The Financial Instability Hypothesis: a Stochastic Microfoundation Framework

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Abstract

This paper examines the dynamics of financial distress and in particular the mechanism of transmission of shocks from the financial sector to the real economy. The analysis is performed by representing the linkages between microeconomic financial variables and the aggregate performance of the economy by means of a microfounded model with firms that have heterogeneous capital structures. The model is solved both numerically and analytically, by means of a stochastic approximation that is able to replicate the numerical solution. These methodologies overcome the restrictions of the representative agent hypothesis which seems to be unsuitable for a context where different financial conditions of firms, and consequently different reactions to external shocks, impact on the macroeconomic dynamics.

Keywords: Financial fragility, complex dynamics, stochastic aggregation.

JEL classification: E12, E22, E44

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1 Introduction

Minsky (1977) defines financial fragility as “...an attribute of the financial system. In a fragile financial system continued normal functioning can be disrupted by some not unusual event”. The two key points highlighted by this definition are the “not unusual event” that may stop the normal functioning of a financial system and that the system in question must display a certain degree of fragility. As regards the former point, there is no shortage of interpretations in this sense of the crises that, at progressively shorter intervals, have hit the capitalist economies in the last quarter of century (Kindleberger, 2005). The idea of an intrinsic instability of the capitalist financial system dates back to Minsky (1963) and has gained increasing attention, especially during the recent financial crisis. As regards the second point, the identification of the degree of systemic fragility, according to Minsky, involves a micro-level analysis, being dependent on the ratio of financially sound to financially distressed firms in the economy. More precisely, in his famous 1963 essay, Minsky classifies firms into hedge, speculative or Ponzi type. The first are the sound firms that can repay their debt and the interest on it. The second type are the ones able to meet only the interest due on outstanding debt while, for the Ponzi firms, their cash flow is insufficient to fulfill either the repayment of capital or the interest due on outstanding debts.

As Taylor and O’Connell (1985) point out “Shifts of firms among classes as the economy evolves in historical time underlie much of its cyclical behavior. This detail is rich and illuminating but beyond the reach of mere algebra”. According to them, this is the main reason for which Minkey’s work has been so far either neglected or formulated in aggregate terms rather than being micro-founded.

Such is no longer the case. In recent years a consistent stream of research has started to deal with the microfoundation of macroeconomics with heterogeneous and evolving agents. Significant results in terms of replication of empirical stylized facts has been reached through the numerical solution of agent based model. From an analytical perspective, the most relevant contribution has been provided by Aoki, whose framework seems to allow a comprehensive analytical development of Minsky’s theory that satisfactorily encompasses its essential microeconomic foundation. Aoki adopts analytical tools originally developed in statistical mechanics. In his view, as the economy is populated by a very large number of dissimilar agents, we cannot know which agent is in which condition at a given time and whether an agent will change its condition, but we can know the present probability of a given state of the world. This approach hence focuses in particular on the evolution of agents’ characteristics through time. The basic idea consists in introducing a meso-level of aggregation, obtained by grouping the agents in clusters according to a measurable

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1 See the working paper series of the Levy Economic Institute of Bard College.
2 See by way of example Axelrod (2002); Axtell et al. (1996); Delli Gatti et al. (2004).
3 Namely, Aoki (1996, 2002); Aoki and Yoshikawa (2006), with a further development provided by Di Guilmi (2008)
variable. The dynamics of the number of firms in each cluster defines as well the evolution of the whole economy, which is identifiable by specifying some general assumptions on the stochastic evolution of these quantities. For example, assuming their dynamics to be a Markov process, it is possible to describe the stochastic evolution of these occupation numbers using the master equation which is a standard tool developed in statistical mechanics to model the evolution of ensembles of particles. Interaction among agent is modelled by means of the mean-field approximation (Aoki, 1996) that, basically, consists in reducing the vector of observations of a variable over a population to a single value. The usefulness and the potential of this approach for analysing Minsky’s theoretical structure appears to be promising.

The aim of this paper is to propose a financial fragility model, along the lines of Minsky (1975) and Taylor and O’Connell (1985), with heterogeneous and interacting firms, using first a numerical simulation of the agent based model and then comparing this solution with the one obtained by means of the stochastic dynamic aggregation technique mentioned above. Besides the technical contribution, such an improvement should allow a deeper insight into the mechanism by which shocks are transmitted from the financial sector to the real economy. This aspect, which is central in Minsky’s approach, is not the main focus of Taylor and O’Connell (1985). In their view the market valuation of shares may differ from the present value of capital, with the difference being absorbed by net worth. Given the substitutability of assets, a shift of investor preferences impacts on firms’ net worth via a different evaluation of capital assets. Therefore, investor expectations of future profits influence, on the one hand, the prices of firms’ equities on the stock market and, on the other hand, the current value of firms’ assets. For example, if the market forecasts a rise in the demand for a certain product, there will be an increase in the evaluation of the machines that produce that good and a contemporaneous rise in the price of shares for the firms that sell them (Wray and Tymoigne, 2008). These two effects shape firms’ decisions on investment and, as a consequence, output and employment levels. At the aggregate level then the economy may experience periods of growth, depression or fluctuations due solely to changes in the market mood and not to its actual productivity. This mechanism, first studied by Keynes (1936), has been subsequently modelled by Kalecki (1971) and Minsky (1975). Taylor and O’Connell (1985) introduce into the original analysis of Minsky an exogenous variable that expresses the level of confidence of the market, isolating the effect of investors’ expectations on the value of a firm’s assets.

