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# Hedging futures options with stochastic interest rates

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## Abstract

This paper presents a simulation study of hedging long-dated futures options, in the Rabinovitch (1989) model which assumes correlated dynamics between spot asset prices and interest rates. Under this model and when the maturity of the hedging instruments match the maturity of the option, forward contracts and futures contracts can hedge both the market risk and the interest rate risk of the options positions. When the hedge is rolled forward with shorter maturity hedging instruments, then bond contracts are additionally required to hedge the interest rate risk. This requirement becomes more pronounced for longer maturity contracts and amplifies as the interest rate volatility increases. Factor hedging ratios are also considered, which are suited for multi-dimensional models, and their numerical efficiency is validated.

*Keywords:* Futures options; Stochastic interest rates; Delta hedging; Interest rate hedging;  
*JEL:* C60, G13

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## 1. Introduction

Typically, the sensitivity of an option's price with respect to the interest rate, that is the partial derivatives of the option price with respect to the interest rate, is an increasing function with time-to-maturity. Thus, interest rate risk should be more relevant to derivatives with longer maturities and models with stochastic interest rates tend to improve pricing and hedging performance on long-dated contracts, see Bakshi, Cao, and Chen (2000) for supporting evidence in equity markets. A representative literature on spot option pricing models with stochastic interest rates includes Rabinovitch (1989), Amin and Jarrow (1992) and Kim and Kunitomo (1999). Scott (1997) demonstrates that stochastic volatility and stochastic interest rates have a significant impact on stock option prices, particularly on the prices of long-dated contracts. Schwartz (1997), Brennan and Crew (1997), Bühler, Korn, and Schöbel (2004) and Shiraya and Takahashi (2012) have considered hedging with futures contracts long-term commodity commitments in spot or forward markets, yet most of these models do not assume stochastic interest rates. Very limited literature has addressed the hedging of long-dated futures options, under a stochastic interest rate economy.

This paper aims to gauge the impact of interest rate risk in futures options positions and examine the hedging of this risk. We employ the two-factor Rabinovitch (1989) model to price options on futures and compute their corresponding hedge ratios. Under this model, the spot asset price process follows a geometric Brownian motion, the stochastic interest rate process is modelled by an Ornstein-Uhlenbeck process and their dynamics are correlated. The model is one of the most tractable spot price models accommodating stochastic interest rates and leads to a modified Black and Scholes (1973) option pricing formula.

The hedging of futures option positions is discussed, where both futures and forward contracts are considered as hedging instruments. Under the assumption of the Rabinovitch (1989) model, we show that forward contracts can hedge both the underlying market risk and the interest rate risk simultaneously, as long as the maturity of the forward contracts coincide with the maturity of the futures option. Furthermore, with a suitable convexity adjustment, futures contracts with the same maturity as the option, can also hedge both the market risk and the interest rate risk of the futures options positions. To gauge the contribution of the stochastic interest rate specifications to hedging long-dated option positions, we consider several hedge strategies including the hedge ratios derived from the deterministic interest rate Black (1976) model and the stochastic interest rate two-factor Rabinovitch (1989) model. We also introduce the factor hedging approach (see Clewlow and Strickland (2000) and Chiarella, Kang, Nikitopoulos, and Tô (2013)) which is well suited for hedging with more general multi-dimensional models and we validate the numerical efficiency of the approach.

A Monte Carlo simulation approach is employed to numerically investigate the hedging performance of long-dated futures options for a variety of hedging schemes such as delta hedging and interest-rate hedging. The stock price process and the interest rate process are generated using the Euler scheme under the historical measure with the market price of risk and the market price of interest rate risk introduced to the drift terms of the processes in the Rabinovitch (1989) model. The contribution of the discretisation error in the proposed hedging schemes is evaluated, as well as the impact of the model parameters such as the interest rate volatility, the long-term mean of interest rates and the correlation between the spot price process and the interest process. We find that when interest rate volatility is high,

the models with stochastic interest rates improve significantly the hedging performance of futures options positions compared to the models with deterministic interest rate specifications. This improvement becomes more pronounced for options with longer maturities.

We finally examine the effect of hedging long-dated options with instruments of shorter maturities. Within the stochastic interest rate model, we numerically validate using forward contracts as hedging instruments with the hedge ratio calculated according to the Black (1976) model (using the volatility of the forward), with a balance in the continuously compounded savings account. We find that this can replicate the forward price of the option in the limit. However, when using short-dated contracts to hedge options with longer maturities, forward or futures contracts alone can no longer hedge the interest rate risk. Adding bond contracts to the hedging portfolio is necessary in order to mitigate the interest rate risk.

The remainder of the paper is structured as follows. Section 2 presents the Rabinovitch (1989) spot price model featuring stochastic interest rates and gives pricing equations for forwards, futures and futures options. Section 3 describes the hedging methodology including a variety of hedge ratios and hedge schemes such as delta hedge and interest-rate hedge. Numerical investigations to assess the contribution of stochastic interest rates to hedging long-dated futures options as well as reflections on the results are presented in Section 4. Section 5 concludes.

## 2. Model description

We consider a filtered probability space  $(\Omega, \mathcal{F}_T, \mathbb{F}, \mathbb{P}), T \in [0, \infty)$  satisfying the usual conditions<sup>1</sup>. Here  $\Omega$  is the state space,  $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0, T]}$  is a set of  $\sigma$ -algebras representing measurable events and  $\mathbb{P}$  is the historical (real-world) probability measure. The Rabinovitch (1989) model with spot asset price process  $S(t)$  and correlated stochastic interest rate process  $r(t)$  is specified by the following dynamics under the historical measure  $\mathbb{P}$ :

$$dS(t) = (r(t) + \Lambda_1 \sigma)S(t)dt + \sigma S(t)dW_1^{\mathbb{P}}(t), \quad (1)$$

$$dr(t) = (\lambda(\bar{r} - r(t)) + \Lambda_2 \theta)dt + \theta dW_2^{\mathbb{P}}(t), \quad (2)$$

$$\rho dt = dW_1^{\mathbb{P}}(t)dW_2^{\mathbb{P}}(t).$$

Each parameter of the set  $\Psi = \{\sigma, \lambda, \bar{r}, \theta, \rho, \Lambda_1 \text{ and } \Lambda_2\}$  is a constant and the initial state variables are  $S(0) = S_0$  and  $r(0) = r_0$ . The market price of spot asset price risk and interest rate risk are  $\Lambda_1$  and  $\Lambda_2$ , respectively. The long-term level of the interest rate process and the rate of reversion to the long-term level of the interest rate process are  $\bar{r}$  and  $\lambda$ , respectively. The volatilities of the spot asset price process and the interest rate process are  $\sigma$  and  $\theta$ , respectively. Thus, under the spot risk-neutral measure  $\mathbb{Q}$ , the dynamics of the spot asset

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<sup>1</sup>The usual conditions satisfied by a filtered complete probability space are: (a)  $\mathcal{F}_0$  contains all the  $\mathbb{P}$ -null sets of  $\mathcal{F}$  and (b) the filtration is right continuous

price are expressed as:

$$dS(t) = r(t)S(t)dt + \sigma S(t)dW_1(t), \quad (3)$$

$$dr(t) = \lambda(\bar{r} - r(t))dt + \theta dW_2(t), \quad (4)$$

$$\rho dt = dW_1(t)dW_2(t).$$

Under these model specifications, derivatives on this underlying asset are priced next, including futures, forwards, options on the spot and options on futures.

### 2.1. Futures price

At time  $t$ , the term structure of futures prices  $\{F(t, T)\}_{T \in [t, T^*]}$  (where  $T^*$  is the maximum maturity under consideration) is determined by the spot risk-neutral expectation of the future spot asset prices<sup>2</sup>:

$$F(t, T) = \mathbb{E}^{\mathbb{Q}}[S(T)|\mathcal{F}_t] \quad (5)$$

$$\begin{aligned} &= S(t) \exp\left(-\frac{1}{2}\sigma^2(T-t)\right) \exp\left(M + \frac{1}{2}V^2\right) \\ &= S(t) \exp\left(M + \frac{1}{2}V_1^2 + V_3^2\right), \end{aligned} \quad (6)$$

where

$$M \triangleq (r(t) - \bar{r})A(t, T, \lambda) + \bar{r}(T-t), \quad (7)$$

$$V^2 \triangleq V_1^2 + V_2^2 + 2V_3^2, \quad (8)$$

with

$$\begin{aligned} V_1^2 &= \frac{\theta^2}{\lambda^2} \left( (T-t) - 2A(t, T, \lambda) + A(t, T, 2\lambda) \right), \\ V_2^2 &= \sigma^2(T-t), \\ V_3^2 &= \frac{\rho\theta\sigma}{\lambda} \left( (T-t) - A(t, T, \lambda) \right), \\ A(t, T, \lambda) &= \frac{1}{\lambda} (1 - e^{-\lambda(T-t)}). \end{aligned} \quad (9)$$

$V_1^2$  denotes the variance of the stochastic interest rates, accumulated from time  $t$  to  $T$ , which is equal to  $\text{var}^{\mathbb{Q}} \left[ \int_t^T r(u) du | \mathcal{F}_t \right]$ .  $V_2^2$  denotes the variance of the logarithm of the spot asset price process contributed by just the market risk  $W_1^{\mathbb{P}}(t)$ , accumulated from time  $t$  to  $T$ , which is equal to  $\text{var}^{\mathbb{Q}} [\sigma W_1^{\mathbb{P}}(T) | \mathcal{F}_t]$  and  $V_3^2$  denotes the cross variance, accumulated from time  $t$  to  $T$ . The details of the derivation of the expectation of the future spot asset price (5) given the dynamics (1) and (2) can be found in Appendix 1.

