceptively difficult problem as it cannot be solved by either the Fourier sine or cosine transform. Methods for its analytical solution can be found in Fokas (2008). They are however well outside the scope of this paper. It is possible to solve it using Lie symmetry methods, similar to those presented here, but we will not do so here.

Here we observe that setting $u(x,t) = e^{-ax/2-a^2t/4}v(x,t)$ reduces problem (12) to

$$v_t = v_{xx},\ v(x,0) = e^{ax/2}f(x), v_x(0,t) = \frac{a}{2}e^{a^2t/4}u(0,t).$$

So we have reduced the PDE to the heat equation, but the resulting boundary value problem cannot be solved as we do not know the value of $u(0,t)$. The moral is that being able to reduce a PDE to the heat equation (or (10)) is not a universal panacea. We can arrive at a problem harder than the original, or indeed one that cannot be solved at all. Therefore it is essential that we have techniques which yield solutions without the need to make any changes of variables.

Note that the equation (8) corresponds to the backward Kolmogorov equations associated to the diffusion which can be defined as a solution of the SDE

$$dX_t = f(X_t)dt + \sqrt{2\sigma}x^{\gamma/2}dW_t,$$

killed at the rate $g(X_t)$. Thus if we define

$$v(x,t) = \mathbb{E}_x\left[e^{-\int_0^t g(X_s)ds}f(X_t)\right],$$

then $v$ is a solution of (8) with $v(.,0) = f$.

If we take $g = 0$ in the argument above and $f(x) = e^{-\lambda x}$ then we have

$$v(x,t) = \mathbb{E}_x\left[e^{-\lambda X_t}\right],$$

which is the Laplace transform of the marginal law of $X_t$. This Laplace transform can in theory be inverted, holding $(x,t)$ fixed, to get the fundamental solution of the PDE which can be interpreted as the transition probability of the diffusion process.

The articles of Craddock and coauthors cited above sought to find solutions of the PDE's with initial datum $f(x) = e^{-\lambda x}$ by exploiting the symmetries of the PDE itself. Specifically, in the conservative case $g = 0$, they were able to find all the symmetries admitted by the PDE and then they noted that there is a symmetry that maps the solution constantly equal to 1 to a solution which is an exponential in $x$ at $t = 0$. In the following section we will pursue a similar goal on an equation arising from a different linear diffusion.

# 3 The Problem

Following Bluman and Kumei (1989) and Olver (1993), we are looking for infinitesimal symmetries of the PDE (2) when $\sigma(t,x) = \sigma(x)$:

$$u_t = \frac{1}{2}\sigma^2(x)u_{xx} - g(x)u, \quad x \in \mathcal{D} \subseteq \mathbb{R},\ \sigma > 0, \tag{13}$$

where $\sigma$ and $g$ are functions defined in the state space domain $\mathcal{D} \subset \mathbb{R}$ which we take to be an interval of the form $\mathcal{D} = (x_l; x_r)$. As noted above, this is the Kolmogorov backward equation associated to a diffusion $X$ satisfying

$$dX_t = \sigma(X_t)dW_t, \tag{14}$$

killed at rate $g(X_t)$, where $W$ is standard Brownian motion defined in a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$.

Local Volatility Models typically include the presence of a time dependence in the volatility, that is $\sigma = \sigma(x,t)$. The special separable case

$$\sigma(x,t) = f(t)h(x), \tag{15}$$

can be managed by the time-change methodology for the Brownian motion, see also Andersen (2011).

The general (not separable) case corresponding to (2) can be managed using the same methodology, but it leads to extremely difficult equations. We prefer to leave this out since little insight can be gained in those cases without a considerable amount of analysis.

*Remark* 3.1. If we specify a boundary condition like $u(x,0) = e^{-\lambda x}$ then, using a Feynman-Kac argument, (13) can be associated to the transform

$$u(x,t) = \mathbb{E}^x\left[\exp\left\{-\lambda X_t - \int_0^t g(X_s)ds\right\}\right].$$

The equation (13) can be transformed to one where the second derivative term has a constant coefficient by introducing a change of variables. Let $y = \int^x \frac{dz}{\sigma(z)} = b(x)$. We suppose that $b$ is invertible, so that $x = b^{-1}(y)$. Then set $u(x,t) = U(b(x),t)$. Then

$$u_x = \frac{1}{\sigma(x)}U_y, \quad u_{xx} = \frac{1}{\sigma^2(x)}U_{yy} - \frac{\sigma'(x)}{\sigma^2(x)}U_y,$$

so that (2) becomes

$$U_t = \frac{1}{2}U_{yy} - \frac{1}{2}\sigma'(b^{-1}(y))U_y - g(b^{-1}(y))U. \tag{16}$$

The more general equation

$$u_t = \frac{1}{2}\sigma^2(x)u_{xx} + k(x)u_x - g(x)u, \tag{17}$$

can be transformed to

$$U_t = \frac{1}{2}U_{yy} + \left(\frac{k(b^{-1}(y))}{\sigma(b^{-1}(y))} - \frac{1}{2}\sigma'(b^{-1}(y))\right)U_y - g(b^{-1}(y))U. \tag{18}$$

Craddock and Lennox (2009) showed that a PDE of the form

$$u_t = \sigma x^\gamma u_{xx} + f(x)u_x - g(x)u,$$

has non-trivial symmetries if and only $h(x) = x^{1-\gamma} f(x)$ satisfies an equation of the form

$$\sigma x h' - \sigma h + \frac{1}{2} h^2 + 2\sigma x^{2-\gamma} g(x) = 2\sigma A x^{2-\gamma} + B \tag{19}$$

$$\sigma x h' - \sigma h + \frac{1}{2} h^2 + 2\sigma x^{2-\gamma} g(x) = \frac{A x^{4-2\gamma}}{2(2-\gamma)^2} + \frac{B x^{2-\gamma}}{2-\gamma} + C, \tag{20}$$

$$\sigma x h' - \sigma h + \frac{1}{2} h^2 + 2\sigma x^{2-\gamma} g(x) = \frac{A x^{4-2\gamma}}{2(2-\gamma)^2} + \frac{B x^{3-\frac{3}{2}\gamma}}{3-\frac{3}{2}\gamma} + \frac{C x^{2-\gamma}}{2-\gamma} - \kappa, \tag{21}$$

with $\kappa = \frac{\gamma}{8}(\gamma - 4)\sigma^2$. See Craddock and Lennox (2009) for the specifics. We remark that on the line, a non-trivial symmetry group will have dimensions of either four or six. Every time homogeneous, linear parabolic PDE on the line has the trivial symmetries $u(x,t) \to cu(x, t+\epsilon)$, which corresponds to a two dimensional symmetry group. A general result describing conditions under which an arbitrary linear parabolic PDE on the real line has non trivial symmetries follows easily.

**Proposition 3.2.** *Let* $F(y) = y\left(\frac{k(b^{-1}(y))}{\sigma(b^{-1}(y))} - \frac{1}{2}\sigma'(b^{-1}(y))\right)$. *Then the PDE* (17) *has non-trivial symmetries if and only if*

$$\frac{1}{2} y F' - \frac{1}{2} F + \frac{1}{2} F^2 + y^2 g(b^{-1}(y)) = A_1 y^4 + B_1 y^2 + C_1 y^{3/2} + D_1,$$

*where* $y = b(x) = \int^x \frac{1}{\sigma(z)} dz$ *and the constants* $A_1, ..., D_1$ *are as in the right side of* (19)-(21), *with* $\gamma = 0$.

This result is at the moment largely of theoretical interest. A more detailed account for higher dimensional equations is in Vu's forthcoming thesis, Vu (2016). For a given $\sigma$, we can obtain $b$ and determine drift functions $k$ for a given $g$ such that the PDE has symmetries and the results of Craddock (2009) can be employed to find fundamental solutions, if the symmetry group is four dimensional. However, if we treat this as an equation for $\sigma$ it is quite difficult to deal with. So here we will derive an explicit equation for $\sigma$ in the case of zero drift and treat in detail a model which arises from this.

# 4 The Determining Equation

By applying the prolongation to the PDE (13) we get

$$\mathrm{pr}^2 \mathbf{v}\left[u_t - \frac{1}{2}\sigma^2 u_{xx} + gu\right] = \phi^t - \sigma(x)\sigma'(x) u_{xx}\xi - \frac{1}{2}\sigma^2(x)\phi^{xx} + g'(x) u\xi + g(x)\phi,$$

which is

$$\phi^t = \sigma(x)\sigma'(x) u_{xx}\xi + \frac{1}{2}\sigma^2(x)\phi^{xx} - g'(x) u\xi - g(x)\phi.$$

Since the PDE is linear, standard arguments (see e.g. Bluman and Kumei (1989)) imply that $\xi$ and $\tau$ will be independent of $u$, that is

$$\xi = \xi(x,t), \ \tau = \tau(x,t),$$

and $\phi$ will be linear in $u$, that is $\phi(x,t,u) = \alpha u + \beta$ for some $\alpha(x,t), \beta(x,t)$. Using this information we obtain

$$\begin{aligned} \beta_t + \alpha_t u + (\alpha - \tau_t) u_t - \xi_t u_x = \ & \sigma(x)\sigma'(x) u_{xx}\xi - g'(x) u\xi - g(x)(\alpha u + \beta) \\ & + \frac{1}{2}\sigma^2(x) \left(\beta_{xx} + \alpha_{xx} u + (2\alpha_x - \xi_{xx}) u_x \right. \\ & \left. - \tau_{xx} u_t + (\alpha - 2\xi_x) u_{xx} - 2\tau_x u_{xt}\right). \end{aligned}$$