We bring three main modifications to this original framework. The most relevant is the microfoundation of the financial fragility approach. As already stressed, the presence of heterogeneous agents and the consequent issues in consistently macrofounding the framework are among the factors that have limited the diffusion through the economics profession of Minsky’s approach. Therefore, with respect to the original models, the equations are expressed with reference to the micro-level. We then study two different macro-dynamics: the first being an agent based one, with the highest degree of heterogeneity, and a stochastic approximation, obtained by means of Aoki’s aggregation tools. The second de-
rives from the observation that, from our perspective, the evaluation of capital assets comes from the stock market, in which investors display heterogeneous expectations about firms’ future profits. In particular we consider as endogenous the new variable introduced by Taylor and O’Connell (1985), linking it to the predominant strategy in the stock market. The third aspect of novelty regards the modelling of endogenous money which in the present treatment is linked to the overall amount of financial assets rather than being a multiple of the amount of money as typically presented in literature. In our opinion this view is more consistent with the Minsky’s idea from two different perspectives. First, from a formal point of view, Minsky, in particular in his later writings seems to connect the degree of liquidity of the system to a wider range of marketable paper, such as for example, securities and derivatives, than the typical monetary aggregates. As a consequence financial innovation, besides having the effect of transferring risk, influences the credit supply also by affecting the endogenous determination of the degree of liquidity in the economy. Second, from a theoretical point of view, by modelling endogenous money as endogenous wealth, the availability of credit is linked to the conditions of the financial system and, in particular, to the expectations of investors, providing a quantitative benchmark for the analytical representation of his idea of increased propensity to supply credit in periods of expansions (Minsky, 1963).

The outline of the paper is as follows. Section 2 describes the general features of the model and outlines the basic structure of firms in the agent based framework. The behavioural hypotheses and equations are the same for the dynamics and for the stochastic approximation and refer to each single firm, without limitation on the endogenous heterogeneity, while the stochastic dynamics consider a representative firm for each cluster. Section 3 defines the hypotheses for investors and capital market. Section 4 discusses the stochastic approximation to a high order heterogeneous model. Section 5 uses simulation to contrast the outcomes of the two dynamics, the agent based model and the stochastic approximation. Section 6 concludes with some discussion of the questions that can be addressed using the framework developed here.

2 Firms

This section presents the structure of the agent based model. Variables are written with the superscript \( j \) when they refer to a generic firm, with the subscript \( z = 1, 2 \) when referring to a microstate, and without any sub- or superscript when indicating aggregate values. The model is set up in continuous time. The hypotheses of the model are listed below.

- Due to informational imperfections in capital markets (Myers and Majluf 1984; Greenwald and Stiglitz 1990), firms prefer to finance their investments \( I^j \) with retained earnings \( F^j \) and, only if they are not sufficient, by

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4For a review see Fontana (2003).
5As in Minsky (2008) about debt securities.
the emission of new equities $E^j$ or with new debt $D^j$.

- Firms are classified into two groups, clustering together the speculative and Ponzi firms of the Minsky (1963) taxonomy. In order to ease the calculations, analogously to Lima and de Freitas (2003), the threshold level of debt is set to 0. Therefore, the classification defines as 
  
  speculative (type 1) the firms that have to finance their investment with debt or new equity and as hedge (type 2) the firms that can finance their investments with retained profits and do not need external sources. Thus firms can be classified into two states, depending on whether or not they display a positive debt in their balance sheet:

  - state 1: $D^j(t) > 0$,
  - state 2: $D^j(t) = 0$.

A generic firm is indicated with the superscript $j$; variables referring to one of the two states are identified by the subscript $z = 1, 2$. Within the two clusters firms are identical.

- A firm decides the level of investment $I^j(t)$ based on the shadow-price (Minsky, 1975; Kalecki, 1971) of its capital $P^j_k(t)$, so that:

$$I^j(t) = a P^j_k(t),$$

where $a$ is a parameter measuring the sensitivity of firms to the current value of capital assets and the shadow price $P^j_k$ is specified below. This formulation recalls the one adopted by Delli Gatti et al. (1999), while the model of Taylor and O’Connell (1985), very much in line with Minsky (1975), takes into account the price differential between the shadow price and the price of furnishing new investment goods. Our choice in (1) is motivated by our desire to keep the computational mechanism as simple as possible. Moreover, the solution adopted by Taylor and O’Connell (1985) would add a factor that might turn out to be too noisy for the identification of the effects of financial markets fluctuations on investment.

- The selling price of the final good is obtained by applying a mark-up $\tau$ on the direct production costs according to

$$P = (1 + \tau)wb,$$

where $w$ is the nominal wage and $b$ is the labour-output ratio, assumed equal for all firms.

- Firms produce a good that can be used either for consumption or investment. The product market is assumed to be in equilibrium.\(^6\)

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\(^6\)The condition of equilibrium implies that the variations in the investment-product ratio are absorbed by the total wage bill. The simulations show negligible variations of this ratio over time that we ignore for simplicity.
• Assuming that the firms adopt a technology with constant coefficients, the amount of labour requested is residually determined once the optimal level of investment, and hence of capital, is quantified. In particular, the production function for all firms is

\[ X^j(t) = G(K^j(t), L^j(t)) \]  

(3)

with \( K \) and \( L \) representing, respectively, physical capital and labour. Assuming \( G \) to be a homogeneous function of \( L \) we have

\[ 1/b = G(K^j/L^j, 1) \equiv g(K^j) \]  

(4)

For the sake of simplicity (and with a slight abuse of notation) from now on we refer to the capital labour ratio simply as capital.

• The rate of profit \( r^j \) is given by

\[ r = r^j(t) = \frac{\tau}{1 + \frac{K^j(t)}{X^j(t)}} \]  

(5)

which is set equal across firms since they are assumed to apply the same mark-up and use the same technology. Final production and physical assets are priced at the level \( P \), as in Taylor and O’Connell (1985), and all profits are retained.

