Dynamic Markets for Lemons: Performance, Liquidity, and Policy Intervention*

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Abstract

We study non-stationary dynamic decentralized markets with adverse selection in which trade is bilateral and prices are determined by bargaining. Examples include labor markets, housing markets, and markets for financial assets. We characterize equilibrium, and identify the dynamics of transaction prices, trading patterns, and the average quality in the market. When the horizon is finite, the surplus in the unique equilibrium exceeds the competitive surplus; as traders become perfectly patient the market becomes completely illiquid at all but the first and last dates, but the surplus remains above the competitive surplus. When the horizon is infinite, the surplus realized equals the static competitive surplus. We study policies aimed at improving market performance, and show that subsidies to low quality or to trades at a low price, taxes on high quality, restrictions on trading opportunities, or government purchases may raise the surplus. In contrast, interventions like the Public-Private Investment Program for Legacy Assets reduce the surplus when traders are patient.

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Notation Chart

A Market for Lemons

τ:	the good's quality, $\tau \in \{H, L\}$.
u^τ:	value to buyers of a unit of τ-quality.
c^τ:	cost to sellers of τ-quality.
q^τ:	fraction of sellers of τ-quality.
$u(q)$:	$= qu^H + (1-q)u^L$.
$\bar{q}$:	$= (c^H - u^L)/(u^H - u^L)$, i.e., $u(\bar{q}) = c^H$.
$\bar{S}$:	$= m^L(u^L - c^L)$.

A Decentralized Market for Lemons

t:	a date at which the market is open, $t \in \{1, \ldots, T\}$.
r_t^τ:	reservation price at date t of sellers of τ-quality.
λ_t^τ:	probability that a seller of τ-quality who is matched at date t trades.
m_t^τ:	stock of τ-quality sellers in the market at date t.
q_t^τ:	fraction of τ-quality sellers in the market at date t.
V_t^τ:	expected utility of a seller of τ-quality at date t.
V_t^B:	expected utility of a buyer at date t.
ρ_t^τ:	probability of a price offer of r_t^τ at date t.
δ:	traders' discount factor.
α:	probability of meeting a partner.
$\hat{q}$:	$= (c^H - c^L)/(u^H - c^L)$, i.e., $u(\hat{q}) - c^H = (1-\hat{q})(u^L - c^L)$.
$\bar{\rho}$:	$= (u^L - c^L)/(c^H - c^L)$.
$\bar{\phi}$:	$= u(\hat{q}) - c^H = (1-\hat{q})(u^L - c^L)$.
ϕ_t:	$= \alpha\delta^{T-t}\bar{\phi}$.
S^{DME}:	$= m^L(u^L - c^L) + m^H\alpha\delta^{T-1}\bar{\phi}$.
$\tilde{S}^{DME}$:	$= m^L(u^L - c^L) + m^H\alpha\bar{\phi}$.

1 Introduction

Adverse selection pervades markets for real goods (e.g., cars, housing, labor) and financial assets (e.g., insurance, stocks). Akerlof's finding that competitive markets for lemons may perform poorly thus has broad welfare implications, and calls for research on fundamental questions that remain open: How do dynamic markets for lemons perform? What is the role of frictions in alleviating adverse selection? What determines market liquidity? Is there a role for government intervention? Our analysis provides answers to these questions.

We study the performance of decentralized markets for lemons in which trade is bilateral and time consuming, and buyers and sellers bargain over prices. These features are common in markets for real goods and financial assets. We characterize the unique decentralized market equilibrium, identify the dynamics of transaction prices, trading patterns, and the market composition (i.e., the fractions of units of the different qualities in the market), and study its asymptotic properties as traders become perfectly patient. Using our characterization of market equilibrium, we identify policy interventions that are welfare improving.

We consider markets in which sellers are privately informed about the quality of the good they hold, which may be high or low, and buyers are homogeneous and value each quality more highly than sellers. Since we are interested in understanding dynamic trading when the lemons problem is severe, we assume that the expected value to buyers of a random unit is below the cost of a high quality unit.[1] The market operates over a number of consecutive dates. All buyers and sellers are present at the market open, and there is no further entry. At each date a fraction of the buyers and sellers remaining in the market are randomly paired. In every pair, the buyer makes a take-it-or-leave-it price offer. If the seller accepts, then the agents trade at that price and exit the market. If the seller rejects the offer, then the agents split and both remain in the market at the next date. There are trading frictions since meeting a partner is time-consuming and traders discount future gains.

[1] Under this assumption, in the unique static competitive equilibrium is inefficient as only low quality trades, and the entire surplus is captured by low quality sellers. We take the payoffs and surplus at this static competitive equilibrium as the competitive benchmark. (We study dynamic competitive equilibrium in Appendix B.)

In this market, equilibrium dynamics are non-stationary and involve a delicate balance: At each date, buyers' price offers must be optimal given the sellers' reservation prices, the market composition, and the buyers' payoff to remaining in the market. While the market composition is determined by past price offers, the sellers' reservation prices are determined by future price offers. Thus, a market equilibrium cannot be computed recursively.

We begin by studying the equilibria of decentralized markets that open over a finite horizon. Perishable goods such as fresh fruit or event tickets, as well as financial assets such as (put or call) options or thirty-year bonds are noteworthy examples. We show that if frictions are not large, then equilibrium is unique, and we calculate it explicitly. The key features of equilibrium dynamics are as follows: at the first date, both a *low* price (accepted only by low quality sellers) and *negligible* prices (rejected by both types of sellers) are offered; at the last date, both a *high* price (accepted by both types of sellers) and a low price are offered; and at all the intervening dates, all three types of prices – high, low and negligible – are offered. Interestingly, as the traders' discount factor approaches one, there is trade only at the first and last two dates, and the market is completely illiquid at all intervening dates.

In contrast to the competitive equilibrium, low quality trades with delay and high quality trades. The surplus realized in the decentralized market equilibrium exceeds the surplus realized in the competitive equilibrium: as we show, the gain realized from trading high quality units more than offsets the loss resulting from trading low quality units with delay. The surplus realized increases as frictions decrease, and thus decentralized markets yield more than the competitive surplus (and traders' payoffs are not competitive) even in the limit as frictions vanish.

In markets that open over an infinite horizon, there are multiple equilibria. We focus on the unique equilibrium that is obtained as the market horizon approaches infinity. In this equilibrium the trading dynamics are simple: at the first date buyers make low and negligible price offers (hence only some low quality sellers trade), and at every date thereafter buyers make only high and negligible price offers in proportions that do not change over time. In contrast to prior results in the literature, each trader obtains his competitive payoff and the competitive surplus is realized even when frictions are significant. Moreover, all units trade eventually, and therefore the surplus lost due to trading low quality with delay exactly equals the surplus realized from trading high quality.

Our characterization of decentralized market equilibrium yields insights into the effectiveness of policies designed to increase market efficiency and market liquidity. We take the liquidity of a good to be the ease with which it is sold, i.e., the equilibrium probability it trades. In markets that open over a finite horizon, the liquidity of high quality decreases as traders become more patient and, somewhat counter-intuitively, as the probability of meeting a partner increases. Indeed, as the discount factor approaches one, trade freezes at all but the first and the last two dates. In markets that open over an infinite horizon, the liquidity of each quality decreases as traders become more patient, and is unaffected by the probability of meeting a partner.

We examine the impact on the market equilibrium of a variety of policies. Taxes and subsidies conditional on the quality of the good may alleviate or aggravate the adverse selection problem. When the horizon is finite, providing a subsidy to buyers or sellers of low quality raises the (net) surplus, although a subsidy to buyers has a greater impact. In contrast, a subsidy to either buyers or sellers of high quality tends to reduce the net surplus – it does so unambiguously when traders are sufficiently patient. Regarding liquidity, a subsidy to buyers or sellers of low quality increases the liquidity of high quality, whereas a subsidy to buyers of high quality has the opposite effect. Remarkably, when the horizon is infinite, a tax on high quality raises revenue without affecting either payoffs or surplus, and hence increases the net surplus.

We also study subsidies conditional on the price at which the good trades. We show that a subsidy conditional on trading at a low price increases the traders' payoffs as well as the net surplus. When the horizon is infinite the subsidy increases the liquidity of both qualities after the first date, as well as the net surplus. A subsidy conditional on trading at the high price increases (decreases) the payoffs of buyers (low quality sellers). Interestingly, the liquidity of high quality decreases. When the horizon is infinite, a subsidy is purely wasteful, whereas a tax raises revenue without affecting payoffs, thus raising the net surplus.

In our setting, a Public-Private Investment Program (PPIP) such as the one implemented for legacy assets is effectively a subsidy to buyers who purchase a low quality unit at the high price. We show that a PPIP has effects analogous to subsidizing buyers of high quality: it increases the payoff of buyers and the surplus, decreases the payoff of low quality sellers and the liquidity of high quality, and as δ approaches one reduces the net surplus.

We study the effect of closing the market for some period of time. Such policies have been studied in the literature – e.g., Fuchs and Skrzypacz (2013) study it in a dynamic competitive setting. Our characterization of the market equilibrium shows that reducing the horizon over which the market opens (so long as the market remains open for at least two dates) increases payoffs and surplus. We show that if the horizon is not too long relative to the traders' discount factor, then closing the market at all dates except the first and the last has no effect on payoffs and surplus. If the horizon is long, however, by closing the market for some period of time separating market equilibria emerge in which the surplus is larger than when the market is open at all dates.

Finally, we show that government purchases increase the payoff of low quality sellers and decrease the payoff of buyers; surplus increases provided the government values low quality nearly as highly as buyers, but decreases otherwise.

Related Literature

The recent financial crisis has stimulated interest in understanding the effects of adverse selection in decentralized markets. Moreno and Wooders (2010) studies markets with stationary entry and shows that payoffs are competitive as frictions vanish. In their setting, and in the present paper, traders only observe their own personal histories. Kim (2011) studies a continuous time version of the model of Moreno and Wooders (2010), and shows that if frictions are small and buyers observe the amount of time that sellers have been in the market, then market efficiency improves, whereas if buyers observe the number of prior offers sellers have rejected, then efficiency is reduced. Thus, Kim (2011)'s results reveal that increased transparency is not necessarily efficiency enhancing, and call for caution when regulating information disclosure. Bilancini and Boncinelli (2011) study a market for lemons with finitely many buyers and sellers, and show that if the number of sellers in the market is public information, then in equilibrium all units trade in finite time.

For markets with one-time entry, the focus of the present paper, Blouin (2003) studies a market open over an infinite horizon in which only one of three exogenously given prices may emerge from bargaining. Blouin (2003) shows that equilibrium payoffs are not competitive even as frictions vanish. Although we address a broader set of questions, on this issue we find that payoffs are competitive even when frictions are non-negligible.

Camargo and Lester (2011) studies a model in which agents' discount factors are randomly drawn at each date from a distribution whose support is bounded away from one, and buyers may make only one of two exogenously given price offers. It shows that in every equilibrium both qualities trade in finite time. Moreover, liquidity, i.e., the fraction of buyers offering the high price, increases with the fraction of high quality sellers initially in the market. In contrast, in our model the unique equilibrium exhibits neither of these features: a positive measure of high quality remains in the market at all times, and marginal changes in the fraction of high quality only affects the liquidity of low quality at date 1. Camargo and Lester (2011) also provides a numerical example demonstrating that a PPIP subsidy for has an ambiguous impact on liquidity as measured by the minimum time at which the market clears (taken over the set of all equilibria). We show that in our setting this policy decreases the liquidity of high quality, and we are able to evaluate its welfare effects.

In contrast to Blouin (2003) and Camargo and Lester (2011) our model imposes no restriction on admissible price offers. Moreover, equilibrium is unique and is characterized in closed-form, which allows for a direct comparative static analysis of the effect of changes in the parameters values on payoffs, social surplus, and liquidity.

The first paper to consider a matching model with adverse selection is Williamson and Wright (1994), who show that money can increase welfare. Inderst and Muller (2002) show that the lemons problem may be mitigated if sellers can sort themselves into different submarkets. Inderst (2005) studies a model where agents bargain over contracts, and shows that separating contracts always emerge in equilibrium. Cho and Matsui (2011) study long term relationships in markets with adverse selection and show that unemployment and vacancy do not vanish even as search frictions vanish. In their model, agents respond strategically to price proposals that are drawn from a uniform distribution. Lauermann and Wolinsky (2011) explore the role of trading rules in a search model with adverse selection, and show that information is aggregated more effectively in auctions than under sequential search by an informed buyer.

Our work also relates to a literature that examines the mini-micro foundations of competitive equilibrium. This literature has established that decentralized trade of homogeneous goods tends to yield competitive outcomes when trading frictions vanish. See, for example, Gale (1987, 1996) or Binmore and Herrero (1988) when bargaining is under complete information, and Moreno and Wooders (2002) and Serrano (2002) when bargaining is under incomplete information.