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<sup>2</sup>See Cox, Ingersoll, and Ross (1981).

### 2.2. Forward price

The price at time  $t$  of a zero-coupon bond maturing at time  $T$  is denoted by  $B(t, T, r(t))$ . If the interest rate  $r(t)$  follows the dynamics (4), then the bond price  $B(t, T, r(t))$  is expressed as

$$B(t, T, r(t)) = \exp \left( -M + \frac{1}{2}V_1^2 \right), \quad (10)$$

where  $M$  and  $V_1^2$  are shown in equations (7) and (9). At time  $t$ , the term structure of forward prices  $\{\text{For}(t, T)\}_{T \in [t, T^*]}$  is determined by the  $\mathbb{T}$ -forward expectation of the future spot asset prices<sup>3</sup>:

$$\text{For}(t, T) = \mathbb{E}^{\mathbb{T}}[S(T)|\mathcal{F}_t] \quad (11)$$

$$= \frac{S(t)}{B(t, T, r(t))} = S(t) \exp \left( M - \frac{1}{2}V_1^2 \right). \quad (12)$$

Equations (6) and (12) reveal that the difference between the futures and forward prices is affected by the interest rate volatility and this is true even when the instantaneously correlation between the spot and the interest rate process is zero. Evidently, futures prices and forward prices do not depend on the variance of the spot asset price process  $V_2^2$ . If the spot asset price process is uncorrelated to the interest rate process (namely  $\rho = 0$ ), there is still a convexity adjustment for futures prices which depends on the variance of the stochastic interest rates. Thus, forward prices are different to the futures prices even when this correlation is zero. From (6) and (12), we consequently obtain the associated convexity adjustment (which is critical to hedging applications) as

$$\frac{\text{For}(t, T)}{F(t, T)} = \exp \left( -V_1^2 - V_3^2 \right). \quad (13)$$

Note that, for zero correlation, the convexity adjustment reduces to  $\exp \left( -V_1^2 \right)$ . However, under deterministic interest rate specifications, futures prices and forward prices would be the same. Thus, under the assumption of stochastic interest rates, the forward price and futures price are different. For a rigorous proof, see Appendix 2.

### 2.3. Option price

We consider next futures options with the underlying futures contract maturing at the same time  $T$  as the maturity of the futures option. At maturity, the futures price converges to the spot asset price. That is, at time  $T$ ,  $F(T, T) = S(T)$ . If  $C_f(F(t, T), r(t), T - t; K)$  and  $C(S(t), r(t), T - t; K)$  denote the prices of call futures option and the call spot option, respectively, then the price of a European call futures option is the expectation of the discounted future payoff at maturity under the spot risk-neutral measure which reduces to the price of a European call spot option:

$$\begin{aligned} C_f(F(t, T), r(t), T - t; K) &= \mathbb{E}^{\mathbb{Q}} \left[ e^{-\int_t^T r(u) du} (F(T, T) - K)^+ | \mathcal{F}_t \right] \\ &= \mathbb{E}^{\mathbb{Q}} \left[ e^{-\int_t^T r(u) du} (S(T) - K)^+ | \mathcal{F}_t \right] \\ &= C(S(t), r(t), T - t; K), \end{aligned}$$

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<sup>3</sup>See Lemma 9.6.2 and equation (9.30) on page 341 in Musiela and Rutkowski (2006).

Similar arguments apply to the price of a put futures option  $P_f(F(t, T), r(t), T - t; K)$  and the price of a put spot option  $P(F(t, T), r(t), T - t; K)$ :

$$\begin{aligned} P_f(F(t, T), r(t), T - t; K) &= \mathbb{E}^{\mathbb{Q}} \left[ e^{-\int_t^T r(u) du} (K - F(T, T))^+ | \mathcal{F}_t \right] \\ &= \mathbb{E}^{\mathbb{Q}} \left[ e^{-\int_t^T r(u) du} (K - S(T))^+ | \mathcal{F}_t \right] \\ &= P(S(t), r(t), T - t; K). \end{aligned}$$

Since both the dynamics of  $e^{-\int_t^T r(u) du} | r(t)$  and  $S(T) | S(t)$  are log-normally distributed and the product of two log-normally distributed random variables is also log-normally distributed, then Black-Scholes-like European option pricing formula can be applied with some modifications. Its derivation can be found in Rabinovitch (1989). When  $S(t)$  follows the dynamics (3) and (4), then the price at time  $t$  of a European call option with payoff  $(S(T) - K)^+$  at time  $T$  is:<sup>4</sup>

$$C(S(t), r(t), T - t; K) = S(t)N(d_1) - KB(t, T, r(t))N(d_2), \quad (14)$$

and the price of the corresponding put is

$$P(S(t), r(t), T - t; K) = KB(t, T, r(t))N(-d_2) - S(t)N(-d_1), \quad (15)$$

with

$$d_1 = \left( \log \left( \frac{S(t)}{KB(t, T, r(t))} \right) + \frac{V^2}{2} \right) / V, \quad (16)$$

$$d_2 = d_1 - V, \quad (17)$$

where  $V^2$  is defined in (8) and  $N(x)$  denotes the standard normal cumulative distribution function.

We discuss next the hedging of risks associated with positions in futures options under the proposed model by employing a variety of hedge ratios and hedge schemes.

### 3. Hedging delta and interest rate risk

We aim to hedge the risk of a position in futures options arising from changes in the underlying asset, known as delta, as well as from changes in the interest rates. Typically, delta hedges require positions in the underlying asset (futures contracts or forward contracts) while hedging the interest rate risk would require positions in interest rate sensitive assets, such as bonds. Under the model specification of Section 2, the following proposition demonstrates that a position in a forward contract can hedge simultaneously both the delta risk and the interest rate risk of the forward price of the option. Note that the forward price of the option is computed by

$$\frac{C(S(t), r(t), T - t; K)}{B(t, T, r(t))}.$$

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<sup>4</sup>Equation (5) of Rabinovitch (1989) should be  $\delta(\tau) = -\nu B(\tau)$  which leads to  $+2\rho\sigma(\tau - B)\nu/q$  in equation (8) of Rabinovitch (1989).

**Proposition 1.** *Under the Rabinovitch (1989) model, a position in forward contracts with a balance of the hedge made up of the continuously compounded savings account replicates the forward price of an option with the same maturity as the forward contract. Thus, no separate hedge of the interest rate risk is required.*

*Proof.* Divide the call option formula (14) by the bond price  $B(t, T, r(t))$  to obtain:

$$\frac{C(S(t), r(t), T - t; K)}{B(t, T, r(t))} = \text{For}(t, T)N(d_1) - KN(d_2). \quad (18)$$

By taking the partial derivatives with respect to the forward price  $\text{For}(t)$ , we obtain:

$$\frac{\partial \left( \frac{C(S(t), r(t), T - t; K)}{B(t, T, r(t))} \right)}{\partial \text{For}(t, T)} = N(d_1). \quad (19)$$

By Itô's Lemma the diffusion part of the stochastic process

$$\frac{C(S(t), r(t), T - t; K)}{B(t, T, r(t))}$$

is exactly the same as the diffusion part of the stochastic process of a position of  $N(d_1)$  in the forward contract  $\text{For}(t, T)$ . Thus, a hedging portfolio consisting of  $N(d_1)$  amount of the forward  $\text{For}(t, T)$ , with the balance of the hedge made up of the continuously compounded savings account, hedges all the risk. There is no need to separately hedge the interest rate risk. The hedging portfolio replicates the forward price of the option that collapses to  $C(S(T), r(T), 0; K) = [S(T) - K]^+$  at maturity.  $\square$

We may also consider futures contracts (instead of forward contracts) as hedging instruments. Futures contracts are exchange-traded contracts with several desirable features over the forward contracts, which are over-the-counter contracts. There is no default risk involved in futures transactions because the exchange acts as an intermediary, guaranteeing delivery and payment by the use of a clearing house. Futures contracts are standardised and futures prices are mark-to-market, which provide a realised profit and loss daily, unlike forward contracts where the the profit and loss will not be realised until the maturity of the contract.

Thus, under the Rabinovitch (1989) model specifications, the forward price of the option (with its underlying futures contract maturing at the same time as the option) can be replicated by holding a position in the forward contract with the balance invested in a continuously compounded savings account. However, it is unknown to what extent holding a position in the futures contract with a balance invested in a savings account can replicate the forward price of the option in the presence of interest rate risk. We answer this question in Section 4, where we conduct a numerical analysis using Monte Carlo simulations.

### 3.1. Three ways to compute delta

We employ three different methods to calculate the number of hedging instruments required to hedge positions in futures options. Firstly, we hedge the futures option under the assumption of deterministic interest rates, i.e. the hedge ratios are the Black-Scholes deltas.

