The Impact of Short-Selling Constraints on Financial Market Stability in a Model with Heterogeneous Agents∗

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Abstract

Recent turmoil on global financial markets has once again led to a discussion on which policy measures should or could be taken to stabilize financial markets. One such a measure that resurfaced is the imposition of short-selling constraints. It is conjectured that these short-selling constraints reduce speculative trading and thereby have the potential to stabilize financial markets. The purpose of the current paper is to investigate this conjecture in a framework of a conventional asset-pricing model with heterogeneous agents. We find that the local stability properties of the fundamental equilibrium do not change when short-selling constraints are imposed. However, when the fundamental equilibrium is unstable, restrictions on short-selling increase the price volatility. Addressing the issue of control of the market stability, we find that imposing the constraints during the downward market movement cushions market drop but does not necessarily stop it.

JEL codes:

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1 Introduction

The practice of short-selling – borrowing a financial instrument from another investor to sell it immediately and close the position in the future by buying and returning the instrument – is widespread in financial markets. In fact, short-selling is the mirror image of a “long position”, where an investor buys an asset which did not belong to him before. While a long position can be thought of as a bet on the increase of the assets’ value (with dividend yield and opportunity costs taken into account), short selling in fact allows investors to bet on a fall in stock prices. Some people have argued that this betting on a fall in stock prices may increase volatility and lead to the incidence of crashes in financial markets, and that the possibility of short selling should therefore be restricted. In this paper we investigate consequences of such restriction in an agent-based model of a financial market.

A historical account of Galbraith (1954) provides an evidence that the short sales were common during the market crash of 1929. As short-sellers were often blamed for the crash, the Securities and Exchange Commission (SEC) introduced in 1938 the so-called “uptick rule”, which prohibited the short sells “on a downtick”, i.e., at prices lower than the previous transaction price. Curiously enough, the uptick rule has been removed on July 6, 2007, right before the market crash of 2008 has began. Since then, the calls to restoring the uptick rule have been recurrent. In the left panel of Fig. 1 we show the evolution of the S&P500 index and also indicate the end of the uptick rule period as well as the statements by different practitioners, authority experts, congressmen and senators for its restore or imposing some other constraints on short sales. The calls for regulation did not remain unanswered. In the fall of 2008 – at the peak of the credit crisis – the SEC temporarily prohibited short selling of securities for 799 different financial companies. The SEC’s chairman, Christopher Cox, argued that: “The emergency order temporarily banning short selling of financial stocks will restore equilibrium to markets”. Similar actions were taken in, for example, the United Kingdom and Austria. However, it is not clear whether such a ban on short selling has actually been helpful in stabilizing financial markets. The index continued to fall during the short-sell ban as well as afterwards, see the right panel of Fig. 1.

The traditional academic view on constraints on short selling is that they may lead to mispricing of the asset. Miller (1977), for example, argues that short sales increase the effective supply of stocks, which lowers their price. Restricted or costly short selling then may lead to mispricing. His predictions have been empirically supported. Jones and Lamont (2002) study data on the costs of short-selling between 1926 and 1933 and find that those assets which were expensive to sell short earned subsequently lower return. Lamont and Thaler (2003) discuss 3Com/Palm and other examples of clear mispricing. They attribute a failure to correct the mispricing to the high cost of selling short. Also laboratory experiments with human subjects suggest that short-selling constraints may lead to considerable mispricing, with important reservation that relaxing the constraints reduces but does not eliminate mispricing.

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2 Apart from the legal constraints on short selling which we focus on, short selling may also be constrained because it may be costly, compared to taking a “long position”. In particular, an investor willing to sell short should eventually deliver the shares to the buyer, and hence is required to “locate” the shares, i.e., to find another investor who is willing to lend these shares. (When shares have not been located, the operation is called “naked short selling”, which is subject to more strict regulations and is often banned, since it is believed to permit price manipulation.) In the absence of a centralized market for lending shares such an operation may be complicated. At least, it is costly, because short selling requires not only paying a standard fee to the broker but also involves a commission (plus dividends) to the actual owner of the stock. Moreover, there is a recall risk at any moment that the lender wants to recall a borrowed stock.

Similarly, theoretical literature argues against any form of short selling constraints. For example, in a model of Harrison and Kreps (1978), rational, risk-neutral investors have different expectations about the dividends of a certain asset. In the absence of short selling constraints, investors with different opinions would tend to take infinitely large opposite positions. When short-selling constraints are imposed, however, such situation is ruled out, and the most optimistic investors will determine the price, which becomes larger than the rational value. In the asymmetric information model of Diamond and Verrecchia (1987) the short-selling constraint prevents some investors from complete trading. This implies that not all private information is fully incorporated into the price. The price will be unbiased only when fully rational investors trade in this market. Such investors should take the short-selling constraints into account, realize that the order flow does not reflect all available information, and trade at the correctly computed prices. Clearly, such strategy requires extraordinary amount of rationality.

Theoretical conclusion about absence of mispricing when constraints are relaxed is not so straightforward, however. Gallmeyer and Hollifield (2008), for example, study the effect of short selling constraints on the equilibrium price in a market with rational, forward-looking optimizers who have heterogeneous priors about the dividend process. They show that, depending on the agents’ preference parameter, imposing short selling constraints can lead to higher or lower volatility. Furthermore, Lecce, Lepone, and Segara (2008) and Setzu and Marchesi (2008) argue that short selling can lead to excess volatility.

Most of the models discussed above are static in nature and assume that investors are fully rational. This assumption of full rationality has been challenged on theoretical as well as empirical grounds. Theoretically, it can be argued that to actually compute rational beliefs, agents would need to know the precise structure and laws of motion for the economy, even though this structure depends on other agents’ beliefs, c.f. Evans and Honkapohja (2001). Empirically, some important markets regularities, such as recurrent periods of speculative bubbles and crashes, fat tails of the return distribution, excess volatility, long memory and volatility clustering are difficult to explain with these theoretical models. Moreover, there is an abundance of experimental evidence that suggests that theoretical models with fully rational
agents do not even provide accurate descriptions of relatively simple laboratory markets (see, e.g., Smith, Suchanek, and Williams, 1988, Lei, Noussair, and Plott, 2001, and Hommes, Sonnemans, Tuinstra, and Velden, 2005).

An alternative approach is to consider models of behavioral finance (see, e.g., Shleifer, 2000 and Barberis and Thaler, 2003 for reviews) or, closely related, heterogeneous agents models (HAMS, see Hommes, 2006 and LeBaron, 2006 for reviews). In HAMS, for example, traders choose between different heuristics or rules of thumb when making an investment decision. Typically, heuristics that turned out to be more successful in the (recent) past will be used by more traders. Such models are also successful empirically (by reproducing many of the empirical regularities discussed above, see Lux, 2009), and therefore become an increasingly accepted alternative to the traditional models with a fully rational, representative agent. In this paper we investigate the impact of introducing short-selling constraints in such a heterogeneous agents model.

We take the well-known asset pricing model with heterogeneous beliefs from Brock and Hommes (1998) as our benchmark model. Traders in this model have to decide every period how much to buy or sell of an inelastically supplied risky asset, and they base their decision on one of a number of behavioral prediction strategies (e.g., a fundamentalist or a trend following/chartist prediction strategy). As new data become available agents not only update their forecasts but also switch from one forecasting technique to another depending on past performances of those techniques. Such a low-dimensional heterogeneous agents model is able to generate the type of dynamics typically observed in financial markets, in particular when traders are very sensitive to differences in profitability between different trading/prediction strategies.

In this framework we investigate the impact of imposing short-selling constraints, by analytical as well as by numerical methods. Specifically we assume that the demand of any trader for the risky asset cannot be smaller than a certain threshold. We find that the imposition of such a short-selling constraint does not affect the local stability properties of the price dynamics, that is, the financial market is neither stabilized nor destabilized due to the constraints. However, if the price dynamics are volatile to begin with, the short-selling constraints affect the global dynamics and lead to even more volatile price dynamics. The intuition for this result is that short-selling constraints reduce the financial market's potential to quickly correct mispricing. It happens not only due to the limited liquidity of

The rest of the paper is organized as follows. Section 2 briefly recalls the framework with heterogeneous beliefs and introduces the short-sell constraints in this framework. Section 3 discusses a simple, but instructive example of heterogeneous beliefs, and derives our main results. Section 4 is devoted to question whether the stability of the market can be managed by introduction of the constraint in a proper moment. The final remarks and directions for the future research are given in Section 5.