Let us now identify the coefficients of the last equation: the constant term gives

$$\beta_t = \frac{1}{2}\sigma^2(x)\beta_{xx} - g(x)\beta,$$

which basically says that $\beta$ satisfies the initial PDE. The coefficient of $u$ yields

$$\alpha_t + \tau_t g(x) = \frac{1}{2}\sigma^2(x)\left(\alpha_{xx} + \tau_{xx} g(x)\right) - g'(x)\xi; \tag{22}$$

while the coefficient of $u_x$ gives

$$-\xi_t = \frac{1}{2}\sigma^2(x)\left(2\alpha_x - \xi_{xx}\right). \tag{23}$$

The coefficient of $u_{xx}$ gives

$$(\alpha - \tau_t)\frac{1}{2}\sigma^2(x) = \sigma(x)\sigma'(x)\xi + \frac{1}{2}\sigma^2(x)\left(-\tau_{xx}\frac{1}{2}\sigma^2(x) + \alpha - 2\xi_x\right) \tag{24}$$

and finally the coefficient of $u_{xt}$ gives

$$2\tau_x = 0, \tag{25}$$

which implies $\tau = \tau(t)$, so that (22) becomes

$$\alpha_t + \tau'(t) g(x) = \frac{1}{2}\sigma^2(x)\alpha_{xx} - g'(x)\xi \tag{26}$$

and from (24) we obtain

$$\xi(x,t) = \sigma(x)\left(\frac{1}{2}\tau'(t)\int^x \frac{1}{\sigma(y)} dy + \rho(t)\right), \tag{27}$$

where $\rho$ is an arbitrary deterministic function[1]. We now differentiate the last expression and we plug the results into equation (23) that becomes

$$\begin{aligned} \alpha_x &= -\frac{1}{\sigma^2(x)}\xi_t(x,t) + \frac{1}{2}\xi_{xx}(x,t) \\ &= -\frac{1}{2}\tau''(t)\frac{1}{\sigma(x)}\int^x \frac{1}{\sigma(y)} dy - \rho'(t)\frac{1}{\sigma(x)} \\ &\quad + \frac{1}{4}\tau'(t)\sigma''(x)\int^x \frac{1}{\sigma(y)} dy + \frac{1}{2}\rho(t)\sigma''(x) + \frac{1}{4}\tau'(t)\frac{\sigma'(x)}{\sigma(x)}, \end{aligned}$$

[1] We do not specify lower boundary of the integral since the constant term can be included into the arbitrary time-dependent function $\rho$.

that we write in a more compact way by dropping the dependence on the arguments:

$$\alpha_x = -\frac{1}{2}\tau''\frac{1}{\sigma}\int\frac{1}{\sigma} - \rho'\frac{1}{\sigma} + \frac{1}{4}\tau'\sigma''\int\frac{1}{\sigma} + \frac{1}{2}\rho\sigma'' + \frac{1}{4}\tau'\frac{\sigma'}{\sigma}. \tag{28}$$

Integrating with respect to $x$ gives

$$\alpha = -\frac{1}{4}\tau''\left(\int\frac{1}{\sigma}\right)^2 - \rho'\int\frac{1}{\sigma} + \frac{1}{4}\tau'\sigma'\int\frac{1}{\sigma} + \frac{1}{2}\rho\sigma' + \eta, \tag{29}$$

where $\eta = \eta(t)$ is an arbitrary function of time. Also, differentiating (28) gives

$$\alpha_{xx} = -\frac{1}{2}\tau''\left(-\frac{\sigma'}{\sigma^2}\int\frac{1}{\sigma} + \frac{1}{\sigma^2}\right) + \rho'\frac{\sigma'}{\sigma^2} + \frac{1}{2}\rho\sigma''' + \frac{1}{4}\tau'\left(\sigma'''\int\frac{1}{\sigma} + 2\frac{\sigma''}{\sigma} - \frac{(\sigma')^2}{\sigma^2}\right), \tag{30}$$

and from (29) we get

$$\alpha_t = -\frac{1}{4}\tau'''\left(\int\frac{1}{\sigma}\right)^2 - \rho''\int\frac{1}{\sigma} + \frac{1}{4}\tau''\sigma'\int\frac{1}{\sigma} + \frac{1}{2}\rho'\sigma' + \eta'. \tag{31}$$

Then after some manipulations (26) becomes

$$\begin{aligned}
-\frac{1}{4}\tau'''\left(\int\frac{1}{\sigma}\right)^2 - \rho''\int\frac{1}{\sigma} + \eta' &= -\frac{1}{4}\tau'' + \rho\left(\frac{\sigma^2}{4}\sigma''' - \sigma g'\right) \\
&\quad + \tau'\left[\frac{1}{2}\left(\frac{\sigma^2}{4}\sigma''' - \sigma g'\right)\int\frac{1}{\sigma} + \frac{1}{8}\left(2\sigma\sigma'' - (\sigma')^2\right) - g\right].
\end{aligned}$$

Now notice that

$$\frac{1}{8}\left(2\sigma\sigma'' - (\sigma')^2\right) - g = \int\left(\frac{\sigma}{4}\sigma''' - g'\right)$$

since $\int\sigma''\sigma' = (\sigma')^2/2$, then we can write the determining equation granting the existence of non trivial symmetries:

$$\begin{aligned}
-\frac{1}{4}\tau'''\left(\int\frac{1}{\sigma}\right)^2 - \rho''\int\frac{1}{\sigma} + \eta' &= -\frac{1}{4}\tau'' + \rho\sigma\left(\frac{\sigma}{4}\sigma''' - g'\right) \\
&\quad + \tau'\left[\frac{\sigma}{2}\left(\frac{\sigma}{4}\sigma''' - g'\right)\int\frac{1}{\sigma} + \int\left(\frac{\sigma}{4}\sigma''' - g'\right)\right].
\end{aligned} \tag{32}$$

*Remark* 4.1. The case $\sigma(x) = \sqrt{2x}$ has been investigated by Craddock and Platen (2004), who considered the following PDE

$$u_t = xu_{xx} + f(x)u_x,$$

with $\mathcal{D} = \mathbb{R}_+$. Taking $g = 0$ in (32) gives

$$-\frac{1}{2}x\tau''' - \sqrt{2x}\tilde{\rho}'' + \frac{\tau''}{4} + \eta' = \frac{3}{16x^{\frac{3}{2}}}\sqrt{2}\tilde{\rho},$$

which agrees with Craddock and Platen (2004) p. 288 when (using their notation) we take $f = 0$; $\rho = \sqrt{2}\tilde{\rho}$; $\sigma'(t) = \eta'(t) + \tau''(t)/4$.

*Remark* 4.2. The case $\sigma(x) = \sqrt{2\tilde{\sigma}}x^{\gamma/2}$, $\gamma \neq 2$, has been investigated by Craddock and Lennox (2007), who considered the following PDE

$$u_t = \tilde{\sigma}x^\gamma u_{xx} + f(x)u_x - \mu x^r u,$$

with $\mathcal{D} = \mathbb{R}_+$, and subsequently by Craddock and Lennox (2009) in the more general case

$$u_t = \tilde{\sigma}x^\gamma u_{xx} + f(x)u_x - g(x)u.$$

In the latter case (32) gives

$$-\frac{x^{2-\gamma}}{2\tilde{\sigma}(2-\gamma)^2}\tau''' - \frac{\sqrt{2}x^{1-\frac{\gamma}{2}}}{\sqrt{\tilde{\sigma}}(2-\gamma)}\tilde{\rho}'' + \frac{\tau''}{4} + \eta' = g'(x)\xi - \tau'(t)g(x) + \frac{1}{4}\tilde{\rho}\tilde{\sigma}\sqrt{2\tilde{\sigma}}\gamma\left(\frac{\gamma}{2}-1\right)\left(\frac{\gamma}{2}-2\right)x^{\frac{3}{2}\gamma-3},$$

which agrees with formula (2.9) in Craddock and Lennox (2009) where we take $f = 0$; $\rho = \tilde{\rho}\sqrt{2\tilde{\sigma}}$.

From (32) we see that in order to match the terms depending on $x$ we have to compare the functional form of $\frac{\sigma}{4}\sigma''' - g'$ with the ones of $\left(\int \frac{1}{\sigma}\right)^2$ and $\int \frac{1}{\sigma}$, or equivalently we have to assume some functional forms for $g$. We then consider separately some cases.

## 5 The Infinitesimal Generators

In this section we show that if $\sigma$ and $g$ satisfy a given integro-differential equation, then the PDE (13) admits a symmetry group whose finite-dimensional part has dimension 6. In the following theorem we state the requirement and find the associated determining equation (32). It is also possible to have Lie symmetry algebras which are two dimensional and four dimensional. As we are interested in the quadratic local volatility model, for brevity, we will focus on the six dimensional case.

**Theorem 5.1.** *The PDE* (13) *admits a six dimensional Lie symmetry group if and only if $\sigma$ and $g$ satisfy*

$$\frac{1}{4}\sigma\sigma''' - g' = A\frac{1}{\sigma} + B\frac{1}{\sigma}\int\frac{1}{\sigma} \tag{33}$$

*for some constants $A, B \in \mathbb{R}$.*

*Proof.* Under (33) the determining equation (32) becomes

$$-\frac{1}{4}\tau'''\left(\int\frac{1}{\sigma}\right)^2 - \rho''\int\frac{1}{\sigma} + \eta' = -\frac{1}{4}\tau'' + A\rho + \left(B\rho + \frac{3}{2}A\tau'\right)\int\frac{1}{\sigma} + B\tau'\left(\int\frac{1}{\sigma}\right)^2. \tag{34}$$

Note that (34) involves only the expression $\int^x 1/\sigma(y)dy$ which is a non constant function of $x$. This allows us to identify the time-dependent coefficients and arrive to the system for $\tau$, $\rho$ and $\eta$:

$$\begin{aligned} -\frac{1}{4}\tau''' &= B\tau'; \\ -\rho'' &= B\rho + \frac{3}{2}A\tau'; \\ \eta' &= -\frac{1}{4}\tau'' + A\rho. \end{aligned}$$

Conversely, if $\sigma, g$ do not satisfy (33), it follows immediately that either $\tau' = 0$, which in turns implies that there is no symmetry group transforming solutions which are constant in time to solutions which are not, see e.g. Baldeaux and Platen (2013) p. 129, or $\rho = 0$ which will produce a four dimensional Lie symmetry algebra.[2] The interested reader may investigate this case. □

In the following we compute the infinitesimal generator of the Lie group admitted by the PDE (13) in the case where $B$ is null. The other cases can be treated similarly and we gather them in Appendix A.