Then we hedge the futures option assuming stochastic interest rates, i.e. the hedge ratios are calculated using the Rabinovitch (1989) model. In the third method, hedge ratios are derived from *factor hedging*, which can be easily generalised to hedge risk generated from multi-factor / multi-dimensional models. To calculate the number of bond contracts needed to hedge the interest rate risk of the option position, when necessary, we use hedge ratios derived from factor hedging.

### 3.1.1. Black-Scholes' delta

Firstly, the number of the forward contracts are determined by the Black and Scholes (1973) delta, denoted by  $\delta_t^{\text{BS,For}}$ , that assumes constant interest rates, and it is given by  $N(d_1^{\text{BS,For}}(t))$  with

$$d_1^{\text{BS,For}}(t) = \frac{\log\left(\frac{S(t)}{K}\right) + (\bar{r} + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad (20)$$

where, under this model, the future interest rates until maturity are assumed to be fixed which equals to the long-term average  $\bar{r}$ . Unlike the forward price of options which have payoff at maturity, futures contracts are mark-to-market with profits or losses realised at the end of each trading day. Because of this, the number of the futures contracts to hold must be adjusted and it is given by:

$$\delta_t^{\text{BS,Fut}} = \delta_t^{\text{BS,For}} e^{-\bar{r}(T-t)}. \quad (21)$$

### 3.1.2. Rabinovitch's delta

Under the stochastic interest rate model of Rabinovitch (1989), the number of forward contracts required for hedging, denoted by  $\delta_t^{\text{R,For}}$ , is equal to  $N(d_1^{\text{R,For}}(t))$  with  $d_1^{\text{R,For}}(t)$  specified by (16). To determine the number of futures contracts required for hedging, we need to additionally multiply  $N(d_1^{\text{R,For}}(t))$  by the convexity adjustment  $e^{-V_1^2 - V_3^2}$ , as shown in equation (13).<sup>5</sup> As a result, we have  $\delta_t^{\text{R,Fut}} = \delta_t^{\text{R,For}} e^{-V_1^2 - V_3^2} B(t, T, r(t))$ .

### 3.1.3. Delta hedge by factor hedging

Hedge ratios derived from the Black-Scholes' and Rabinovitch's delta are suited for one-factor model specifications, similar to the ones treated in this paper. However, for more general multi-dimensional models (that provide better fit to market data, see Cheng, Nikitopoulos, and Schlögl (2016b)), we propose hedge ratios obtained from factor hedging. In this paper, we only consider one-dimensional models, so we reduce the factor hedging specifications to one factor only. Nevertheless, these investigations will allow us to compare the performance of hedge ratios from factor hedging with alternative hedge ratios and validate their efficiency. To delta hedge an option's position by the factor hedging method, we attempt to immunise the profits and losses (hereafter P/L) of a forward call option due to a small movement from the spot asset price by adding some position in forward contracts. On the trading day  $t_k$ , we determine  $\delta_k^{\text{F,For}}$  units of the forward contracts required for delta hedging as follows. We determine two spot asset prices  $S^u$  and  $S^d$  for the next trading day

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<sup>5</sup>We note that we do not adjust our Black-Scholes' delta for futures by this convexity adjustment because in the Black-Scholes interest rates are deterministic, hence the futures price equals the forward price.

$t_{k+1}$  given a one-standard-deviation up and down shock:<sup>6</sup> to the spot asset price dynamics (1), respectively. Thus<sup>7</sup>

$$\begin{aligned} S^u &\triangleq S_k \exp \left( (r_k + \Lambda_1 \sigma - \frac{1}{2} \sigma^2) \Delta t + \sigma \sqrt{\Delta t} \right), \\ S^d &\triangleq S_k \exp \left( (r_k + \Lambda_1 \sigma - \frac{1}{2} \sigma^2) \Delta t - \sigma \sqrt{\Delta t} \right), \end{aligned}$$

where  $\Delta t = 1/252$  represents the time interval of one trading day. Then the delta hedge ratio  $\delta_k^F$  is determined by:

$$\begin{aligned} \delta_k^{\text{F,For}} &= \left( \frac{C(S^u, r_k)}{B(t_k, T, r_k)} - \frac{C(S^d, r_k)}{B(t_k, T, r_k)} \right) \bigg/ \left( \frac{S^u}{B(t_k, T, r_k)} - \frac{S^d}{B(t_k, T, r_k)} \right) \\ &= \frac{C(S^u, r_k, T - t_k; K) - C(S^d, r_k, T - t_k; K)}{S^u - S^d}. \end{aligned} \quad (22)$$

Similarly to Black-Scholes' delta and Rabinovitch's delta, to determine the number of futures contracts required for hedging, we need to adjust the factor hedging delta in (22) by the convexity adjustment  $e^{-V_1^2 - V_3^2}$ :

$$\delta_k^{\text{F,Fut}} = \delta_k^{\text{F,For}} e^{-V_1^2 - V_3^2} B(t_k, T, r_k). \quad (23)$$

The replicating portfolio consisting of a position in the forward contracts (or futures contract) and money in the savings account would be able to replicate the forward price of the option at all time and provide a payoff that matches exactly the payoff of the spot option. Thus, the forward (or futures) contracts hedge both the delta risk and the interest rate risk of the options positions. This holds under the assumption that the maturity of the hedging instruments is the same as the maturity of the option to be hedged.

### 3.2. Hedging with short-dated hedging instruments

In this section, we consider hedging of long-dated options with short-dated contracts. Then, short-dated forwards and futures hedge the delta risk of the longer-dated option positions, yet bonds are additionally required to hedge the interest rate risk. Thus, we assume the following hedging instruments: shorter maturity forward and futures contracts, while the maturity of the bonds is assumed to be the same as the maturity of the futures option.<sup>8</sup> This also allows us to gauge the impact of the basis risk emerging due to rolling the hedge forward with shorter maturity forward or futures contracts.

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<sup>6</sup>Chiarella et al. (2013) perform a simulation over a number of shocks instead of using a deterministic one-standard-deviation jumps. We have compared both methods and we do not find any noticeable difference between them in the present context, so we use one-standard-deviation up and down shocks for the sake of simplicity and computational efficiency

<sup>7</sup>We define the shorthanded notation of  $S(t_k)$  by  $S_k$  and  $r(t_k)$  by  $r_k$ .

<sup>8</sup>In practice, long-dated bond contracts are liquid thus we may assume that there is no need to use short-dated bonds as hedging instruments.

### 3.2.1. Delta hedging

For delta hedging, we adjust the position of the hedging instruments by a factor equal to the bond price maturing at the same time as the hedging instrument, further divided by the bond price maturing at the same time as the futures option (see Appendix 3 for details). We specifically denote as  $T$  the maturity of the option, and  $T_F$  the maturity of the hedging instrument. To reduce confusion in the notation, in this section, we explicitly show the dependency of the maturity and current time in  $V^2, V_1^2, V_2^2$  and  $V_3^2$  in equation (8) by  $V^2(t, T), V_1^2(t, T), V_2^2(t, T)$  and  $V_3^2(t, T)$ . The Rabinovitch delta for forward contracts from Section 3.1.2 is generalised to:

$$\delta_t^{\text{R,For}} = N(d_1^{\text{R,For}}(t)) \frac{B(t, T_F, r(t))}{B(t, T, r(t))}$$

and the Rabinovitch delta for futures contracts is given by:

$$\begin{aligned} \delta_t^{\text{R,Fut}} &= \delta_k^{\text{R,For}} e^{-V_1^2(t, T_F) - V_3^2(t, T_F)} B(t, T, r(t)) \\ &= N(d_1^{\text{R,For}}(t)) e^{-V_1^2(t, T_F) - V_3^2(t, T_F)} B(t, T, r(t)) \frac{B(t, T_F, r(t))}{B(t, T, r(t))} \\ &= N(d_1^{\text{R,For}}(t)) e^{-V_1^2(t, T_F) - V_3^2(t, T_F)} B(t, T_F, r(t)). \end{aligned}$$

We note that  $d_1^{\text{R,For}}$  is defined exactly as in equation (16). The deltas for forward and futures contracts calculated using factor hedging method are generalised as

$$\begin{aligned} \delta_k^{\text{F,For}} &= \frac{B(t_k, T_F, r_k)}{B(t_k, T, r_k)} \left( \frac{C(S^u, r_k)}{B(t_k, T, r_k)} - \frac{C(S^d, r_k)}{B(t_k, T, r_k)} \right) \bigg/ \left( \frac{S^u}{B(t_k, T, r_k)} - \frac{S^d}{B(t_k, T, r_k)} \right) \\ &= \left( \frac{C(S^u, r_k)}{B(t_k, T, r_k)} - \frac{C(S^d, r_k)}{B(t_k, T, r_k)} \right) \bigg/ \left( \frac{S^u}{B(t_k, T_F, r_k)} - \frac{S^d}{B(t_k, T_F, r_k)} \right), \end{aligned} \quad (24)$$

and

$$\delta_t^{\text{F,Fut}} = \delta_k^{\text{F,For}} e^{-V_1^2 - V_3^2} B(t_k, T, r_k),$$

respectively. Interest rate hedging is discussed next by adding bonds to the hedging portfolios.