• \( P^j_k \) is determined according to

\[ P^j_k(t) = \frac{(r + \rho^j(t))P}{i(t)} \]  

(6)

where \( i \) is the interest rate and \( \rho^j \) is the expected difference of return to capital for the firm \( j \) with respect to the average level \( r \). The variable \( \rho \) is introduced by Taylor and O’Connell (1985) in their analysis of the original Minsky model in order to link investors’ expectations to the investment decision; it plays a decisive role in their treatment as well as in the present one. Here we make it endogenous, considering it as a function of the prevailing strategy on the financial market. This quantity is therefore the key variable in the mechanism of transmission of shocks from the financial markets to the real economy. The mechanism by which the process occurs is fully detailed in section 3. Combining (1) with (6) we obtain

\[ I^j(t) = a \left[ \frac{(r + \rho^j(t))P}{i(t)} \right]. \]  

(7)

• Firms finance the part of investment that cannot be covered with internal funds by a fraction \( \phi \) of equities, where \( \phi > 0 \) is a parameter, and then the rest with debt, the dependence on the interest rate reflecting the

\[^7\]In the simulations a control is introduced in order to ensure that \( \phi \leq 1 \).
fact that in periods with a high interest rate equities would be preferred. The price of the new capital goods is assumed to be equal to the final goods price $P_e$. The sum of retained profits is indicated by $F_j$. Thus, the variations of $E_j$ and $D_j$ at an instant of time are given by

$$dE_j(t) = \phi(t) \left[ \frac{PI_j(t) - F_j(t)}{P_e} \right] dt \quad (8)$$

$$dD_j(t) = \left[ 1 - \phi(t) \right] \left[ PI_j(t) - F_j(t) \right] dt \quad (9)$$

where $P_e$ is the price of equities for speculative firms to be defined in the following section.

• The timeline of the whole process over successive time intervals is shown in figure 1. In the first stage market expectations determine the shadow price of capital and the desired level of investment. In the following unit of time firms make effective the decision, modifying the capital stock and producing the final good. The product is then sold in the following period, originating a profit.

![Figure 1: Timeline of the investment process.](image)

• The balance sheet of a typical firm has the structure shown in Table 1. We use $A$ to indicate the difference in the market valuations of assets and shares, less the eventual debt. Adopting the terminology of Taylor and O’Connell (1985) we term it as net worth. Actually, in terms of accountability, retained profits are a component of firms net worth and therefore they should be added to the latter. We indicate them separately in order to quantify the cash flow that can be used to finance future investment.

• Capital depreciates in each period at a constant rate.

• Profits are given by

$$\pi_j(t) = PX_j(t) - wb - i(t)D_j(t) = \tau wb X_j(t) - i(t)D_j(t). \quad (10)$$

• Accordingly, the variation in retained profits, or cash flow, for a hedge firm is

$$\frac{dF_j(t)}{dt} = \pi_j(t) - P_k D_j(t). \quad (11)$$
<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{r + \rho}{i} K^j )</td>
<td>( P_e E^j )</td>
</tr>
<tr>
<td></td>
<td>( D^j ) (or ( F^j ))</td>
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<td></td>
<td>( A )</td>
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Table 1: Structure of a generic firm’s balance sheet

If, at time \( t \), \( F^j(t) < I^j(t) \), the firm becomes speculative and \( D^j(t) = F^j(t) - I^j(t) \) will be financed with new equities and debt according to equations (8) and (9).

- A firm fails if its debt level exceeds some multiple of its capital stock, that is if
  \[
  D^j(t) > c K^j(t)
  \]
  (12)
  with \( c > 1 \). The probability of a new firm entering is directly proportional to the variation in the aggregate production with respect to the previous period. Thus in boom periods failed firms are rapidly replaced whilst in periods of distress this process can take quite a deal longer.

## 3 Investors

### 3.1 Behavioural hypothesis

Even though a complete modelling of stock markets would go beyond the aim of the present analysis, some behavioural assumptions on investors are needed for the internal consistency of the framework. The preferences of investors are modelled in a Keynesian fashion, assuming that a share of wealth is kept liquid. Minsky (1975) gives to the usual Keynesian motives (transaction, precautionary, speculative) a formal representation, modelling the demand for money as a function of income, interest rate, asset price, firm debt and near money supply. We shall model demand for money in a similar fashion. Moreover, we shall assume that the financial operators act according to a bounded rationality paradigm. Consequently, we classify them into the two broad categories of chartists and fundamentalists, within an approach that has an established tradition in the literature. It has been demonstrated (Aoki and Yoshikawa, 2006, ch. 9) that this classification accounts for almost the totality of different possible strategies. We adopt this assumption that turns out to be particularly suitable in this framework. Indeed we can reasonably assume that, on average, fundamentalists, focusing on the real value of firms, will favour investment in hedge firms, while chartists, who base their decisions on extra-balance sheet information, may prefer riskier equities. We assume that all investors maximize

\cite[See for example]{Zeeman:1974, Chiarella_and_Ho:2003, Chiarella_et_al:2009}.
a CARA utility function in order to avoid the distinction between chartists’ and fundamentalists’ wealth. Since our focus is on how changes in investors’ expectations impact on the real economy, we assume that variations in the proportion of the types of operators are not dependent on firms’ performance and are simply governed by a stochastic law. This also allows for a wider range of possible outcomes and behaviours as a result of the multiplicity of exogenous factors (not related to the economy) that influence the markets.

3.2 The determination of $\rho$

As anticipated, the variable $\rho$ plays a key role in the entire story, as it incorporates expectations that emerge in financial markets into the decision process of firms about investment. Taylor and O’Connell (1985) introduce it in order to better isolate the effect of the difference between the anticipated return and the current profit rate, an effect that in the original treatment of Minsky (1975) is directly incorporated in the shadow price $P_k$. They are not interested in the impact of financial markets and hence they do not explicitly model $\rho$, assuming independence between the behaviour of investors and firms. On the contrary, since our perspective is mainly focused on the transmission of shocks from the financial sector, the role of $\rho$ recalls Tobin’s $q$ (Tobin, 1969), that is connected to equity values. In this sense our work constitutes a bridge between these two approaches and an extension of them.