2 Heterogeneous Beliefs Framework without and with Short-Selling Constraints

We start with refreshing a stylized model of heterogeneous beliefs in Brock and Hommes (1998). There are two assets in the market, risky ("stock") and riskless ("bond"). The riskless asset is in perfect elastic supply at gross return $R = 1 + r_f$, its price is always normalized to 1 and it plays a role of the numeraire. The risky asset has ex-dividend price $p_t$ and pays random
dividend \( y_t \) at a period \( t \). The dividend is assumed to be i.i.d. with mean value \( \bar{y} \), this is a common knowledge. Total supply of the risky asset is constant, \( S \geq 0 \).\(^3\)

Market is populated by a large number of traders with the mean-variance demand.\(^4\) We assume that agents have homogeneous expectations about price variance, denoted as \( \sigma^2 \), but heterogeneous beliefs of the price. Assuming that there are \( H \) types of traders, and denoting the expectations of type \( h \in \{1, \ldots, H\} \) as \( \mathbb{E}_t,h[p_{t+1}] \), the demand function for type \( h \) is given by

\[
A_{t,h}(p) = \mathbb{E}_{t,h}[p_{t+1}] + \bar{y} - (1 + r_f)p - a \sigma^2 \bar{S},
\]

where \( a \) is the risk aversion coefficient common between agents. In deriving (2.1) it is assumed that all agents have correct expectations about dividend process, so that the only source of heterogeneity lies in price expectations.

The market for the risky asset is cleared every period. At such “temporary equilibrium” the price \( p_t \) is determined from the equality between demand and supply. Denote the fraction of agents of type \( h \) at time \( t \) as \( n_{t,h} \). Then the equilibrium equation is given by

\[
\sum_{h=1}^{H} n_{t,h} A_{t,h}(p_t) = \bar{S},
\]

where \( \bar{S} \geq 0 \) is the supply of the asset per trader. Using (2.1), the pricing equation above gives the price at the temporary equilibrium as

\[
p_t = \frac{1}{1 + r_f} \left( \frac{S}{\sum_{h=1}^{H} n_{t,h} \mathbb{E}_{t,h}[p_{t+1}] + \bar{y} - a \sigma^2 \bar{S}} \right).
\]

When the forecasting rules \( \mathbb{E}_{t,h}[p_{t+1}] = f_h(I_{t-1}) \) are specified as some deterministic functions of past prices, equation (2.3) describes well-defined price dynamics, as soon as evolution of fractions, \( n_{t,h} \), is given.

The evolution of fractions constitutes the last ingredient of the model. At the end of every trading round agents update their forecasting type. The choice of forecasting type is based upon the performance measure given by the realized excess profit of a type. It is computed as a product of holdings of the risky asset at the end of round \( t - 1 \) and its excess return

\[
U_{t,h} = A_{t-1,h} \left( p_t + y_t - (1 + r_f)p_{t-1} \right).
\]

The fraction of agents choosing the type \( h \) at the next period \( t + 1 \) is described by

\[
n_{t+1,h} = \frac{\exp \left( \beta(U_{t,h} - C_h) \right)}{\sum_{h'=1}^{H} \exp \left( \beta(U_{t,h'} - C_{h'}) \right)},
\]

\(^3\)Original Brock and Hommes (1998) model had, for the sake of simplicity, \( S = 0 \), whereas, e.g., Hommes, Huang, and Wang (2005) assume that \( S > 0 \). The former case implies a short selling of some agents along any non-equilibrium path, while the latter not.

\(^4\)Appendix A shows how the “Large Market Limit” model explained in this Section can be derived from the agent-based setting. In this paper we restrict our attention only on the Large Market Limit version. It allows us to study the dynamics analytically, to some extent, but does not let us to address different potentially interesting questions, as e.g., an effect for liquidity constraints. An analysis of the agent-based version is left for the future research.
where $C_h$ is the cost of strategy $h$. Parameter $\beta \geq 0$ is the *intensity of choice*, measuring the sensitivity of agents with respect to the difference in past performances of the two strategies. If the intensity of choice is infinite, the traders always switch to the historically most successful strategy. At the opposite extreme, $\beta = 0$, agents are equally distributed between different forecasting types independent of their past performances.

**Rational Expectations**

In a special case of rational expectations, the model outlined above can be solved. Under rational (and, hence, homogeneous) expectations, price dynamics (2.3) implies that

$$p_t(1 + r_f) = p_{t+1} + \bar{y} - a \sigma^2 \bar{S},$$

The only steady state of the dynamics under rational expectations is given by

$$p_f = \frac{\bar{y}}{r_f} - a \sigma^2 \bar{S}.$$  \hfill (2.5)

The standard asset-pricing model says that price of the risky asset is the discounted value of the future stream of the dividends. If the discount factor is equal to $1/(1 + r_f)$, this value is equal to $\bar{y}/r_f$. However, under positive supply, risk-averse investors will ask risk premium which is reflected in the last term of (2.5). We will refer to $p_f$ as the *fundamental price*. More precisely, it is a unique non-bubble solution of the market equation in the case of homogeneous agents with rational expectations.

Whether the fundamental price is a natural outcome of the dynamics of the model with heterogeneous expectations is an important theoretical question. Brock and Hommes (1998) analyze a number of special cases (corresponding to the ecologies with different sets of forecasting rules) and find that for a large region in the parameter space, the dynamics of the model depends on the value of the intensity of choice, parameter $\beta$. When the intensity of choice is small, the dynamics converges to the equilibrium steady-state with the fundamental price. However, as the intensity of choice becomes larger, the fundamental steady state exhibits a bifurcation and loses its stability. Consequently, for larger $\beta$, dynamics may converge to the non-fundamental steady-state with some price $p^* \neq p_f$, or price dynamics can oscillate. Eventually, when $\beta$ increases further, the oscillations become wilder and dynamics exhibits the property of sensitive dependence of initial conditions, typical for chaotical systems.

**Imposing the short-sell constraints**

Let us consider an agent of type $h$ during the trading session $t$. The demand for the risky asset of such agent is given by (2.1). At price $p_t$ an agent will be a net buyer or a net seller, depending on whether the amount in (2.1) is bigger or smaller than the agent’s previous holdings. However, a final position of the agent is always (i.e., independent of the agent’s type in the previous period) given by $A_{t,h}(p)$. For high price $p_t$, this expression is negative, and investor will end up having negative amount of shares. Such situation is called *short position* (in the risky asset). As explained in the Introduction, a popular policy is to restrict investors from having too many shares short. In our setting it means that the holdings can never become smaller than some fixed, non-positive amount.

Let us introduce $\bar{A} \geq 0$ and assume that $-\bar{A}$ represents the lowest boundary for the amount of shares. In other words, when the short selling are constrained, the demand of an agent of
type $h$ becomes

$$A_{t,h}(p) = \max \left\{ -\bar{A}, \frac{E_{t,h}[p_{t+1}] + y - (1 + r_f)p}{a \sigma^2} \right\}, \quad \text{where} \ \bar{A} \geq 0. \quad (2.6)$$

Such restriction may, of course, change a clearing price on the Walrasian market, but the position of every agent at the market clearing price is still equal to his desired position at this price:

$$A_{t,h} = A_{t,h}(p_t) = \max \left\{ -\bar{A}, \frac{E_{t,t}[p_{t+1}] + y - (1 + r_f)p_t}{a \sigma^2} \right\}. \quad (2.6)$$

Our contribution in this paper is to study the model with heterogeneous expectations, under the presence of the constraints on the short-selling. The only formal difference with a standard framework is that we employ the demand function (2.6) instead of the function (2.1). The goals of an investigation of the short-sell constraints in the heterogeneous beliefs framework can be formulated as follows:

1. to compare the price dynamics for two cases: the no-constraint case, when demand functions are given by (2.1), and constraint case, when demand is given by (2.6);
2. to study an impact of such variables as restrictions’ boundary $\bar{A}$ and the total supply of assets $\bar{S}$ on the price dynamics stability;
3. to study the dynamical effects of imposing constraints (2.6) at different points of time evolution;
4. to capture the effects of imperfect market for borrowing shares, resulting in possible rationing of short-sellers;
5. to study all these effects in the presence of restrictions on borrowing the riskless asset.

The goals 1 and 2 are reached in Section 3 where we study the low-dimensional Large Market Limit model. This approximation is adequate under the short-sell restriction because the individual demand functions (2.6) are still identical for all agents of a given type. Thus, the agent-based model can be always written as $H$-types low-dimensional model, even when the constraints are imposed. The third goal is addressed in Section 4, where we simulate dynamics and introduce the constraints at different points along dynamics. The last two goals are more ambitious and require the agent-based simulations, see Appendix A for a discussion.