### 3.2.2. Delta-IR hedging

To hedge the delta and interest rate risk by factor hedging,<sup>9</sup> we firstly immunise the P/L of a forward option due to a small movement in the market risk, e.g. delta hedge by factor hedging. The step to calculate the amount of forward or futures contracts required to hedge is exactly the same as in equation (24). We further immunise the residual risk due to a small

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<sup>9</sup>Aiming to generalise this hedging application to multi-dimensional models, we examine delta-IR hedging by using the factor hedging since it can be easily extended to more general models.

movement in the interest rate risk. We calculate the up and down movements of the interest rate process  $r^u$  and  $r^d$  by:

$$\begin{aligned} r^u &= r_k + \lambda(\bar{r} - r_k + \Lambda_2\theta)\Delta t + \theta\sqrt{\Delta t}, \\ r^d &= r_k + \lambda(\bar{r} - r_k + \Lambda_2\theta)\Delta t - \theta\sqrt{\Delta t} \end{aligned}$$

from these two movements, the up and down interest-rate shocked bond and forward prices (with maturity equal  $T_F$  that can be different to the maturity of the futures option  $T$ ) are:

$$\begin{aligned} B^{ur} &= B(t_k, T_F, r^u), \\ B^{dr} &= B(t_k, T_F, r^d), \\ \text{For}^{ur} &= \frac{S_k}{B^{ur}}, \\ \text{For}^{dr} &= \frac{S_k}{B^{dr}}. \end{aligned}$$

We also let:

$$\begin{aligned} C^{ur} &= C(S_k, r^u, T - t_k; K), \\ C^{dr} &= C(S_k, r^d, T - t_k; K). \end{aligned}$$

The change of the time- $T$  forward value of this hedging portfolio (long a call and short  $\delta$  forward) solely due to the up and down interest rate shocks in forward time  $T$  is:

$$\frac{C^{ur}}{B^{ur}} - \frac{C^{dr}}{B^{dr}} - \delta_k^{\text{F,For}} (\text{For}^{ur}(t_k, T_F) - \text{For}^{dr}(t_k, T_F)), \quad (25)$$

where  $\delta_k^{\text{F,For}}$  is given in equation (24). This change of the time- $T$  forward value of the hedging portfolio due to the interest rate shocks is to be further immunised by holding  $\Phi$  number of bonds maturing at time  $T$ . The change of the time- $T$  forward value of  $\Phi$  number of bonds (maturing at  $T$ ) due to the up and down interest rate shocks is:

$$\Phi_k \frac{B(t_k, T, r^u) - B(t_k, T, r^d)}{B(t_k, T, r_k)}. \quad (26)$$

The amount of bond contracts  $\Phi$  required for hedging is computed by equating equations (25) and (26):

$$\Phi_k = \left( \frac{\frac{C^{ur}}{B^{ur}} - \frac{C^{dr}}{B^{dr}} - \delta_k^{\text{F,For}} (\text{For}^{ur}(t_k, T_F) - \text{For}^{dr}(t_k, T_F))}{B(t_k, T, r^u) - B(t_k, T, r^d)} \right) B(t_k, T, r_k). \quad (27)$$

We note that the maturity of the bond is the same as the maturity of the futures option. Only the futures or forward contract as hedging instrument is allowed to have a shorter maturity. The formula (27) should be adjusted by the suitable convexity when futures contracts are used for hedging.

#### 4. Numerical investigations

In this section, we perform numerical investigations to gauge the impact of stochastic interest rates on hedging long-dated futures options. Firstly, we assess the contribution of the discretisation error in our hedging applications. Secondly, we compare the performance of different hedging schemes, including hedging ratios from deterministic interest rate specifications, stochastic interest rate specifications and factor hedging. For each hedging scheme, we consider different hedging instruments, for instance, forwards, futures and bonds with varying maturities. Lastly, the impact of the model parameters such as the long-term mean of interest rates ( $\bar{r}$ ) and the interest rate volatility ( $\theta$ ) is also evaluated.

Let the set of trading days be  $\{t_k\}$ ,  $k = 1, \dots, N$  where  $N$  is the total number of trading days and the option, futures and forwards all have the same maturity, denoted by  $T > T_N$ . Given a hedge frequency  $h$  representing the frequency that the number of futures, forwards or bond contracts is allowed to change in a day, in each simulation, we can calculate the standard deviations of the profits and losses of the forward call option over a period of  $N$  trading days by:

$$SD_j = \sqrt{\sum_{k=2}^{N \times h} (P/L_{j,k})^2}, \quad (28)$$

$$P/L_{j,k} \triangleq CB_{t_k}^j - CB_{t_{k-1}}^j, \quad (29)$$

where  $CB_t^j = \frac{C_t^j}{B_t^j}$  denotes the forward price of the option and  $j = 1, \dots, M$  denotes the  $j^{\text{th}}$  realisation of the Monte Carlo simulation. Further, we let  $C_{t_k}^j = C(S^j(t_k), r^j(t_k), T - t_k; K)$  to denote the simulated option price at time  $t_k$  using the  $j^{\text{th}}$  simulated spot asset price  $S^j(t_k)$  and the  $j^{\text{th}}$  simulated interest rate  $r^j(t_k)$  under the historical measure as specified in (1) and (2) where both spot asset prices and interest rates evolve stochastically. Similarly we let  $B_{t_k}^j = B(t_k, T, r^j(t_k))$  to be the simulated bond price at time  $t_k$  using the simulated interest rate  $r^j(t_k)$  under the historical measure.

The  $j^{\text{th}}$  realised standard deviation of a hedging portfolio using Black-Scholes' delta is defined similarly to (28) with a time- $T$  forward value<sup>10</sup> of  $P/L_k$  defined as follows:

$$\begin{aligned} P/L_{j,k}^{\text{BS,For}} &\triangleq C_{t_k}^j e^{\bar{r}(T-t_k)} - C_{t_{k-1}}^j e^{\bar{r}(T-t_{k-1})} - \delta_{t_{k-1}}^{\text{BS,For}} \left( \text{For}(t_k, T) - \text{For}(t_{k-1}, T) \right) \\ &\quad - C_{t_{k-1}}^j e^{\bar{r}(T-t_k)} (e^{\bar{r}(t_k-t_{k-1})} - 1) \\ &= C_{t_k}^j e^{\bar{r}(T-t_k)} - C_{t_{k-1}}^j e^{\bar{r}(T-t_{k-1})} - \delta_{t_{k-1}}^{\text{BS,For}} \left( \text{For}(t_k, T) - \text{For}(t_{k-1}, T) \right) \\ &\quad - C_{t_{k-1}}^j (e^{\bar{r}(T-t_{k-1})} - e^{\bar{r}(T-t_k)}) \end{aligned} \quad (30)$$

and

$$\begin{aligned} P/L_{j,k}^{\text{BS,Fut}} &\triangleq C_{t_k}^j e^{\bar{r}(T-t_k)} - C_{t_{k-1}}^j e^{\bar{r}(T-t_{k-1})} \\ &\quad - \delta_{t_{k-1}}^{\text{BS,Fut}} \left( F(t_k, T) - F(t_{k-1}, T) \right) e^{\bar{r}(T-t_k)} - C_{t_{k-1}}^j (e^{\bar{r}(T-t_{k-1})} - e^{\bar{r}(T-t_k)}). \end{aligned} \quad (31)$$

<sup>10</sup>To get the forward value under the Black-Scholes world, we assume the interest rate is fixed till maturity. So the forward value of \$1 is \$  $e^{r_k(T-t_k)}$ .

The P/L at time  $t_k$  of a  $j^{\text{th}}$  realised standard deviation of a hedging portfolio using Rabinovitch's delta is:

$$\begin{aligned} \text{P/L}_{j,k}^{\text{R,For}} \triangleq & CB_{t_k}^j - CB_{t_{k-1}}^j - \delta_{t_{k-1}}^{\text{R,For}} \left( \text{For}(t_k, T) - \text{For}(t_{k-1}, T) \right) \\ & - \frac{C_{t_{k-1}}^j}{B(t_k, T, r_k)} \left( 1/B(t_{k-1}, t_k, r_{k-1}) - 1 \right). \end{aligned} \quad (32)$$

or for futures contracts as hedging instruments, we have:

$$\begin{aligned} \text{P/L}_{j,k}^{\text{R,Fut}} \triangleq & CB_{t_k}^j - CB_{t_{k-1}}^j - \delta_{t_{k-1}}^{\text{R,Fut}} \left( F(t_k, T) - F(t_{k-1}, T) \right) / B(t_k, T, r_k) \\ & - \frac{C_{t_{k-1}}^j}{B(t_k, T, r_k)} \left( 1/B(t_{k-1}, t_k, r_{k-1}) - 1 \right). \end{aligned} \quad (33)$$

The P/L at time  $t_k$  of a  $j^{\text{th}}$  realised standard deviation of a hedging portfolio using delta calculated by the factor hedging method,  $\text{P/L}_{j,k}^{\text{F,For}}$ , is defined similarly to (32), but with  $\delta_{t_{k-1}}^{\text{R,For}}$  replaced by  $\delta_{t_{k-1}}^{\text{F,For}}$ .  $\text{P/L}_{j,k}^{\text{F,Fut}}$  is defined similarly to (33) using  $\delta_{t_{k-1}}^{\text{F,Fut}}$ .