There is also a growing literature studying dynamic competitive (centralized) markets with adverse selection. Janssen and Roy (2002) study a market that operates in discrete time and in which there is a continuum of qualities, and show that competitive equilibria may involve intermediate dates without trade before the market clears in finite time. Fuchs and Skrzypacz (2013) study a market that operates in continuous time, and show that interrupting trade always raises surplus, while infrequent trade generates more surplus under some conditions. Philippon and Skreta (2012) and Tirole (2012) examine optimal government interventions in asset markets. In Appendix B we study the properties of dynamic competitive equilibria in our setting, compare the performance of centralized and decentralized markets, and discuss the differential effects of policy interventions.

2 A Decentralized Market for Lemons

Consider a market for an indivisible commodity whose quality can be either high (H) or low (L). There is a positive measure of buyers and sellers. The measure of sellers with a unit of quality $\tau \in \{H, L\}$ is $m^\tau > 0$. For simplicity, we assume that the measure of buyers (m^B) is equal to the measure of sellers, i.e., $m^B = m^H + m^L$.[2] Each buyer wants to purchase a single unit of the good. Each seller owns a single unit of the good. A seller knows the quality of his good, but quality is unobservable to buyers prior to purchase.

Preferences are characterized by values and costs: the value to a buyer of a unit

[2] This assumption is standard in the literature, e.g., it is made in all the related papers discussed in the Introduction. It simplifies the analysis (without it the matching probability is endogenous and varies over time), but involves some loss of generality.

of high (low) quality is u^H (u^L); the cost to a seller of a unit of high (low) quality is c^H (c^L). Thus, if a buyer and a seller trade at price p, the buyer obtains a utility of $u - p$ and the seller obtains a utility of $p - c$, where $u = u^H$ and $c = c^H$ if the unit traded is of high quality, and $u = u^L$ and $c = c^L$ if it is of low quality. A buyer or seller who does not trade obtains a utility of zero.

We assume that both buyers and sellers value high quality more than low quality (i.e., $u^H > u^L$ and $c^H > c^L$), and that buyers value each quality more highly than sellers (i.e., $u^H > c^H$ and $u^L > c^L$). Also we restrict attention to markets in which the lemons problem is severe; that is, we assume that the fraction of sellers of τ-quality in the market, denoted by

$$q^\tau := \frac{m^\tau}{m^H + m^L},$$

is such that the expected value to a buyer of a randomly selected unit of the good, given by

$$u(q^H) := q^H u^H + (1 - q^H) u^L,$$

is below the cost of high quality, c^H. Equivalently, we may state this assumption as

$$q^H < \bar{q} := \frac{c^H - u^L}{u^H - u^L}.$$

Note that $q^H < \bar{q}$ implies $c^H > u^L$.

Therefore, we assume throughout that $u^H > c^H > u^L > c^L$ and $q^H < \bar{q}$. Under these parameter restrictions only low quality trades in the unique static competitive equilibrium, even though there are gains to trade for both qualities. For future reference, we describe this equilibrium in Remark 1 below.

Remark 1. *The market has a unique static competitive equilibrium. In equilibrium all low quality units trade at the price u^L, and no high quality unit trades. Thus, the surplus, given by*

$$\bar{S} = m^L(u^L - c^L), \tag{1}$$

is captured by low quality sellers.

In our model of decentralized trade, the market is open for T consecutive dates. All traders are present at the market open, and there is no further entry. Traders discount utility at a common rate $\delta \in (0, 1]$, i.e., if at date t a unit of quality τ trades at price p, then the buyer obtains a utility of $\delta^{t-1}(u^\tau - p)$ and the seller obtains a utility of $\delta^{t-1}(p - c^\tau)$. At each date every buyer (seller) in the market meets a randomly selected seller (buyer) with probability $\alpha \in (0, 1]$. In each pair, the buyer offers a price at which to trade. If the offer is accepted by the seller, then the agents trade and both leave the market. If the offer is rejected by the seller, then the agents split and both remain in the market at the next date. A trader who is unmatched at the current date also remains in the market at the next date. An agent observes only the outcomes of his own matches.

In this market, the behavior of buyers at each date t may be described by a *c.d.f.* λ_t with support on $\mathbb{R}_+$ specifying a probability distribution over price offers. Likewise, the behavior of sellers of each quality is described by a probability distribution with support on $\mathbb{R}_+$ specifying their reservation prices. Given a sequence $\lambda = (\lambda_1, \ldots, \lambda_T)$ describing buyers' price offers, the maximum expected utility of a seller of quality $\tau \in \{H, L\}$ at date $t \in \{1, ..., T\}$ is defined recursively as

$$V_t^\tau = \max_{x \in \mathbb{R}_+} \left\{ \alpha \int_x^\infty (p - c^\tau)\, d\lambda_t(p) + \left(1 - \alpha \int_x^\infty d\lambda_t(p)\right) \delta V_{t+1}^\tau \right\},$$

where $V_{T+1}^\tau = 0$. In this expression, the payoff to a seller of quality τ who receives a price offer p is $p - c^\tau$ if p is at least his reservation price x, and it is δV_{t+1}^τ, his continuation utility, otherwise. Since all sellers of quality τ have the same maximum expected utility, then their equilibrium reservation prices are identical. Therefore we restrict attention to strategy distributions in which all sellers of quality $\tau \in \{H, L\}$ use the same sequence of reservation prices $r^\tau = (r_1^\tau, ..., r_T^\tau) \in \mathbb{R}_+^T$.

Let (λ, r^H, r^L) be a *strategy distribution.* For $t \in \{1, \ldots, T\}$, the probability that a matched seller of quality $\tau \in \{H, L\}$ trades, denoted by λ_t^τ, is

$$\lambda_t^\tau = \int_{r_t^\tau}^\infty d\lambda_t(p). \tag{2}$$

The stock of sellers of quality τ in the market at date $t + 1$, denoted by m_{t+1}^τ, is

$$m_{t+1}^\tau = (1 - \alpha\lambda_t^\tau)\, m_t^\tau,$$

where $m_1^\tau = m^\tau$. The fraction of sellers of high quality in the market at date t, denoted by q_t^H, is

$$q_t^H = \frac{m_t^H}{m_t^H + m_t^L}$$

if $m_t^H + m_t^L > 0$, and $q_t^H \in [0,1]$ is arbitrary otherwise.[3] The fraction of sellers of low quality in the market at date t, denoted by q_t^L, is

$$q_t^L = 1 - q_t^H.$$

The maximum expected utility of a buyer at date $t \in \{1, ..., T\}$ is defined recursively as

$$V_t^B = \max_{x \in \mathbb{R}_+} \left\{ \alpha \sum_{\tau \in \{H,L\}} q_t^\tau I(x, r_t^\tau)(u^\tau - x) + \left(1 - \alpha \sum_{\tau \in \{H,L\}} q_t^\tau I(x, r_t^\tau) \right) \delta V_{t+1}^B \right\},$$

where $V_{T+1}^B = 0$. Here $I(x,y)$ is the indicator function whose value is 1 if $x \geq y$, and 0 otherwise. In this expression, the payoff to a buyer who offers the price x is $u^\tau - x$ when matched to a τ-quality seller who accepts the offer (i.e., when $I(x, r_t^\tau) = 1$), and it is δV_{t+1}^B, her continuation utility, otherwise.

Definition. A strategy distribution (λ, r^H, r^L) is a *decentralized market equilibrium (DME)* if for each $t \in \{1, \ldots, T\}$:

$$r_t^\tau - c^\tau = \delta V_{t+1}^\tau \qquad (DME.\tau)$$

for $\tau \in \{H, L\}$, and for almost all p in the support of λ_t

$$\alpha \sum_{\tau \in \{H,L\}} q_t^\tau I(p, r_t^\tau)(u^\tau - p) + \left(1 - \alpha \sum_{\tau \in \{H,L\}} q_t^\tau I(p, r_t^\tau) \right) \delta V_{t+1}^B = V_t^B. \qquad (DME.B)$$

Condition $DME.\tau$ ensures that each type τ seller is indifferent between accepting or rejecting an offer of his reservation price. Condition $DME.B$ ensures that price offers that are made with positive probability are optimal.

The *surplus* realized in a decentralized market equilibrium can be calculated as

$$S^{DME} = m^B V_1^B + m^H V_1^H + m^L V_1^L. \qquad (3)$$

[3] Evaluating payoffs requires specifying a value for q_t^H for all t. Lemma 2, part 1, implies that $m_t^H > 0$ for all t, and thus how q_t^H is specified when $m_t^H + m_t^L = 0$ does not affect equilibrium.

3 Decentralized Market Equilibrium

Proposition 1 establishes basic properties of decentralized market equilibria.

Proposition 1. *Assume that $T < \infty$ and $\delta < 1$, and let $t \in \{1, ..., T\}$. In a DME:*

(P1.1) $r_t^H = c^H > r_t^L$, $V_t^H = 0$, and $q_{t+1}^H \geq q_t^H$.

(P1.2) Only the high price $p_t = r_t^H$, or the low price $p_t = r_t^L$, or negligible prices $p_t < r_t^L$ may be offered with positive probability.

The intuition for these results is straightforward. Since the payoff of a seller who does not trade at date T is zero, sellers' reservation prices at date T are equal to their costs, i.e., $r_T^\tau = c^\tau$. Thus, price offers above c^H are suboptimal at date T, and are made with probability zero. Therefore the expected utility of high quality sellers at date T is zero, i.e., $V_T^H = 0$, and hence $r_{T-1}^H = c^H$. Also, since $\delta < 1$, i.e., delay is costly, low quality sellers accept price offers below c^H, i.e., $r_{T-1}^L < c^H$. A simple induction argument shows that $r_t^H = c^H > r_t^L$ for all t.

Obviously, prices above r_t^H, which are accepted by both types of sellers, or prices in the interval (r_t^L, r_t^H), which are accepted only by low quality sellers, are suboptimal, and are therefore made with probability zero. Moreover, since $r_t^H > r_t^L$ then the proportion of high quality sellers in the market (weakly) increases over time (i.e., $q_{t+1}^H \geq q_t^H$) as low quality sellers accept offers of both r_t^H and r_t^L, and therefore exit the market at least as fast as high quality sellers, who only accept offers of r_t^H.

In equilibrium, at each date a buyer may offer a *high price* $p = r_t^H$, which is accepted by both types of sellers, or a *low price* $p = r_t^L$, which is accepted by low quality sellers and rejected by high quality sellers, or a *negligible price* $p < r_t^L$, which is rejected by both types of sellers. For $\tau \in \{H, L\}$ denote by ρ_t^τ the probability of a price offer equal to r_t^τ. Since prices greater than r_t^H are offered with probability zero, then the probability of a high price offer is $\rho_t^H = \lambda_t^H$. (Recall that λ_t^τ is the probability that a matched τ-quality seller trades at date t – see equation (2).) And since prices in the interval (r_t^L, r_t^H) are offered with probability zero, then the probability of a low price offer is $\rho_t^L = \lambda_t^L - \lambda_t^H$. Thus, the probability of a negligible price offer is $1 - (\rho_t^H + \rho_t^L) = 1 - \lambda_t^L$.

Proposition 1 thus allows a simpler description of a DME. Henceforth we describe a DME by a collection (ρ^H, ρ^L, r^L), where $\rho^\tau = (\rho_1^\tau, \ldots, \rho_T^\tau)$ for $\tau \in \{H, L\}$, and thus ignore the distribution of negligible price offers, which is inconsequential. Also we omit the reservation price of high quality sellers which is $r_t^H = c^H$ for all t by $P1.1$.

Proposition 2 establishes additional properties of DME.

Proposition 2. *Assume that $T < \infty$ and $\delta < 1$. Then in a DME:*

(P2.1) At every date $t \in \{1, \ldots, T\}$ either high or low prices are offered with positive probability, i.e., $\rho_t^H + \rho_t^L > 0$.

(P2.2) At date 1 high prices are offered with probability zero, i.e., $\rho_1^H = 0$.

(P2.3) At date T negligible prices are offered with probability zero, i.e., $1 - \rho_T^H - \rho_T^L = 0$.

The intuition for $P2.2$ is clear: Since at date 1 the expected utility of a random unit is less than c^H by assumption, then high price offers are suboptimal, i.e., $\rho_1^H = 0$. The intuition for $P2.3$ is also simple: At date T the sellers' reservation prices are equal to their costs. Thus, buyers obtain a positive payoff by offering either the low price $r_T^L = c^L$ (when $q_T^H < 1$), or the high price $r_T^H = c^H$ (when $q_T^H = 1$). Since a buyer who does not trade obtains zero, then negligible price offers are suboptimal, i.e., $\rho_T^H + \rho_T^L = 1$. The intuition for $P2.1$ is as follows: Suppose to the contrary that all buyers make negligible offers at date t, i.e., $\rho_t^H = \rho_t^L = 0$. Let t' be the first date following t where a buyer makes a non-negligible price offer. Since there is no trade between t and t', then the distribution of qualities is the same at t and t', i.e., $q_t^H = q_{t'}^H$. Thus, an impatient buyer is better off by offering at date t the price she offers at t', which implies that negligible prices are suboptimal at t; i.e., $\rho_t^H + \rho_t^L = 1$. Hence $\rho_t^H > 0$ and/or $\rho_t^L > 0$.