Model in deviations

It will be convenient to study the model in deviations from the fundamental value: $x_t = p_t - p_f$. Independently of the expectation rules, the realized return is given by

$$r_{t+1} = p_{t+1} + y_{t+1} - (1 + r_f)p_t = x_{t+1} - (1 + r_f)x_t + y_{t+1} - r_f p_f =$$

$$= x_{t+1} - (1 + r_f)x_t + \delta_{t+1} + a \sigma^2 \bar{S}, \quad (2.7)$$

where $\delta_{t+1} = y_{t+1} - \bar{y}$ is a shock due to the dividend realization. Thus, at the fundamental steady-state expected return is given by $a \sigma^2 \bar{S}$. When the supply is positive, the return is also positive, due to the risk premium required by the risk-averse investors. At the non-fundamental
steady-state, if it exists, the return is equal to $a\sigma^2 \bar{S} - r_f x^*$. As expected, it is smaller (bigger) than fundamental in the steady-state where the asset is overvalued (undervalued).

The demand functions (2.1) of agents in deviations are simplified to

$$A_{t,h}(p) = \frac{E_{t,h}[p_{t+1}] + \bar{y} - (1 + r_f)p}{a \sigma^2} = \frac{E_{t,h}[x_{t+1}] - (1 + r_f)x + \bar{S}}{a \sigma^2}.$$  \hspace{1cm} (2.8)

The term $\bar{S}$ reflects the equal distribution of the shares between agents of different types at the steady-state, where agents do not expect excess return. Of course, agents tend to have larger positions if they expect larger return.

The price dynamics (2.3) in deviation is given by

$$x_t = \frac{1}{1 + r_f} \sum_{h=1}^{H} n_{t,h} E_{t,h}[x_{t+1}].$$  \hspace{1cm} (2.9)

Finally, the performance of a given rule computed at the end of period $t$ is given by

$$A_{t-1,h} r_t = \left( \frac{E_{t-1,h}[x_t] - (1 + r_f)x_{t-1}}{a \sigma^2} + \bar{S} \right) \left( x_t - (1 + r_f)x_{t-1} + \delta_t + a\sigma^2 \bar{S} \right).$$

Thus the supply $\bar{S}$ matters for the fraction determination but not for the price determination.

### 3 Short Sell Constraints under Heterogeneous Beliefs

We consider the simplest model within the heterogeneous beliefs framework with $H = 2$ types of agents. The two types we consider are “fundamentalists”, who always predict fundamental price

$$E_{1,t}[p_{t+1}] = p_f,$$  \hspace{1cm} (3.1)

but pay costs $C > 0$ for such prediction, and “chartists” whose forecast is conditioned on the price deviation from the past fundamental value

$$E_{2,t}[p_{t+1}] = p_f + g (p_{t-1} - p_f),$$  \hspace{1cm} (3.2)

where $g > 0$ is the extrapolation coefficient. The forecast of chartists is available for free.\textsuperscript{5} This behavioral ecology was also analyzed in Brock and Hommes (1998), and its main advantage lies in the clarity of the dynamics. However, the precise ecology does not matter for the main conclusions, as it will be clear from the discussion below, and is confirmed by our simulations with other ecologies. One of such alternative ecologies is considered in Section 4, where the short-sell constraints are introduced in 2-types specification, “fundamentalists vs. trend-followers” studied earlier in Gaunersdorfer and Hommes (2007) and Anufriev and Panchenko (2009).

\textsuperscript{5}Notice that the model is written entirely in deviations from the fundamental price. A natural interpretation of the situation which we consider, is that the fundamental price is changing every time period and new fundamentalists should spend some efforts to find out the new price. The chartists, on the other hand, base their forecast on the deviation from the fundamental price of the previous period, which is assumed to be known to everybody.
3.1 Dynamics in the Absence of Constraints

Without short-selling constraints the price dynamics is given by (2.3), which, in deviations, becomes

\[ x_t = n_{2,t} \frac{g}{1 + r_f} x_{t-1}, \]  

(3.3)

with fraction of the chartists given by

\[ n_{2,t} = \frac{1}{\exp\left(-\beta \left[ \frac{gx_{t-1}}{a_0^2} (x_{t-1} - (1 + r_f)x_{t-2} + \delta_{t-1} + a_0^2 \bar{S}) + C \right] \right) + 1}. \]  

(3.4)

Thus, the price deviation from fundamental values depends on the previous price deviation and the current fraction of trend-followers: the larger this fraction is, the stronger previous deviations propagates to the new period.

The following result describes the dynamics of the heterogeneous belief model with fundamentalists (3.1) and chartists (3.2). It is a generalization of Lemma 2 from Brock and Hommes (1998) for the case of positive supply of shares, \( \bar{S} > 0 \).

**Proposition 3.1.** Consider system (3.3)-(3.4) with supply \( \bar{S} \geq 0 \) and \( \delta_t = 0 \), i.e., when \( y_t = \bar{y} \). Let \( n^q_2 = 1/(\exp(-\beta C) + 1) \), \( n^f_2 = (1 + r_f)/g \) and \( x_+ \) and \( x_- \) are solutions of

\[ \frac{1}{n^q_2} = 1 + \exp \left[-\beta \left( \frac{g x}{a_0^2} (-r_f x + a_0^2 \bar{S}) + C \right) \right]. \]

Then:

1. for \( 0 < g < 1 + r_f \), the fundamental steady-state \( E_1 = (0, n^q_2) \) is the unique, globally stable steady-state.
2. for \( g > 2(1 + r_f) \), there exist three steady-states \( E_1 = (0, n^q_2) \), \( E_2 = (x_+, n^q_2) \) and \( E_3 = (x_-, n^q_2) \); the fundamental steady-state \( E_1 \) is unstable.
3. for \( 1 + r_f < g < 2(1 + r_f) \) there are the following possibilities:
   (a) \( 0 \leq \beta < \beta^* \): the fundamental steady-state \( E_1 \) is stable;
   (b) \( \beta = \beta^* \): the fundamental steady-state \( E_1 \) loses its stability, where
   \[ \beta^* = \frac{1}{C} \ln \frac{g - 1 - r_f}{1 + r_f}, \]
   (c) \( \beta > \beta^* \): the fundamental steady-state \( E_1 \) is unstable and two other steady-states \( E_2 = (x_+, n^q_2) \) and \( E_3 = (x_-, n^q_2) \) exist;
   (d) \( \beta > \beta^{**} \): all the steady-states are unstable, and the non-fundamental steady-states lose its stability through the Neimark-Sacker bifurcations.

The dynamics (3.3)-(3.4) can have multiple steady-states. However, if the chartists extrapolate not too strong, their destabilizing efforts are not sufficient to break the stability of the fundamental steady-state. On the other extreme, if chartists are strong extrapolators, dynamics diverge. Consequently, we will concentrate on the most interesting, third case, when the coefficient of extrapolation takes intermediate values. For such parameters the result of
Figure 2: Bifurcation diagrams for the model with fundamentalists and chartists without short-selling constraints. **Upper panel:** Zero supply per investor, $\bar{S} = 0$. **Lower panel:** Positive supply per investor, $\bar{S} = 0.1$. For each $\beta \in (0, 12)$, 1000 points after 1000 transitory periods are shown. The red points correspond to the trajectory started below the fundamental steady-state, while the blue points show the trajectory started above the steady-state. The remaining parameters are: $r = 0.1$, $\bar{y} = 10$, $g = 1.2$, and $C = 1$. For this parameters the fundamental steady-state loses its stability at $\beta^* \approx 2.398$, shown by the dotted line. The second dotted line corresponds to $\beta = 4$, whose dynamics is illustrated in Fig. 2.
Proposition 3.1 is illustrated on the bifurcation diagrams on Fig. 2 both for the zero supply case and for the positive supply case. Two colors of the bifurcation diagrams correspond to different attractors for the same intensity of choice, see caption.