Two basic assumptions are at the root of the formulation of $\rho$: the first is its dependence on the relative proportion of chartists and fundamentalists in the market; the second concerns the formation of expectations. Since fundamentalists look at the balance sheet of firms while chartists focus only on the evolution of returns, we can assume that an increment in the proportion of chartists fuels the expectations about indebted firms that, on the contrary, reduce when the share of fundamentalists is bigger. Accordingly, $\rho^j$ is determined differently depending on whether a firm is in state 1 or in state 2, namely

$$f_1(n^c) = \rho_1^n = \frac{n^c}{\tilde{\omega}^j},$$

$$f_2(n^c) = \rho_2^n = 1 - \frac{n^c}{\tilde{\omega}^j},$$

(13)

where $\tilde{\omega}^j$ is an idiosyncratic random variable specified over a positive support. Since this random variable has the same support for each firm, on average a bigger fraction of chartists in the market leads firms in state 1 to increase their investments, their production and their debt. At the same time, the growing demand for credit puts pressure on interest rates. Therefore the system experiences a debt driven expansion that makes it vulnerable to sudden changes in investors expectations (a diminution in the number of chartist and firm bankruptcies, in the present treatment).
3.3 Equilibrium in the capital market

The equities of the two different types of firms can be correspondingly sorted into two classes with different associated risk. Investors will allocate part of their wealth between the two classes of shares according to the market expectation at the time of choice. In order to define an allocation criteria, we assume that they consider the two indicative values $\rho_1$ and $\rho_2$, calculated as a statistic of the $\rho$ within each cluster of firms. The variables $\rho_1$ and $\rho_2$ represent our mean-field values.

The wealth $W$ of investors is the sum of shares, bonds and money, so that

$$W(t) = P_{e1}(t)E_1(t) + P_{e2}(t)E_2(t) + D(t) + M(t).$$

(14)

where $M(t)$ is the nominal demand for money. Wealth evolves over time according to:

$$\frac{dW}{dt} = \frac{dP_{e1}}{dt}E_1(t) + \frac{dP_{e2}}{dt}E_2(t) + P_{e1}\frac{dE_1}{dt} + P_{e2}\frac{dE_2}{dt} + \frac{dD}{dt} + \frac{dM}{dt}$$

(15)

An initial endowment of money is assumed. Variations in total wealth are then due to capital gains, which in this framework constitute high-powered wealth.

Investors allocate their wealth among equities, firms’ bonds and money according to the proportions: $\epsilon_1(i, \rho_1, \rho_2, \psi)$, $\epsilon_2(i, \rho_1, \rho_2, \psi)$, $\beta(i, \rho_1, \rho_2, \psi)$ and $\Psi(i, \rho_1, \rho_2, \psi)$ that satisfy the constraint $\epsilon_1 + \epsilon_2 + \beta + \Psi = 1$. The parameter $\psi$ reflects the propensity toward liquid assets and it is assumed to be constant over time. Given the structure of the equilibrium conditions, and in particular the fact that the demand for credit is always accommodated, the parameter may be interpreted as a proxy for the capacity of the system to generate endogenous money. The proportions of the two kinds of strategies influence $\rho$ and through this the allocation of wealth between the two assets. The equilibrium conditions on equities and credit markets are (time indexes are omitted):

$$\frac{\epsilon_1(i, r + \rho_1)}{P_{e1}}W = E_1,$$

$$\frac{\epsilon_2(i, r + \rho_2)}{P_{e2}}W = E_2,$$

$$\beta(i, r + \rho_1)W = D,$$

$$\Psi(i, W, \psi)W = M,$$

$$W = P_{e1}E_1 + P_{e2}E_2 + D + M.$$

(16)

9Actually, in each period, only speculative firms issue equities, given that hedge firms can finance all their investment with retained profits. Anyway in the market there are also the equities of firms that were speculative and became hedge, which are assessed differently by investors.

10The introduction of $\psi$ allows us to provide a functional form for the demand of money according to the formulation of Minsky (1975, chap. 4). Namely in his treatment it is given by the combined liquidity effects of the income $Y$, the interest rate $r$ and the shadow price of capital $P_k$, the outstanding private financial commitments $F$ and the supply of near money activities $NM$:

$$M = L_1(Y) + L_2(r, P_k) + L_3(F) - L_4(NM).$$
The system (16) may be solved for the value of asset prices, interest rate, demand of money and aggregate rentiers’ wealth. This latter turns out to be endogenously determined within the system in order to (partially) accommodate the demand for credit.

4 Stochastic dynamics

Our discussion so far has been in terms of single firms, referring all the variables to the agent level, and only in the last section have we introduced the mean-field approximations $\rho_z$. These variables allow us to set up the tools for the analytical solution of the model. Equations (1) and (6) can be computed starting from the mean-field values $\rho_z$ in order to calculate the variables $I_z$ that refer to two representative firms, one for each state. With this approximation, using the techniques of Aoki (2002) and Di Giulmi (2008), it is possible to obtain an exhaustive analytical description of the system’s dynamics, starting from the micro level probabilities. Therefore, as explained in section 5 below, the model is able to generate dynamics in two different ways: an agent based approach with $N$ different agents and a stochastic approximation, with two different firms: one “good” and one “stressed”.