**Theorem 5.2.** *If $\sigma$ and $g$ satisfy* (33) *with $B = 0$ then the PDE*

$$u_t = \frac{1}{2}\sigma^2(x)u_{xx} - g(x)u, \quad x \in \mathcal{D}$$

*admits a Lie symmetry group whose finite dimensional part has dimension 6. The corresponding Lie algebra is generated by the following infinitesimal symmetries:*

$$\begin{aligned}
\mathbf{v}_1 &= \sigma(x)\left(\frac{t}{2}\int^x \frac{1}{\sigma(y)}dy - \frac{A}{4}t^3\right)\partial_x + \frac{1}{2}t^2\partial_t \\
&\quad + \left(-\frac{1}{4}\left(\int^x \frac{1}{\sigma(y)}dy\right)^2 + \left(\frac{3}{4}At^2 + \frac{t}{4}\sigma'(x)\right)\int^x \frac{1}{\sigma(y)}dy - \frac{\sigma'(x)}{8}At^3 - \frac{t}{4} - \frac{1}{16}A^2t^4\right)u\partial_u; \\
\mathbf{v}_2 &= \left(\frac{\sigma(x)}{2}\int^x \frac{1}{\sigma(y)}dy - \frac{3}{8}\sigma(x)At^2\right)\partial_x + t\partial_t \\
&\quad + \left(\frac{3}{4}At\int^x \frac{1}{\sigma(y)}dy + \frac{\sigma'(x)}{4}\int^x \frac{1}{\sigma(y)}dy - \frac{3}{16}At^2\sigma'(x) - \frac{1}{8}A^2t^3\right)u\partial_u; \\
\mathbf{v}_3 &= \partial_t; \\
\mathbf{v}_4 &= \sigma(x)t\partial_x + \left(-\int^x \frac{1}{\sigma(y)}dy + \frac{t}{2}\sigma'(x) + \frac{1}{2}At^2\right)u\partial_u; \\
\mathbf{v}_5 &= \sigma(x)\partial_x + \left(\frac{1}{2}\sigma'(x) + At\right)u\partial_u; \\
\mathbf{v}_6 &= u\partial_u.
\end{aligned}$$

*All other symmetries of* (13) *have infinitesimal generators of the form $\mathbf{v}_\beta = \beta\frac{\partial}{\partial u}$, where $\beta$ is an arbitrary solution of the PDE. These correspond to the superposition of symmetries. i.e. $u \to u + \beta$ is a solution when $u$ and $\beta$ are solutions.*

*Remark* 5.3. In the case $g = A = 0$ we recover $\sigma''' = 0$, corresponding to the Quadratic Normal Volatility Model. We will obtain fundamental solutions in this case below.

[2] The dimension of the Lie algebra is determined by the number of constants of integration. The function $\rho$ appears as a second derivative, so it yields two constants of integration. If we set $\rho = 0$, we therefore reduce the dimension of the Lie algebra from six to four.

*Proof.* If $B=0$, the system for $\tau$, $\rho$ and $\eta$ admits the following solution:

$$\begin{aligned}
\tau(t) &= \frac{C_1}{2}t^2 + C_2 t + C_3;\\
\rho(t) &= -\frac{C_1}{4}At^3 - \frac{3}{8}C_2 A t^2 + C_4 t + C_5;\\
\eta(t) &= -\frac{C_1}{16}A^2t^4 - \frac{C_2}{8}A^2t^3 + \frac{C_4}{2}At^2 + \left(C_5 A - \frac{C_1}{4}\right)t + C_6,
\end{aligned}$$

where $C_1, ..., C_6$ are arbitrary constants. From (27) we have

$$\xi(x,t) = \sigma(x)\left(\frac{C_1 t + C_2}{2}\int^x \frac{1}{\sigma(y)}dy - \frac{A}{4}t^3 C_1 - \frac{3}{8}At^2 C_2 + C_4 t + C_5\right), \tag{35}$$

and from (29) we get

$$\begin{aligned}
\alpha(x,t) &= -\frac{C_1}{4}\left(\int^x \frac{1}{\sigma(y)}dy\right)^2 - \left(-\frac{3}{4}C_1 A t^2 - \frac{3}{4}C_2 A t + C_4\right)\int^x \frac{1}{\sigma(y)}dy\\
&\quad + \frac{1}{4}(C_1 t + C_2)\sigma'(x)\int^x \frac{1}{\sigma(y)}dy + \frac{1}{2}\left(-\frac{C_1}{4}At^3 - \frac{3}{8}C_2 A t^2 + C_4 t + C_5\right)\sigma'(x)\\
&\quad - \frac{C_1}{16}A^2t^4 - \frac{C_2}{8}A^2t^3 + \frac{C_4}{2}At^2 + \left(C_5 A - \frac{C_1}{4}\right)t + C_6.
\end{aligned}$$

Recall that the latter equation for $\alpha$ determines $\phi$ as $\phi = \alpha u + \beta$.

We are looking for infinitesimal symmetries whose vector field has the following form:

$$\mathbf{v} = \xi\partial_x + \tau\partial_t + \phi\partial_u,$$

then we arrive at

$$\begin{aligned}
\mathbf{v} &= \sigma(x)\left(\frac{C_1 t + C_2}{2}\int^x \frac{1}{\sigma(y)}dy - \frac{A}{4}t^3 C_1 - \frac{3}{8}At^2 C_2 + C_4 t + C_5\right)\partial_x\\
&\quad + \left(\frac{1}{2}C_1 t^2 + C_2 t + C_3\right)\partial_t\\
&\quad + \Bigg\{u\Bigg[-\frac{C_1}{4}\left(\int^x \frac{1}{\sigma(y)}dy\right)^2 - \left(-\frac{3}{4}C_1 A t^2 - \frac{3}{4}C_2 A t + C_4\right)\int^x \frac{1}{\sigma(y)}dy\\
&\quad + \frac{1}{4}(C_1 t + C_2)\sigma'(x)\int^x \frac{1}{\sigma(y)}dy + \frac{1}{2}\left(-\frac{C_1}{4}At^3 - \frac{3}{8}C_2 A t^2 + C_4 t + C_5\right)\sigma'(x)\\
&\quad - \frac{C_1}{16}A^2t^4 - \frac{C_2}{8}A^2t^3 + \frac{C_4}{2}At^2 + \left(C_5 A - \frac{C_1}{4}\right)t + C_6\Bigg] + \beta\Bigg\}\partial_u.
\end{aligned}$$

Now taking the coefficients of the arbitrary constants yields the result. □

# 6 The Quadratic Normal Volatility Model

The preceding material provides the tools that are needed to obtain analytical results for a wide class of models. Our aim now it to demonstrate how such an analysis can proceed, by studying a well known case.

In this section we therefore consider the case $B = 0$, corresponding to $1/4\sigma\sigma''' - g' = A/\sigma$, in the specification where $g = A = 0$, which leads to

$$\sigma(x) = \alpha + \beta x + \gamma x^2 \tag{36}$$

with $\gamma \neq 0$ (the case $\gamma = 0$ corresponds to the so called shifted lognormal model). This corresponds to the Quadratic Normal Volatility model, where the underlying process $X_t$ satisfies the following SDE:

$$dX_t = (\alpha + \beta X_t + \gamma X_t^2)dW_t,$$

that is a local volatility model deeply investigated in Lipton (2002), Zuhlsdorff (2002) and recently re-discovered by Andersen (2011) and Carr et al. (2013). The Quadratic Normal Volatility model admits closed form formulas for the price of vanilla options and it is in general highly tractable. The following subsections constitute an analytical counterpart of the probabilistic arguments of Carr et al. (2013) in support of the analytical tractability of the model. It admits indeed non trivial symmetries that are crucial in the determination of the fundamental solutions to the PDE corresponding to the probability density of the underlying.

### 6.1 Distinct Real Roots

We will consider here the case where the polynomial admits two distinct real roots $l < m$. In this case we can write w.l.o.g.

$$\sigma(x) = \frac{(x-m)(x-l)}{m-l}, \quad m \neq l, \tag{37}$$

corresponding to

$$dX_t = \frac{(X_t - m)(X_t - l)}{m-l} dW_t, \quad X_0 \in \mathbb{R}.$$

*Remark* 6.1. The presence of a deterministic function $f(t)$ such that

$$\sigma(x,t) = f(t)\frac{(x-m)(x-l)}{m-l}$$

can be managed by the time-change methodology for the Brownian motion, see (15).

This diffusion has the property that it will be absorbed when it hits either $m$ or $l$, by the Markov property.

In this case we have

$$\int^x \frac{1}{\sigma(y)} dy = \ln \frac{x-m}{x-l}$$

and

$$\sigma'(x) = \frac{2x - m - l}{m - l},$$

and we can exploit Theorem 5.2 to get the infinitesimal symmetries parametrized by the six arbitrary coefficients $C_1, ..., C_6$:

$$
\begin{aligned}
\mathbf{v} &= \xi\partial_x + \tau\partial_t + \phi\partial_u \\
&= C_1\left[\frac{t}{2}\frac{(x-m)(x-l)}{m-l}\ln\frac{x-m}{x-l}\partial_x + \frac{1}{2}t^2\partial_t\right. \\
&\quad \left. + \left(-\frac{1}{4}\ln^2\frac{x-m}{x-l} + \frac{t}{4}\frac{2x-m-l}{m-l}\ln\frac{x-m}{x-l} - \frac{t}{4} - \frac{1}{2}t^2\right)u\partial_u\right] \\
&\quad + C_2\left[\frac{1}{2}\frac{(x-m)(x-l)}{m-l}\ln\frac{x-m}{x-l}\partial_x + t\partial_t + \frac{1}{4}\frac{2x-m-l}{m-l}\ln\frac{x-m}{x-l}u\partial_u\right] \\
&\quad + C_3\partial_t \\
&\quad + C_4\left[t\frac{(x-m)(x-l)}{m-l}\partial_x + \left(-\ln\frac{x-m}{x-l} + \frac{t}{2}\frac{2x-m-l}{m-l}\right)u\partial_u\right] \\
&\quad + C_5\left[\frac{(x-m)(x-l)}{m-l}\partial_x + \frac{1}{2}\frac{2x-m-l}{m-l}u\partial_u\right] \\
&\quad + C_6 u\partial_u,
\end{aligned}
$$

together with the usual (infinite dimensional) symmetries generated by$\beta u\partial_u$.