For small $\beta$ the fundamental steady-state is globally stable. At this steady-state, the pre-cost performances of both types are the same, and the positions of all the agents are identical and equal to $\bar{S}$. However, the agents are distributed unevenly, with $n^*_2 > 1/2 > n^*_1$, due to the positive costs paid by fundamentalists. The fundamental steady-state loses stability, when the intensity of choice is high enough. The precise bifurcation scenario depends on $\bar{S}$. When the supply is zero the stability is lost through the pitchfork bifurcation. For some range of parameters $\beta \in (\beta^*, \beta^{**})$ the two symmetric, non-fundamental steady-states exist, so that the system converges to one of them, depending on the initial conditions. They simultaneously lose their stability at $\beta = \beta^{**}$, and for the higher intensity of choice the oscillations with growing amplitudes persist. When the supply is positive, the two non-fundamental steady-states, stable and unstable, emerge through the saddle-node bifurcation for $\beta_{sn} < \beta^*$. At $\beta = \beta^*$ the fundamental steady-state loses its stability through the transcritical bifurcation and if $\bar{S}$ is not too high there exists some parameter range for which both non-fundamental steady-states are locally stable. When $\beta$ increases further, both these steady-states loses stability through the Neimark-Sacker bifurcation leading to the quasi-cyclic behavior.

Notice that when $\beta$ is relatively high, this model, even in its deterministic version, is able to reproduce the pattern of price bubbles and crashes. Consider the illustration of the dynamics for $\beta = 4$ in Fig. 3. Each of the four panels shows the evolution of price $p_t$, return $r_t$, fractions of two types $n_{t,h}$ and positions of two types $A_{t,h}$.

Two upper panels correspond to the case of zero asset supply, $\bar{S} = 0$. The upper left panel shows the dynamics at the upper attractor, when price is above $p_f = 100$. In the first half of simulation the price is growing, but this growth is not sufficient to overcome the risk-free interest rate and the resulting return is eventually negative. The relative fraction of chartists is high, because near the fundamental steady-state both types perform similarly, but fundamentalists pay the costs. Since the asset is overvalued, fundamentalists sell it, and the trend-followers buy. Notice that the positions of two types are very different. Indeed, as a group, fundamentalists hold short exactly the same amount of shares as chartists hold long. However, the relative fractions of two groups are very different and thus the per trader positions of fundamentalists are much larger (in absolute value) than the per trader position of chartists. This asymmetry will have important consequences when short-sell constraints are introduced. The nature of crash is well-understood in this model. First of all, recall that we consider the case of moderately extrapolating trend-followers. It guarantees that at a certain point along the bubble, the capital gain is not sufficient to overcome the negative effect of asset overpricing on the excess return (see (2.7)), and so the total return becomes negative. Second, when the return is negative, the positive fundamentalists’ pre-cost performance is clearly better than the negative performance of chartists. Therefore, the proportion of fundamentalists increases, slowing the price growth and decreasing the return even further. Due to the feedback of the model, this effect is self-fulfilling. Eventually it leads to a relatively fast market crash, when the fundamentalists outperform chartists so significantly that they temporarily dominate the market. The story then repeats itself.

The upper right panel shows the dynamics near the lower attractor, where the asset is

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6Without taking the capital gain into account, the higher is the price, the smaller is the return, and also the bigger is the short position of fundamentalists, see (2.8). If the return is negative, both these effects work in favor of fundamentalists by increasing performance of this group.
Figure 3: Dynamics of the model with fundamentalists and chartists without short-selling constraints. **Upper panels:** $\bar{S} = 0$. **Lower panels:** $\bar{S} = 0.1$. **Left panels:** Upper attractor. **Right panels:** Lower attractor. $\bar{S} = 1$. The parameters are $\beta = 4$, $r = 0.1$, $\bar{g} = 10$, $g = 1.2$, and $C = 1$. The vertical lines show the period of maximum price deviation.
undervalued. The situation is symmetric with respect to the upper attractor, and the negative bubble is observed. The only important difference is that now, the stabilizing part of the population (fundamentalists) are long in the risky asset.

Two lower panels show the dynamics in the case of positive asset supply for the same value of the intensity of choice, $\beta = 4$. On the upper attractor, shown in the lower left panel, the situation is similar to the zero supply case: the bubble pattern of growing price is followed by a short period of crash. Notice, however, the effect of positive supply on different variables. The return depends positively on the supply and during the bubble it remains positive for a long time. Since fundamentalists take a short position, their performance is actually much worse than of trend-followers. However, once again the extrapolating effect of trend-followers is not sufficient to keep price growing. When the return becomes negative the crash is inevitable: the fundamentalists with their large short positions get high return, increase their proportion, slow the price growth even further, and eventually overcome the market. On the lower attractor, shown in the right lower panel, the dynamics is also similar to the zero supply case. The symmetry with respect to the upper attractor, which was observed in zero supply case, is not present any longer. The reason lies in the effect of the positive supply on the return. According to (2.7) the return increases with the supply, and on the upper attractor it temporarily prevented the fundamentalists from playing sufficient role to stabilize market. But on the lower attractor, to the contrary, higher return plays a positive role. The fundamentalists who hold the positive positions get higher performance with higher returns, helping market to stabilize faster. This also explains the peculiar difference between two bifurcation diagrams shown in Fig. 2. Namely, controlling for the intensity of choice, the amplitude of oscillations of the positive bubbles is larger for positive supply case than for zero supply case, and the opposite situation is observed for the lower attractor. Ceteris paribus the positive supply increases return, which, given their relative positions, favors destabilizing chartists on the upper attractor and stabilizing fundamentalists on the lower attractor.

3.2 Dynamics with the Short-Sell Constraints

Let us now turn to the case of short-sale constraints and consider the same fundamentalists vs. chartists model with demand functions given by (2.6). The threshold, $\bar{A} > 0$, sets the restriction on the short-sells, the smaller this number is, the stricter the restrictions are. The bifurcation diagrams for two values of the threshold are shown in Fig. 4 for both zero (upper panels) and positive (lower panels) supply cases. On the basis of these diagrams the following four hypothesis can be formulated:

**H1.** The critical value of the primary bifurcation of the fundamental steady-state, $\beta^*$, is not affected by the threshold $\bar{A}$.

**H2.** The critical value of the secondary bifurcations of the non-fundamental steady-states are affected by the threshold. Namely, for $\bar{A}$ small enough, i.e., for strong enough restrictions, the system exhibits secondary bifurcation for smaller value of the intensity of choice.

**H3.** Mispricing (measured as an amplitude of oscillations) increases for a given $\beta > \beta^{**}$: the smaller $\bar{A}$ is, i.e., the stronger restrictions are, the bigger amplitudes of oscillations are.

---

7 The bifurcation structure is slightly different for the positive supply case. In particular, the Neimark-Sacker bifurcation of the lowest (undervalued) steady-state is exhibited by the system for the higher value of $\beta$. When $\beta = 4$ the oscillations of the system did not yet reach the “full” range, i.e., price does not return to the vicinity of the fundamental steady-state.
Figure 4: Bifurcation diagrams for the model with fundamentalists and chartists with short-selling constraints. Upper panels: Zero supply per investor, $\bar{S} = 0$. Lower panels: Positive supply per investor, $\bar{S} = 0.1$. Left panels: $\bar{A} = 2$. Right panels: More stringent constraint, $\bar{A} = 1$. For each $\beta \in (0, 12)$, 1000 points after 1000 transitory periods are shown. The red points correspond to the trajectory started below the fundamental steady-state, while the blue points show the trajectory started above the steady-state. The remaining parameters are: $r = 0.1$, $\bar{y} = 10$, $g = 1.2$, and $C = 1$.

H4. A symmetry between upper and lower attractors breaks down also for the zero supply case: the positive bubbles have bigger amplitudes.

These four hypothesis turns out to be correct. Below we provide a heuristic proof for these hypothesis and extend their validity for other ecologies.

**Hypotheses H1 and H2: Short-sell Constraints and Local Stability**

The first hypothesis says that the stability properties of the fundamental steady-state are independent from the constraints. The result holds, because the positions of both fundamentalists and chartists at the fundamental steady-state are equal to $\bar{S} \geq 0$. Thus, the short-sell constraints with positive threshold are never binding, and do not affect the local stability of the steady-state. In other words, the short-selling restrictions can neither stabilize nor destabilize the market, in the sense that they do not affect the value of bifurcation of the fundamental
steady-state. Hypothesis H1 remains to be true for any market ecology.

On the other hand, as it follows from the second hypothesis, the constraints do affect the stability of the mispriced market. The intuition is straight-forward. Suppose, for the sake of simplicity, that $\bar{S} = 0$ and consider the upper steady-state with positive deviation $x_+$. At this steady-state fundamentalists are short in the risky asset and, as simple computations show, their position decreases monotonically with $\beta$. So far as their equilibrium position is greater than $-\bar{A}$ the constraints are not binding and the steady-state remain to be stable. However, when the position is smaller than given $-\bar{A}$, the equilibrium steady-state becomes unfeasible and the dynamics settles at the quasi-cycle around this steady-state. Clearly, if $\bar{A}$ is small enough, the secondary bifurcation takes place for smaller $\beta$, then in the absence of the constraint. Hypothesis H2 is valid for any market ecology, when the two steady-states coexist after the primary bifurcation.