4.1 Transition probabilities

The probability for a firm to transition from state 2 to state 1 depends upon its level of investment and retained profits. A hedge firm becomes speculative if its level of net worth does not cover the desired investment. Therefore the probability $\zeta$ for a firm to move from state 2 to state 1 is equal to

$$\zeta(t) = Pr\{P I_2(t) \geq F_2(t)\} = Pr\left[\frac{a(r+f(n))P(t)}{i(t)} \geq F_2(t)\right].$$

With regard to speculative firms, they can move to state 2 if they are able to generate a level of profit sufficient to repay their debt; so that the relative probability of transition $\nu$ is given by

$$\nu(t) = Pr\{\tau wb[\tauwbX_1(t) \geq D_1(t)(1 + i(t))]\} = Pr\{\tau wb[K_1(t)] \geq D_1(t)(1 + i(t))\} = Pr\left\{\tau wb\left[K_1(t - \delta t) + \frac{(r+f(n'))P(t)}{i(t)}\right] \geq D_1(t)(1 + i(t))\right\}. \quad (18)$$

Let us denote with $\eta$ the a-priori probability for a firm to be in state 1, taking it as exogenous at this stage. The transition rates will be then given by:

$$\lambda(t) = (1 - \eta)\zeta(t),$$

$$\mu(t) = \eta\nu(t).$$

As demonstrated in Aoki (2002) the solution of the master equation provides also a possible endogenous formulation for the probability $\eta$. We do not analytically develop further this point as it is not essential the present study.
4.2 The system dynamics

We assume that firms switch from one state to another according to a Markov jump process. We have already defined the micro-states of the process, that correspond to states 1 and 2 for the firms. In order to define the macro dynamics we are interested in the occupation numbers, that is in the number of firms that are in one of the states at a given time. These occupation numbers identify the macro-states of the process, that, accordingly, are given by all the possible combination of $N_1$ and $N_2$ with the constraint $N_1 + N_2 = N$. In this way, their stochastic dynamics can be conveniently described by a master equation (Kubo et al., 1978; Aoki, 2002). Using $N_z$ to denote the occupation number for the state $z$, the master equation can be expressed as

$$
\frac{dPr(N_z, t)}{dt} = \lambda Pr(N_z - 1, t) + \mu Pr(N_z + 1, t) - [(\lambda + \mu)Pr(N_z, t)] \quad (21)
$$

where $Pr(N_z, t)$ indicates the probability of observing an occupation number equal to $N_z$ in state $z$ at time $t$. This ordinary differential difference equation for $Pr(N_z, t)$ allows us to describe the stochastic dynamics of the occupation numbers by identifying the components of the stochastic process that governs their evolution. To this end Aoki (2002) suggests splitting the state variable $N_z$ into its drift ($m$) and diffusion ($s$) components, according to

$$
N_z(t) = Nm + \sqrt{Ns}. \quad (22)
$$

At this stage it is possible to apply the method detailed in Di Guilmi (2008) and Landini and Uberti (2008) to obtain the dynamics for $m$ and $s$. First, by means of lead and lag operators, probability fluxes in and out the states can be treated as homogeneous. Then, the Taylor series expansion of the modified master equation identifies a Fokker-Planck equation for the transition density of the spread $Q(s, \tau)$ depending on the trend and the diffusion of the process according to

$$
\frac{\partial Q}{\partial \tau} - N^{1/2} \frac{dm}{d\tau} \frac{\partial Q}{\partial s} \approx \left[ -N^{1/2} \frac{\partial}{\partial s} \alpha_1(m) + \frac{1}{2} \left( \frac{\partial}{\partial s} \right)^2 \alpha_2(m) \right] Q(s, \tau), \quad (23)
$$

where:

- $Q(s, \tau)$ is the transition density function of the spread $s$ denoted with respect to $\tau$, which denotes the time rescaled by the factor $N$, so that $\tau = tN$;
- $\alpha_n$ is the $n$th-moment of the stochastic process for $s$;
- $m = \frac{N_z}{N}$ is the state variable, indicating the proportion of firms of type $z$ in the total population of firms.

\[12\text{See equation (6.43) in Di Guilmi (2008, 73).}\]
The asymptotic solution of the (23) leads to the system of coupled equations

\[
\frac{dm}{d\tau} = \lambda m - (\lambda + \mu)m^2, \quad (24)
\]

\[
\frac{\partial Q}{\partial \tau} = [2(\lambda + \gamma)m - \lambda] \frac{\partial}{\partial s}(sQ(s)) + \left[\frac{\lambda m(1 - m) + \gamma m^2}{2}\right] \left( \frac{\partial}{\partial s} \right)^2 Q(s), \quad (25)
\]

where the first is an ordinary differential equation the solution of which is the drift of the process \(N_2\), while the partial differential equation (25) describes the evolution of the density of the random spread \(s\) around the drift. As one can see from (24), \(m\) is convergent to the steady state value \(m^*\) given by

\[
m^* = \frac{\lambda}{\lambda + \mu}. \quad (26)
\]

Then, directly integrating equation (24) we find that

\[
m(\tau) = \frac{\lambda}{(\lambda + \mu) - \omega e^{-\vartheta\tau}} \quad : \quad (27)
\]

where

\[
\begin{cases}
\omega = 1 - \frac{m^*}{m(0)}, \\
\vartheta = \frac{(\lambda + \mu)^2}{\lambda}.
\end{cases} \quad (28)
\]

Equation (27) describes the evolution of the fraction \(m\) of firms and we see that it is fully dependent on transition rates. The solution of the equation for the density of the spread component yields the limit distribution function \(Q(s) = \lim_{\tau \to \infty} Q(s, \tau)\) for the spread \(s\), determining, in this way, the long run probability distribution of fluctuations, namely

\[
Q(s) = C \exp \left(\frac{\sigma^2}{2\sigma^2}\right) \quad \text{where} \quad \sigma^2 = \frac{\lambda \mu}{(\lambda + \mu)^2}. \quad (29)
\]

Equation (29) is a Gaussian density whose parameters are dependent on the transition rates.