In a market that opens for a single date, i.e., $T = 1$, the sellers' reservation prices are their costs. The fraction of high quality sellers

$$\hat{q} := \frac{c^H - c^L}{u^H - c^L},$$

makes a buyer indifferent between an offer of c^H and an offer of c^L. It is easy to see that $\bar{q} < \hat{q}$. Since $q^H < \bar{q}$ by assumption, then $q^H < \hat{q}$. Thus, if $T = 1$ only low price offers are made (i.e., $\rho_1^H = 0$ and $\rho_1^L = 1$) and only low quality trades, as implied by $P2.1$ and $P2.2$. Remark 2 states these results.

Remark 2. *Assume that $T = 1$ and $\delta < 1$. Then the unique DME is $(\rho_1^H, \rho_1^L, r_1^L) = (0, 1, c^L)$. In equilibrium some low quality units trade at the price c^L, and no high* *quality unit trades. Thus, the surplus realized, which is $\alpha m^L(u^L - c^L)$, is captured by buyers.*

Proposition 3 below establishes that when frictions are not large a decentralized market that opens over a finite horizon $T > 1$ has a unique DME. We say that *frictions are not large* when α and δ are sufficiently near one that the following inequalities hold:

$$\frac{\bar{\rho}}{\alpha\delta} < \min\left\{\frac{c^H - u^L}{(1+\alpha\delta)(1-\delta)(c^H - c^L)}, 1\right\}, \tag{F.1}$$

and

$$\frac{(1-\bar{\rho}/\delta)q^H}{(1-\bar{\rho}/\delta)q^H + (1-\alpha)(1-q^H)} > \hat{q}, \tag{F.2}$$

where

$$\bar{\rho} := \frac{u^L - c^L}{c^H - c^L}.$$

Inequality $F.1$ requires α and δ be sufficiently close to one that a low quality seller prefers to wait one period and trade with probability α at the price c^H rather than trading immediately at the price u^L. The left hand side of $F.1$, $\bar{\rho}/\alpha\delta$, is an upper bound of the probability that a high price is offered at any date as we show in Lemma 2, part 6, in the Online Appendix. It is easy to see that $F.1$ holds for α and δ near one.

Inequality $F.2$ requires that if all matched low quality sellers trade and at most a fraction $\bar{\rho}/\alpha\delta$ of matched high quality sellers trade, then the fraction of high quality sellers in the market at the next date is above $\hat{q}$. In Lemma 2, part 2, in the Online Appendix we show that this inequality implies that the low price is never offered with probability one. Obviously, this inequality holds for α near one.

Write

$$\bar{\phi} := (1-\hat{q})(u^L - c^L),$$

and for $t \in \{1, \ldots, T\}$ let

$$\phi_t = \alpha\delta^{T-t}\bar{\phi}.$$

Clearly ϕ_t is increasing in α and δ, and approaches $\alpha\bar{\phi}$ as δ approaches one; and ϕ_t is decreasing in T, and approaches zero as T approaches infinity.

Proposition 3 establishes that when frictions are not large a market that opens over a finite horizon has a unique DME, and provides a complete characterization of this equilibrium.

Proposition 3. *Assume that $1 < T < \infty$, $\delta < 1$, and inequalities F.1 and F.2 hold (i.e., frictions are not large). Then the unique DME is given by:*

(P3.1) High Price Offers: $\rho_1^H = 0$,

$$\rho_t^H = \frac{1-\delta}{\alpha\delta}\frac{u^L - c^L}{c^H - u^L + \phi_t}$$

for all $1 < t < T$, and

$$\rho_T^H = \frac{u^L - c^L - \alpha\delta\bar{\phi}}{\alpha\delta(c^H - c^L)}.$$

(P3.2) Low Price Offers:

$$\rho_1^L = \frac{\phi_2 + c^H - u(q^H)}{\alpha(1-q^H)(c^H - u^L + \phi_2)},$$

and $\rho_T^L = 1 - \rho_T^H$. If $T > 2$, then

$$\rho_t^L = (1 - \alpha\rho_t^H)\frac{(1-\delta)\phi_{t+1}}{\alpha\left(c^H - u^L + \phi_{t+1}\right)}\frac{u^H - u^L}{u^H - c^H - \phi_t}$$

for all $1 < t < T-1$, and

$$\rho_{T-1}^L = (1 - \alpha\rho_{T-1}^H)\frac{(1-\alpha\delta)(u(\hat{q}) - c^H)}{\alpha\hat{q}(u^H - c^H - \phi_{T-1})}.$$

(P3.3) Reservation Prices: $r_t^L = u^L - \phi_t$ for all $t < T$, and $r_T^L = c^L$.

In equilibrium, the payoff to a buyer is $V_1^B = \phi_1$, and the payoffs to sellers are $V_1^H = 0$ and $V_1^L = u^L - c^L - \phi_1$. Thus, the payoff to a buyer (low quality seller) is above (below) his competitive payoff, decreases (increases) with T and increases (decreases) with α and δ. Moreover, the surplus, given by

$$S^{DME} = m^L(u^L - c^L) + m^H\alpha\delta^{T-1}\bar{\phi},$$

is above the competitive surplus $\bar{S}$, decreases with T, and increases with α and δ.

It is easy to describe the equilibrium trading patterns: at the first date only low and negligible prices are offered, and thus some low quality sellers trade, but no high quality seller trades (i.e., $\rho_1^H = 0 < \rho_1^L < 1$). At intermediate dates, high, low and negligible prices are offered (i.e., $\rho_t^H, \rho_t^L > 0$ and $1 - \rho_t^H - \rho_t^L > 0$), and thus some sellers of both types trade. At the last date only high and low prices are offered (i.e., $\rho_T^H + \rho_T^L = 1$), and thus all matched low quality sellers and some high quality sellers trade.

Thus, both qualities trade with delay. Nevertheless, the surplus generated in the DME is greater than the competitive equilibrium surplus, $\bar{S}$: the gain from trading high quality units more than offsets the loss from trading low quality units with delay. In contrast, in a market for a homogenous good the competitive equilibrium surplus is an upper bound to the surplus that can be realized in a DME – e.g., Moreno and Wooders (2002) show that this bound is achieved as frictions vanish.

Price dispersion is a key feature of equilibrium: At every date but the first there is trade at more than one price since both high and low prices are offered with positive probability. To see that price dispersion is essential, suppose instead that the high price c^H is offered with probability one at some date t. Since α and δ are near one, this implies that the reservation price of low quality sellers prior to t is near c^H, and hence above the value of low quality u^L (recall that $u^L < c^H$). Thus, prior to t a low price offer (which if accepted buys a unit of low quality) is suboptimal, and therefore low price offers are made with probability zero. Therefore sellers of both qualities leave the market at the same rate, and hence the fraction of high quality sellers remains constant, i.e., $q_t^H = q^H$. Since $q^H < \bar{q}$, a high price offer is suboptimal at t, which is a contradiction. Hence high price offers are made with probability less than one at every date.

Likewise, suppose that the low price is offered with probability one at some date t. Then at date t all matched low quality sellers trade, and no high quality seller trades. Since α is near one, this implies that the fraction of high quality sellers in the market at date $t+1$ is near one, and since this sequence in non-decreasing over time, the fraction of high quality sellers at the last date is above $\hat{q}$. (Recall that $\hat{q}$ is the fraction of high quality sellers that makes buyers indifferent between offering the high and the low price at date T.) This implies that offering c^H is uniquely optimal and hence the high price is offered with probability one at date T, which is a contradiction. Thus, low price offers are made with probability less than one at every date.

A more involved argument establishes that all three types of price offers (high, low, and negligible) are made with positive probability at every date except the first and last – see the proof of Lemma 2, part 7, in the Online Appendix.

Identifying the probabilities (ρ_t^H, ρ_t^L) is delicate: Their past values determine the current market composition, q_t^H, and their future values determine the reservation price of low quality sellers at date t. In equilibrium, at intermediate dates the market composition and the sellers' reservation prices must make buyers indifferent between offering high, low or negligible prices, i.e., the equation

$$u(q_t^H) - c^H = (1 - q_t^H)(u^L - r_t^L) + q_t^H \delta V_{t+1}^B = \delta V_{t+1}^B$$

holds. We show in the proof of Proposition 3 in Appendix A that the system formed by these equations (together with the analogous equations for dates 1 and T, and the boundary conditions) admits a single solution. Establishing uniqueness of equilibrium requires showing that these properties are common to all market equilibria – see Lemma 2 in the Online Appendix.

The comparative static properties of equilibrium relative to α, δ and T are intuitive: Since negligible price offers are optimal at every date except the last, the payoff to buyers is just their discounted payoff at the last date. Consequently, the payoff to a buyer increases with α and δ, and decreases with T. Low quality sellers capture surplus whenever high price offers are made, i.e., at every date except the first. The probability of a high price offer decreases with both α and δ, and increases with T, and thus the payoff to low quality sellers decreases with α and δ, and increases with T. The surplus increases with α and δ.

Somewhat counter-intuitively, the surplus decreases with T, i.e., shortening the horizon over which the market opens is advantageous (so long as $T > 2$): Our assumption that frictions are small implies that in equilibrium buyers must be willing to offer negligible prices at every date but the last date. Hence their payoff is just their discounted expected utility at the last date.[4] Thus, a longer horizon provides no advantage in screening sellers, and reduces the buyers' payoff. The payoff to low quality sellers increases with T because the high price is offerred with higher probability at every date (except at the last date, at which it is offerred with a probability independent of T). Further, since buyers' must remain willing to offer the low price, the increase in the payoff of low quality sellers exactly matches the decrease in the payoff of buyers. Therefore the surplus decreases with T since there are more buyers than low quality sellers, and is maximal when $T = 2$.

[4] In contrast, if traders are sufficiently impatient, then there is an equilibrium in which buyers offer r_1^L at date 1, and then offer c^H at every subsequent date. In this equilibrium, lengthening the horizon increases surplus when $\alpha < 1$.

A striking feature of equilibrium in decentralized markets is that the surplus realized exceeds the competitive equilibrium surplus: decentralized markets are more efficient than centralized ones. While in a centralized market all units trade at a single market-clearing price, in a decentralized market several prices are offered with positive probability, and different units trade at different prices. When $\alpha = 1$, for example, low quality units trade for sure – some at the high price and some at the low price – while high quality units trade with probability less than one. Thus decentralized trade generates an allocation closer to the surplus maximizing allocation, in which low quality sellers trade for sure, and high quality sellers trade with positive probability (less than one).[5]

Proposition 4 identifies the limiting DME as traders become perfectly patient. A remarkable feature of the limiting equilibrium is that the market *freezes* at intermediate dates, and both qualities are completely illiquid: Low quality trades at the first and last two dates, and high quality trades only at the last date. The surplus is independent of the duration of the market.

Proposition 4. *Assume that $1 < T < \infty$, $\delta < 1$, and inequalities F.1 and F.2 hold (i.e., frictions are not large). Then as δ approaches one the unique DME approaches $(\tilde{\rho}^H, \tilde{\rho}^L, \tilde{r}^L)$ given by:*

(P4.1) High Price Offers: $\tilde{\rho}_t^H = 0$ for all $t < T$, and

$$\tilde{\rho}_T^H = \frac{u^L - c^L - \alpha\bar{\phi}}{\alpha(c^H - c^L)}.$$

(P4.2) Low Price Offers:

$$\tilde{\rho}_1^L = \frac{\alpha\bar{\phi} + c^H - u(q^H)}{\alpha(1 - q^H)(c^H - u^L + \alpha\bar{\phi})},$$

and $\tilde{\rho}_T^L = 1 - \tilde{\rho}_T^H$. If $T > 2$, then $\tilde{\rho}_t^L = 0$ for all $1 < t < T - 1$ and

$$\tilde{\rho}_{T-1}^L = \frac{(1-\alpha)(u(\hat{q}) - c^H)}{\alpha\hat{q}(u^H - c^H - \alpha\bar{\phi})}.$$

[5] The (static) surplus maximizing menu contract is $\{(p^H, Z^H), (p^L, Z^L)\}$, where $p^H = c^H$, $Z^H = (1 - q^H)(u^L - c^L)/[c^H - c^L - q^H(u^H - c^L)]$, $p^L = c^L + Z^H(c^H - c^L)$ and $Z^L = 1$. Here p^τ is the money transfer from seller to buyer and Z^τ is the probability that the seller transfers the unit of good to the buyer, when the seller reports type τ. Even if $\alpha = 1$, in the DME high quality sellers trade with probability less than Z^H.