In order to obtain fundamental solutions we need useful symmetries. For the fundamental solution on the line, we shall require a symmetry yielding a Fourier transform. To obtain such a symmetry we exponentiate $\mathbf{v}_4$.

**Theorem 6.2.** *Consider the PDE*

$$u_t = \frac{1}{2}\sigma^2(x)u_{xx}, \quad x \in \mathcal{D}, \tag{38}$$

*with*

$$\sigma(x) = \frac{(x-m)(x-l)}{m-l}$$

*and $\mathcal{D} = \mathbb{R}$. Then the PDE* (38) *has a symmetry of the form*

$$U_\epsilon(x,t) = u\left(f_\epsilon(x,t), t\right)\exp\left\{-\left(\ln\left|\frac{x-m}{x-l}\right| - \frac{t}{2}\right)\epsilon + \frac{t}{2}\epsilon^2\right\}\frac{1-\frac{x-m}{x-l}e^{-\epsilon t}}{1-\frac{x-m}{x-l}}, \tag{39}$$

*where $f_\epsilon : \mathcal{D}\times\mathbb{R}_+ \to \mathbb{R}$ is defined by*

$$f_\epsilon(x,t) = \frac{m - l\left(\frac{x-m}{x-l}\right)e^{-\epsilon t}}{1-\left(\frac{x-m}{x-l}\right)e^{-\epsilon t}}.$$

*That is, for $\epsilon$ sufficiently small , $U_\epsilon$ is a solution of* (38) *whenever $u$ is.*

*Proof.* We take the symmetry $\mathbf{v}_4$ which is given by

$$\mathbf{v}_4 = t\frac{(x-m)(x-l)}{m-l}\partial_x + \left(-\ln\frac{x-m}{x-l} + \frac{t}{2}\frac{2x-m-l}{m-l}\right)u\partial_u,$$

and we exponentiate the symmetry, that is we look for the solution to the following system:

$$\begin{aligned}
\frac{d\tilde{x}}{d\epsilon} &= \tilde{t}\frac{(\tilde{x}-m)(\tilde{x}-l)}{m-l}, \quad \tilde{x}(0)=x;\\
\frac{d\tilde{t}}{d\epsilon} &= 0, \quad \tilde{t}(0)=t;\\
\frac{d\tilde{u}}{d\epsilon} &= \left(-\ln\left|\frac{\tilde{x}-m}{\tilde{x}-l}\right| + \frac{\tilde{t}}{2}\frac{2\tilde{x}-m-l}{m-l}\right)\tilde{u}, \quad \tilde{u}(0)=u.
\end{aligned}$$

The solution of this system is given by

$$\frac{\tilde{x}-m}{\tilde{x}-l} = \frac{x-m}{x-l}e^{\epsilon t} \tag{40}$$

$$\tilde{t} = t \tag{41}$$

$$\tilde{u} = u\exp\left\{-\left(\ln\left|\frac{x-m}{x-l}\right| - \frac{t}{2}\right)\epsilon - \frac{t}{2}\epsilon^2\right\}\frac{1-\frac{x-m}{x-l}}{1-\frac{x-m}{x-l}e^{\epsilon t}}. \tag{42}$$

Replacing the new parameters gives the result. □

### 6.1.1 Obtaining Fundamental solutions

Fundamental solutions are not unique and with the methods we present here it is possible to exhibit multiple fundamental solutions for certain PDEs. This is discussed in depth in Craddock and Lennox (2009). In this work we restrict ourselves to exhibiting fundamental solutions which are also positive and continuous probability densities.

We observe that the Lie algebras for our PDEs are all six dimensional, so we should look for a Fourier rather than a Laplace transform. This follows from a result in Craddock and Dooley (2010). In the case of a four dimensional Lie algebra we look for a Laplace transform of a fundamental solution.

To obtain a Fourier transform we consider (39) and make the replacement $\epsilon \to i\epsilon$ and take $u = 1$. This gives

$$U_\epsilon(x,t) = \exp\left\{-\left(\ln\left|\frac{x-m}{x-l}\right| - \frac{t}{2}\right)i\epsilon - \frac{1}{2}\epsilon^2 t\right\}\frac{1-\frac{x-m}{x-l}e^{-i\epsilon t}}{1-\frac{x-m}{x-l}}. \tag{43}$$

At $t = 0$ we have

$$U_\epsilon(x,0) = \exp\left(-i\epsilon\ln\left|\frac{x-m}{x-l}\right|\right).$$

Let us illustrate how we may find a fundamental solution with this solution. We suppose that $x \in (l, m)$. In this case

$$\ln\left|\frac{x-m}{x-l}\right| = \ln\left(\frac{m-x}{x-l}\right).$$

We seek a fundamental solution $p$ such that

$$\int_l^m \exp\left(-i\ln\left(\frac{m-y}{y-l}\right)\epsilon\right)p(t,x,y)dy = \exp\left\{-\left(\ln\left(\frac{m-x}{x-l}\right) - \frac{t}{2}\right)i\epsilon - \frac{t}{2}\epsilon^2\right\}\frac{1-\frac{x-m}{x-l}e^{-i\epsilon t}}{1-\frac{x-m}{x-l}}.$$

Observe that taking $\epsilon = 0$ we get

$$\int_l^m p(t,x,y)dy = U_0 = 1,$$

so that $p$ integrates to one and hence is a probability density.

Putting $z = \ln\left(\frac{m-y}{y-l}\right)$ and noting that $dy = -\frac{e^z(m-l)}{(e^z+1)^2}dz$, we look for a fundamental solution such that

$$\int_{-\infty}^{\infty} e^{-iz\epsilon} p(t,x,z)\frac{e^z(m-l)}{(e^z+1)^2}dz = U_\epsilon(x,t).$$

Fourier inversion gives

$$p(t,x,z)\frac{e^z(m-l)}{(e^z+1)^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\epsilon y}U_\epsilon(x,t)d\epsilon,$$

and this is a straightforward Gaussian integral. Inverting and making the appropriate substitution back to the $y$ variables gives the fundamental solution

$$p(t,x,y) = \frac{(m-l)(x-l)\left(\frac{m-x}{x-l}\right)^{\frac{1}{t}\ln\left(\frac{m-y}{y-l}\right)+\frac{1}{2}}}{\sqrt{2\pi t}(y-l)^2(y-m)}\exp\left(-\frac{\left(2\ln\left(\frac{m-y}{y-l}\right)+t\right)^2+4\ln^2\left(\frac{m-x}{x-l}\right)}{8t}\right), \tag{44}$$

for the equation (38) with $l < x < m$. Observe that for $t > 0$, $\lim_{y\to l,m} p(t,x,y) = 0$, so the apparent singularities at $m, l$ are in fact zeroes. It is also clear that this fundamental solution is positive and continuous.

Observe also that a solution given by

$$u(x,t) = \int_l^m f(y)p(t,x,y)dy,$$

with $p(t,x,y)$ given by (44), will satisfy the boundary conditions $u(m,t) = u(l,t) = 0$ which corresponds to the fact that these are absorbing points for the diffusion.

It is worth asking what happens if we take a different initial solution? There is no general rule. For some PDEs we obtain another fundamental solution that is not a probability density. In other cases we arrive back at the same fundamental solution that our first choice of initial solution produces. This first case is discussed extensively with examples in Craddock and Lennox (2009). The reader may check that taking the stationary solution $u_0(x) = x$ in this case actually returns the same fundamental solution.

We can use the same symmetry to obtain other fundamental solutions. We illustrate by obtaining such a solution on $(m,\infty)$. We do so by obtaining a Fourier cosine transform. Taking the real part of (43) we see that

$$V_\epsilon(x,t) = \frac{\exp\left(-\frac{1}{2}\epsilon^2 t\right)}{m-l}\left[(x-l)\cos\left(\epsilon\left(g(x)-\frac{t}{2}\right)\right)+(m-x)\cos\left(\epsilon\left(g(x)+\frac{t}{2}\right)\right)\right],$$

with $g(x) = \ln\left(\frac{m-x}{l-x}\right)$, is also a solution of (38). Observe that $V_\epsilon(x,0) = \cos\left(\ln\left(\frac{x-m}{x-l}\right)\right)$.

We look for a solution of (38) by setting

$$\int_m^\infty \cos\left(\epsilon \ln\left(\frac{y-m}{y-l}\right)\right) p(t,x,y)dy = V_\epsilon(x,t).$$

As before taking $\epsilon = 0$ gives $\int_m^\infty p(t,x,y)dy = 1$. Now the substitution $z = -\ln\left(\frac{y-m}{y-l}\right)$ produces the Fourier cosine transform

$$\int_0^\infty \cos(\epsilon z) p\left(t, x, \frac{me^z - l}{e^z - 1}\right) \frac{(m-l)e^z}{(e^z-1)^2} dz = V_\epsilon(x,t).$$

We therefore have

$$p\left(t, x, \frac{me^z - l}{e^z - 1}\right) \frac{(m-l)e^z}{(e^z-1)^2} = \frac{2}{\pi} \int_0^\infty \cos(\epsilon z) V_\epsilon(x,t) d\epsilon.$$

Evaluating the inverse cosine transform and replacing with the original variables we arrive at the fundamental solution

$$p(t,x,y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{\left(\ln\left(\frac{y-m}{y-l}\right) - \frac{t}{2}\right)^2 + \ln^2\left(\frac{x-m}{x-l}\right)}{2t}\right) K(t,x,y) \tag{45}$$

in which

$$K(t,x,y) = \frac{(m-l)(x-l)\left(\left(\frac{m-x}{l-x}\right)^{\frac{2\ln\left(\frac{m-y}{y-l}\right)}{t}} - 1\right)\left(\frac{x-m}{x-l}\right)^{\frac{1}{2} - \frac{\ln\left(\frac{y-m}{y-l}\right)}{t}}}{(y-l)(y-m)^2}.$$

It is not hard to see that $\lim_{y\to m} p(t,x,y) = 0$ and if

$$u(x,t) = \int_m^\infty f(y) p(t,x,y) dy$$

then $u(m,t) = 0$. We may find other fundamental solutions for this case by these methods. For example we can obtain a fundamental solution on $(-\infty, l)$. However we shall leave this to the interested reader.