### 4.3 Analytical description of the model

At this point we are able to identify the two dynamical variables that drive the dynamics of the economy: the first is capital accumulation that reflects investors’ expectations and animal spirits, and the second is the underlying stochastic dynamics of the proportion of speculative firms. These two dynamical variables are connected since the transition rate \(\lambda\) is a function of the level of investment \(I_2\) and the aggregate investment depends on the shares of the two types of firms. Taking as state variable the share of speculative firms \(n_1 = \frac{N_1}{N}\) we can write

\[
\begin{cases}
\frac{dn_1(t)}{dt} = (\lambda n_1(t) - (\lambda + \mu)[n_1(t)]^2)dt + \sigma dW, \\
\frac{dK(t)}{dt} = I(t)dt = N \left[ aP_{k_1}(t)[n_1(t) + [aP_{k_2}(t)][1 - n_1(t)] \right] dt
\end{cases} \quad (30)
\]

13
where $dW$ is a stationary Wiener increment and $\sigma dW$ is the stochastic fluctuation component in the proportion of speculative firms, coming from the distribution (29). These dynamics can then also identify the evolution of employment and aggregate output.

In order to put in major emphasis the role of the proportions of the two types of investors and firm we can substitute equations (6) and (13) the second equation in (30). Considering that $E[\bar{\pi}] = 1$, this equation may be expressed as:

$$I(t) = \frac{N_1 \rho}{i(t)} [r + n_2(t) + n_1(t)(2n_c(t) - 1)]$$ (31)

Equation (31) sheds light on the dynamics of the agent based model and, in particular, it analytically represents the fact that the effect of the proportion of the number of speculative firms depends on market expectations. As long as the chartists are the majority, a rise in the proportion of speculative units causes a positive variation in investment for the expected rise in their assets price combined with the high market valuation of their equities. When expectations change and the fundamentalist investors prevail, the proportion of speculative firms has a negative effect on investment.

A flow chart of the model is displayed in figure 4.3. As the chart shows, the key variable in all the story is $\rho$. The left part of chart summarizes the stages trough which it influences the equilibrium prices in the financial market. The right side details the determination of firm profits and thus their capacity to finance future investment with internal funds. Both sides influence the dynamics of aggregate capital.
Figure 2: Flowchart of the model.
5 Simulations

5.1 Specification of functional forms

The production function (4) is assumed to be of the form

\[ X^i(t) = \varphi K^i(t) \]  

(32)

with \( \varphi > 0 \) as constant parameter.

The random variable \( n^c \) and \( \tilde{\varpi} \) are assumed to have a uniform distribution. As a consequence of these hypotheses the transition probabilities can be specified in term of the known probability function of \( n^c \) as

\[ \zeta(t) = F(n^c) = Pr \left\{ n^c(t) \leq \varpi - \frac{F_i \varpi}{p_a} + 1 \right\} \]  

(33)

\[ \nu(t) = 1 - F(n^c) = Pr \left\{ n^c(t) > \varpi \left[ \frac{D(t)(1+i)}{\tau_{wb}\varphi} - K_1(t - \delta t) \right] - r \right\} \]  

(34)

where \( \varpi = \mathbb{E}[\tilde{\varpi}] \). Equations (33) and (34) come from (17) and (18) with \( \rho_z \) substituted by (13). The expression on the right hand sides are the critical levels in the proportion of chartists \( n^c \) for having the shift of a firm from one group to the other. The functions \( \epsilon_z \) and \( \beta \) of system (16) are formulated as logistic functions, in order to ensure a meaningful value of the proportions of wealth invested in the different activities. The shares of wealth invested in hedge firm equities, speculative firm equities, debt and money are assumed to be positively related to, respectively, \( \rho_2, \rho_1 \), the rate of interest \( i \) and the general propensity to liquid activities \( \psi \). The shares are thus given by

\[ \epsilon_1(t) = \frac{1}{1 + e^{i(t) + \rho_2(t) + \psi - \rho_1(t)}} \]  

(35)

\[ \epsilon_2(t) = \frac{1}{1 + e^{i(t) + \rho_1(t) + \psi - \rho_2(t)}} \]  

(36)

\[ \beta(t) = \frac{1}{1 + e^{\rho_1(t) + \rho_2(t) + i(t)}} \]  

(37)

\[ \Psi(t) = \frac{1}{1 + e^{i(t) + \rho_1(t) + \rho_2(t) - \psi}} \]  

(38)

The parameter \( \psi \) is kept fixed.

Therefore the system (16) becomes:

\[
\begin{align*}
P_{c1}(t)E_1(t) &= \frac{W(t)}{1 + e^{(i(t) + \rho_2(t) + \psi - \rho_1(t))}}, \\
P_{c2}(t)E_2(t) &= \frac{W(t)}{1 + e^{(i(t) + \rho_1(t) + \psi - \rho_2(t))}}, \\
D(t) &= \frac{W(t)}{1 + e^{(i(t) + \rho_1(t) + \psi - \rho_2(t))}}, \\
M(t) &= \frac{W(t)}{1 + e^{(i(t) + \rho_1(t) + \psi - \rho_2(t))}}, \\
W(t) &= P_{c1}(t)E_1(t) + P_{c2}(t)E_2(t) + D(t) + M(t).
\end{align*}
\]  

(39)
The mechanism for entry of new firms is stochastic. In every period a random number drawn from a uniform distribution with support $[0, 1]$ is assigned to each potential new firm; if this number is bigger than the normalized variation of aggregate output observed in the previous period the firm becomes active. The variation is normalized such that a variation of +5% is equal to 0 and of −5% is equal to 1. The typical configuration of parameters is (where not otherwise indicated): $n^e \in [0, 1]$; $\tilde{\sigma} \in [0.01, 1.99]$; $a = 4$; $\phi = 1$; $c = 7$; $\psi = 0.3$. The values of $\rho_z$ are the mean of the $\rho^s$ included between the 10th and the 90th percentiles within each cluster of firms. As regards the starting values, all firms have an initial endowment of internal funds and the initial interest rate is set to 0.1. Further details on the setting on the parameters are provided below.