Hypotheses H3 and H4: Short-sell Constraints and Price Determination

In order to understand the last two hypotheses, we need to analyze the price determination of the model. The formal details of this analysis are left for Appendix B, while here we give a geometric illustration for the model with fundamentalists and chartists.

Consider the temporary equilibrium of time $t$ in the market with fundamentalists and chartists as illustrated in the left panel of Fig. 5. Here we show the so-called “adjusted” demand and supply curves defined in (B.1) and (B.2) for both groups of traders. The abscissa shows the quantity demanded or supplied in deviation from $\bar{S}$, while the ordinate shows the price deviation from the fundamental value. All individual demand schedules have the same slope, $1/(1 + r_f)$, in our model, and hence all the adjusted demand and supply curves have also the same slope (in absolute value). Fundamentalists forecast zero price deviation for the next period, so that their adjusted demand is positive when $x_t < 0$, and their adjusted supply is positive otherwise. The dashed green lines show individual adjusted demand (supply) schedules of fundamentalists. The non-trading point of the fundamentalists is $x_{NT}^f = 0$, when both adjusted demand and adjusted supply curves coincide. The dashed blue lines show the individual adjusted demand (supply) schedules of chartists. In this example we supposed that the past deviation of price from the fundamental value was positive, and so the chartists expect positive deviation denoted as $x_c$. This corresponds to the non-trading price of the chartists, i.e., $x_{NT}^c = x_c$. Assume that there are 5 fundamentalists and 5 trend-followers in the market. Horizontal summation of five corresponding curves gives the aggregate adjusted demand and aggregate adjusted supply curves, both shown by the thick red lines. The aggregate curves intersect in the point labeled $E_U$, which is the equilibrium price deviation without short-selling constraints. All the agents trade at the equilibrium price. The actual traded volume depends on the agents’ holdings before the trade, thus it cannot be seen from the illustration. The quantity which we observe in the horizontal axis is the deviation from an average $\bar{S}$ of the total demanded quantity, which is given by $\sum_n \bar{D}_{t,n}(x_t)$ equal to $\sum_n \bar{S}_{t,n}(x_t)$. In some sense it gives a measure of “inequality” of the distribution of the assets among the traders’ types.

Notice that in this Fig. 5, the equilibrium price lies on a half way between $x_{NT}^f$ and $x_{c NT}$. This is, of course, a consequence of equal amount of fundamentalists and chartists. In general, the following holds

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8We do not investigate it in this paper, but notice that in the presence of noise the dynamical property of the stochastic system should be affected by the constraints.

9When the supply is zero, $\bar{S} = 0$, these are simply the net demand and the net supply curves.
Figure 5: Effect of short-sale constraints on the demand function and market clearing. **Left panel:** Short-sales are allowed. **Right panel:** The positions of the risky asset are bounded by amount $-\bar{A} = -1$ share per agent.

**Proposition 3.2.** Suppose that there are two types of heterogeneous beliefs in the market, i.e., $H = 2$. Then:

1. The price at the trading session $t$ is getting closer to the non-trading price of a given type, when proportion of this type, $n_{t,h}$, increases.

2. The optimistic type (with the highest no-traded price) has positive adjusted demand, $A_{t,h}(x_t) - \bar{S}$, whereas the pessimistic type (whose no-traded price is lower) has positive adjusted supply, $-(A_{t,h}(x_t) - \bar{S})$.

3. The adjusted position of an agent of a given type is larger, when the fraction of this type is smaller.

4. The relative adjusted position of a given type (computed as the adjusted position of an agent of this type times the relative fraction, $|A_{t,h}(x_t) - \bar{S}|n_{t,h}$) reaches its maximum when the agents are equally divided between two types, and decreases to zero when $n_{t,h}$ decreases to zero or increases to one.

This result is expected from the illustration and is proven in Appendix B. As follows from the statement 2 in the equilibrium the optimistic type (i.e., chartists in the geometric example before) has higher amount of the risky asset than $\bar{S}$. Therefore, this optimistic type will be necessarily long in the asset. The pessimistic type should have positive supply, which means that when $\bar{S}$ is not too high, this type goes short. The validity of statement 3 have been observed in Fig. 3 where, e.g., during the positive bubble, when fundamentalists are in minority, the position of every fundamentalists is much larger than the position of a chartist. The statement 4 geometrically equivalent to the observation that the abscissa of the equilibrium point $E_U$ on Fig. 5 reaches its maximum when the agents are equally divided.

Let us now impose the constraints on short selling, so that

$$\frac{E_{t,n}[x_{t+1} - (1 + r_f)x]}{a \sigma^2} > -\bar{A} - \bar{S}.$$ 

The right-hand side of this inequality is always negative, hence the short-sell constraints are never binding for the adjusted demand curve, as we have already anticipated. However,
they are binding on the adjusted supply curve, which becomes inelastic at quantity $\bar{A} + \bar{S}$ as illustrated in the right panel of Fig. 5.\textsuperscript{10} Any individual adjusted supply curve becomes vertical when

$$ x_t = \frac{E_{t,n}[x_{t+1}] + a\sigma^2(\bar{A} + \bar{S})}{1 + r_f}. $$

Imposed short-sell constraints change the aggregate supply schedule, as well as the equilibrium point $E_C$. In this example the equilibrium price under short-selling constraints is larger than the equilibrium price without constraints, while the quantity traded is smaller. Such outcome is expected, given Proposition 3.2. Fundamentalists represent a pessimistic type, and for them the constraints are binding. Since supply is restricted, the price goes up. 

Now the hypotheses $\textbf{H3}$ and $\textbf{H4}$ formulated above become obvious. When the short-sell constraints are imposed, and the dynamics follow the positive trend, the fundamentalists are short. The equilibrium level of price at a given period becomes higher due to insufficient liquidity. Such an effect is well understood in the theoretical literature on short-selling at least since Miller (1977). In our model, however, there are also dynamic consequences. The higher price growth rate (with respect to the no-constrained benchmark model) has a positive impact on the return. At the same time, the fundamentalists are restricted in their negative positions. Both these factors contribute to worsening the performance of fundamentalists. Consequently, the mispricing will be larger at the next period, and even larger if the constraints are imposed at that period as well. Overall, the bubble can last longer. It explains hypothesis $\textbf{H3}$.

Finally, there exists an asymmetry on the impact of the short-selling constraints between dynamics on the upper and on the lower attractors. In both cases chartists dominate the market, but on the upper attractor their type dominates, while and on the lower attractor the other type dominates. Part 3 of Proposition 3.2 implies that the same constraints are more binding for the type which is in minority. It means that the constraint will always have larger consequences for the dynamics on the upper attractor. In particular, it explains hypothesis $\textbf{H4}$, that is the asymmetry breaking in the zero supply case.

We illustrate the dynamics in the case of short-selling constraints in Fig. 6 which can be directly compared with Fig. 3. We again show four simulations, but this time in all the simulations the short-sell constraint $\bar{A} = 1$ is imposed. The left panels show the dynamics on the upper attractor for zero (upper panel) and positive (lower panel) supply, whereas the right panels show the corresponding dynamics on the lower attractor. Comparison with Fig. 3 reveals that the short-sell constraints significantly changes the dynamics on the upper attractor. The dynamics on the lower attractor is practically unchanged. This supports hypothesis $\textbf{H4}$.

On the upper attractor the constraint becomes binding for the fundamentalists around period 30. As a consequence the price grows faster afterwards, and the return at each period is higher than the return in the no-constraint case. Notice that the return is still decreasing, so that the ultimate crash can be expected. Indeed, for larger deviations return becomes smaller, while even when the constraints are imposed the price growth rate is restricted by the coefficient of the chartists extrapolation, and cannot compensate for lowing return. In any case, the amplitude of oscillation is higher for the constraint case, which supports hypothesis $\textbf{H3}$.