The P/L at time  $t_k$  of a  $j^{\text{th}}$  realised standard deviation of a hedging portfolio using both forwards and bonds, with the maturity of the short-dated forwards as  $T_F < T$ , is:

$$\begin{aligned} \text{P/L}_{j,k}^{\text{F,For,IR}} \triangleq & CB_{t_k}^j - CB_{t_{k-1}}^j - \delta_{t_{k-1}}^{\text{R,For}} \left( \text{For}(t_k, T_F) - \text{For}(t_{k-1}, T_F) \right) \\ & - \Phi_{k-1} \left( B(t_k, T, r_k) - B(t_{k-1}, T, r_{k-1}) \right) / B(t_k, T, r_k) \\ & - \frac{C_{t_{k-1}}^j - \Phi_{k-1} B(t_{k-1}, T, r_{k-1})}{B(t_k, T, r_k)} \left( 1/B(t_{k-1}, t_k, r_{k-1}) - 1 \right) \\ = & CB_{t_k}^j - CB_{t_{k-1}}^j - \delta_{t_{k-1}}^{\text{R,For}} \left( \text{For}(t_k, T_F) - \text{For}(t_{k-1}, T_F) \right) \\ & - \Phi_{k-1} \left( 1 - \frac{B(t_{k-1}, T, r_{k-1})}{B(t_k, T, r_k) B(t_{k-1}, t_k, r_{k-1})} \right) \\ & - \frac{C_{t_{k-1}}^j}{B(t_k, T, r_k)} \left( 1/B(t_{k-1}, t_k, r_{k-1}) - 1 \right) \end{aligned}$$

and  $\text{P/L}_{j,k}^{\text{F,Fut,IR}}$  at time  $t_k$  of a hedging portfolio using both futures and bonds is defined similarly.

Proposition 1 demonstrates that a hedging position consisting of forward or futures contracts (with a maturity equal to the futures option's maturity) and the balance of the hedge in a continuously compounded savings account is enough to replicate the forward price of the option. One requirement for this hedging portfolio to exactly replicate the forward price of the option is that it has to be re-balanced continuously, which is impossible to execute in practice. If the portfolio cannot be continuously re-balanced, hedging error due to discretisation exists. We assess the contribution of discretisation error when the forward price of the option is replicated by using different hedging schemes and their performance is assessed as the hedging frequency increases. Furthermore, we assess hedging performance when the maturity of the hedging instruments match with the maturity of the option to be hedged and when their maturities do not match (e.g., use shorter maturity hedging instruments).

Table 1: Parameters Values

Parameters	Values	Notes
$M$	1000	Number of simulations
$N$	1000	Number of trading days
$T$	1500	Option maturity
$T_F$	1500	Hedging futures / for maturity
$\lambda$	0.4	mean-reversion rate
$\Lambda_1$	0	market price of risk
$\Lambda_2$	0	market price of interest rate risk
$\sigma$	varying	volatility
$\bar{r}$	{5%, 1%}	long-term mean level of interest rate
$\theta$	{8%, 3%, 1%}	volatility of interest rate
$\rho$	0%	interest rate correlation
$S(0)$	\$50	initial stock price
$r(0)$	$\bar{r}$	initial interest rate
$K$	at-the-money forward	strike
Type	Call	option type

#### 4.1. Hedging with matching maturity

In this section, we assume that the maturity of the option and the hedging instruments (forward and futures) is the same and is set to 1,500 trading days for our numerical investigations. The number  $M$  of the simulations is 1000 and the option is hedged from day 1 to day 1000. For consistency and comparability,  $\sigma$  are chosen such that  $V^2$  in equation (8) remains the same as we vary  $\theta$ . The option price is struck at-the-money forward (that is  $K = \text{For}(t, T)$ ) so  $d_1$  in equation (16) reduces to  $\frac{V}{2}$  and with  $V$  remaining the same across different  $\theta$ , the price of the initial options with different  $\theta$  are the same. However, the forward price of the options will differ slightly because the bond prices change for different values of  $\theta$ . Table 1 provides the parameter values used in the numerical hedging analysis and we consider the following six different hedging schemes:

1. Black-Scholes delta with forward contracts as hedging instrument,
2. Black-Scholes delta with futures contracts as hedging instrument,
3. Rabinovitch delta using forwards contracts as hedging instrument,
4. Rabinovitch delta with futures contracts as hedging instrument,
5. Factor hedging method with forward contracts as hedging instrument,
6. Factor hedging method with futures contracts as hedging instrument.

Tables 2 and 3 show the simulated hedging performance, measured by the average of the standard deviation of the forward price of the hedging portfolios over 1000 paths consisting of an option and a number of futures or forwards derived by using three different hedging schemes as described above and for increasing hedging frequency. Table 2 considers a long-term mean of interest rates of 5% (around the last 12-year empirical average), while Table 3 considers a long-term mean of interest rates around 1% which is the currently observed level of interest rates. According to Proposition 1, the hedging portfolio consisted of a position

$\bar{r} = 5\%, \theta = 8\%$							
Hedge Frequency	Call/Bond	BS For	BS Fut	Rab For	Rab Fut	Factor For	Factor Fut
1	18.9006	7.8440	7.0645	0.3866	0.3418	0.3866	0.3418
10	18.2189	7.3175	6.5420	0.1130	0.0977	0.1130	0.0977
100	18.8607	7.5770	6.7804	0.0364	0.0318	0.0364	0.0318
$\bar{r} = 5\%, \theta = 3\%$							
Hedge Frequency	Call/Bond	BS For	BS Fut	Rab For	Rab Fut	Factor For	Factor Fut
1	16.4546	2.0149	1.9708	0.2328	0.2223	0.2329	0.2223
10	15.9600	1.9582	1.9172	0.0723	0.0688	0.0723	0.0688
100	16.1199	1.9334	1.8933	0.0227	0.0216	0.0227	0.0216
$\bar{r} = 5\%, \theta = 1\%$							
Hedge Frequency	Call/Bond	BS For	BS Fut	Rab For	Rab Fut	Factor For	Factor Fut
1	16.0233	0.6620	0.6598	0.2105	0.2092	0.2105	0.2093
10	15.5760	0.6152	0.6136	0.0654	0.0650	0.0654	0.0650
100	15.8806	0.6237	0.6222	0.0210	0.0209	0.0210	0.0209

Table 2: **Discretisation hedging error with increasing hedging frequency when  $\bar{r} = 5\%$  and volatility of interest rates are 8%, 3% and 1%, respectively.** The table displays the discretisation hedging error, when a 1500-day option is hedged by using forwards/futures with the same maturity. The option is hedged over 1000 days.

in futures or forwards, calculated using the Rabinovitch delta or the factor hedging method and the balance in the continuously compounded savings account is enough to replicate the forward price of the option. Thus, the errors that we see in the tables above by using the Rabinovitch or the factor hedging delta are due to discretisation only, and as expected, the errors are reduced by a factor of  $\sqrt{10}$  as we increase the hedging frequency from 1 to 10 and from 10 to 100. The variance obtained from the hedged position with the number of futures or forward contracts calculated by using the Black-Scholes delta does not perform well. Indeed, the errors do not decrease as the hedging frequency increases. This simulation exercise demonstrates that using futures or forward contracts alone, with their maturities matching the maturity of the option, can hedge both the underlying market risk and the interest risk and as a consequence, additional interest rate sensitive contracts such as bonds are not required.

Next, we compare the hedging performance of the Rabinovitch model that assumes stochastic interest rates to the Black model that assumes deterministic interest rates. In a very high interest rate volatility environment the improvement provided by the stochastic interest rate model over the deterministic interest rate model is significant. From Table 2 and when interest rate volatility is at 8%, the hedging error of using forwards as hedging instruments reduces from 7.8440 in the Black model to 0.3866 in the Rabinovitch model; that is a reduction of over 95%. However, when interest rate volatility is at 1%, the reduction is only 68% ( $1 - 0.2105/0.6620$ ). Similar conclusions can be drawn using results from Table 3 when the long-term level of the interest rates is low. We also observe that the long-term level of the interest rates  $\bar{r}$  does not have any structural impact to the hedging performance.

Finally, both tables validate the robustness of the hedge ratios computed by the factor hedging since the errors from Rabinovitch hedging are identical to the errors from factor delta hedging.