### 5.2 Simulations results

Simulations are performed by implementing two separate procedures, one agent based and the other for the stochastic dynamics, that each produce their own dynamics of proportions of firms and capital accumulation. The two procedures are linked as the agent based one provides the mean-field variables $\rho_z$, with $z = 1, 2$. These values are the inputs for the stochastic approximation. Then the procedure is replicated with the two representative firms for each state, obtaining dynamics driven by the stochastic system (30). The transition probabilities are normalized taking the theoretical maximum and minimum values of the right hand side of the inequalities in (33) and (34). The bankruptcy condition provides an upper limit for debt, while it is not possible to set a limit for $k_z$ and $F_z$. We performed simulations for different numbers of periods and the results have been verified by running 1000 Monte Carlo replications for each simulation. A period can be considered as a year and the average interest rate for each replication is about 6 – 7%.

The model replicates some quantitative features of a real economy. The results for the macro level results are displayed in table 2.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Empirical evidence</th>
<th>Simulations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Correlation market cap.-GDP</td>
<td>0.70</td>
<td>0.70</td>
</tr>
<tr>
<td>(1929-2000)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean market cap.-GDP ratio</td>
<td>0.81 ± 0.36</td>
<td>0.74 ± 0.36</td>
</tr>
<tr>
<td>(1929-2000)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Correlation debt-GDP</td>
<td>0.97</td>
<td>0.86</td>
</tr>
<tr>
<td>(1950-2007)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean debt-GDP ratio</td>
<td>0.52 ± 0.12</td>
<td>0.55 ± 0.13</td>
</tr>
<tr>
<td>(1950-2007)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Variance in GDP fluctuations</td>
<td>0.0036</td>
<td>0.0040</td>
</tr>
<tr>
<td>(percentage, 1950-2007)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Results from model’s simulation and comparable evidence for US.

The simulations results satisfactorily reproduce the behaviour of US economy as regards the correlation between market capitalization and aggregate product, debt to aggregate product ratio and variance of GDP fluctuations. The mean of market capitalization ratio and the correlation between debt and aggregate product are below the level empirically observed. As regards the second variable, this may be due to the higher volatility of the model with respect to US dynamics which displays a constant and almost smooth growing trend in the period in exam. A tentative explanation is provided here in the following.

The model is able to replicate some statistical regularities that are observed in real data. The distribution of variations in the aggregate output is well approximated by a Weibull distribution (Di Guilmi et al., 2005) and the distribution of firms’ growth rates is well fitted by an exponential PDF (Stanley et al., 1996). The size of speculative firms, which account for the larger percentage, is distributed according to a Pareto law while the hedge firms are larger and more dispersed. The overall distribution of size appears therefore as a bimodal distribution.

Figure 3 reports the dynamics of the capital and the share speculative firms. For both variables the stochastic approximation satisfactorily mimics the results produced by the agent based simulation. The dynamics of capital (and consequently of aggregate production) displays a long term upward trend. Within this trend, long cycles of a duration varying between 20 and 40 periods and smaller variations (from one period to another) are identifiable. The length and the amplitude of the cycles are determined by the underlying debt cycle. During periods of accelerated growth, the proportion of speculative firms and aggregate debt rise. Consistently with Minsky’s model, growth and the accumulation of debt increase until the most indebted speculative firms begin to fail, reducing the amount of capital and the aggregate wealth in the system. The amount of available credit reduces causing the demise of other speculative units. This downward spiral ceases when all the firms in the relatively worst financial condition have collapsed, allowing the cycle to start again. The dynamics of bankruptcies and capital are compared in figure 4 for a different simulation run.

The period that we considered relevant as an empirical benchmark is the post world war two. Interestingly in this period the US economy displayed a more regular pattern than the model. Given the quantitative similarities for series correlations, aggregate production dynamics and growth rates of firms distribution, the major fluctuations and the long periods of depression that the model displays can be interpreted as the contribution of households and public sectors which are taken into consideration in the model. In particular, this contribution historically consisted in the shifting of debt from the productive and financial sectors to the household and public sectors, according to a pattern that has become evident during the last global financial crisis. In recent decades in the US, and to a lesser extent in all other developed economies, the growth of demand has been sustained by a remarkable increase of household debt. Due to

13 Only for market capitalisation does the series start in 1929.
14 Note also that in the present model the level of productivity is assumed to be constant.
the restrictions in the credit market in the wake of the crisis and to the ensuing recession, firms and households started a process of deleveraging which forced national governments to directly take on part of the private sector’s liabilities or to sustain demand, shifting the private debt to the public sector. This feature is further highlighted by figure 5 with displays the series of debt generated by the model, highlighting the periods of negative variation in production, and how this contrasts with comparable data for the US. In both cases the beginning of a deleveraging process marks a recession. In the real world household and private sector provide a safety net, while in the model deleveraging and depression last until all the weakest productive units exit from the market.

The model shows that in a context of systemic financial fragility traditional policies of stabilization may be less effective than regulatory policies or limitations to the speculative activity. The application, for example, of the Taylor rule may cause undesired effects in the case of a wrong identification of the actual trend of the aggregate production. Indeed, during a boom phase, it is particularly tricky to verify if the economy is growing along its long run path or if it is overheated by the excess of liquidity, since these periods of speculation may last for several years. On the other hand, as well explained in Tonveronachi and Montanaro (2009), the identification of an optimal interest rate during these expansionary periods, is difficult as it should cool the system down without worsening the condition of speculative units and starting a bankruptcy wave. Even if price dynamics is not modelled here, the model provides an endogenous kind of Taylor rule as the interest rate rises during phases of accelerated growth and decreases during downturns.