(P4.3) Reservation Prices: $\tilde{r}_t^L = u^L - \alpha\bar{\phi}$ *for all* $t < T$*, and* $\tilde{r}_T^L = c^L$*.*

Moreover, $(\tilde{\rho}^H, \tilde{\rho}^L, \tilde{r}^H, \tilde{r}^L)$ *is a DME of the market when* $\delta = 1$*. In equilibrium, the payoff to a buyer is* $\tilde{V}_1^B = \alpha\bar{\phi}$*, and the payoffs to sellers are* $\tilde{V}_1^H = 0$ *and* $\tilde{V}_1^L = [1 - \alpha(1 - \hat{q})](u^L - c^L)$*. Thus, the payoff to a buyer (low quality seller) remains above (below) his competitive payoff. The surplus, given by*

$$\tilde{S}^{DME} = m^L(u^L - c^L) + m^H \alpha\bar{\phi},$$

is independent of T *and remains above the competitive surplus.*

When $\delta = 1$, time can no longer be used as a screening device (until the very last period), and the market freezes at all dates but the last two. The DME identified in Proposition 4 is not the unique market equilibrium. For example, there are DME in which buyers mix over low and negligible prices at dates prior to T in such a way that the total measure of low quality sellers that trades prior to T is the same as in the DME identified in Proposition 4; then buyers offer high and low prices at date T with probabilities $\tilde{\rho}_T^H$ and $\tilde{\rho}_T^L$, respectively.

We illustrate our findings in propositions 3 and 4 with an example.

Example 1

Consider a market in which $u^H = 1$, $c^H = .6$, $u^L = .4$, $c^L = .2$, $m^H = .2$, $m^L = .8$, and $T = 10$. The graphs in the top row of Figure 1 show the evolution of the stocks of high quality sellers m_t^H in the market, and the fraction of high price offers ρ_t^H for several different combinations of α and δ. The graphs in the middle row show the evolution of m_t^L and ρ_t^L. The bottom graph shows the evolution of the fraction of high quality sellers in the market q_t^H. These graphs illustrate several features of equilibrium as frictions become small: high quality trades more slowly; low quality trades more quickly at the first date and at the last date, but trades more slowly at intermediate dates; the fraction q_t^H increases more quickly, but equals $\hat{q} = .5$ at the market close regardless of the level of frictions.

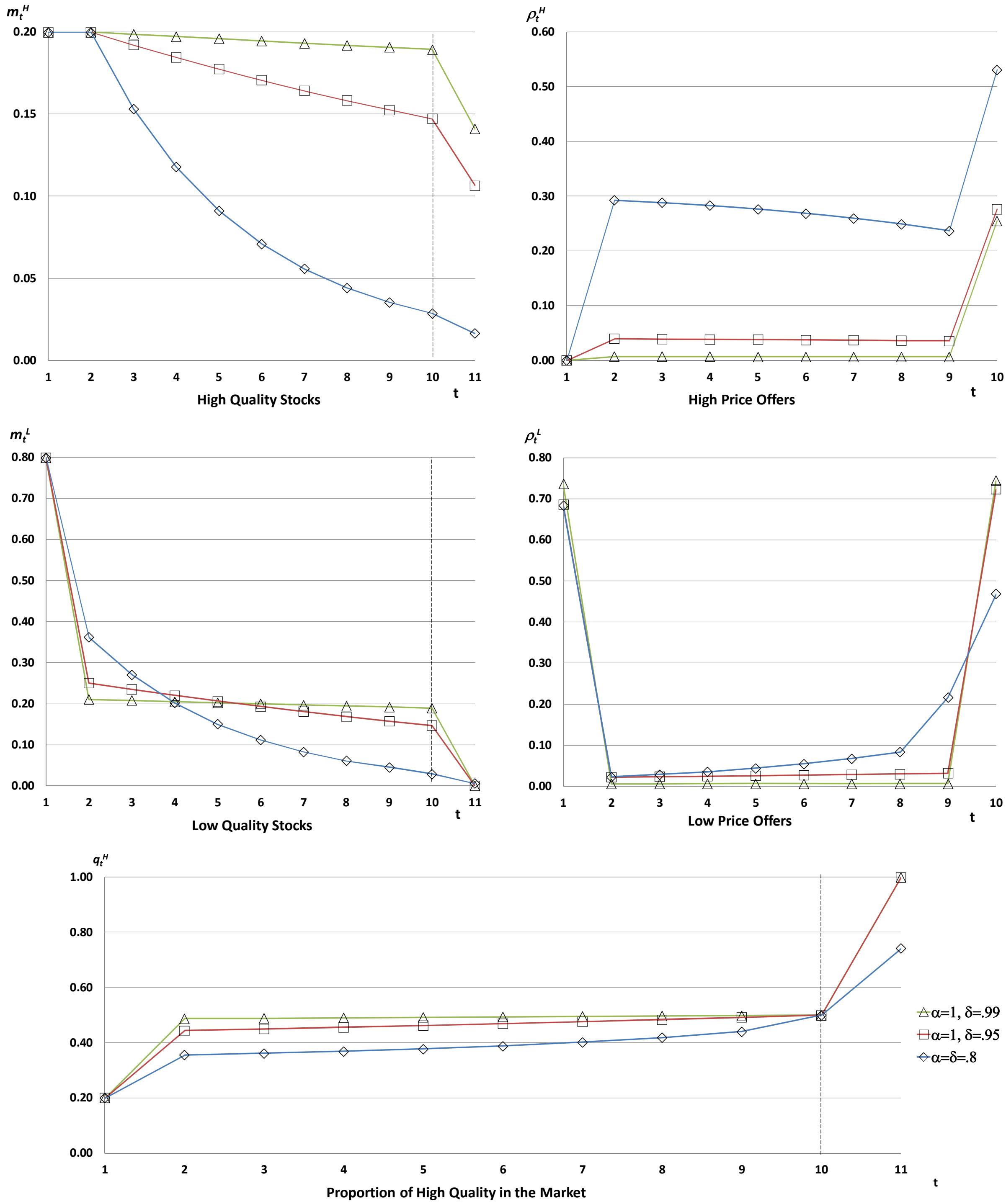

Figure 1: Equilibrium Dynamics in a Decentralized Market

Decentralized Market Equilibria when the Horizon is Infinite

We now consider decentralized markets that open over an infinite horizon. In these markets, given a strategy distribution one calculates the maximum expected utility of each type of trader at each date by solving a dynamic optimization problem. The definition of DME remains otherwise the same.

Proposition 5 identifies the limiting DME as T approaches infinity, and establishes it this limit is a DME of the market that opens over an infinite horizon. In relating the formulae in propositions 3 and 5, it is useful to observe that ϕ_t approaches zero as T approaches infinity.

Proposition 5. *Assume that $\delta < 1$, and inequalities F.1 and F.2 hold (i.e., frictions are not large). Then as T approaches infinity the unique DME approaches $(\hat{\rho}^H, \hat{\rho}^L, \hat{r}^L)$ given by:*

(P5.1) High Price Offers: $\hat{\rho}_1^H = 0$, and for all $t > 1$,

$$\hat{\rho}_t^H = \frac{1-\delta}{\alpha\delta}\frac{u^L - c^L}{c^H - u^L}.$$

(P5.2) Low Price Offers:

$$\hat{\rho}_1^L = \frac{\bar{q} - q^H}{\alpha\bar{q}\left(1 - q^H\right)} \text{ and } \hat{\rho}_t^L = 0 \text{ for all } t > 1.$$

(P5.3) Reservation Prices: $\hat{r}_t^L = u^L$ for all t.

Moreover, if $T = \infty$ then $(\hat{\rho}^H, \hat{\rho}^L, \hat{r}^H, \hat{r}^L)$ is a DME. In equilibrium, the traders' payoffs are the competitive payoffs, i.e., $\hat{V}_1^B = 0$, $\hat{V}_1^H = 0$ and $\hat{V}_1^L = u^L - c^L$, and the surplus is the competitive surplus $\bar{S}$.

As the horizon becomes infinite, all units trade eventually. At the first date, some low quality units trade but no high quality units trade. At subsequent dates, units of both qualities trade with the same constant probability. In the limit, the traders' payoffs are competitive independently of α and δ, and hence so is the surplus, even if frictions are non-negligible. Kim (2011) obtains an analogous result in a stationary setting. In contrast, the previous literature has established that payoffs are competitive only as frictions vanish, e.g., Gale (1987), Binmore and Herrero (1988), and Moreno and Wooders (2002) for homogenous goods markets, and Moreno and Wooders (2010) for markets with adverse selection.

The intuition for these results is simple: in the DME of a market that opens over a finite horizon, the payoff to a buyer at the last date is $V_T^B = \alpha\bar{\phi} > 0$, independently of the horizon T. Since negligible prices are optimal at every date except the last, the payoff to a buyer is his discounted payoff at the last date, $\alpha\delta^{T-1}\bar{\phi}$, which approaches zero as the horizon approaches infinity. Thus, in a market that opens over an infinite horizon the payoff to a buyer is zero. Hence low price offers, if made with positive probability, must yield a payoff equal to zero, which implies that $r_t^L = u^L > c^L$. Then high prices must be offered with positive probability at some dates. At these dates the proportion of high quality must be $\bar{q}$ in order for the expected payoff to a buyer offering the high price to be zero. In a stationary equilibrium, the equation $r_t^L = u^L$ pins down the rate at which high price offers are made, and $q_2^H = \bar{q}$ pins down the proportion of low price offers at date 1. Since the payoffs of buyers is zero, the proportion of high quality sellers in the market can not rise above $\bar{q}$, and thus low price offers are made with probability zero after date 1.

When $T = \infty$ there are multiple equilibria. Uniqueness of equilibrium when the horizon is finite justifies focusing on the limiting DME identified in Proposition 5.[6]

4 Policy Intervention

Our results allow an assessment of the impact of policies aimed at improving market efficiency, such as subsidies, taxes, programs like the Public-Private Investment Program for Legacy Assets, or other interventions like closing the market for some period of time.

Taxes and Subsidies Conditional on Quality

Suppose that the government provides a per unit subsidy of $\sigma_B^L > 0$ to buyers of low quality. Then the instantaneous payoff to a buyer who purchases a unit of

[6]When $T = \infty$ there is a continuum of DME that share the basic properties identified in Proposition 5: $\rho_1^H = 0$, $\rho_1^L > 0$ is such that $q_2^H = \bar{q}$, and $r_1^L = u^L \leq r_t^L$ for all $t > 1$. In these DME, payoffs are competitive: $V_1^B = 0$ implies $r_1^L = u^L$, and thus $V_1^L = \delta V_2^L = r_1^L - c^L = u^L - c^L$. In fact, we conjecture that payoffs are competitive in all DME. This conjecture is based on the idea that in all DME buyers make negligible price offers with positive probability at every date, which implies that their payoff would diverge if it was positive. (Proving this conjecture requires establishing versions of lemmas 1 and 2 for a market with $T = \infty$.)

low quality at price p is $u^L + \sigma_B^L - p$ rather than $u^L - p$. The impact of the subsidy may therefore be evaluated as an increase in the value of low quality, u^L. Likewise, if the government provides a per unit subsidy of $\sigma_S^L > 0$ to sellers of low quality, then the instantaneous payoff to a seller who sells a unit of low quality at price p is $p - (c^L - \sigma_S^L)$ rather than $p - c^L$, and therefore the impact of the subsidy may be evaluated as a decrease in the cost of low quality, c^L. Such subsidies are feasible provided that quality is verifiable following purchase. Taxes are negative subsidies.

When $T < \infty$, the effect of a subsidy on the market equilibrium may be determined using the formulae given in Proposition 3. For example, subsidizing buyers of low quality increases the *net* surplus: a marginal subsidy increases (gross) surplus by

$$\frac{\partial S^{DME}}{\partial u^L} = m^L + m^H \alpha \delta^{T-1} \frac{d\bar{\phi}}{du^L} = m^L + m^H \alpha \delta^{T-1} (1 - \hat{q}),$$

whereas the present value of the subsidy is at most m^L since at most m^L units receive the subsidy. Subsidizing sellers of low quality increases the net surplus as well since

$$\frac{\partial S^{DME}}{\partial c^L} = -m^L + m^H \alpha \delta^{T-1} \frac{d\bar{\phi}}{dc^L} = -m^L - m^H \alpha \delta^{T-1} (1 - \hat{q}) \frac{u^H - u^L}{u^H - c^L} < -m^L.$$

Comparing these two expressions reveals that subsidizing buyers has a larger effect on surplus, i.e., $\partial S^{DME}/\partial u^L > |\partial S^{DME}/\partial c^L|$, since $(u^H - u^L)/(u^H - c^L) < 1$. Corollary 1 below summarizes the effect of subsidies to low quality on payoffs and surplus. Its proof, which follows from differentiating the formulae given in Proposition 3, is omitted.

Corollary 1. *Under the assumptions of Proposition 3, a subsidy to either buyers or sellers of low quality increases the payoffs of buyers and low quality sellers, as well as the net surplus. However, subsidizing buyers has a larger effect on the payoff of buyers and on the surplus S^{DME}, and a smaller effect on the payoff of low quality sellers, than subsidizing sellers.*

The intuition for the result that subsidies to low quality raise surplus is as follows: A subsidy, whether to buyers or sellers, raises the payoff to buyers at the last date, V_T^B, and therefore raises their payoff at every date, V_t^B. Consider a subsidy to buyers. Since buyers must remain indifferent between low and negligible price offers prior to date T, i.e.,

$$(1 - q_t^H)(u^L - r_t^L) + q_t^H \delta V_{t+1}^B = \delta V_{t+1}^B,$$

(equivalently, $u^L - r^L_t = \delta V^B_{t+1}$), then the reservation price of low quality sellers must increase.[7] Hence the payoff to low quality sellers must increase, which requires that high price offers be made more frequently at every date (except the first). Thus, a greater measure of high quality trades and a greater surplus is realized. A subsidy to buyers yields a greater increase in the payoff to buyers at the last date than does an equal-sized subsidy to sellers, and therefore a subsidy to buyers leads to a greater increase in surplus.