Also, one could exponentiate symmetries generated by other vector fields and get different fundamental solutions. In Appendix B we discuss this possibility by taking $\mathbf{v}_1$.

## 6.2 Single Real Root

We now consider the case where the polynomial admits one single root $a$ of multiplicity 2. In this case we can write w.l.o.g.

$$\sigma(x) = (x-a)^2, \tag{46}$$

corresponding to

$$dX_t = (X_t - a)^2 dW_t, \quad X_0 \in \mathbb{R}.$$

If we impose the initial condition $X_0 = a$, then $X_t = a$ is the unique solution of this SDE by the Yamada-Watanabe Theorem, see Klebaner (2005). Thus by the Markov property, if $X_t$ reaches $a$, then it will remain at $a$. Of course one may condition the diffusion to be, say reflected left or right at $a$, but we will not consider these cases here.

So we have

$$\int^x \frac{1}{\sigma(y)} dy = -\frac{1}{x-a}$$

and

$$\sigma'(x) = 2(x-a),$$

and we can exploit theorem 5.2 again to get the infinitesimal symmetries parametrized by the six arbitrary coefficients $C_1, ..., C_6$.

If we exponentiate $\mathbf{v}_4$ we have a symmetry that can be used to produce a fundamental solution.

**Theorem 6.3.** *Consider the PDE*

$$u_t = \frac{1}{2}\sigma^2(x) u_{xx}, \quad x \in \mathcal{D}, \tag{47}$$

*with*

$$\sigma(x) = (x-a)^2$$

*and* $\mathcal{D} = \{x \in \mathbb{R}\}$*. Then the PDE* (47) *has a symmetry of the form*

$$U_\epsilon(x,t) = u\left(f_\epsilon(x,t), t\right) \exp\left\{\frac{1}{x-a}\epsilon + \frac{t}{2}\epsilon^2\right\}\left(1 + (x-a)\epsilon t\right), \tag{48}$$

*where* $f_\epsilon : \mathcal{D} \times \mathbb{R}_+ \to \mathbb{R}$ *is defined by*

$$f_\epsilon(x,t) = \frac{x-a}{1 + (x-a)\epsilon t}.$$

*That is, for* $\epsilon$ *sufficiently small,* $U_\epsilon$ *is a solution of* (47) *whenever* $u$ *is.*

*Proof.* We take the symmetry $\mathbf{v}_4$ which is given by

$$\mathbf{v}_4 \;=\; t(x-a)^2 \partial_x + \left(\frac{1}{x-a} + t(x-a)\right) u \partial_u$$

and we exponentiate the symmetry, that is we look for the solution to the following system:

$$\begin{aligned}
\frac{d\tilde{x}}{d\epsilon} &= \tilde{t}(\tilde{x}-a)^2, \quad \tilde{x}(0) = x; \\
\frac{d\tilde{t}}{d\epsilon} &= 0, \quad \tilde{t}(0) = t; \\
\frac{d\tilde{u}}{d\epsilon} &= \left(\frac{1}{\tilde{x}-a} + \tilde{t}(\tilde{x}-a)\right)\tilde{u}, \quad \tilde{u}(0) = u.
\end{aligned}$$

The solution of this system is given by

$$\begin{aligned}
\tilde{x} - a &= \frac{x-a}{1-(x-a)\epsilon t}; \\
\tilde{t} &= t; \\
\tilde{u} &= u \exp\left\{\frac{1}{x-a}\epsilon - \frac{t}{2}\epsilon^2\right\} \frac{1}{1-(x-a)\epsilon t}.
\end{aligned}$$

Replacing the new parameters $\tilde{x}, \tilde{t}$ gives

$$\begin{aligned}
t &= \tilde{t}; \\
x - a &= \frac{\tilde{x}-a}{1+(\tilde{x}-a)\epsilon \tilde{t}}; \\
x &= f_\epsilon(\tilde{x}, \tilde{t}),
\end{aligned}$$

therefore the function $\tilde{u}$ can be written as follows

$$\tilde{u}(\tilde{x}, \tilde{t}) = u(f(\tilde{x}, \tilde{t}), \tilde{t}) \exp\left\{\frac{1}{\tilde{x}-a}\epsilon + \frac{\tilde{t}}{2}\epsilon^2\right\} (1 + (\tilde{x}-a)\epsilon \tilde{t}).$$

We relabel the parameters and arrive at (48). □

Using the same type of Fourier transform argument as before, we obtain the fundamental solution. We let

$$u(x,t) = \exp\left\{-i\frac{1}{x-a}\epsilon - \frac{1}{2}\epsilon^2 t\right\} (1 - i(x-a)\epsilon t),$$

be a solution and we look for a fundamental solution such that

$$\int_{-\infty}^{\infty} \exp\left(-i\frac{\epsilon}{y-a}\right) p(t,x,y)dy = \exp\left\{-i\frac{1}{x-a}\epsilon - \frac{1}{2}\epsilon^2 t\right\} (1 - i(x-a)\epsilon t). \qquad (49)$$

We make the change of variables $z = \frac{1}{y-a}$. Writing $\int_{-\infty}^{\infty} = \int_{-\infty}^{a^-} + \int_{a^+}^{\infty}$ we see that under the change of variables $\int_{-\infty}^{a^-} \to \int_0^{-\infty}(H(z)(-dz)$ and $\int_{a^+}^{\infty} \to \int_{\infty}^{0} H(z)(-dz)$ so that (49) becomes

$$\int_{-\infty}^{\infty} e^{-i\epsilon z} p\left(t, x, \frac{az+1}{z}\right) \frac{dz}{z^2} = \exp\left\{-i\frac{1}{x-a}\epsilon - \frac{1}{2}\epsilon^2 t\right\} (1 - i(x-a)\epsilon t).$$

Inverting the Fourier transform and converting back to the original variables produces the fundamental solution

$$p(t,x,y) = \frac{(x-a)}{\sqrt{2\pi t}(y-a)^3} \exp\left(-\frac{(x-y)^2}{2t(x-a)^2(y-a)^2}\right),$$

for the PDE

$$u_t = \frac{1}{2}(x-a)^4 u_{xx}. \qquad (50)$$

Taking $\epsilon = 0$ in (49) shows that this fundamental solution is also a probability density, since the right side is equal to one at $\epsilon = 0$ and the left is the integral of the fundamental solution. It is also not hard to check that taking the solution

$$u_1(x,t) = \exp\left\{-i\frac{1}{x-a}\epsilon - \frac{1}{2}\epsilon^2 t\right\}(x-a),$$

leads to the same fundamental solution. Observe that a solution defined by

$$u(x,t) = \int_{-\infty}^{\infty} \varphi(y)\frac{(x-a)}{\sqrt{2\pi t}(y-a)^3}\exp\left(-\frac{(x-y)^2}{2t(x-a)^2(y-a)^2}\right)dy,$$

has the property $u(a,t) = 0$. So this density should correspond to a process which is absorbed at $x = a$, which is true of this process. One can also check that $\lim_{y\to a} p(t,x,y) = 0$. So we have a positive fundamental solution, which is a continuous probability density.

Now let us study the process restricted to $(a,\infty)$. We do this by obtaining a Fourier cosine transform. Taking the real part of (49) we see that

$$U_\xi(x,t) = \exp\left(-\frac{\xi^2 t}{2}\right)\left(\cos\left(\frac{\xi}{x-a}\right) - \xi t(x-a)\sin\left(\frac{\xi}{x-a}\right)\right)$$

is a solution of (50). Observe that $U_\xi(x,0) = \cos\left(\frac{\xi}{x-a}\right)$. We therefore look for a solution such that

$$\int_a^\infty \cos\left(\frac{\xi}{y-a}\right)p(t,x,y)dy = \exp\left(-\frac{\xi^2 t}{2}\right)\left(\cos\left(\frac{\xi}{x-a}\right) - \xi t(x-a)\sin\left(\frac{\xi}{x-a}\right)\right).$$

Taking $\xi = 0$ gives $\int_a^\infty p(t,x,y)dy = 1$, so such a fundamental solution will be a probability density.

Setting $z = \frac{1}{y-a}$ gives

$$\int_0^\infty \cos(\xi z)p(t,x,1/z+a)\frac{dz}{z^2} = \exp\left(-\frac{\xi^2 t}{2}\right)\left(\cos\left(\frac{\xi}{x-a}\right) - \xi t(x-a)\sin\left(\frac{\xi}{x-a}\right)\right).$$

This is a Fourier cosine transform. We can then recover the fundamental solution by

$$p(t,x,1/z+a)/z^2 = \frac{2}{\pi}\int_0^\infty \cos(\xi z)\exp\left(-\frac{\xi^2 t}{2}\right)\left(\cos\left(\frac{\xi}{x-a}\right) - \xi t(x-a)\sin\left(\frac{\xi}{x-a}\right)\right)d\xi.$$

This leads to

$$\begin{aligned} p(t,x,y) = {} & \frac{1}{\sqrt{2\pi t}}\frac{x-a}{(y-a)^2}\left[\exp\left(\frac{(x+y-2a)^2}{2t(x-a)^2(y-a)^2}\right) - \exp\left(\frac{(x-y)^2}{2t(x-a)^2(y-a)^2}\right)\right] \\ & \times \exp\left(-\frac{1}{t(x-a)^2} - \frac{1}{t(y-a)^2}\right). \end{aligned}$$

A fundamental solution on $(-\infty,a)$ can also be obtained. We leave this to the interested reader.