$\bar{r} = 1\%, \theta = 8\%$							
Hedge Frequency	Call/Bond	BS For	BS Fut	Rab For	Rab Fut	Factor For	Factor Fut
1	14.2534	5.7584	5.1510	0.2445	0.2053	0.2445	0.2053
10	14.4185	5.7653	5.1542	0.0785	0.0659	0.0785	0.0659
100	14.1518	5.8260	5.2197	0.0246	0.0206	0.0246	0.0206
$\bar{r} = 1\%, \theta = 3\%$							
Hedge Frequency	Call/Bond	BS For	BS Fut	Rab For	Rab Fut	Factor For	Factor Fut
1	12.5001	1.5301	1.4959	0.1571	0.1478	0.1571	0.1478
10	12.5569	1.5211	1.4894	0.0492	0.0464	0.0492	0.0464
100	12.1457	1.4881	1.4564	0.0156	0.0147	0.0156	0.0147
$\bar{r} = 1\%, \theta = 1\%$							
Hedge Frequency	Call/Bond	BS For	BS Fut	Rab For	Rab Fut	Factor For	Factor Fut
1	12.3066	0.5083	0.5063	0.1464	0.1453	0.1464	0.1453
10	12.2661	0.4812	0.4799	0.0460	0.0457	0.0460	0.0457
100	11.6745	0.4680	0.4669	0.0144	0.0143	0.0144	0.0143

Table 3: **Discretisation hedging error with increasing hedging frequency when  $\bar{r} = 1\%$  and volatility of interest rates are 8%, 3% and 1%, respectively.** The table displays the discretisation hedging error, when a 1500-day option is hedged by using forwards/futures with the same maturity. The option is hedged over 1000 days.

The hedging exercise performed in this section sheds some light on making a decision of a model assuming stochastic interest rates which is more involved mathematically and also computationally demanding or assuming the much simpler deterministic interest rate model. We conclude that the interest rate volatility is the single most decisive factor when one considers stochastic interest rates. In the next section, a numerical investigation is conducted where the hedging instruments used have shorter maturities from the maturity of the futures option.

#### 4.2. Hedging with maturity mismatch

In practice, when hedging long-dated futures options it is not always possible to hedge with the underlying futures contract. Long-dated options are hedged with shorter maturity futures or forward contracts and then the hedge is rolled over to another contract. Since only market risk and interest rate risk are present in our model, the interest rate risk is the risk that reflects the basis risk introduced when we roll the hedge forward. The following numerical investigations allow us to gauge the level of deterioration in the hedging performance as the frequency of rolling the hedge forward is increasing.

Table 4 displays the parameter values, and the maturities of the option, futures and forward used for the numerical hedging analysis conducted when the hedge is rolled forward. We assume that the call option in this simulation experiment matures in  $T = 2000$  trading days, with hedging instruments (futures or forwards) maturing in 2000, 1800, 1200 and 600 trading days denoted as  $T_{F1}$ ,  $T_{F2}$ ,  $T_{F3}$  and  $T_{F4}$ , respectively. We hedge the option for 500 trading days, so there is no rolling-over required in the futures or forward contracts. The following six hedging schemes are considered in these investigations:

1. Rabinovitch delta with forward contracts as hedging instrument,
2. Rabinovitch delta with futures contracts as hedging instrument,

3. Factor delta hedging with forward contracts as hedging instrument,
4. Factor delta hedging with futures contracts as hedging instrument,
5. Factor delta and interest rate hedging with forward contracts and bonds as hedging instruments,
6. Factor delta and interest rate hedging with futures contracts and bonds as hedging instruments,

Table 5 and Table 6 present the hedging performance of these hedging schemes with increasing hedging frequency, maturity mismatches and zero correlation coefficient between the asset price process and the interest rate process. In particular, Table 5 considers a relatively high interest rate volatility of 8%, while Table 6 considers a low interest rate volatility environment with interest rate volatility of 1%. By comparing the results in these two tables, we conclude that adding a position in bond contracts provides a substantial improvement especially when interest rates have high volatility.<sup>11</sup> For instance, in Table 5 with maturities of hedging instrument equal to 1200 trading days ( $T_{F3}$ ) and with hedging frequency equals to 1 day, the standard deviation of the portfolio with forwards alone is 2.3242 and 0.2550 when bond contracts are added. This reduction in the standard deviation is nearly by a factor of 10. However, in Table 6, the corresponding results are 0.3173 and 0.1502, with a reduction of factor by about 2. Consequently, when the portfolio is not re-balanced frequently (e.g. daily) and when the volatility of the interest rates  $\theta$  is lower than 1%, adding bond contracts as hedging instruments will provide only a marginal improvement.

From the results of the scenarios where the maturities of futures or forwards match the option's maturity ( $T_{F1}$ ), futures or forward contracts alone can hedge both market and interest rate risks (because their standard deviation are reduced as we increase the hedging frequency). Furthermore, as expected, adding bond contracts in the hedging portfolio does not further reduce the standard deviation.

When the maturities of futures and forwards equal to 1800 trading days ( $T_{F2}$ ), we observe that there is a reduction in the standard deviation as the hedging frequency increases but the reduction factor is no longer  $\sqrt{10}$  (for example, from the Rabinovitch forward column in Table 5, the standard deviation is reduced from 0.4226 to 0.3373 to 0.3295 and similarly to futures contracts). However, there is a significant reduction of the variance when the hedging portfolio consists of bond contracts and futures (or forward) contracts. As the hedging frequency increases, the hedging portfolio consisting of bond contracts and futures (or forward) contracts reduces the standard deviation by a factor of  $\sqrt{10}$ . Thus, we conclude that when futures or forwards contracts with shorter maturities are used as hedging instruments, the interest rate risk cannot be hedged by futures or forward contracts alone. The results of the scenarios with hedging instruments maturing in 1200 trading days and 600 trading days are also consistent with these findings. Comparing the hedging performance of using forward contracts that mature in day 1800 ( $T_{F2}$ ) and in day 1200 ( $T_{F3}$ ) over a high interest rate volatility environment ( $\theta = 8\%$ ) and with a hedge frequency of 1, the hedging performance

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<sup>11</sup>This does not hold, when the maturity of the hedging instruments match the option's maturity. Using only forward or futures can hedge both the market risk and interest rate risk so bonds are not required, see Section 4.1.

Table 4: Parameters Values

Parameters	Values	Notes
$M$	1000	Number of simulations
$N$	500	Number of trading days
$T$	2000	Option maturity
$T_{F1}$	2000	Hedging fut / for maturity 1
$T_{F2}$	1800	Hedging fut / for maturity 2
$T_{F3}$	1200	Hedging fut / for maturity 3
$T_{F4}$	600	Hedging futures maturity 4
$T_B$	2000	Hedging bond maturity
$\lambda$	0.4	mean-reversion rate
$\Lambda_1$	0	market price of risk
$\Lambda_2$	0	market price of interest rate risk
$\sigma$	varying	volatility
$\bar{r}$	5%	long-term mean level of interest rate
$\theta$	{8%, 1%}	volatility of interest rate
$\rho$	{0%, -50%}	interest rate correlation
$S(0)$	\$50	initial stock price
$r(0)$	$\bar{r}$	initial interest rate
$K$	at-the-money forward	strike
Type	Call	option type

deteriorates by more than five-fold, from 0.4226 to 2.3242. However, during a lower interest rate volatility environment ( $\theta = 1\%$ ) the hedging performance deteriorates only by over twofold, from 0.1555 to 0.3173.

Tables 7 and 8 display the results of similar investigations as in Tables 5 and 6 by allowing though the correlation  $\rho$  to be  $-50\%$ . Overall, the correlation has no impact to the hedging performance of the option positions. Similarly, a maturity mismatch introduces an interest rate risk that cannot be hedged by futures contracts only and as a result, bond contracts should be added to reduce this interest rate risk.

## 5. Conclusion

In this paper, we analyse the impact of interest rate risk on futures options positions and examine the hedging of this risk. Due to its tractability, the Rabinovitch (1989) model is considered with correlated dynamics between spot asset prices and interest rates. The market risk and the interest rate risk of a position in futures options is hedged by using several hedging schemes and compared to deterministic interest rate specifications. In particular, we study the impact of the discretisation error and the parameter values (such as interest rate volatility) to the hedging applications. We also examine hedging long-dated options with shorter maturity hedging instruments.

We show mathematically and numerically that, under the Rabinovitch (1989) model, the forward price of the option can be replicated by forward contracts that have the same maturity as the option. We show that forward contracts can hedge both the interest rate risk

$T_{F1} = 2000, \theta = 8\%, \rho = 0\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	15.1403	0.2520	0.2220	0.2521	0.2220	0.2521	0.2220
10	14.8047	0.0764	0.0654	0.0764	0.0654	0.0764	0.0654
100	14.8824	0.0237	0.0204	0.0237	0.0204	0.0237	0.0204
$T_{F2} = 1800, \theta = 8\%, \rho = 0\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	15.1403	0.4226	0.3996	0.4219	0.3987	0.2521	0.2070
10	14.8047	0.3373	0.3348	0.3372	0.3348	0.0765	0.0644
100	14.8824	0.3295	0.3293	0.3295	0.3293	0.0237	0.0198
$T_{F3} = 1200, \theta = 8\%, \rho = 0\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	15.1403	2.3242	2.3222	2.3234	2.3214	0.2550	0.2246
10	14.8047	2.2492	2.2490	2.2491	2.2489	0.0778	0.0695
100	14.8824	2.2522	2.2522	2.2522	2.2522	0.0241	0.0215
$T_{F4} = 600, \theta = 8\%, \rho = 0\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	15.1403	7.4311	7.4309	7.4308	7.4306	0.2760	0.2683
10	14.8047	7.2242	7.2242	7.2242	7.2242	0.0845	0.0823
100	14.8824	7.2375	7.2375	7.2375	7.2375	0.0263	0.0256

Table 5: **Hedging performance with increasing hedging frequency, maturity mismatch and  $\theta = 8\%$ .** The forward price of a long-dated option (Call/Bond) is hedged by using futures or forwards with various maturities. The four different maturities  $T_{F1}, T_{F2}, T_{F3}$  and  $T_{F4}$  represent the maturities of the hedging futures and forwards with 2000, 1800, 1200 and 600 trading days, respectively. In all cases, the maturity of the bond contracts is 2000 trading days.

and the underlying market risk of options positions and as a consequence there is no need to separately hedge interest rate risk. This also holds when we use futures contracts as hedging instruments, but convexity adjustment must be considered in calculating the hedging ratio of futures contracts.