Next we describe the impact of subsidies to buyers and sellers of high quality. When $T < \infty$, the effect of such subsidies on payoffs and on the surplus may be assessed using the formulae of Proposition 3 as changes in the value or cost of high quality. Their impact on the net surplus is unclear in general as it is difficult to calculate the present value of the subsidy, but as δ approaches one the effect is clear from Proposition 4: A subsidy to buyers of high quality affects surplus through its impact on $\hat{q}$:

$$\frac{\partial \tilde{S}^{DME}}{\partial u^H} = -m^H \alpha (u^L - c^L) \frac{\partial \hat{q}}{\partial u^H} = m^H \alpha (u^L - c^L) \frac{c^H - c^L}{(u^H - c^L)^2}.$$

Since high quality trades only at the last date, the marginal cost of the subsidy approaches $m^H \alpha \tilde{\rho}^H_T$. Thus the marginal effect on the net surplus approaches

$$\begin{aligned} \frac{\partial \tilde{S}^{DME}}{\partial u^H} - m^H \alpha \tilde{\rho}^H_T &= m^H \alpha (u^L - c^L) \frac{c^H - c^L}{(u^H - c^L)^2} - m^H \frac{u^L - c^L - \alpha \bar{\phi}}{(c^H - c^L)} \\ &\leq m^H \frac{u^L - c^L}{u^H - c^L} \left(\frac{c^H - c^L}{u^H - c^L} - 1 \right) \\ &< 0, \end{aligned}$$

where the weak inequality holds since $\alpha \leq 1$. A subsidy to sellers of high quality also reduces the net surplus since

$$\frac{\partial \tilde{S}^{DME}}{\partial c^H} = -m^H \alpha (u^L - c^L) \frac{\partial \hat{q}}{\partial c^H} = -m^H \alpha \frac{u^L - c^L}{u^H - c^L},$$

and therefore

$$\left| \frac{\partial \tilde{S}^{DME}}{\partial c^H} \right| - m^H \alpha \tilde{\rho}^H_T = -(1 - \alpha) m^H \frac{u^L - c^L}{u^H - c^L} \frac{u^H - c^L}{c^H - c^L} \leq 0.$$

[7] A subsidy of σ^L_B increases δV^B_{t+1} by less than σ^L_B, whereas u^L increases by σ^L_B. Hence r^L_t must increase in order to preserve the equality.

We state these results in Corollary 2.

Corollary 2. *Under the assumptions of Proposition 3, subsiding either buyers or sellers of high quality has the same qualitative effects: the payoff of buyers and the surplus increase, and the payoff of low quality sellers decreases. However, subsidizing sellers has a larger effect on payoffs and surplus. As δ approaches one, either subsidy reduces the net surplus.*

Table 1 below illustrates the effect of several policies for the market described in Example 1 when $\alpha = \delta = .95$. The second row describes the effect of a subsidy to buyers of low quality, $\sigma_B^L = .05$. Relative to the equilibrium without any subsidy or tax (first row), the volume of high quality sellers that trades increases 11.05 percentage points, and the net surplus increases 4% from .1720 to .1790. The third row shows the effect of the subsidy on sellers of low quality. The differential effects of these two subsidies are consistent with Corollary 1.

Policy	Vol. Trade %		Payoffs		Surplus		Policy
($\sigma = .05$)	H	L	$m^B V_1^B$	$m^L V_1^L$	S^{DME}	Net	Cost
None	47.90	99.09	.0599	.1121	.1720	.1720	.0000
Sub. Buyer $\tau = L$	58.95	99.20	.0748	.1401	.2150	.1790	.0360
Sub. Seller $\tau = L$	54.30	99.24	.0704	.1436	.2140	.1777	.0363
Sub. Buyer $\tau = H$	46.60	98.96	.0634	.1093	.1727	.1693	.0034
Sub. Seller $\tau = H$	51.20	98.88	.0673	.1061	.1735	.1697	.0038
Sub. Buyer $\tau \in \{H, L\}$	57.40	99.09	.0792	.1366	.2159	.1761	.0398
Tax Buyer $\tau = H$	49.40	99.20	.0559	.1153	.1712	.1748	$-.0036$
PPIP	46.45	98.95	.0639	.1089	.1728	.1681	.0047
Sub. Low Price	60.50	99.30	.0704	.1796	.2141	.1821	.0320
Sub. High Price	45.17	98.81	.0673	.1061	.1735	.1702	.0033

Table 1: Policy Effects

The fourth and fifth rows of Table 1 describe the effects of subsidies to buyers and sellers of high quality, respectively. Both subsidies decrease the payoff of low quality sellers and increase the payoff of buyers and the (gross) surplus. Consistent with Corollary 2, these effects are stronger for the subsidy to sellers than the subsidy to buyers. In the example, the negative effect on net surplus of the subsidy to sellers is smaller.

The sixth row of Table 1 reports the effects of an unconditional subsidy to buyers. (If quality is not verifiable after purchase, then a subsidy conditional on the quality of the good is not feasible.) The unconditional subsidy has a smaller positive effect on the net surplus than a subsidy on buyers of low quality alone. The seventh row of Table 1 shows the effect of a tax on buyers of high quality. Its effects are opposite of a subsidy. In particular, it increases the measures of trade of both qualities and the net surplus.

Next we address the effects of taxes and subsidies in a market that opens over an infinite horizon. In such markets the effects of subsidies on either quality are easily assessed by differentiating the formulae provided in Proposition 5. Inspecting these formulae leads to an interesting first observation: in these markets *subsidizing either buyers or sellers of τ-quality has identical effects on payoffs and surplus.* Corollary 3 describes the effects of subsidizing low quality.

Corollary 3. *Assume that $T = \infty$ and the assumptions of Proposition 5 hold. Subsidizing low quality has no effect on the payoff of buyers, and increases the payoff of low quality sellers and the net surplus. As δ approaches one, the subsidy has no effect on the net surplus and amounts to a transfer to low quality sellers.*

While a subsidy σ^L to low quality raises the surplus by $\sigma^L m^L$, the present value of the subsidy is less than $\sigma^L m^L$, and therefore the net surplus increases. Establishing that as δ approaches one a subsidy σ^L to low quality amounts to a transfer to low quality sellers requires showing that the present value of the subsidy approaches $\sigma^L m^L$ – see the proof of Corollary 3 in Appendix A.

Interestingly, a tax on high quality raises revenue without affecting either payoffs or surplus, thereby increasing net surplus. A tax on buyers of high quality, for example, increases $\hat{\rho}_1^L$ while leaving $\hat{\rho}_t^L$ and $\hat{\rho}_t^H$ unchanged for $t > 1$, thus accelerating trade. We state this result in Corollary 4.

Corollary 4. *Assume that $T = \infty$ and the assumptions of Proposition 5 hold. A tax on high quality raises revenue without affecting payoffs or surplus, thereby increasing* *the net surplus.*

Market Liquidity

Liquid assets are those that can be easily bought or sold. In our setting, we define the liquidity of a good to be the equilibrium probability that it trades. In equilibrium, at each date t high quality trades with probability $\alpha\rho_t^H$, and low quality trades with probability $\alpha(\rho_t^H + \rho_t^L)$. Since high quality is always illiquid at date 1, we focus on its liquidity at dates $t > 1$.

Corollary 5 describes the effects on liquidity of subsidies, taxes, and market frictions in a market that opens over a finite horizon. These results, which are provided without proof, are obtained by differentiating the formulae in Proposition 3. Perhaps counter-intuitively, high quality is less liquid as the probability of meeting a partner α increases or as traders become more patient. Indeed, both qualities become completely illiquid at intermediate dates as δ approaches one – see Proposition 4.

Corollary 5. *Under the assumptions of Proposition 3 the liquidity of high quality decreases monotonically as frictions vanish. It also increases if either buyers of low quality or sellers of either quality are subsidized, and decreases if buyers of high quality are subsidized.*

The intuition for the effects of subsidies to low quality were discussed in connection to Corollary 1. A subsidy to buyers of high quality raises the payoffs of buyers at the last date, and therefore raises their payoff at every date, V_t^B. Since buyers must remain indifferent between low and rejected price offers prior to date T, i.e., $u^L - r_t^L = \delta V_{t+1}^B$, and since u^L is unaffected by the subsidy, then the reservation price (and payoff) of low quality sellers must decrease. Hence high price offers are made less frequently, i.e., the liquidity of high quality decreases.

The effects of subsidies to high quality sellers are more subtle: A subsidy σ_S^H raises V_t^B at every date, and since it does not affect u^L, then r_t^L (and V_{t+1}^L) must decrease. At the same time, the subsidy reduces the high price offer, which becomes $c^H - \sigma_S^H$, and therefore directly reduces the payoff of low quality sellers. The effect on the frequency of high price offers is thus ambiguous, and must be determined by signing a derivative. It turns out this derivative is positive, i.e., the (now smaller) high price offer is made *more* frequently with the subsidy, and therefore the liquidity of high quality increases.

Corollary 6 establishes results on liquidity for a market that opens over an infinite horizon. These results follow from differentiating the formulae in Proposition 5. These formulae reveal that the liquidity of low quality at date 1 is independent of the discount factor, decreases with a subsidy on high quality, decreases with a subsidy to buyers of low quality, and is unaffected by subsidies to low quality sellers. Since low prices are offered only at the first date (i.e., $\hat{\rho}_t^L = 0$ for all $t > 1$), the liquidity of both qualities for $t > 1$ is $\alpha\hat{\rho}_t^H$. Note that $\alpha\hat{\rho}_t^H$ is independent of α, i.e., the liquidities of both goods are unaffected by changes in the probability of meeting a partner α.

Corollary 6. *Assume that $T = \infty$ and the assumptions of Proposition 5 hold. The liquidities of both qualities at dates $t > 1$ approach zero monotonically as the traders' become perfectly patient, increase with a subsidy on low quality, decrease with a subsidy to high quality sellers, and are unaffected by subsidies to buyers of high quality.*

The Public-Private Investment Program for Legacy Assets

This program was designed to draw new private capital into the market for troubled real estate-related assets, comprised of legacy loans and securities, by providing equity co-investment and public financing. Its main objective was to reduce the perceived excessive liquidity discounts in legacy asset prices. The program provided private investors with non-recourse loans to purchase legacy assets. Investors had to provide only a small amount of equity (a fraction $\gamma = 1/14$ of the purchase price). Thus, under this program an investor who purchased a low quality asset may choose to default on the loan and surrender the asset, losing only her equity (i.e., the fraction γ of the price paid for the asset).

This policy may be framed in our setting as a subsidy to buyers who pay the high price c^H for a low quality unit: under this program a buyer who purchases at the high price, upon observing the quality of the unit acquired faces the choice to keep the unit and pay the loan, which is optimal if it is of high quality since

$$u^H > (1 - \gamma)\, c^H,$$

or default and surrender the unit, which is optimal when it is of low quality provided

$$u^L < (1-\gamma)\, c^H.$$

Assuming that γ is sufficiently small that this inequality holds, the payoff to a buyer offering the high price c^H when the fraction of high quality in the market is q, denoted by P^H, is

$$P^H = q\left(u^H - c^H\right) + (1-q)\left(-\gamma c^H\right).$$

This payoff may be written as

$$\begin{aligned} P^H(q,s) &= qu^H + (1-q)\, u^L - c^H + (1-q)s \\ &= u(q) - c^H + (1-q)s, \end{aligned}$$

where $s = (1-\gamma)c^H - u^L$ is effectively a subsidy to buyers who purchase a low quality unit at the high price c^H.

Of course, the lemons problem can be solved altogether by setting a subsidy sufficiently large. Evaluating the impact of a small subsidy is somewhat more complex than a comparative statics exercise. However, the introduction of a small PPIP subsidy does not change the basic properties of equilibrium, and the formulae provided in propositions 3 to 5 describing the DME can be readily modified to show the impact of this policy.

Reviewing the proof of Proposition 3 reveals how the introduction of a subsidy s affects the DME: The formulae describing the sequences of probabilities of high price offers and reservation prices of low quality sellers, as well as the traders' payoffs and surplus, are not affected directly by the subsidy, but only indirectly via its impact on the fraction $\hat{q}(s)$ of high quality sellers in the market at the last date. Of course, $\hat{q}(s)$ affects in turn the entire sequence q_t^H, and the functions $\phi_t(s) = \alpha\delta^{T-t}\bar{\phi}(s)$, where $\bar{\phi}(s) = (1-\hat{q}(s))\left(u^L - c^L\right)$. However, the subsidy appears explicitly in the formulae describing the sequence of probabilities of low price offers – we provide these formulae in the Online Appendix.