The possible areas of effective intervention for the policy maker are identifiable in the limitations to the creation of liquid assets and in the regulation of bankruptcy. These two areas are represented respectively by the parameters $\psi$, which quantifies the capacity of the system to create endogenous money, and $c$, which is the maximum debt ratio allowed to avoid failure. The parameters have been calibrated in order to obtain the best performance in terms of replication of empirical data (see figure 6).

As shown by figure 7, the degree of financial innovation and the consequent capacity of the system to create endogenous money plays a key role in generating the booms. The availability of easily tradable financial instruments pumps liquidity into the system augmenting the level of wealth $W(t)$. Credit becomes cheaper and the accumulation of capital is faster. A bigger stock of liquid assets in the market makes possible the accumulation of a larger stock of capital in the long run. The downside of this is the much larger volatility and the associated long periods of contraction. In this perspective a regulatory framework for securitization and in general for financial derivatives may reduce $\psi$ and be effective in reducing the volatility. Interestingly, for values of $\psi$ close to 0.5, and therefore slightly higher than the value that gives the best fit to empirical data, as shown in table 2, the system becomes unstable and subjected to waves of failures that involve almost all the active firms.

15For up to date documentation and analysis see www.debtdeflation.com/blog.
A larger value of the parameter $c$ corresponds to an easing in the position of heavily indebted firms\textsuperscript{16} or to a temporary support to those in a critical condition. The model demonstrates that larger debt ratios can lengthen the positive trend but, as a consequence, the period of distress is longer, increasing the uncertainty in the system (figure 8). These results are consistent with the findings of Suarez and Sussman (2007) that a softening in the bankruptcy law ensures a faster growth at the expense of long term stability.

The degree of uncertainty in the present model can be quantified by the parameter $a$, which measures the reactions of firms to market expectations, and by the possible changes in investor strategies, captured by the distribution of $n^c$. As $a$ grows the cycles become more regular, longer and of larger amplitude. Over a certain threshold ($a > 10$) the positive long run trend virtually disappears and only long fluctuations are observable. A reduction in the support for the distribution of $n^c$ or a less dispersed distribution (truncated normal rather than a uniform distribution) reduces the interest rate, allowing a more sustained growth with a slower accumulation of debt.

As regards the choice of firms between equity and debt as source of financing, figure 9 shows that, as the proportion of investment financed with equity increases, the system becomes more stable. The accumulation of capital improves despite the fact that for low shares of debt financing the private wealth shrinks.

6 Concluding remarks

In this paper we study the transmission of shocks from the financial sector to the real economy, along the lines of Minsky and Taylor and O’Connell (1985). We provide a consistent microfoundation for Minsky’s theory using two different methods, one numerical and the other analytical, to deal with the issue of heterogeneity. This feature has prevented so far a wider diffusion of Minsky’s ideas, since, until recent times, this class of models could be elaborated only in macroeconomic terms. The present model involves firms which are heterogeneous for size and financial conditions and can generate two types of dynamics. One is agent based and allows a numerical solution which replicates some stylized facts of a real economy; the other is a stochastic approximation that can be solved analytically. This latter turns out to be capable of satisfactorily mimicking the outcomes of an agent based model with a much higher degree of heterogeneity.

The simulations results show a dynamics consistent with Minsky’s intuition: there are boom periods, in which the economy grows at a rate significantly higher than its long run trend. At the same time the availability of credit leads firms to take on more debt. When the units in the worst condition begin to fail the process reverses, causing a depression. This pattern is reproduced cyclically, revealing a structural fragility of the system. The economy can be stabilised

\textsuperscript{16}The art. 11 in USA and other similar legislations in other country may be regarded as an example.
reducing its capacity to create endogenous money and the maximum debt ratio allowed.

Such an approach is not possible within the traditional economic paradigm. Indeed, even though in principle allowing a bottom-up approach, it is by construction unsuitable to deal with this problem for two main reasons. The first is the representative agent hypothesis, which is hardly compatible with the Financial Instability Hypothesis (that is formulated considering different types of firms with respect to their financial structure) and cannot involve phenomena like insolvency and bankruptcy. The second reason is that, in the neoclassical view, financial markets are not even potentially a factor of instability as, on the contrary, they are supposed to stabilize the economy, absorbing temporary disequilibria in real markets by means of derivatives and futures.

The framework appears then to be an efficient tool to analyse the effects of instability in financial markets on the real sector of the economy and, in particular, through the modelling of the generation of investors’ expectations, to identify the conditions under which the system generates speculative bubbles and how they burst. This basic framework may be extended to include various forms of speculative behaviour and the banking sector in the intermediation of credit. Another further development may include institutional aspects such as government policies, fiscal and monetary, and the study of the possible effects of a regulatory framework.

References


Tonveronachi, M. and Montanaro, E. (2009): Some preliminary proposals for re-regulating financial systems, Department of Economics University of Siena 553, Department of Economics, University of Siena.


Figure 3: Different dynamics of capital (upper panel) and share of speculative firms (lower panel). Simulation of agent based model (black continuous line) and endogenous stochastic dynamics (red dashed line).

Figure 4: Rate of bankruptcy and aggregate capital (right axis). Simulation of agent based model.
Figure 5: Upper panel: Business debt dynamics for US (billion of dollars) and recessions (grey areas). Lower panel: comparable results from simulations of agent based model.
Figure 6: Correlation between aggregate product and debt for different values of $c$ (upper panel) and $\psi$ (lower panel). Monte Carlo simulation of agent based model.

Figure 7: Dynamics of aggregate capital for different values of $\psi$. Monte Carlo simulation of agent based model.
Figure 8: Average aggregate capital, variance of fluctuations, interest rate and final wealth for different values of $c$. Monte Carlo simulation of agent based model.

Figure 9: Average aggregate capital, variance of fluctuations, interest rate and final wealth for different values of $\phi$. Monte Carlo simulation of agent based model.