Numerical results demonstrate that when the maturities of forward (or futures) contracts and the maturity of the option differ slightly (maturity of 2000 trading days for options and 1800 for futures/forwards), the hedging performance deteriorates noticeably. To improve the hedging performance, the interest rate risk must be hedged separately by holding positions in bond contracts. Furthermore, when the interest rate volatility is high, adding bond contracts improves the hedging performance substantially and the improvement builds up as we increase the hedging frequency. Finally, the long-term mean level of interest rates and the correlation between the spot process and the interest rate process do not affect the hedging performance. We also validate the numerical efficiency of factor hedging that is suited for multi-dimensional models, similarly to the ones considered in Cheng, Nikitopoulos, and Schlögl (2016a).

These numerical investigations provide useful insights on the sources and the nature of the interest rate risk present in option futures positions, with the objective to better understand it and manage it. We conclude that the interest rate volatility should be the decisive factor, when one considers stochastic interest rate models. Motivated by the significant in-

$T_{F1} = 2000, \theta = 1\%, \rho = 0\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	12.8962	0.1499	0.1486	0.1499	0.1486	0.1499	0.1486
10	12.7605	0.0471	0.0467	0.0471	0.0467	0.0471	0.0467
100	12.9293	0.0150	0.0149	0.0150	0.0149	0.0150	0.0149
$T_{F2} = 1800, \theta = 1\%, \rho = 0\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	12.8962	0.1555	0.1543	0.1555	0.1544	0.1499	0.1488
10	12.7605	0.0621	0.0618	0.0621	0.0618	0.0471	0.0467
100	12.9293	0.0434	0.0434	0.0434	0.0434	0.0150	0.0149
$T_{F3} = 1200, \theta = 1\%, \rho = 0\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	12.8962	0.3173	0.3169	0.3174	0.3170	0.1502	0.1494
10	12.7605	0.2781	0.2781	0.2781	0.2781	0.0472	0.0469
100	12.9293	0.2788	0.2788	0.2788	0.2788	0.0150	0.0150
$T_{F4} = 600, \theta = 1\%, \rho = 0\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	12.8962	0.9077	0.9077	0.9078	0.9077	0.1514	0.1512
10	12.7605	0.8816	0.8816	0.8816	0.8816	0.0476	0.0475
100	12.9293	0.8947	0.8947	0.8947	0.8947	0.0152	0.0151

Table 6: **Hedging performance with increasing hedging frequency, maturity mismatch and  $\theta = 1\%$ .** The forward price of a long-dated option (Call/Bond) is hedged by using futures or forwards with various maturities. The four different maturities  $T_{F1}, T_{F2}, T_{F3}$  and  $T_{F4}$  represent the maturities of the hedging futures and forwards with 2000, 1800, 1200 and 600 trading days, respectively. In all cases, the maturity of the bond contracts is 2000 trading days.

crease in trading volumes of longer-dated commodity derivatives, especially crude oil options, subsequent work by Cheng et al. (2016a) empirically analyses the hedging performance on long-dated crude oil futures options market data.

$T_{F1} = 2000, \theta = 8\%, \rho = -50\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	12.8207	0.1893	0.2048	0.1894	0.2047	0.1894	0.2048
10	12.7255	0.0582	0.0629	0.0582	0.0629	0.0582	0.0629
100	12.6190	0.0183	0.0197	0.0183	0.0197	0.0183	0.0197
$T_{F2} = 1800, \theta = 8\%, \rho = -50\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	12.8207	0.3693	0.3592	0.3693	0.3591	0.1898	0.1692
10	12.7255	0.3181	0.3170	0.3181	0.3170	0.0584	0.0523
100	12.6190	0.3097	0.3096	0.3097	0.3096	0.0183	0.0165
$T_{F3} = 1200, \theta = 8\%, \rho = -50\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	12.8207	2.1702	2.1695	2.1702	2.1694	0.1960	0.1850
10	12.7255	2.1419	2.1418	2.1419	2.1418	0.0604	0.0571
100	12.6190	2.1184	2.1184	2.1184	2.1184	0.0190	0.0180
$T_{F4} = 600, \theta = 8\%, \rho = -50\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	12.8207	6.9531	6.9532	6.9532	6.9533	0.2385	0.2418
10	12.7255	6.8812	6.8812	6.8812	6.8812	0.0737	0.0747
100	12.6190	6.8075	6.8075	6.8075	6.8075	0.0231	0.0234

Table 7: **Hedging performance with increasing hedging frequency, maturity mismatch,  $\theta = 8\%$  and  $\rho = -50\%$ .** In this table the forward price of a long-dated option is hedged by using bonds and futures or forwards with various maturities. The four different maturities  $T_{F1}, T_{F2}, T_{F3}$  and  $T_{F4}$  represent the maturities of the hedging futures and forwards with 2000, 1800, 1200 and 600 trading days respectively. In all cases, the maturity of the bond contracts is 2000 trading days.

$T_{F1} = 2000, \theta = 1\%, \rho = -50\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	12.5936	0.1425	0.1522	0.1425	0.1523	0.1425	0.1523
10	12.6937	0.0454	0.0485	0.0454	0.0485	0.0454	0.0485
100	12.6386	0.0142	0.0152	0.0142	0.0152	0.0142	0.0152
$T_{F2} = 1800, \theta = 1\%, \rho = -50\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	12.5936	0.1482	0.1521	0.1481	0.1520	0.1424	0.1465
10	12.6937	0.0611	0.0620	0.0611	0.0620	0.0453	0.0466
100	12.6386	0.0430	0.0431	0.0429	0.0431	0.0142	0.0146
$T_{F3} = 1200, \theta = 1\%, \rho = -50\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	12.5936	0.3114	0.3130	0.3110	0.3126	0.1418	0.1454
10	12.6937	0.2806	0.2808	0.2805	0.2807	0.0451	0.0463
100	12.6386	0.2772	0.2772	0.2772	0.2772	0.0142	0.0145
$T_{F4} = 600, \theta = 1\%, \rho = -50\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	12.5936	0.8977	0.8980	0.8973	0.8976	0.1408	0.1430
10	12.6937	0.8904	0.8904	0.8903	0.8904	0.0448	0.0455
100	12.6386	0.8897	0.8897	0.8897	0.8897	0.0141	0.0143

Table 8: **Hedging performance with increasing hedging frequency, maturity mismatch,  $\theta = 1\%$  and  $\rho = -50\%$ .** In this table the forward price of a long-dated option is hedged by using bonds and futures or forwards with various maturities. The four different maturities  $T_{F1}, T_{F2}, T_{F3}$  and  $T_{F4}$  represent the maturities of the hedging futures and forwards with 2000, 1800, 1200 and 600 trading days respectively. In all cases, the maturity of the bond contracts is 2000 trading days.

## Appendix 1 Derivation of the futures prices

In this section we derive an expression for the futures prices given the following dynamics of the spot asset price and the spot interest rate under the spot risk-neutral measure:

$$dS(t) = r(t)S(t)dt + \sigma S(t)dW_1(t), \quad (\text{A.1.1})$$

$$dr(t) = \lambda(\bar{r} - r(t))dt + \theta dW_2(t), \quad (\text{A.1.2})$$

$$\rho dt = dW_1(t)dW_2(t).$$

The futures price with a maturity  $T \geq t$  is the expectation at time  $t$  of the future spot asset price at time  $T$  under the risk neutral measure, i.e.,

$$\begin{aligned} F(t, T) &= \mathbb{E}^{\mathbb{Q}}[S(T)|\mathcal{F}_t] \\ &= S(t) \exp\left(-\frac{1}{2}\sigma^2(T-t)\right) \mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_t^T r(u) du + \sigma W^1(T-t)\right)|\mathcal{F}_t\right] \\ &= S(t) \exp\left(-\frac{1}{2}\sigma^2(T-t)\right) \mathbb{E}^{\mathbb{Q}}[\exp(Y)|\mathcal{F}_t] \end{aligned}$$

where

$$Y = \int_t^T r(u) du + \sigma W^1(T-t).$$

Since the random variable  $Y$  is a sum of an integral of Gaussian random variables and a Wiener process which also has a Gaussian distribution,  $Y$  also has a Gaussian distribution. So we need to find the expectation and the variance of  $Y$  and then use the moment generating function to get an expression of  $F(t, T)$ . Now, we re-write the two correlated Wiener processes into two independent Wiener processes as follow,

$$\begin{aligned} dW_1(t) &= dB^1(t), \\ dW_2(t) &= \rho dB^1(t) + \sqrt{(1-\rho^2)}dB^2(t), \\ 0 &= dB^1(t)dB^2(t) \end{aligned}$$

so the dynamics of the interest rate process becomes

$$dr(t) = \lambda(\bar{r} - r(t))dt + \theta\rho dB^1(t) + \theta\sqrt{(1-\rho^2)}dB^2(t).$$