Intuitively, the impact of this policy is as follows: In equilibrium, at date T buyers are indifferent between offering the high or the low price, and therefore the fraction of high quality sellers q_T^H must be such that

$$P^H(q_T^H, s) = (1-q_T^H)(u^L - c^L).$$

The solution to this equation, $q_T^H = \hat{q}(s)$, is decreasing in s. Hence introducing a PPIP subsidy s decreases the fraction of high quality in the market, and increases the buyers' expected utility, at the last date.

It is easy to see the effects of a PPIP subsidy in a market that opens only for two dates: the measure of low quality sellers that trades at date 1 decreases. Moreover, since buyers are indifferent between trading at date 1 or at date 2, and their expected utility is greater with the subsidy, then the reservation price of low quality sellers at date 1 decreases with the subsidy, which in turn implies that the probability of a high price offer at date 2 decreases, and the measure of high quality sellers who trade decreases as well. Thus, this policy reduces the net surplus and makes both qualities less liquid.

The analysis of the impact of a PPIP subsidy for a market that opens for more than two dates is more complex. However, its qualitative effects, as well as the intuition for how it affects the DME, are analogous to a subsidy to buyers of high quality (see corollaries 2 and 5). We summarize our conclusions in Corollary 7.

Corollary 7. *Under the assumptions of Proposition 3, a PPIP subsidy increases the payoff of buyers, and decreases the payoff of low quality sellers and the liquidity of high quality. Moreover, it reduces the net surplus as δ approaches one.*

For the market in Example 1, the row in Table 1 labeled PPIP shows the impact of this program: Its effects are qualitatively the same as a subsidy to buyers of high quality – see the fourth row – but the PPIP program leads to a larger reduction of the net surplus due to its larger cost.

In a market that opens over an infinite horizon, the only impact of a PPIP program is to decrease the probability of low price offers at the first date. Since surplus is unaffected, it is purely wasteful: it causes an increase of the cost of delay in trading low quality that exactly offsets the subsidy.

Camargo and Lester (2014) study the impact of the PPIP program on liquidity as measured by the minimum time (taken over the set of all equilibria) required for the market to clear. They present numerical examples showing that in their model this policy has an ambiguous effect on liquidity: it may increase it when the lemons problem is very severe (i.e., q^H is very small), but may decrease it when it is not so severe.

Taxes and Subsidies Conditional on Price

Subsidies conditional on the quality of the good are feasible only if quality is verifiable following purchase. Hence it is useful to study the effects of taxes and subsidies conditional on the price at which the good trades. The effect of a small subsidy may also be assessed by modifying the formulae provided in propositions 3 to 5. It is interesting to observe that unlike subsidies conditional on the quality of the good, the effects of a subsidy conditional on the price at which the good trades are the same whether it is given to buyers or sellers.

Subsidies conditional on trading either at the high price c^H or at a low price (i.e., a price below c^H), affect the fraction of high quality sellers in the market at the last date, which becomes a function of the subsidy, as well as the functions $\bar{\phi}$ and ϕ_t involved in the formulae describing the DME. The formulae describing the probabilities of high and low price offers must be modified appropriately – see the Online Appendix. These formulae reveal the effects of these subsidies on traders' payoffs, surplus, and market liquidity.

With a subsidy conditional on trading at a low price, for example, at the last date the payoff to offering the low price increases, and therefore the fraction of high quality in the market needed to preserve the indifference between high and low price offers increases. This has an impact on the probabilities of offering high, low and negligible prices, as well as on the reservation prices of low quality sellers, at every date. Corollary 8 describes the impact of such a subsidy. The intuition for these results is analogous to that of a subsidy to low quality – see Corollary 1.

Corollary 8. *Under the assumptions of Proposition 3, a subsidy conditional on trading at a low price increases the payoffs of buyers and low quality sellers, as well as the net surplus. The liquidity of low (high) quality at date* 1 *(T) increases. When $T = \infty$ the subsidy increases the liquidity of both qualities after the first date, as well as the net surplus.*

The next to the last row of Table 1 describes the effects of a subsidy conditional on trading at a low price for the market described in Example 1. This policy is the most effective: relative to the DME without intervention (first row), the volume of trade of high quality increases 12.6 percentage points, and the net surplus increases by 5.9% from .1720 to .1821. Low quality sellers are the main beneficiaries as their payoff increases by 60%, while the payoff of buyers increases by 17.5%.

The effects of a subsidy conditional on trading at the high price on payoffs, surplus and liquidity are summarized in Corollary 9. This subsidy has effects analogous to those of subsidies to buyers or sellers of high quality – see Table 1, rows 4, 5 and 10.

Corollary 9. *Under the assumptions of Proposition 3, a subsidy conditional on trading at the high price increases (decreases) the payoffs of buyers (low quality sellers). The liquidity of high quality decreases. When $T = \infty$ the subsidy is purely wasteful, whereas a tax raises revenue without affecting payoffs, thereby increasing the net surplus.*

Restricting trading opportunities

Our results allow assessing other policies studied in the literature such as closing the market for some period of time – see, e.g., Fuchs and Skrzypacz (2013). Consider a market which opens for $T > 2$ dates. Since surplus is decreasing in T by Proposition 3, closing the market altogether after date 2 increases the buyers' payoff and the surplus, and decreases the payoff of low quality sellers.

Suppose instead that the market is open only at dates 1 and T, i.e., the market is closed at all intermediate dates. If δ is sufficiently large and/or T is sufficiently small that inequalities $F.1$ and $F.2$ hold when δ is replaced by δ^{T-1}, then the formulae of Proposition 3, particularized for $T = 2$ and a discount factor equal to δ^{T-1}, describe the market outcome. An inspection of these formulae reveals that this intervention does not affect payoffs and surplus.

The intuition for this result is as follows: Since buyers make negligible offers at every date except the last, their payoff is their discounted utility at the last date, i.e., $\alpha\delta^{T-1}(1-\hat{q})(u^L - c^L)$, and is the same whether the market is open at intermediate dates or not. Furthermore, since in both markets buyers obtain the same payoff and are indifferent between low and negligible price offers at date 1, this implies that low quality sellers have the same reservation price at date 1, and thus the same payoff in both markets.

Closing the market may increase surplus if the time horizon is long. For simplicity, assume that $\alpha = 1$. Let $T \geq \hat{t}$, where $\hat{t}$ is sufficiently large that

$$u^L - c^L \geq \delta^{\hat{t}-1}(u^H - c^L). \tag{4}$$

(Hence $F.1$ fails if δ is replaced by δ^{T-1}.) If the market opens at date 1, closes at dates $t \in \{2, \ldots, \hat{t} - 1\}$, and re-opens at dates $t \in \{\hat{t}, \ldots, T\}$, there is an equilibrium in which all buyers offer $r_1^L = \delta^{\hat{t}-1}(c^H - c^L) + c^L$ at date 1, and offer c^H at every date $t \in \{\hat{t}, \ldots, T\}$. It easy to verify that the surplus realized in this equilibrium,

$$m^L(u^L - c^L) + m^H \delta^{\hat{t}-1}(u^H - c^H),$$

is greater than the surplus in the DME when the market is always open, whether $T < \infty$ or $T = \infty$. Thus, for markets in which T is large or infinite, closing the market after the first date for sufficiently long that the (separating) equilibrium described above can be sustained, raises welfare: Closing the market prevents the wasteful delay that results when low quality sellers attempt to pool with high quality sellers. When the market is closed sufficiently long, low quality sellers are not willing to wait for high price offers at date $\hat{t}$.

We summarize these results in Corollary 10.

Corollary 10. *Under the assumptions of Proposition 3, if $F.1$ and $F.2$ hold when δ is replaced by δ^{T-1}, then closing the market for dates $2, \ldots, T-1$ has no effect on payoffs or surplus. If α is close to one and $u^L - c^L > \delta^{T-1}(u^H - c^L)$, then closing the market for some period of time may increase the surplus.*

Government Purchases

We discuss the impact of government purchases. Such policies have been studied in the literature – see, e.g., Philippon and Skreta (2012) and Tirole (2012). Assume that at the market open the government offers to buy β units of the good, e.g., via a uniform price auction. In equilibrium, the government acquires β units of low quality at a price equal to the reservation price of low quality sellers in the market that follows, i.e., r_1^L. Our assumption that the matching probability is constant over time, and equal for buyers and sellers, is no longer appropriate since after the government intervention there are more buyers than sellers in the market. Let assume instead that the buyers' matching probability is a function of the *market tightness*, i.e., the ratio $\theta_t = (m_t^H + m_t^L)/m_t^B$. Since equal measures of buyers and sellers trade and leave the market every date, θ_t decreases over time. Hence a direct effect of this program is to decrease the buyers' matching probability at every date.

Assuming that β is not so large as to alter the structure of the DME, the buyers' payoff at the last date conditional on being matched is unaffected since buyers must remain indifferent between offering the high and the low price. However, since the buyers' matching probability is smaller, then their payoff at the last date decreases. Further, since buyers must be willing to offer high, low and negligible prices at every date but the first and last, and their expected utility at the last date is smaller, then in order to reduce the payoff of low (high) price offers in line to the decrease in the payoff of negligible price offers, the sequences of reservation prices of low quality sellers increases (the fraction of high quality sellers in the market decreases). Hence the payoff of low quality sellers increases, which implies that high price offers are made with greater probability. Thus, a positive impact of the program is to increase the volume of trade of high quality. If government purchases crowd out private trade and the government's value for low quality is less than the buyers' value, then the program also has a negative effect on surplus, and the overall effect is unclear.

We examine the impact of this policy in a market that operates over two dates, and in which at every date t buyers are matched with probability $\alpha\theta_t$. In this market, since the fraction of high quality in the market at date 2 is the same with and without the government purchases, the measure of low quality sellers who sell their good at date 1 (either to the government to private buyers) is also the same; and since all matched low quality sellers trade at date 2, the liquidity of low quality, and hence the volume of trade of low quality, are unaffected. Since buyers' payoff at date 2 is smaller, for buyers to be willing to offer negligible prices at date 1 the payoff to offering the low price must decrease, which implies that the reservation price of low quality sellers increases, and therefore that the high price is offered with a greater probability at date 2. Hence the volume of trade of high quality increases. The effect on the net surplus depends on whether the surplus gained from the increase in the volume of trade of high quality is greater or less than the surplus loss due to the smaller value of low quality to the government. We summarize these conclusions in Corollary 11. The formal analysis is presented in the Online Appendix.

Corollary 11. *If $T = 2$ and the assumptions of Proposition 3 hold, then government purchases at the market open increase the payoff of low quality sellers and the liquidity of high quality, and decrease the payoff of buyers. If the value of low quality to the government is close to the buyers' value, then the net surplus increases.*

5 Discussion

When the horizon is finite and frictions are not large, in the equilibrium of a decentralized market most low quality units as well as some high quality units trade, and the surplus is above the competitive surplus. When the horizon is infinite all units of both qualities trade, although with delay, and payoffs and surplus are competitive.

Appendix B studies the market described in Section 2 but where trade is centralized, i.e., trade is multilateral and agents are price takers. We show in Proposition 6 that if the horizon is finite and traders are patient (i.e., their discount factor is not too small), then in a *dynamic* competitive equilibrium (DCE) all low quality units trade at the first date and no high quality units ever trade. Hence the surplus realized is the same as in the *static* competitive equilibrium. We show that subsidies, which are effective in decentralized markets, are ineffective in centralized markets. Moreover, high (low) quality is more (less) liquid in decentralized markets than in centralized ones. These features hold even as frictions vanish. These results suggest that when the horizon is finite, decentralized markets perform better than centralized markets.

We also show that if traders are sufficiently impatient or the horizon is infinite, there are dynamic competitive equilibria in which all low quality units trade immediately at a low price and all high quality units trade with delay at a high price. These separating DCE, in which different qualities trade at different dates, yield a surplus greater than the static competitive surplus. Consequently, when the horizon is infinite, centralized markets may perform better than decentralized markets.

Interestingly, we show in Proposition 7 that as frictions vanish the surplus at a separating DCE of a market that opens over an infinite horizon equals the surplus in the equilibrium of a decentralized market that opens over a finite horizon. Intuitively this result holds since the same incentive constraints operate in both markets. In a separating DCE high quality trades with sufficiently long delay that low quality sellers are willing to trade immediately at a low price rather than waiting to trade at a high price. Likewise, in a DME high price offers are made with a sufficiently small probability that low quality sellers are willing to immediately accept a low price, rather than waiting for a high price.

6 Appendix A: Proofs

We begin by establishing a number of lemmas. In the proofs, we refer to previous results established in lemmas or propositions by using the letter L and P, respectively, followed by the number. The proof of lemma 1 and straightforward, and is provided in the Online Appendix.

Lemma 1. *Assume that* $1 < T < \infty$ *and* $\delta < 1$*, and let* (λ, r^H, r^L) *be a DME. Then for each* $t \in \{1, \ldots, T\}$:

$(L1.1)$ $\lambda_t(\max\{r_t^H, r_t^L\}) = 1$.