## 6.3 Distinct Complex Roots

We now consider the case where the polynomial admits two distinct complex roots $a \pm ib$ and the process starts at $X_0 \in \mathbb{R}$. In this case we can write without loss of generality

$$\sigma(x) = b\left(1 + \left(\frac{x-a}{b}\right)^2\right)$$

corresponding to

$$dX_t = b\left(1 + \left(\frac{X_t-a}{b}\right)^2\right) dW_t, \quad X_0 \in \mathcal{D},$$

with $\mathcal{D} = \mathbb{R}$. In this case we have

$$\int^x \frac{1}{\sigma(y)} dy = \arctan \frac{x-a}{b}$$

and

$$\sigma'(x) = 2\frac{x-a}{b},$$

and we can exploit Theorem 6.3.2 again to get the infinitesimal symmetries parametrized by the six arbitrary coefficients $C_1, ..., C_6$.

If we exponentiate $\mathbf{v}_4$ we have a symmetry that leads to the Fourier transform of a fundamental solution, just as in the two distinct roots case.

**Theorem 6.4.** *Consider the PDE*

$$u_t = \frac{1}{2}\sigma^2(x)u_{xx}, \quad x \in \mathcal{D}, \tag{51}$$

*with*

$$\sigma(x) = b\left(1 + \left(\frac{x-a}{b}\right)^2\right)$$

*and $\mathcal{D} = \mathbb{R}$. Then the PDE* (51) *has a symmetry of the form*

$$U_\epsilon(x,t) = u\left(f_\epsilon(x,t), t\right) \exp\left\{-\arctan\left(\frac{x-a}{b}\right)\epsilon + \frac{t}{2}\epsilon^2\right\} \frac{\cos\left(\arctan\left(\frac{x-a}{b}\right) - \epsilon t\right)}{\cos\left(\arctan\left(\frac{x-a}{b}\right)\right)}, \tag{52}$$

*where $f_\epsilon : \mathcal{D} \times \mathbb{R}_+ \to \mathbb{R}$ is defined by*

$$f_\epsilon(x,t) = a + b\tan\left(\arctan\left(\frac{x-a}{b}\right) - \epsilon t\right).$$

*That is, for $\epsilon$ sufficiently small, $U_\epsilon$ is a solution of* (51) *whenever $u$ is.*

*Proof.* We take the non trivial symmetry associated to $\mathbf{v}_4$ which is given by

$$\mathbf{v}_4 = b\left(1 + \left(\frac{x-a}{b}\right)^2\right) t\partial_x + \left(-\arctan\left(\frac{x-a}{b}\right) + t\frac{x-a}{b}\right) u\partial_u \tag{53}$$

and we exponentiate the symmetry, that is we look for the solution to the following system:

$$\begin{aligned}\frac{d\tilde{x}}{d\epsilon} &= b\left(1+\left(\frac{\tilde{x}-a}{b}\right)^2\right)\tilde{t}, \quad \tilde{x}(0)=x;\\ \frac{d\tilde{t}}{d\epsilon} &= 0, \quad \tilde{t}(0)=t;\\ \frac{d\tilde{u}}{d\epsilon} &= \left(-\arctan\left(\frac{\tilde{x}-a}{b}\right)+\tilde{t}\frac{\tilde{x}-a}{b}\right)\tilde{u}, \quad \tilde{u}(0)=u.\end{aligned}$$

The solution of this system is given by

$$\begin{aligned}\frac{\tilde{x}-a}{b} &= \tan\left(\arctan\left(\frac{x-a}{b}\right)+\epsilon t\right);\\ \tilde{t} &= t;\\ \tilde{u} &= u\exp\left\{-\arctan\left(\frac{x-a}{b}\right)\epsilon-\frac{t}{2}\epsilon^2\right\}\frac{\cos\left(\arctan\frac{x-a}{b}\right)}{\cos\left(\arctan\frac{x-a}{b}+\epsilon t\right)}.\end{aligned}$$

Replacing the new parameters $\tilde{x},\tilde{t}$ gives

$$\begin{aligned} t &= \tilde{t};\\ \frac{x-a}{b} &= \tan(\arctan\frac{\tilde{x}-a}{b}-\epsilon t);\\ x &= f_\epsilon(\tilde{x},\tilde{t}),\end{aligned}$$

therefore the function $\tilde{u}$ can be written as follows

$$\tilde{u}(\tilde{x},\tilde{t}) = u(f(\tilde{x},\tilde{t}),\tilde{t})\exp\left\{-\arctan\left(\frac{x-a}{b}\right)\epsilon+\frac{\tilde{t}}{2}\epsilon^2\right\}\frac{\cos\left(\arctan\frac{\tilde{x}-a}{b}-\epsilon\tilde{t}\right)}{\cos\left(\arctan\frac{\tilde{x}-a}{b}\right)}. \tag{54}$$

We relabel the parameters and we arrive at (52). □

### 6.3.1 A Fourier Series Representation of the Fundamental Solution

This case is interesting in that we actually obtain a Fourier series representation of our fundamental solution, which is of course the Fourier transform on the circle. As this is a situation that has not been investigated previously in the literature, we will present the details.

From the previous symmetry we are able to exhibit the solution

$$u(x,t)=\frac{1}{b}\sqrt{(x-a)^2+b^2}e^{-\frac{1}{2}\epsilon^2 t-i\epsilon\arctan\left(\frac{x-a}{b}\right)}\cosh\left(\epsilon t+i\arctan\left(\frac{x-a}{b}\right)\right), \tag{55}$$

by taking our initial solution in equation (52) to be $u=1$ and making the replacement $\epsilon\to i\epsilon$. We have

$$u(x,0)=\exp\left(-i\epsilon\arctan\left(\frac{x-a}{b}\right)\right). \tag{56}$$

We therefore seek a fundamental solution $p(t,x,y)$ such that

$$\int_{-\infty}^{\infty} u(y,0)p(t,x,y)dy = \frac{1}{b}\sqrt{(x-a)^2+b^2}e^{-\frac{1}{2}\epsilon^2 t - i\epsilon\arctan\left(\frac{x-a}{b}\right)}\cosh\left(\epsilon t + i\arctan\left(\frac{x-a}{b}\right)\right). \tag{57}$$

It is clear from the fact $U_0 = 1$ that this fundamental solutions satisfies $\int_{-\infty}^{\infty} p(t,x,y)dy = 1$ and so is a probability density.

We make the change of variables $z = \arctan\left(\frac{y-a}{b}\right)$ and this becomes

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-i\epsilon z}p(t,x,a+b\tan z)\sec^2 z dz = \frac{1}{b^2}\sqrt{(x-a)^2+b^2}e^{-\frac{1}{2}\epsilon^2 t - i\epsilon\arctan\left(\frac{x-a}{b}\right)}\\ \times\cosh\left(\epsilon t + i\arctan\left(\frac{x-a}{b}\right)\right).\end{aligned}$$

Putting $\epsilon = 2n$, $z \to z/2$ leads to the expression

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-niz}p(t,x,a+b\tan(z/2))\sec^2(z/2)dz = \frac{1}{b^2\pi}\sqrt{(x-a)^2+b^2}e^{-2n^2 t - 2ni\arctan\left(\frac{x-a}{b}\right)}\\ \times\cosh\left(2nt + i\arctan\left(\frac{x-a}{b}\right)\right),\end{aligned}$$

which is the $n$-th Fourier coefficient of the kernel function. Fourier inversion then gives

$$\begin{aligned}p(t,x,a+b\tan(z/2)) = \frac{\cos^2(z/2)}{b^2\pi}\sum_{n=-\infty}^{\infty} e^{niz}\sqrt{(x-a)^2+b^2}e^{-2n^2 t - 2ni\arctan\left(\frac{x-a}{b}\right)}\\ \times\cosh\left(2nt + i\arctan\left(\frac{x-a}{b}\right)\right).\end{aligned}$$

From this one easily obtains a Fourier series for the fundamental solution

$$\begin{aligned}p(t,x,y) &= \frac{\cos^2\left(\arctan\left(\frac{y-a}{b}\right)\right)}{\pi b^2}\sum_{n=-\infty}^{\infty} e^{2ni\arctan\left(\frac{y-a}{b}\right)}\sqrt{(x-a)^2+b^2}e^{-2n^2 t}\\ &\quad\times e^{-2ni\arctan\left(\frac{x-a}{b}\right)}\cosh\left(2nt + i\arctan\left(\frac{x-a}{b}\right)\right)\\ &= \frac{1}{\pi(a^2+(y-b)^2)}\sum_{n=-\infty}^{\infty} e^{2ni\arctan\left(\frac{b(y-x)}{1+(y-a)(x-a)}\right)}e^{-2n^2 t}\\ &\quad\times\left[b\cosh(2nt) - i(a-x)\sinh(2nt)\right].\end{aligned}$$

## 7 Other tractable models

It is possible to obtain other models from this which are also tractable. In fact using our methods we can generate them quite easily. We consider the equation

$$\frac{1}{4}\sigma\sigma''' - g' = A\frac{1}{\sigma} + B\frac{1}{\sigma}\int\frac{1}{\sigma}. \tag{58}$$

Suppose we set $A = B = 0$ then integrate. This produces the nonlinear ODE

$$\frac{1}{4}\sigma\sigma'' - \frac{1}{8}(\sigma')^2 - g = C, \tag{59}$$

where $C$ is a constant of integration. For specific choices of $g$ this equation can be solved. For example, $g(x) = \mu/x$ implies that $\sigma(x) = 4\sqrt{2\mu/3}\sqrt{x}$. And one can perform the analysis we considered here in that case. However we can also reverse the process. If we specify $\sigma$ in equation (58) in advance and determine $g$ from this. For this choice of $g$ and $\sigma$ we can then compute appropriate fundamental solutions. Of course one might not have any specific financial applications in mind for every functional $\mathbb{E}\left[\exp\left(-\int_0^t g(X_s)ds\right)\right]$, but potentially other useful models may be found in this way.