From A.1.1, we can derive the future stock price at time  $T \geq t$  given the current stock price at time  $t$  as:

$$S(T) = S(t) \exp\left(-\frac{1}{2}\sigma^2(T-t) + \int_t^T r(u) du + \sigma B^1(T-t)\right).$$

We can further obtain an expression for  $\int_t^T r(s) ds, s \geq t$  by firstly deriving  $r(s)$ :

$$\begin{aligned} r(s) &= r(t)e^{-\lambda(s-t)} + \bar{r}(1 - e^{-\lambda(s-t)}) + \theta\rho \int_t^s e^{-\lambda(s-u)} dB^1(u) \\ &\quad + \theta\sqrt{(1-\rho^2)} \int_t^s e^{-\lambda(s-u)} dB^2(u) \end{aligned}$$

then by integrating  $r(s)$  from  $t$  to  $T$  and by applying Fubini's theorem we obtain:

$$\begin{aligned} \int_t^T r(s) ds &= (r(t) - \bar{r})A(t, T, \lambda) + \bar{r}(T - t) + \theta\rho \int_t^T A(u, T, \lambda) dB^1(u) \\ &\quad + \theta\sqrt{(1 - \rho^2)} \int_t^T A(u, T, \lambda) dB^2(u) \end{aligned}$$

where

$$A(t, T, \lambda) = \frac{1}{\lambda}(1 - e^{-\lambda(T-t)}).$$

The expectation of  $Y$  is:

$$\begin{aligned} M &\triangleq \mathbb{E}^{\mathbb{Q}}[Y|\mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}\left[\int_t^T r(u) du + \sigma W^1(T - t)|\mathcal{F}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\int_t^T r(u) du|\mathcal{F}_t\right] \\ &= (r(t) - \bar{r})A(t, T, \lambda) + \bar{r}(T - t). \end{aligned}$$

The variance of  $Y$  is:

$$\begin{aligned} V^2 &\triangleq \text{var}^{\mathbb{Q}}[Y|\mathcal{F}_t] = \text{var}^{\mathbb{Q}}\left[\int_t^T r(u) du|\mathcal{F}_t\right] + \text{var}^{\mathbb{Q}}[\sigma W^1(T - t)|\mathcal{F}_t] \\ &\quad + 2\text{cov}^{\mathbb{Q}}\left(\int_t^T r(u) du, \sigma W^1(T - t)|\mathcal{F}_t\right) \\ &= V_1^2 + V_2^2 + 2V_3^2 \end{aligned}$$

where,

$$\begin{aligned} V_1^2 &= \text{var}^{\mathbb{Q}}\left[\int_t^T r(u) du|\mathcal{F}_t\right] \\ V_2^2 &= \text{var}^{\mathbb{Q}}[\sigma W^1(T - t)|\mathcal{F}_t] \\ V_3^2 &= \text{cov}^{\mathbb{Q}}\left(\int_t^T r(u) du, \sigma W^1(T - t)|\mathcal{F}_t\right) \end{aligned}$$

Now, we have:

$$\begin{aligned} V_1^2 &= \text{var}^{\mathbb{Q}}\left[\theta \int_t^T A(u, T, \lambda) dW_2(u)|\mathcal{F}_t\right] \\ &= \theta^2 \mathbb{E}^{\mathbb{Q}}\left[\left(\int_t^T A(u, T, \lambda) dW_2(u)\right)^2|\mathcal{F}_t\right] \\ &= \theta^2 \int_t^T A^2(u, T) du \quad (\text{Itô isometry}) \\ &= \frac{\theta^2}{\lambda^2} \int_t^T (1 - e^{-\lambda(T-u)})^2 du \\ &= \frac{\theta^2}{\lambda^2} \left((T - t) - 2A(t, T, \lambda) + A(t, T, 2\lambda)\right). \end{aligned}$$

and

$$V_2^2 = \sigma^2(T - t)$$

Finally:

$$\begin{aligned} V_3^2 &= \text{cov}^{\mathbb{Q}} \left( \theta \rho \int_t^T A(u, T, \lambda) dB^1(u), \sigma W^1(T - t) | \mathcal{F}_t \right) \\ &= \rho \theta \sigma \text{cov}^{\mathbb{Q}} \left( \int_t^T A(u, T, \lambda) dB^1(u), W^1(T - t) | \mathcal{F}_t \right) \\ &= \rho \theta \sigma \mathbb{E}^{\mathbb{Q}} \left[ \int_t^T A(u, T, \lambda) dB^1(u) \int_t^T dW_1(u) | \mathcal{F}_t \right] \\ &= \rho \theta \sigma \int_t^T A(u, T, \lambda) du \\ &= \frac{\rho \theta \sigma}{\lambda} ((T - t) - A(t, T, \lambda)). \end{aligned}$$

Putting all together we have:

$$\begin{aligned} F(t, T) &= \mathbb{E}^{\mathbb{Q}}[S(T) | \mathcal{F}_t] \\ &= S(t) \exp \left( -\frac{1}{2} \sigma^2(T - t) \right) \exp \left( M + \frac{1}{2} V^2 \right) \\ &= S(t) \exp \left( M + \frac{1}{2} V_1^2 + V_3^2 \right). \end{aligned}$$

## Appendix 2 Forward and futures prices under stochastic interest rates

From Musiela and Rutkowski (2006), we have that<sup>12</sup>:

$$F(t, T) = \mathbb{E}^{\mathbb{Q}}[S(T) | \mathcal{F}_t] \tag{A.2.3}$$

and<sup>13, 14</sup>

$$\begin{aligned} \text{For}(t, T) &= \mathbb{E}^{\mathbb{T}}[S(T) | \mathcal{F}_t] \\ &= \frac{S(t)}{B(t, T, r(t))}. \end{aligned} \tag{A.2.4}$$

The futures price at time  $t$  maturing at time  $T$  is:

$$\begin{aligned} F(t, T) &= \mathbb{E}^{\mathbb{Q}}[S(T) | \mathcal{F}_t] \\ &= \mathbb{E}^{\mathbb{Q}} \left[ S(t) \exp \left( -\frac{1}{2} \sigma^2(T - t) + \int_t^T r(u) du + \sigma W_1(T - t) \right) | \mathcal{F}_t \right] \\ &= S(t) \exp \left( M + \frac{1}{2} V_1^2 + V_3^2 \right), \end{aligned}$$

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<sup>12</sup>See Definition 11.5.1 on page 418, Musiela and Rutkowski (2006)

<sup>13</sup>See Lemma 9.6.2 on page 342, Musiela and Rutkowski (2006)

<sup>14</sup>See equation 9.30, on page 341, Musiela and Rutkowski (2006)

We also have<sup>15</sup>:

$$\begin{aligned} B(t, T, r(t)) &= \mathbb{E}^{\mathbb{Q}} \left[ \exp \left( - \int_t^T r(u) du \right) | \mathcal{F}_t \right] \\ &= \exp \left( -M + \frac{1}{2} V_1^2 \right). \end{aligned}$$

So  $F(t, T) \neq \text{For}(t, T)$ . The quotient of forward and futures is called the convexity adjustment that we would need to consider when we hedge an option using futures contracts and it is given by:

$$\frac{\text{For}(t, T)}{F(t, T)} = e^{-V_1^2 - V_3^2}. \quad (\text{A.2.5})$$

When  $\rho = 0$ , we have

$$\frac{\text{For}(t, T)}{F(t, T)} = e^{-V_1^2}. \quad (\text{A.2.6})$$

### Appendix 3 Using short-dated forward to hedge long-dated options

Let the maturity of the option be  $T$  and the maturity of the hedging forward contract be  $T_F$ . Let the forward price of the underlying forward contract of the option be  $F(t, T)$  and the forward price of the hedging forward contract be  $F(t, T_F)$ . We assume that  $T_F \leq T$ .

$$F(t, T) = \frac{S(t)}{B(t, T)} \Leftrightarrow S(t) = B(t, T)F(t, T)$$

and

$$F(t, T_F) = \frac{S(t)}{B(t, T_F)} \Leftrightarrow S(t) = B(t, T_F)F(t, T_F).$$

We have:

$$F(t, T) = \frac{B(t, T_F)F(t, T_F)}{B(t, T)}$$

and

$$\frac{\partial F(t, T)}{\partial F(t, T_F)} = \frac{B(t, T_F)}{B(t, T)}.$$

If  $\delta$  amount of forward contracts with maturity  $T$  is required to hedge the option, we would need to use  $\delta \frac{B(t, T_F)}{B(t, T)}$  amount of forward contracts with maturity  $T_F$  to hedge.

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<sup>15</sup>See equation (9.14) on page 333, Musiela and Rutkowski (2006)

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