$(L1.2)$ $r_t^H = c^H > r_t^L$, $V_t^H = 0 < V_t^B$, *and* $V_t^L \leq c^H - c^L$.

$(L1.3)$ $q_{t+1}^H \geq q_t^H$.

$(L1.4)$ $\lambda_t(c^H) = 1$.

$(L1.5)$ $\lambda_t(p) = \lambda_t(r_t^L)$ *for all* $p \in [r_t^L, c^H)$.

With these results in hand we prove propositions 1 and 2.

Proof of Proposition 1. $P1.1$ follows from $L1.2$ and $L1.3$, and $P1.2$ follows from $L1.4$ and $L1.5$. □

Proof of Proposition 2. We prove $P2.3$. Suppose by way of contradiction that $\rho_T^H + \rho_T^L < 1$. Then negligible prices are optimal, and therefore $V_T^B = \delta V_{T+1}^B = 0$, which contradicts $L1.2$.

We prove $P2.1$. Suppose contrary to $P2.1$ that there is k such that $\rho_k^H + \rho_k^L = 0$. By $P2.3$, $k < T$. Let k be the largest such date. Then $\rho_{k+1}^H + \rho_{k+1}^L > 0$ and $q_{k+1}^\tau = q_k^\tau$ for $\tau \in \{H, L\}$. If $\rho_{k+1}^H > 0$, i.e., offering r_{k+1}^H is optimal, then

$$V_{k+1}^B = \alpha(q_{k+1}^H u^H + q_{k+1}^L u^L - c^H) + (1 - \alpha)\, \delta V_{k+2}^B.$$

Since $V_{k+1}^B \geq \delta V_{k+2}^B$ (because the payoff to offering a negligible price is δV_{k+2}^B), then

$$q_{k+1}^H u^H + q_{k+1}^L u^L - c^H \geq V_{k+1}^B.$$

And since $q_{k+1}^\tau = q_k^\tau$ for $\tau \in \{H, L\}$, $V_{k+1}^B > 0$ (by $L1.2$) and $\delta < 1$, then

$$q_k^H u^H + q_k^L u^L - c^H = q_{k+1}^H u^H + q_{k+1}^L u^L - c^H \geq V_{k+1}^B > \delta V_{k+1}^B.$$

Therefore a negligible price offer at k is not optimal, which contradicts that $\rho_k^H + \rho_k^L = 0$. Hence $\rho_{k+1}^H = 0$, and thus $\rho_{k+1}^L > 0$ and

$$V_{k+1}^L = \alpha\rho_{k+1}^L(r_{k+1}^L - c^L) + (1 - \alpha\rho_{k+1}^L)\delta V_{k+2}^L = \delta V_{k+2}^L.$$

Therefore

$$r_k^L = c^L + \delta V_{k+1}^L \leq c^L + V_{k+1}^L = c^L + \delta V_{k+2}^L = r_{k+1}^L.$$

Since $\rho_{k+1}^L > 0$, i.e., price offers of r_{k+1}^L are optimal at date $k+1$, we have

$$q_{k+1}^L(u^L - r_{k+1}^L) + (1 - q_{k+1}^L)\delta V_{k+2}^B \geq \delta V_{k+2}^B.$$

Hence

$$\delta V_{k+2}^B \leq u^L - r_{k+1}^L$$

and

$$V_{k+1}^B = \alpha q_{k+1}^L(u^L - r_{k+1}^L) + (1 - \alpha q_{k+1}^L)\delta V_{k+2}^B \leq u^L - r_{k+1}^L.$$

Since $\rho_k^H + \rho_k^L = 0$, then the payoff to a negligible offer at date k is greater or equal to the payoff to a low price offer at date k, i.e.,

$$\delta V_{k+1}^B \geq \alpha q_k^L(u^L - r_k^L) + (1 - \alpha q_k^L)\delta V_{k+1}^B.$$

Thus $u^L - r_k^L \leq \delta V_{k+1}^B$. Since $V_{k+1}^B > 0$ (by $L1.2$) and $\delta < 1$, then

$$u^L - r_k^L \leq \delta V_{k+1}^B < V_{k+1}^B \leq u^L - r_{k+1}^L,$$

i.e., $r_{k+1}^L < r_k^L$, which is a contradiction. Hence $\rho_k^H + \rho_k^L > 0$ for all k, which establishes $P2.1$.

We prove $P2.2$. Since $q_1^H = q^H < \bar{q}$ by assumption and $V_2^B > 0$ by $L1.2$, then

$$q_1^H u^H + q_1^L u^L - c^H < 0 < \delta V_2^B.$$

Hence offering c^H at date 1 is not optimal; i.e., $\rho_1^H = 0$. Therefore $\rho_1^L > 0$ by $P2.1$. $\square$

Lemmas 2 establishes properties that a DME has when frictions are not large. Recall that by assumption $q^H < \bar{q} < \hat{q} < 1$. When $\rho_t^H + \rho_t^L = 1$ at some date t, then the fraction of high quality sellers in the market at date $t+1$ is

$$q_{t+1}^H = \frac{m_{t+1}^H}{m_{t+1}^H + (1-\alpha)\, m_{t+1}^L} = \frac{(1-\alpha\rho_t^H)q_t^H}{(1-\alpha\rho_t^H)q_t^H + (1-\alpha)(1-q_t^H)} = g(q_t^H, \rho_t^H),$$

where the function g, given by

$$g(x,y) := \frac{(1-\alpha y)x}{(1-\alpha y)x + (1-\alpha)(1-x)},$$

is increasing in x and decreasing in y, and satisfies $g(q^H, \bar{\rho}/\alpha\delta) > \hat{q}$ by $F.2$.

Lemma 2. *Assume that $1 < T < \infty$, $\delta < 1$, and the inequalities $F1$ and $F2$ are satisfied (i.e., frictions are not large), and let $(\rho^H, \rho^L, r^H, r^L)$ be a DME. Then for all $t \in \{1, \dots, T\}$:*

- $(L2.1)$ $\rho_t^H < 1$.
- $(L2.2)$ $\rho_t^L < 1$.
- $(L2.3)$ $\rho_T^H > 0$, $\rho_T^L > 0$, *and* $q_T^H = \hat{q}$.
- $(L2.4)$ $V_t^L > 0$.
- $(L2.5)$ $\rho_t^L > 0$.
- $(L2.6)$ $\rho_t^H < \dfrac{\bar{\rho}}{\alpha\delta}$.
- $(L2.7)$ *If* $t < T$, *then* $\rho_t^L + \rho_t^H < 1$ *and* $\rho_{t+1}^H > 0$.

The proof of lemma 2 is provided in the Online Appendix. Now we prove Proposition 3.

Proof of Proposition 3. We show first that if $(\rho^H, \rho^L, r^H, r^L)$ is a DME, then it is given by $P3.1$ to $P3.4$, and the payoffs and surplus are as given in Proposition 3.

Since $q_T^H = \hat{q}$ by $L2.3$, then a buyer's expected utility at T is

$$V_T^B = \alpha(1-\hat{q})(u^L - c^L) = \phi_T.$$

By $L2.7$ negligible offers are optimal for all $t < T$, i.e., $1 - \rho_t^H - \rho_t^L > 0$. Then $V_t^B = \delta V_{t+1}^B$ for $t < T$ by $DME.B$, and therefore for all t we have

$$V_t^B = \phi_t. \tag{5}$$

By $L1.2$

$$r_t^H = c^H \tag{6}$$

for all t. Since $\rho_t^H > 0$, and $1 - \rho_t^H - \rho_t^L > 0$ for $1 < t < T$ by $L2.7$, and $\delta\phi_{t+1} = \phi_t$ then

$$q_t^H u^H + (1 - q_t^H)u^L - c^H = \delta V_{t+1}^B = \phi_t$$

by $DME.B$. Hence for $1 < t < T$ we have

$$q_t^H = \frac{c^H - u^L + \phi_t}{u^H - u^L}. \tag{7}$$

Since $\rho_t^L > 0$ by $L2.5$ and $1 - \rho_t^H - \rho_t^L > 0$ for $t < T$ by $L2.7$, then by $DME.B$

$$\alpha q_t^L(u^L - r_t^L) + (1 - \alpha q_t^L)\delta V_{t+1}^B = \delta V_{t+1}^B,$$

i.e.,

$$u^L - r_t^L = \delta V_{t+1}^B = \phi_t.$$

Hence for $t < T$ we have

$$r_t^L = u^L - \phi_t. \tag{8}$$

Moreover, since $r_T^L - c^L = \delta V_{T+1}^L$ by $DME.L$, then

$$r_T^L = c^L. \tag{9}$$

We calculate the expected utility of low quality sellers. Since $r_t^L - c^L = \delta V_{t+1}^L$ for all t by $DME.L$, then equation (8) yields

$$u^L - \phi_t - c^L = \delta V_{t+1}^L$$

for $t < T$. Reindexing we get

$$V_t^L = \frac{1}{\delta}\left(u^L - c^L - \phi_{t-1}\right) = \frac{u^L - c^L}{\delta} - \phi_t, \tag{10}$$

for $t \in \{2, \ldots, T\}$. And since $\rho_1^H = 0$ by $P2.2$, then

$$V_1^L = \delta V_2^L = u^L - c^L - \delta\phi_2 = u^L - c^L - \phi_1. \tag{11}$$

Next we calculate the probabilities of high price offers ρ^H. Since $r_t^L - c^L = \delta V_{t+1}^L$ for all t by $DME.L$, we can write the expected utility of a low quality seller as

$$V_t^L = \alpha \rho_t^H (c^H - c^L) + (1 - \alpha \rho_t^H) \delta V_{t+1}^L,$$

i.e.,

$$V_t^L - \delta V_{t+1}^L = \alpha \rho_t^H (c^H - c^L - \delta V_{t+1}^L).$$

For $1 < t < T$, since $\delta \phi_{t+1} = \phi_t$, then $\delta V_{t+1}^L = u^L - c^L - \phi_t$ by equation (10), and therefore

$$V_t^L - \delta V_{t+1}^L = \frac{1-\delta}{\delta}(u^L - c^L).$$

Hence

$$\frac{1-\delta}{\delta}(u^L - c^L) = \alpha \rho_t^H (c^H - c^L - (u^L - c^L - \phi_t)),$$

and solving for ρ_t^H yields

$$\rho_t^H = \frac{1-\delta}{\alpha\delta} \frac{u^L - c^L}{c^H - u^L + \phi_t} \tag{12}$$

for $1 < t < T$. Clearly $\rho_t^H > 0$. Moreover, since

$$\begin{aligned}
\alpha\delta(c^H - u^L + \phi_t) &> \alpha\delta(c^H - u^L) \\
\text{(by } F.1) &> (1 + \delta\alpha)(1 - \delta)\,\bar{\rho}\left(c^H - c^L\right) \\
&= (1 + \delta\alpha)(1 - \delta)\left(u^L - c^L\right) \\
&> (1 - \delta)\left(u^L - c^L\right),
\end{aligned}$$

then $\rho_t^H < 1$.

Recall that $\rho_1^H = 0$ by $P2.2$. We calculate ρ_T^H. Since $r_T = c^L$ by $DME.L$, then

$$V_T^L = \alpha \rho_T^H (c^H - c^L).$$

Hence using (10) for $t = T$ we have

$$\frac{u^L - c^L}{\delta} - \phi_T = \alpha \rho_T^H (c^H - c^L).$$

Solving for ρ_T^H and using $\delta\phi_T = \phi_{T-1} = \alpha\delta\,(1 - \hat{q})\left(u^L - c^L\right)$ yields

$$\rho_T^H = \frac{u^L - c^L - \phi_{T-1}}{\alpha\delta\left(c^H - c^L\right)} = (1 - \alpha\delta\,(1 - \hat{q}))\,\frac{1}{\alpha\delta}\frac{u^L - c^L}{c^H - c^L} = (1 - \alpha\delta\,(1 - \hat{q}))\,\frac{\bar{\rho}}{\alpha\delta}. \tag{13}$$

Substituting $\phi_{T-1} = \alpha\delta\,(1-\hat{q})\,(u^L - c^L)$ in this expression we get

$$\rho_T^H = (1 - \alpha\delta\,(1-\hat{q}))\,\frac{1}{\alpha\delta}\frac{u^L - c^L}{c^H - c^L} = (1 - \alpha\delta\,(1-\hat{q}))\,\frac{\bar{\rho}}{\alpha\delta},$$

and therefore $\rho_T^H > 0$. Moreover, since $\bar{\rho}/\alpha\delta < 1$ by $F.1$, then $\rho_T^H < 1$.

We calculate the probabilities of low prices offers ρ^L. For each t we have

$$q_{t+1}^H = \frac{(1-\alpha\rho_t^H)q_t^H}{(1-\alpha\rho_t^H)q_t^H + (1-\alpha(\rho_t^L + \rho_t^H))q_t^L}.$$

Solving for ρ_t^L we obtain

$$\rho_t^L = (1-\alpha\rho_t^H)\frac{q_{t+1}^H - q_t^H}{\alpha q_{t+1}^H\,(1-q_t^H)} \tag{14}$$

for all t. Since $q_{t+1}^H \geq q_t^H$ by $L1.3$ and $\rho_t^H < 1$, then $\rho_t^L \geq 0$. For $t = 1$ we have $\rho_1^H = 0$ by $P2.2$, and therefore

$$\rho_1^L = \frac{\phi_2 - (u(q^H) - c^H)}{\alpha(1-q^H)(c^H - u^L + \phi_2)} > 0, \tag{15}$$

where the inequality follows since $u(q^H) - c^H < 0$.