\textsuperscript{10}In this case we assume that the short selling constraint is imposed with $\bar{A} = 1 - \bar{S}$. Therefore, such figure is valid only for $\bar{S} < 1$. When $\bar{S} = 1$ such figure corresponds to the case when agent is not allowed to go short at all.
Figure 6: Dynamics of the model with fundamentalists and chartists with short-selling constraints $\bar{A} = 1$. **Upper panels:** $\bar{S} = 0$. **Lower panels:** $\bar{S} = 0.1$. **Left panels:** Upper attractor. **Right panels:** Lower attractor. $\bar{S} = 1$. The parameters are: $\beta = 4$, $r = 0.1$, $\bar{y} = 10$, $g = 1.2$, and $C = 1$. The vertical lines show the period of maximum price deviation for dynamics without constraints, see Fig. 3.
Figure 7: Crash of the market with fundamentalists and chartists with short-selling constraints \( \bar{A} = 1 \). **Left panel:** \( \bar{S} = 0 \). **Right panel:** \( \bar{S} = 0.1 \). The parameters are: \( \beta = 4 \), \( r = 0.1 \), \( g = 10 \), \( g = 1.2 \), and \( C = 1 \). The vertical lines show the period of maximum price deviation for dynamics without constraints, see Fig. 3.

The dynamics on the lower attractor is always unaffected by the constraints. For example for the positive supply case (lower right panel) the constraints are never binding. The chartists are short in the asset and they represent the majority in the market. Consequently, the position of every of them is relatively high. Actually, the lowest position per chartist is observed after the negative bubble “crashes”, when the population of chartists shrinks. For the zero supply case (upper right panel) this is a unique moment when the constraints are binding. When the constraints are imposed the fundamentalists represent even larger population than without constraints, which helps market to correct faster.

Let us now return to the upper attractor and consider the effects of the constraints on the crash, see Fig. 7. The left panel illustrates the case of zero supply and the right panel illustrates the case of positive supply. In both cases the crash occurs much later than in the case of no-constraints (compare two horizontal lines indicating the maximum price deviation). The constraints do not prevent the crash, but as a consequence of the constraints the crash is much more pronounce, and is characterized by the lower return. The evolution of fractions shows that much larger relative fraction of fundamentalists is needed to stop the price growth.

The analysis above also suggests how the dynamics is affected by parameters \( \bar{A} \) and \( \bar{S} \). Return to Fig. 5 and notice that increase in the assets’ supply and increase of the threshold \( \bar{A} \) (i.e., relaxing the constraints) have equal effects on the geometry of restrictions. In both cases the inelastic part of supply curve shifts right. Effectively, an increase of the total supply makes the constraints less strict. The amplitude of dynamics will decrease in this case given \( \beta \).
Indeed, comparing Figs. 7 and 3 one can see that equal amount of the short-selling constraints leads to relatively bigger increase in amplitude of oscillations for the zero supply case.

4 Managing Price Dynamics by Short-Selling Constraints

In the previous Section we have established two important results. On the one hand, we have found the short-selling constraints do not matter for the intensity of choice smaller than the bifurcation value of the fundamental steady-state. On the other hand, we have found that after the bifurcation, the amplitude of oscillations increases. For such values of the intensity of choice, the short-sell constraints destabilize dynamics.

One can now ask, however, what would the effects of the constraints be, if they would be imposed only on the downward part of the price dynamics. This would be similar to the three weeks short-sell ban imposed in October 2008, as it was discussed in the Introduction. Furthermore, in the example of Section 3, the chartists did not extrapolate the trend, and so they could not potentially worsen the crash of the market. Motivated by these remarks we will consider now the model with alternative two types of traders.

The fundamentalists have the rule

\[ E_{t,1}[p_{t+1}] = p_f + v (p_{t-1} - p_f), \quad v \in [0, 1], \] (4.1)

which is similar to (3.1). Fundamentalists predict that any price deviation from the fundamental level will be (partially) corrected. In one limiting case, \( v = 0 \), immediate correction is expected, while in another limiting case, \( v = 1 \), agents rely on the market, expecting that the last observed price is the best predictor. The trend-following forecasting rule

\[ E_{t,2}[p_{t+1}] = p_{t-1} + g (p_{t-1} - p_{t-2}), \quad g > 0, \] (4.2)

predicts that past trends in the price will hold. It extrapolates the past trend with coefficient \( g \). Notice that the fundamental forecasting rule (as opposed to the trend-following forecasting rule) requires a knowledge of fundamental value. Consequently, we assume that to use the fundamental rule the agent has to pay cost \( C > 0 \) per period, whereas the second rule is available for free.

In the absence of the short-sell constraints the model is described by one equation of the fourth order (or, equivalently by the 4-dimensional system) consisting of the market clearing equation coupled with an update of the fractions of fundamentalists:

\[
\begin{align*}
\begin{cases}
x_{t+1} = \frac{1}{R} \left( v x_t n_{1,t+1} + (x_t + g(x_t - x_{t-1})) (1 - n_{1,t+1}) \right) + \varepsilon_{t+1} \\
n_{1,t+1} = \exp \left( \beta \frac{(v x_{t-2} - R x_{t-1})}{a \sigma^2} + s \right) \left( x_t - R x_{t-1} + \delta_t + a \sigma^2 s \right) - C \right) / Z_{t+1},
\end{cases}
\end{align*}
\] (4.3)

where normalization factor

\[ Z_{t+1} = \exp \left( \beta(U_{1,t} - C) \right) + \exp \left( \beta U_{2,t} \right) = 
\begin{equation}
= \exp \left( \beta \frac{(v x_{t-2} - R x_{t-1})}{a \sigma^2} + s \right) \left( x_t - R x_{t-1} + \delta_t + a \sigma^2 s \right) - C \right) + + \exp \left( \beta \left( x_{t-2} + g(x_{t-3} - x_{t-2}) - R x_{t-1} \right) / a \sigma^2 + s \right) \left( x_t - R x_{t-1} + \delta_t + a \sigma^2 s \right) \right). \] (4.4)

20
When $g > 1 + r_f$, this system has two possible dynamics as function of the intensity of choice. When the intensity of choice $\beta$ is low, the fundamental steady-state is locally stable. This steady-state loses stability, and for higher intensity of choice the oscillations persist.

Fig. 8 compares the bifurcation diagrams without and with short-selling constraints. Even if the structure of the dynamics is different with respect to the model in Section 3, the effect of short-sell constraints is the same. Also in this “fundamentalists vs. trend-followers” model the short-sell constraints leads to larger amplitude of fluctuations. The main reason of such larger amplitude is the limitation of fundamentalists in the mispricing correction. The situation is also illustrated in the left panel of Fig. 9. There we show how the price dynamics changes when the short-sell constraints are imposed at a period $t = 40$, when the market is on the upward part of the price trend. Similarly to the previous model, fundamentalists are constrained and the bubble can last longer.

Now suppose that the constraints are introduced on the downward part of dynamics. The simulation in the right panel of Fig. 9 shows that the market crash, indeed, slows with the help of the constraints. However, the lowest point of the price dynamics is not changing. In other words, the constraints does not allow to avoid the crash.

5 Conclusion

In this paper we have analyzed the quantitative consequences of the short-selling constraints for the asset-pricing dynamics. Existing literature mostly points out that the short-selling restrictions can lead to systematic overvaluation of the security. The intuition for that was provided by Miller (1977) who shows, in the two-period setting, that a diversity of expectations among investors leads to overpricing. Our model formalizes this intuition and extends it for a dynamical setting.

In our model demand of myopic investors depends on their expectations of future price. The expectations are heterogeneous and agents are allowed to switch between different forecasting rules over time. As it is well known, the dynamics of such model depends on the intensity of
Figure 9: Price dynamics in the fundamentalists vs. trend-followers model. **Left panel:** Price dynamics without constraints (blue) is compared with price dynamics when the short-selling constraints with $A = 7$ are imposed at period $t = 40$ (red). **Right panel:** The same price dynamics are compared with price dynamics when the short-selling constraints with $A = 5$ are imposed at period $t = 50$ (purple). Parameters are: $r = 0.1$, $\bar{y} = 10$, $v = 0.1$, $g = 1.2$, $C = 1$ and $\bar{S} = 0$.

choice. For low values of this parameter the dynamics converge to the fundamental steady-state. When the constraints are introduced in this scenario, the local dynamics will not change, since the constraints will never be binding in the steady-state. For high values of the intensity of choice dynamics do not converge to the fundamental steady-state. Instead, the model exhibits price oscillations with excess volatility. The relevance of short-sell constraints is especially important for the dynamics of such scenario.