# 8 Conclusions

In this paper we gave conditions on $\sigma$ for the tractability of a local volatility model and in the special case of the Quadratic Normal Volatility model we were able to explicitly exponentiate the admitted Lie group of transformations to find the symmetries of the PDE and the corresponding fundamental solutions. By doing so we provided an analytical counterpart to the probabilistic justification of the tractability of this model given in Carr et al. (2013). In the future, we aim at finding symmetries of more general models (e.g. in the case where $B$ is not zero) and at inverting the resulting integral transforms to provide fundamental solutions for such models.

# A Infinitesimal generator for $B \neq 0$

In this appendix we compute the infinitesimal generator of the Lie group admitted by the PDE (13) when the constant $B$ in (33) is not null.

## A.1 Case $B > 0$

**Theorem A.1.** *If $\sigma$ and $g$ satisfy (33) with $B > 0$ then the PDE*

$$u_t = \frac{1}{2}\sigma^2(x)u_{xx} - g(x)u, \quad x \in \mathcal{D}$$

*admits a Lie symmetry group whose finite dimensional part has dimension 6. The corresponding Lie algebra is generated by the following infinitesimal symmetries:*

$$\begin{aligned}
\mathbf{v}_1 \;=\; & \sigma(x)\left[-\sqrt{B}\sin(2\sqrt{B}t)\int^x \frac{1}{\sigma(y)}dy - \frac{A}{\sqrt{B}}\sin(2\sqrt{B}t)\right]\partial_x + \left[\cos(2\sqrt{B}t)\right]\partial_t \\
& + \Bigg[B\cos(2\sqrt{B}t)\left(\int^x \frac{1}{\sigma(y)}dy\right)^2 + 2A\cos(2\sqrt{B}t)\int^x \frac{1}{\sigma(y)}dy \\
& - \frac{\sqrt{B}}{2}\sin(2\sqrt{B}t)\sigma'(x)\int^x \frac{1}{\sigma(y)}dy - \frac{A}{\sqrt{B}}\sin(2\sqrt{B}t)\sigma'(x) \\
& + \frac{\sqrt{B}}{2}\sin(2\sqrt{B}t) - \frac{A^2}{2B}\cos(2\sqrt{B}t)\Bigg]u\partial_u;
\end{aligned}$$

$$
\begin{aligned}
\mathbf{v}_2 &= \sigma(x)\left[\sqrt{B}\cos(2\sqrt{B}t)\int^x \frac{1}{\sigma(y)}dy + \frac{A}{\sqrt{B}}\cos(2\sqrt{B}t)\right]\partial_x + \left[\sin(2\sqrt{B}t)\right]\partial_t \\
&+ \left[B\sin(2\sqrt{B}t)\left(\int^x \frac{1}{\sigma(y)}dy\right)^2 + 2A\sin(2\sqrt{B}t)\int^x \frac{1}{\sigma(y)}dy\right. \\
&+\frac{\sqrt{B}}{2}\cos(2\sqrt{B}t)\sigma'(x)\int^x \frac{1}{\sigma(y)}dy - \frac{A}{\sqrt{B}}\cos(2\sqrt{B}t)\sigma'(x) \\
&\left.+\frac{A^2}{2B}\sin(2\sqrt{B}t) - \frac{\sqrt{B}}{2}\cos(2\sqrt{B}t)\right]u\partial_u; \\
\mathbf{v}_3 &= \partial_t; \\
\mathbf{v}_4 &= \sigma(x)\cos(\sqrt{B}t)\partial_x \\
&+ \left[\sqrt{B}\sin(\sqrt{B}t)\int^x \frac{1}{\sigma(y)}dy + \frac{1}{2}\cos(\sqrt{B}t)\sigma'(x) + \frac{A}{\sqrt{B}}\sin(\sqrt{B}t)\right]u\partial_u; \\
\mathbf{v}_5 &= \sigma(x)\sin(\sqrt{B}t)\partial_x \\
&+ \left[-\sqrt{B}\cos(\sqrt{B}t)\int^x \frac{1}{\sigma(y)}dy + \frac{1}{2}\sin(\sqrt{B}t)\sigma'(x) - \frac{A}{\sqrt{B}}\cos(\sqrt{B}t)\right]u\partial_u; \\
\mathbf{v}_6 &= u\partial_u.
\end{aligned}
$$

*In both cases, there is an infinitesimal symmetry $\mathbf{v}_\beta = \beta\frac{\partial}{\partial u}$, making the Lie algebra infinite-dimensional.*

*Proof.* If $B > 0$, the system for $\tau$, $\rho$ and $\eta$ admits the following solution:

$$
\begin{aligned}
\tau(t) &= C_1\cos(2\sqrt{B}t) + C_2\sin(2\sqrt{B}t) + C_3; \\
\rho(t) &= C_4\cos(\sqrt{B}t) + C_5\sin(\sqrt{B}t) + \frac{A}{\sqrt{B}}C_2\cos(2\sqrt{B}t) - \frac{A}{\sqrt{B}}C_1\sin(2\sqrt{B}t); \\
\eta(t) &= \left(\frac{\sqrt{B}}{2}C_1 + \frac{A^2}{2B}C_2\right)\sin(2\sqrt{B}t) - \left(\frac{\sqrt{B}}{2}C_2 - \frac{A^2}{2B}C_1\right)\cos(2\sqrt{B}t) \\
&+C_4\frac{A}{\sqrt{B}}\sin(\sqrt{B}t) - C_5\frac{A}{\sqrt{B}}\cos(\sqrt{B}t) + C_6;
\end{aligned}
$$

where $C_1, ..., C_6$ are arbitrary constants. From (27) we have

$$
\begin{aligned}
\xi(x,t) &= \sigma(x)\left[\sqrt{B}\left(-C_1\sin(2\sqrt{B}t) + C_2\cos(2\sqrt{B}t)\right)\int^x \frac{1}{\sigma(y)}dy\right. \\
&\left.+C_4\cos(\sqrt{B}t) + C_5\sin(\sqrt{B}t) + \frac{A}{\sqrt{B}}C_2\cos(2\sqrt{B}t) - \frac{A}{\sqrt{B}}C_1\sin(2\sqrt{B}t)\right] \quad (60)
\end{aligned}
$$

and from (29) we get

$$
\begin{aligned}
\alpha(x,t) \;=\;& B\left(C_1\cos(2\sqrt{B}t)+C_2\sin(2\sqrt{B}t)\right)\left(\int^x \frac{1}{\sigma(y)}dy\right)^2 \\
&+\left(\sqrt{B}C_4\sin(\sqrt{B}t)-\sqrt{B}C_5\cos(\sqrt{B}t)+2AC_2\sin(2\sqrt{B}t)+2AC_1\cos(2\sqrt{B}t)\right)\times \\
&\int^x \frac{1}{\sigma(y)}dy+\frac{\sqrt{B}}{2}\left(-C_1\sin(2\sqrt{B}t)+C_2\cos(2\sqrt{B}t)\right)\sigma'(x)\int^x \frac{1}{\sigma(y)}dy \\
&+\frac{1}{2}\left(C_4\cos(\sqrt{B}t)+C_5\sin(\sqrt{B}t)+\frac{A}{\sqrt{B}}C_2\cos(2\sqrt{B}t)-\frac{A}{\sqrt{B}}C_1\sin(2\sqrt{B}t)\right)\sigma'(x) \\
&+\left(\frac{\sqrt{B}}{2}C_1+\frac{A^2}{2B}C_2\right)\sin(2\sqrt{B}t)-\left(\frac{\sqrt{B}}{2}C_2-\frac{A^2}{2B}C_1\right)\cos(2\sqrt{B}t) \\
&+C_4\frac{A}{\sqrt{B}}\sin(\sqrt{B}t)-C_5\frac{A}{\sqrt{B}}\cos(\sqrt{B}t)+C_6.
\end{aligned}
$$

Now taking the coefficients of the arbitrary constants yields the result. □

### A.2 Case $B < 0$

**Theorem A.2.** *If $\sigma$ and $g$ satisfy* (33) *with $B < 0$ then the PDE*

$$u_t = \frac{1}{2}\sigma^2(x)u_{xx} - g(x)u, \quad x \in \mathcal{D}$$

*admits a Lie symmetry group whose finite dimensional part has dimension 6. The corresponding Lie algebra is generated by the following infinitesimal symmetries:*

$$
\begin{aligned}
\mathbf{v}_1 \;=\;& \sigma(x)\left[-\sqrt{B}\exp(-2\sqrt{-B}t)\int^x \frac{1}{\sigma(y)}dy+\frac{A}{\sqrt{-B}}\exp(-2\sqrt{-B}t)\right]\partial_x \\
&+\left[B\exp(-2\sqrt{-B}t)\left(\int^x \frac{1}{\sigma(y)}dy\right)^2+2A\exp(-2\sqrt{-B}t)\int^x \frac{1}{\sigma(y)}dy\right. \\
&-\frac{\sqrt{-B}}{2}\exp(-2\sqrt{-B}t)\sigma'(x)\int^x \frac{1}{\sigma(y)}dy+\frac{A}{\sqrt{-B}}\exp(-2\sqrt{-B}t)\sigma'(x) \\
&\left.+\left(\frac{\sqrt{-B}}{2}+\frac{A^2}{2B}\right)\exp(-2\sqrt{-B}t)\right]u\partial_u+\left[\exp(-2\sqrt{-B}t)\right]\partial_t;
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{v}_2 &= \sigma(x)\left[\sqrt{-B}\exp(2\sqrt{-B}t)\int^x \frac{1}{\sigma(y)}dy - \frac{A}{\sqrt{-B}}\exp(2\sqrt{-B}t)\right]\partial_x \\
&\quad + \left[B\exp(2\sqrt{-B}t)\left(\int^x \frac{1}{\sigma(y)}dy\right)^2 - 2A\exp(2\sqrt{-B}t)\int^x \frac{1}{\sigma(y)}dy\right. \\
&\quad + \frac{\sqrt{-B}}{2}\exp(2\sqrt{-B}t)\sigma'(x)\int^x \frac{1}{\sigma(y)}dy - \frac{A}{\sqrt{-B}}\exp(2\sqrt{-B}t)\sigma'(x) \\
&\quad \left. - \left(\frac{\sqrt{-B}}{2} - \frac{A^2}{2B}\right)\exp(2\sqrt{-B}t)\right]u\partial_u + \left[\exp(2\sqrt{-B}t)\right]\partial_t; \\
\mathbf{v}_3 &= \partial_t; \\
\mathbf{v}_4 &= \sigma(x)\exp(-\sqrt{-B}t)\partial_x + \left[\sqrt{-B}\exp(-\sqrt{-B}t)\int^x \frac{1}{\sigma(y)}dy + \frac{1}{2}\exp(-\sqrt{-B}t)\sigma'(x)\right. \\
&\quad \left. - \frac{A}{\sqrt{-B}}\exp(-\sqrt{-B}t)\right]u\partial_u; \\
\mathbf{v}_5 &= \sigma(x)\exp(\sqrt{-B}t)\partial_x + \left[-\sqrt{-B}\exp(\sqrt{-B}t)\int^x \frac{1}{\sigma(y)}dy + \frac{1}{2}\exp(\sqrt{-B}t)\sigma'(x)\right. \\
&\quad \left. + \frac{A}{\sqrt{-B}}\exp(\sqrt{-B}t)\right]u\partial_u; \\
\mathbf{v}_6 &= u\partial_u.
\end{aligned}
$$