It turns out that the constraints increase volatility of the price. This is the outcome of two effects, *liquidity effect*, which limits the stabilizing force of those traders whose evaluations are the closest to the fundamental value, and *market ecology effect*, according to which the performance of such stabilizing traders is getting worse when the short-sell constraints are imposed, and consequently more time is needed for the market correction. As a result, inevitable crash is more severe in the presence of the short-sell constraints. We have also investigated the consequence of the restrictions introduced along the downward price trend, when the constrained investors are the destabilizing trend-followers. Similarly to the dynamics observed during 2008 market crash, such constraints would slow the downward price movement, but would not stop it.

To study the effect of short-sell constraints, we have deliberately chosen a model with heterogeneous expectations, capable to generate the patterns of bubbles and crashes. Some features of this model may however, influence our findings. Consequently, in the future research we would like to extend the model in different directions. First, the myopic agents of the model do not take the short-sell constraints into account while forming their expectations. While we believe that such assumption is quite reasonable in the framework of boundedly rational agents, it would be also interesting to analyze the model with some fraction of rational agents, who take the short-sell constraints into account. Second, the constraints which we analyzed were individual, while on the real markets there are many aggregate constraints. For example, a total amount of shares available for the short sales is, in reality, limited. The effect of such types of constraint can be analyzed within the agent-based version of the model outlined in Appendix A. Finally, the effect of short selling constraints is closely related to the role
of margin requirements. Indeed, in a real market selling a share short requires providing some collateral to the broker. If the price of an asset rises, the investor who is short should cover his nominal losses to the extent which depends on the margin requirement. It is not surprising that among the most important questions discussed in the literature on marginal requirement is their role in market volatility and preventing bubbles. Two opposing points of views can be found in the literature on margin requirements. On the one hand, Seguin and Jarrell (1993) and Hsieh and Miller (1990) argue that margin requirements are empirically irrelevant for price behavior, whereas, e.g., Garbade (1982) and Hardouvelis and Theodossiou (2002) provide theoretical arguments why an increase in margin requirements is beneficial for market stability. Again, with the agent-based extension of the model presented here we plan to analyze the joint effect of short-sell constraints and market requirements on the dynamics.

References


APPENDIX

A Agent-Based Version of the Model

Denote the trader with sub-index \( n \in \{1, \ldots, N\} \) and consider the evolution of the trader’s position. Assume that agent \( n \) after the trading session at time \( t \) has \( A_{t,n} \) shares of the risky asset and \( B_{t,n} \) of the numeraire (i.e., the riskless assets). The wealth of investor at the end of the period is then given by

\[
W_{t,n} = B_{t,n} + p_t A_{t,n}.
\]

In the beginning of the next period agent receives a dividend \( y_{t+1} \) for every share of the risky asset and the interest on the riskless possession. We assume that agent does not consume and receives all the payments in terms of numeraire. In addition, the agent trades at period \( t+1 \) and his position in the risky asset after the trade is given by \( A_{t+1,n} \) shares. The trade balance is also received in terms of numeraire. Simple accounting implies that

\[
B_{t+1,n} = (1 + r_f) B_{t,n} + y_{t+1} A_{t,n} + p_{t+1} A_{t,n} - p_{t+1} A_{t+1,n},
\]

where the penultimate term in the RHS shows revenue from selling the previous possessions on market prices, while the last term gives the spending on the new shares. With simple algebra we derive the standard equation for wealth evolution

\[
W_{t+1,n} = B_{t+1,n} + p_{t+1} A_{t+1,n} = \left(1 + r_f\right) B_{t,n} + y_{t+1} A_{t,n} + p_{t+1} A_{t,n} - p_{t+1} A_{t+1,n} = \left(1 + r_f\right) W_{t,n} + A_{t,n}(y_{t+1} + p_{t+1} - p_{t+1} A_{t+1,n} - (1 + r_f)p_t).
\]

In the absence of short-selling and liquidity constraints, the mean-variance optimization problem gives us the demand function

\[
A_{t,n}(p) = \frac{E_{t,n}[p_{t+1}] + \bar{y} - (1 + r_f)p}{a \sigma^2}.
\]

Notice that this is the same function as in (2.1), written for agent \( n \) of type \( h \). The demand is a decreasing linear function of price, which represents an amount of shares an agent wishes to have at the end of period \( t \).

Depending on the market clearing mechanism the trader’s demand can be satisfied or the trader can be rationed. For example, in the body of this paper we consider the evolution under Walrasian market clearing, when price \( p_t \) is such that every agents’ demand is satisfied. Similarly, under Market Maker Scenario a special agent, market-maker, quotes the price \( p_t \) at the beginning of the day and every trader’s position is given by his realized demand at this price. Thus in both cases

\[
A_{t,n} = A_{t,n}(p_t) = \frac{E_{t,n}[p_{t+1}] + \bar{y} - (1 + r_f)p_t}{a \sigma^2}.
\]

Instead, rationing can occur in some circumstances, as, e.g., under the order-based market clearing mechanism as studied in Anufriev and Panchenko (2009). In the Walrasian market the rationing can also occur if the short selling constraint should hold not individually, but in aggregate.

The choice of forecasting type by agent \( n \) is based upon the commonly observed deterministic component reflecting the past performances of the rules and stochastic error component reflecting the measurement error or imperfect computations of agents. The choice is modeled as follows. At
the end of trading round $t$, first, an individual realized excess profit is computed as a product of holdings of the risky asset at the end of round $t - 1$ and its excess return, that is

$$A_{t-1,n} \left( p_t + y_t - (1 + r_f) p_{t-1} \right).$$

(A.2)

The position of agent $A_{t-1,n}$ reflects the choice of the forecasting rule made before the $t - 1$-trade. Once individual profits have been computed, the performances of any rule, $U_{t,h}$, are defined as the averages of the individual realized excess profits over all agents who used a given rule. From (A.2) it is clear that performance of the rule is the average holdings of the followers of this rule times the excess return. Thus, if the risky asset has earned a positive (negative) return, then the performance of the group with larger average possessions of the asset is higher (smaller).

Finally, agent $n$ chooses the predictor for which the following maximum is realized

$$\max_h \left( U_{t,h} - C_h + \xi_{t,n,h} \right),$$

where $C_h$ is the cost of rule $h$, and $\xi_{t,n,h}$ are random variables independent over agents and rules. The choice can be rewritten in terms of probabilities for the special case of a Gumbel distribution of error terms. In this case, individual $n$ chooses the predictors with discrete choice probabilities

$$\pi_{h,t+1} = \frac{\exp \left( \beta (U_{h,t} - C_h) \right)}{\sum_{h'}^{H} \exp \left( \beta (U_{h',t} - C_{h'}) \right)},$$

(A.3)

with a subscript indicating that these probabilities shape the population of trades at period $t + 1$. Parameter $\beta \geq 0$ is the intensity of choice, which is inversely related to the variance of the noise term $\xi_{t,h,t}$.

**Large Market Limit**

We presented above our model in an agent-based way, so that one has to keep track on the position of every agent to analyze the price dynamics. However, the dynamics can be considerably simplified under certain conditions, and the “Large Market Limit” model in the main text of this paper represents such simplification. To derive the “Large Market Limit” recall that under Walrasian market-clearing the demand of every agent is satisfied, so that the individual positions $A_{t,n}$ become identical for all the agents with the same forecasting rule. The dimensionality of dynamics can be then reduced by considering the level of forecasting rules instead of the agents’ level. This forecasting rule level was referred as “type” in the main text.

In the Walrasian framework the equilibrium price $p_t$ is the solution of the market clearing equation:

$$\sum_{n=1}^{N} A_{t,n}(p_t) = S.$$

(A.4)

If we divide both parts of this equality on the number of agents, $N$, we obtain the same equilibrium equation in terms of fractions $n_{t,h} = N_{t,h}/N$ of investors using the rule $h$

$$\sum_{h=1}^{H} n_{t,h} A_{t,h}(p) = \bar{S}.$$

11Our specification of the error terms is common in the literature on random utility models; see Anderson, de Palma, and Thisse (1992). Implied probabilities are used to model a choice in a number of theoretical models with a different range of applications, see, e.g., Brock (1993), Brock and Hommes (1997), Camerer and Ho (1999), and Weisbuch, Kirman, and Herreiner (2000).

12Essentially, the difference between agent-based approach as reviewed in LeBaron (2006) and HAMs as reviewed in Hommes (2006) lies in the dimensionality of dynamical systems, which are low in the latter case, even if agents’ heterogeneity is present.
This is the same equation as in (2.2). Now suppose that \( N \to \infty \) and \( S \to \infty \) in such a way that the supply per investor, \( \bar{S} = S/N \), is constant. Then equation (2.2) gives the price dynamics. Furthermore, as \( N \) becomes large, the probabilistic choice of the strategies as in (A.3) results, according to the Law of Large Numbers, in the fraction of the type \( h \) equal to the probability. In other words, \( n_{h,t+1} = \pi_{h,t+1} \), which implies the evolution (2.4). Thus, the Large Market Model is obtained under Walrasian scenario, when the number of agents becomes large.