*Proof.* If $B < 0$, the system for $\tau$, $\rho$ and $\eta$ admits the following solution:

$$
\begin{aligned}
\tau(t) &= C_1\exp(-2\sqrt{-B}t) + C_2\exp(2\sqrt{-B}t) + C_3; \\
\rho(t) &= C_4\exp(-\sqrt{-B}t) + C_5\exp(\sqrt{-B}t) + \frac{A}{\sqrt{-B}}C_1\exp(-2\sqrt{-B}t) - \frac{A}{\sqrt{-B}}C_2\exp(2\sqrt{-B}t); \\
\eta(t) &= \left(\frac{\sqrt{-B}}{2} + \frac{A^2}{2B}\right)C_1\exp(-2\sqrt{-B}t) - \left(\frac{\sqrt{-B}}{2} - \frac{A^2}{2B}\right)C_2\exp(2\sqrt{-B}t) \\
&\quad - C_4\frac{A}{\sqrt{-B}}\exp(-\sqrt{-B}t) + \frac{A}{\sqrt{-B}}C_5\exp(\sqrt{-B}t);
\end{aligned}
$$

where $C_1, ..., C_6$ are arbitrary constants. From (27) we have

$$
\begin{aligned}
&\xi(x,t) = \sigma(x)\left[-\sqrt{-B}\left(C_1\exp(-2\sqrt{-B}t) - C_2\exp(2\sqrt{-B}t)\right)\int^x \frac{1}{\sigma(y)}dy\right. \\
&\left. + C_4\exp(-\sqrt{-B}t) + C_5\exp(\sqrt{-B}t) + \frac{A}{\sqrt{-B}}C_1\exp(-2\sqrt{-B}t) - \frac{A}{\sqrt{-B}}C_2\exp(2\sqrt{-B}t)\right],
\end{aligned}
$$

and from (29) we get

$$
\begin{aligned}
\alpha(x,t) \;=\;& B\left(C_1\exp(-2\sqrt{-B}t)+C_2\exp(2\sqrt{-B}t)\right)\left(\int^x\frac{1}{\sigma(y)}dy\right)^2\\
&+\left(\sqrt{-B}C_4\exp(-\sqrt{-B}t)-\sqrt{-B}C_5\exp(\sqrt{-B}t)\right.\\
&\left.+2AC_1\exp(-2\sqrt{-B}t)-2AC_2\exp(2\sqrt{-B}t)\right)\int^x\frac{1}{\sigma(y)}dy\\
&-\frac{\sqrt{-B}}{2}\left(C_1\exp(-2\sqrt{-B}t)-C_2\exp(2\sqrt{-B}t)\right)\sigma'(x)\int^x\frac{1}{\sigma(y)}dy\\
&+\frac{1}{2}\left(C_4\exp(-\sqrt{-B}t)+C_5\exp(\sqrt{-B}t)\right.\\
&\left.+\frac{A}{\sqrt{-B}}C_1\exp(-2\sqrt{-B}t)-\frac{A}{\sqrt{-B}}C_2\exp(2\sqrt{-B}t)\right)\sigma'(x)\\
&+\left(\frac{\sqrt{-B}}{2}+\frac{A^2}{2B}\right)C_1\exp(-2\sqrt{-B}t)-\left(\frac{\sqrt{-B}}{2}-\frac{A^2}{2B}\right)C_2\exp(2\sqrt{-B}t)\\
&-C_4\frac{A}{\sqrt{-B}}\exp(-\sqrt{-B}t)+\frac{A}{\sqrt{-B}}C_5\exp(\sqrt{-B}t).
\end{aligned}
$$

Now taking the coefficients of the arbitrary constants yields the result for the case $B < 0$. $\square$

# B Another symmetry in the distinct real roots case

In this appendix we analyze another symmetry for the PDE (38) starting from another vector field generating the Lie symmetry.

**Theorem B.1.** *Consider the PDE* (38) *with* $\mathcal{D} = \{x \in \mathbb{R} : \; x > m > l\}$. *Then the PDE* (38) *has a symmetry of the form*

$$
U_\epsilon(x,t) = \frac{(x-m)\left(\left(\frac{x-l}{x-m}\right)^{\frac{1}{1+4\epsilon t}}-1\right)}{(m-l)\sqrt{1+4\epsilon t}}\exp\left(-\frac{\epsilon\left(2\log\left(\frac{x-m}{x-l}\right)+t\right)^2}{2(1+4\epsilon t)}\right)u\left(f_\epsilon(x,t),\frac{t}{1+4\epsilon t}\right), \tag{61}
$$

*where* $f_\epsilon : \mathcal{D}\times\mathbb{R}_+ \to \mathbb{R}$ *is defined by*

$$
f_\epsilon(x,t) = \frac{m - l\left(\frac{x-m}{x-l}\right)^{\frac{1}{1+4\epsilon t}}}{1-\left(\frac{x-m}{x-l}\right)^{\frac{1}{1+4\epsilon t}}}.
$$

*That is, for* $\epsilon$ *sufficiently small,* $U_\epsilon$ *is a solution of* (38) *whenever* $u$ *is.*

*Proof.* We take the vector field by

$$\begin{aligned}\mathbf{v}_1 &= 4t\frac{(x-m)(x-l)}{m-l}\ln\frac{x-m}{x-l}\partial_x + 4t^2\partial_t \\ &\quad + \left(-\frac{1}{2}t^2 - 2\ln^2\frac{x-m}{x-l} + 2t\frac{2x-m-l}{m-l}\ln\frac{x-m}{x-l} - 2t\right)u\partial_u\end{aligned}$$

and we exponentiate the symmetry, that is we look for the solution to the following system:

$$\frac{d\tilde{x}}{d\epsilon} = 4\tilde{t}\frac{(\tilde{x}-m)(\tilde{x}-l)}{m-l}\ln\frac{\tilde{x}-m}{\tilde{x}-l}, \quad \tilde{x}(0)=x; \tag{62}$$

$$\frac{d\tilde{t}}{d\epsilon} = 4\tilde{t}^2, \quad \tilde{t}(0)=t; \tag{63}$$

$$\frac{d\tilde{u}}{d\epsilon} = \left(-\frac{1}{2}\tilde{t}^2 - 2\ln^2\frac{\tilde{x}-m}{\tilde{x}-l} + 2\tilde{t}\frac{2\tilde{x}-m-l}{m-l}\ln\frac{\tilde{x}-m}{\tilde{x}-l} - 2\tilde{t}\right)\tilde{u}, \quad \tilde{u}(0)=u. \tag{64}$$

The calculations are straightforward, though somewhat tedious.

From (63) we immediately get

$$\tilde{t} = \frac{t}{1-4\epsilon t},$$

then (62) becomes

$$\frac{(m-l)d\tilde{x}}{(\tilde{x}-m)(\tilde{x}-l)\ln\frac{\tilde{x}-m}{\tilde{x}-l}} = \frac{4t}{1-4\epsilon t}d\epsilon, \quad \tilde{x}(0)=x,$$

which through the change of variable $y=(\tilde{x}-m)/(\tilde{x}-l)$ transforms into

$$\frac{dy}{y\ln y} = -d\ln(1-4\epsilon t),$$

thus giving

$$\frac{\tilde{x}-m}{\tilde{x}-l} = \left(\frac{x-m}{x-l}\right)^{\frac{1}{1-4\epsilon t}}.$$

Then (64) becomes

$$\frac{d\tilde{u}}{\tilde{u}} = \left(\frac{-\frac{1}{2}t^2}{(1-4\epsilon t)^2} - \frac{2}{(1-4t\epsilon)^2}\ln^2\frac{x-m}{x-l} + \frac{2t}{(1-4t\epsilon)^2}\frac{1+\left(\frac{x-m}{x-l}\right)^{\frac{1}{1-4\epsilon t}}}{1-\left(\frac{x-m}{x-l}\right)^{\frac{1}{1-4\epsilon t}}}\ln\frac{x-m}{x-l} - \frac{2t}{1-4t\epsilon}\right)d\epsilon,$$

with $\tilde{u}(0)=u$. Integrating and relabeling the parameters completes the proof. □

The symmetry can be extended in a straightforward way to values of $x$ to the left of $m$ by taking absolute values.

We can obtain a fundamental solution from this symmetry. The style of argument is essentially the same as in for example Craddock and Lennox (2007). If we look for a fundamental solution on $(m,\infty)$ using this symmetry then we are lead to a Laplace transform, which yields the fundamental solution (45). We also leave this to the interested reader. In fact we will obtain the fundamental solution already obtained for the case $x>m$, so we do not pursue it here.

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