Notice, that the key assumption in simplification was that the demand of every trader is satisfied. Thus, one can also consider the Large Market Model under the market-maker scenario. In such scenario, the market-maker adjusts price on the basis of the excess demand/supply. It leads to the following (linear) adjustment rule

\[
p_{t+1} = p_t + \mu \left( \sum_{n=1}^{N} A_{t,n}(p_t) - \bar{S} \right),
\]

where positive coefficient \( \mu \) measures the speed of adjustment.\(^{13}\) Again, if \( N \to \infty \) and \( S \to \infty \) but the ratio \( \bar{S} = S/N \) is constant, then (A.5) becomes

\[
p_{t+1} = p_t + \mu \left( \sum_{h=1}^{H} n_{h,t} A_{t,h}(p_t) - \bar{S} \right).
\]

**Further Directions of Research**

In the text of Section 2 of the paper we have formulated five different objections for our research. Three of them can be successfully reached within the Large Market Limit, but the remaining two would require the agent-based approach.

The fourth objective was to study consequences of the imperfection of the market for borrowing shares. Let us illustrate it with the following aggregate restriction

\[
\sum_{n: A_{t,n} < 0} A_{t,n} \geq -S,
\]

which says that the total amount of borrowed shares cannot exceed the total supply of shares. When such restriction is violated at least some short-selling agents will fail to deliver. Therefore, imposing a restriction (A.6) can be interpreted as a ban on the naked short sellings. Of course, very strong short sell restrictions (i.e., those with a very small \( \bar{A} \)) will automatically rule out the naked short sellings in this sense, but it also could be that every agent is allowed to go short to some extent (or until infinity) but aggregate restriction as (A.6) is imposed. Independent of how the agents’ rationing is modeled, the fulfillment of the 4th goal would clearly require an agent-based simulations.

The 5th objection refers on the borrowing constraints, which are symmetric with respect to the short-sell constraints. Recall the evolution of bond given in (A.1) and assume that, similarly with restriction on the short sell of the risky asset we impose a condition

\[
B_{t,n} > -\bar{B},
\]

where \( \bar{B} > 0 \). Notice that \( B_{t,n} \) is determined by all the past history of the market, since it depends not only on an initial endowment, \( B_{0,n} \) of agent \( n \) but also on past prices, dividends, and agents’ past positions \( \{A_{\tau,n}\}_{\tau=0}^{t} \). Thus, again, the agent-based simulations are necessary to study the effect of the short-sell constraints on the riskless asset. In general, constraint (A.7) can be rewritten as

\[
B_{t-1,n}(1 + r_f) + A_{t-1,n}(p_t + d_t) - p_tA_{t,n} > -\bar{B}.
\]

Thus at time \( t \) it can be unsatisfied due to three reasons (see the corresponding terms above):

\(^{13}\)If \( A_{t-1,n} \) are the holdings of the risky asset in previous period, then the agents’ orders submitted to the market maker during the trade session at time \( t \) are given by \( q_{t,n}(p_t) = A_{t,n}(p_t) - A_{t-1,n} \). Market maker satisfy all these orders and adjust price as \( p_{t+1} = p_t + \mu \sum_{i} q_{t,n}(p_t) \), which leads to (A.5).
• short position in the riskless asset last period, which is costly since the interest should be repaid to the bond owner. This happens when $B_{t-1,n}$ were already negative.

• short position in the risky asset last period, which is costly since the dividend and current market value should be repaid to the stock owner. This happens when $A_{t-1,n}$ were already negative.

• very high demand for the risky asset this period, which requires high spending. This happens when $A_{t,n}$ is high and positive.

B Formal Analysis of the Price Determination

For a given demand function $A_{t,n}(p)$ let us introduce the following functions:

$\tilde{D}_{t,n}(p) = \max(0, A_{t,n}(p) - \bar{S}) = \begin{cases} 
0 & \text{if } A_{t,n}(p) - \bar{S} < 0 \\
A_{t,n}(p) - \bar{S} & \text{if } A_{t,n}(p) - \bar{S} > 0
\end{cases}$

and

$\tilde{S}_{t,n}(p) = \max(0, A_{t,n}(p) - \bar{S}) = \begin{cases} 
0 & \text{if } A_{t,n}(p) - \bar{S} > 0 \\
\bar{S} - A_{t,n}(p) & \text{if } A_{t,n}(p) - \bar{S} < 0
\end{cases}$

Notice that the first function is decreasing, the second function is increasing, they both take only non-negative values and that the equilibrium condition $\sum_{n=1}^{N} A_{t,n}(p) = S$ can be rewritten as

$\sum_{n=1}^{N} \tilde{D}_{t,n}(p) = \sum_{n=1}^{N} \tilde{S}_{t,n}(p)$,

or, in terms of fractions and types

$\sum_{h=1}^{H} n_{t,h} \tilde{D}_{t,h}(p) = \sum_{h=1}^{H} n_{t,h} \tilde{S}_{t,h}(p)$.

For an agent $n$ the first function, $\tilde{D}_{t,n}(p)$, can be called an “adjusted demand function”, and the second function, $\tilde{S}_{t,n}(p)$, an “adjusted supply function”. They are adjusted by $\bar{S}$ in order to take the positive supply of the risky asset into account. Similarly to the original function $A_{t,n}(p)$, these two functions express the agents’ final positions at the end of the period with respect to the reference point $\bar{S}$, and, also separate positive from negative position.

In deviations from the fundamental price the net demand function is given by

$\tilde{D}_{t,n}(x) = \max \left(0, \frac{E_{t,n}[x_{t+1}] - (1 + r_f)x}{a\sigma^2} \right)$, \hspace{1cm} (B.1)

while the net supply function is given by

$\tilde{S}_{t,n}(x) = -\min \left(0, \frac{E_{t,n}[x_{t+1}] - (1 + r_f)x}{a\sigma^2} \right)$. \hspace{1cm} (B.2)

Both the adjusted demand and adjusted supply functions have “kink” in the point where

$x_n^{NT} = \frac{E_{t,n}[x_{t+1}]}{1 + r_f}$. \hspace{1cm} (B.3)

The following results hold
Proposition B.1. The price deviation $x_t$ belongs to the interval between the smallest and the largest no-trade price deviation. That is, for a given session $t$, if $x_{t,h,NT}^N$ is the no-trade price deviation for type $h$, then

$$\min_h x_{t,h,NT}^N \leq x_t \leq \max_h x_{t,h,NT}^N.$$ 

Proof. Eq. (2.9) says that the deviation $x_t$ is a weighted average of the no-trade deviated prices of the different types, with the weights $n_{t,h} \in [0,1]$. □

Proposition B.2. The agents whose “no-trade” price deviation is above $x_t$ have positive adjusted demand, those whose “no-trade” price deviation is below $x_t$ have positive adjusted supply.

Proof. Consider the adjusted position of agent $n$

$$A_{t,n}(x_t) - \bar{S} = \frac{1 + rf}{a\sigma^2} (x_{t,n}^{NT} - x_t).$$

Thus, for all those agents whose no-trade price lies above $x_t$ the resulting position is positive, indicating positive adjusted demand. For all those agents whose no-trade price lies below $x_t$ the resulting position is negative, indicating positive adjusted supply. □

Finally, let us consider Proposition 3.2. Everything is pretty obvious. Consider the adjusted position of type $h$ and denote the other type as $h'$. Then

$$A_{h,t}(x_t) - \bar{S} = \frac{1 + rf}{a\sigma^2} (x_{h,t}^{NT} - (n_{h,t}x_{h,t}^{NT} + n_{h',t}x_{h',t}^{NT})) = \frac{1 + rf}{a\sigma^2} (1 - n_{h,t})(x_{h,t}^{NT} - x_{h',t}^{NT}).$$

This decreases when $n_{h,t}$ increases. On the other hand, a relative position of a type is given by

$$n_{h,t}(A_{h,t}(x_t) - \bar{S}) = \frac{1 + rf}{a\sigma^2} n_{h,t}(1 - n_{h,t})(x_{h,t}^{NT} - x_{h',t}^{NT}).$$

is maximized when $n_{h,t} = 1/2$. 

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