Since $\rho_T^H + \rho_T^L = 1$ by $P2.3$, then

$$\rho_T^L = 1 - \rho_T^H = 1 - \frac{u^L - c^L - \phi_{T-1}}{\alpha\delta\,(c^H - c^L)}. \tag{16}$$

Since $\rho_T^H < 1$ as shown above, we have $\rho_T^L > 0$.

If $T > 2$, then for $t \in \{2, \ldots, T-2\}$, using equation (7) yields

$$\rho_t^L = (1-\alpha\rho_t^H)\frac{(1-\delta)\phi_{t+1}}{\alpha\left(c^H - u^L + \phi_{t+1}\right)}\frac{u^H - u^L}{u^H - c^H - \phi_t} > 0. \tag{17}$$

Also $q_T^H = \hat{q}$ and equation (7) yields

$$\rho_{T-1}^L = (1-\alpha\rho_{T-1}^H)\frac{u(\hat{q}) - c^H - \phi_{T-1}}{\alpha\hat{q}(u^H - c^H - \phi_{T-1})}.$$

Since

$$(1-\hat{q})(u^L - c^L) = u(\hat{q}) - c^H,$$

then

$$\begin{aligned}
u^H - c^H - \phi_{T-1} &= u^H - c^H - \alpha\delta(1-\hat{q})(u^L - c^L) \\
&= u^H - c^H - \alpha\delta\left(u(\hat{q}) - c^H\right) \\
&> u^H - c^H - \alpha\delta\left(u^H - c^H\right) \\
&= (1-\alpha\delta)\left(u^H - c^H\right) > 0,
\end{aligned}$$

and

$$u(\hat{q}) - c^H - \phi_{T-1} = u(\hat{q}) - c^H - \alpha\delta(1-\hat{q})(u^L - c^L) = (1-\alpha\delta)\left(u(\hat{q}) - c^H\right) > 0.$$

Hence

$$\rho^L_{T-1} = (1-\alpha\rho^H_{T-1})(1-\alpha\delta)\frac{u(\hat{q}) - c^H}{\alpha\hat{q}(u^H - c^H - \phi_{T-1})} > 0. \tag{18}$$

We show that $\rho^H_t + \rho^L_t < 1$ for $t < T$. We first show $\rho^H_1 + \rho^L_1 < 1$. Since $g(x, y)$ is decreasing in y, $q^H_1 = q^H$, and $g(q^H, \bar{\rho}/\alpha\delta) > \hat{q}$ (by $F.2$) then

$$g(q^H_1, 0) = \frac{q^H_1}{q^H_1 + (1-\alpha)(1-q^H_1)} > g(q^H, \bar{\rho}/\alpha\delta) > \hat{q}.$$

Hence $\alpha\hat{q}(1-q^H_1) > \hat{q} - q^H_1$. Then $\rho^H_1 = 0$ by $P2.2$, $(x - q^H_1)/[\alpha x(1-q^H_1)]$ is increasing in x, and $q^H_2 \leq q^H_T = \hat{q}$ by $L2.3$ and $L1.3$, imply

$$\rho^H_1 + \rho^L_1 = \frac{q^H_2 - q^H_1}{\alpha q^H_2(1-q^H_1)} < \frac{\hat{q} - q^H_1}{\alpha\hat{q}(1-q^H_1)} < 1.$$

For $t \in \{2, \ldots, T-2\}$, from equation (12) we have

$$\rho^H_t < \frac{1-\delta}{\alpha\delta}\frac{u^L - c^L}{c^H - u^L}. \tag{19}$$

Also using equation (7), for $1 < t < T-1$ we have

$$\frac{q^H_{t+1} - q^H_t}{\alpha q^H_{t+1}(1-q^H_t)} = (1-\delta)\frac{\phi_{t+1}}{\alpha(c^H - u^L + \phi_{t+1})}\frac{u^H - u^L}{u^H - c^H - \phi_t}.$$

Since $\phi_t < \alpha(1-\hat{q})(u^L - c^L)$ for all t, and the ratio $\phi_{t+1}/(c^H - u^L + \phi_{t+1})$ is increasing in ϕ_{t+1}, we have

$$\begin{aligned}
\frac{q^H_{t+1} - q^H_t}{\alpha q^H_{t+1}(1-q^H_t)} &< (1-\delta)\frac{(1-\hat{q})(u^L - c^L)}{c^H - u^L + \alpha(1-\hat{q})(u^L - c^L)}\frac{u^H - u^L}{u^H - c^H - \alpha(1-\hat{q})(u^L - c^L)} \\
&< (1-\delta)\frac{(1-\hat{q})(u^L - c^L)}{c^H - u^L}\left(\frac{u^H - u^L}{u^H - c^H - (1-\hat{q})(u^L - c^L)}\right) \\
&= (1-\delta)\frac{u^L - c^L}{c^H - u^L},
\end{aligned}$$

where the equality is obtained by substituting $\hat{q} = \left(c^H - c^L\right) / \left(u^H - c^L\right)$. Using this inequality and inequality (19) above we have

$$\begin{aligned}
\rho_t^H + \rho_t^L &= \rho_t^H + (1-\alpha\rho_t^H)\frac{q_{t+1}^H - q_t^H}{\alpha q_{t+1}^H (1-q_t^H)} \\
&< \rho_t^H + (1-\alpha\rho_t^H)(1-\delta)\frac{u^L - c^L}{c^H - u^L} \\
&= \rho_t^H \left(1 - \alpha(1-\delta)\frac{u^L-c^L}{c^H-u^L}\right) + (1-\delta)\frac{u^L-c^L}{c^H-u^L} \\
&< \frac{1-\delta}{\alpha\delta}\frac{u^L-c^L}{c^H-u^L}\left(1-\alpha(1-\delta)\frac{u^L-c^L}{c^H-u^L}\right) + (1-\delta)\frac{u^L-c^L}{c^H-u^L} \\
&= \frac{1-\delta}{\alpha\delta}\frac{u^L-c^L}{c^H-u^L}\left(1-\alpha(1-\delta)\frac{u^L-c^L}{c^H-u^L} + \alpha\delta\right) \\
&< \frac{1-\delta}{\alpha\delta}\frac{u^L-c^L}{c^H-u^L}(1+\alpha\delta) \\
&= \frac{(1+\alpha\delta)(1-\delta)\left(c^H-c^L\right)}{c^H-u^L}\frac{\bar{\rho}}{\alpha\delta} \\
(\text{by } F.1) &< 1.
\end{aligned}$$

As for $t = T-1$, we have

$$\rho_{T-1}^H + \rho_{T-1}^L = \rho_{T-1}^H + (1-\alpha\rho_{T-1}^H)\frac{u(\hat{q}) - c^H - \phi_{T-1}}{\alpha\hat{q}(u^H - c^H - \phi_{T-1})}.$$

Rearranging yields

$$\rho_{T-1}^H + \rho_{T-1}^L = \rho_{T-1}^H\left(1 - \frac{u(\hat{q}) - c^H - \phi_{T-1}}{\hat{q}(u^H - c^H - \phi_{T-1})}\right) + \frac{u(\hat{q}) - c^H - \phi_{T-1}}{\alpha\hat{q}(u^H - c^H - \phi_{T-1})}.$$

Substituting for ρ_{T-1}^H from equation (12) and using that $\bar{\phi} = (1-\hat{q})\left(u^L - c^L\right) = u(\hat{q}) - c^H$ and $\phi_{T-1} = \alpha\delta\bar{\phi}$

$$\begin{aligned}
\rho_{T-1}^H + \rho_{T-1}^L &= \frac{1-\delta}{\alpha\delta}\frac{u^L-c^L}{c^H-u^L+\alpha\delta\bar{\phi}}\left(\frac{\hat{q}(u^H-c^H-\alpha\delta\bar{\phi}) - \left(u(\hat{q})-c^H-\alpha\delta\bar{\phi}\right)}{\hat{q}(u^H-c^H-\alpha\delta\bar{\phi})}\right) \\
&\quad + \frac{\bar{\phi}-\alpha\delta\bar{\phi}}{\alpha\hat{q}(u^H-c^H-\alpha\delta\bar{\phi})}.
\end{aligned}$$

Since

$$\begin{aligned}
\hat{q}(u^H - c^H - \alpha\delta\phi) - (u(\hat{q}) - c^H - \alpha\delta\phi) &= \hat{q}(u^H - c^H - \alpha\delta\phi) - (\hat{q}u^H + (1-\hat{q})u^L) + c^H + \alpha\delta\phi \\
&= (1-\hat{q})(c^H - u^L + \alpha\delta\phi),
\end{aligned}$$

then

$$\begin{aligned}
\rho_{T-1}^H + \rho_{T-1}^L &= \frac{1-\delta}{\alpha\delta\hat{q}(u^H-c^H-\alpha\delta\bar{\phi})}\left(\frac{\left(u^L-c^L\right)(1-\hat{q})(c^H-u^L+\alpha\delta\bar{\phi})}{c^H-u^L+\alpha\delta\bar{\phi}} + \delta\bar{\phi}(1-\alpha\delta)\right) \\
&= \frac{(1-\delta)\left[1+\delta(1-\alpha\delta)\right]\bar{\phi}}{\alpha\delta\hat{q}(u^H-c^H-\alpha\delta\bar{\phi})}.
\end{aligned}$$

Hence $\rho^H_{T-1}+\rho^L_{T-1}<1$ if and only if

$$(1-\delta)\left[1+\delta(1-\alpha\delta)\right]\bar{\phi}<\alpha\delta\hat{q}(u^H-c^H-\alpha\delta\bar{\phi})$$

i.e.,

$$[1-\alpha\delta^2(1-\alpha\hat{q})]\bar{\phi}<\alpha\delta\hat{q}\left(u^H-c^H\right).$$

Since

$$\frac{\hat{q}}{\bar{\phi}}\left(u^H-c^H\right)=\frac{\hat{q}}{1-\hat{q}}\frac{u^H-c^H}{u^L-c^L}=\frac{c^H-c^L}{u^H-c^H}\frac{u^H-c^H}{u^L-c^L}=\frac{1}{\bar{\rho}},$$

then this inequality becomes

$$1-\alpha\delta^2(1-\alpha\hat{q})<\frac{\alpha\delta}{\bar{\rho}},$$

which holds since $\alpha\delta/\bar{\rho}>1$ by $F.1$ and $0<\alpha\delta^2(1-\alpha\hat{q})<1$.

The surplus can be calculated using (5), (11), and $L1.2$ as

$$\begin{aligned} S^{DME} &= m^BV_1^B+m^HV_1^H+m^LV_1^L \\ &= (m^L+m^H)\phi_1+m^L(u^L-c^L-\phi_1) \\ &= m^H\phi_1+m^L(u^L-c^L). \end{aligned} \tag{20}$$

Equations (12), (13) and $P2.2$ identify ρ^H as given in $P3.1$. Equations (15), (17) and (16) identify ρ^L as given in $P3.2$. Equation (6) identifies r^H as given in $P3.3$. Equations (8) and (9) identify r^L as given in $P3.4$. The traders' payoffs are identified in equations (5) and (11), and in $L1.2$. The surplus is given in equation (20).

Finally, as the construction above shows, the profile defined in $P3.1$ to $P3.4$ of Proposition 3 is indeed a DME. □

Proof of Proposition 4. The unique DME as well as the traders' payoffs and the surplus are given in Proposition 3. By $P3.1$

$$\lim_{\delta\to1}\rho_1^H=0=\tilde{\rho}_1^H,$$

and

$$\lim_{\delta\to1}\rho_t^H=\lim_{\delta\to1}\frac{1-\delta}{\alpha\delta}\frac{u^L-c^L}{c^H-u^L+\alpha\delta^{T-t}(1-\hat{q})(u^L-c^L)}=0=\tilde{\rho}_t^H,$$

for $1<t<T$, and also

$$\lim_{\delta\to1}\rho_T^H=\lim_{\delta\to1}\frac{u^L-c^L-\phi_{T-1}}{\alpha\delta\left(c^H-c^L\right)}=\frac{u^L-c^L-\alpha\bar{\phi}}{\alpha(c^H-c^L)}=\tilde{\rho}_T^H.$$

Since $u^H > u^L > c^L$ by assumption, then $0 < \tilde{\rho}_T^H < 1$.

From equation (7) we have

$$\lim_{\delta\to 1} q_t^H = \lim_{\delta\to 1} \frac{c^H - u^L + \phi_t}{u^H - u^L} = \frac{c^H - u^L + \alpha\bar{\phi}}{u^H - u^L}.$$

for $1 < t < T$. Also $q_T^H = \hat{q}$ implies

$$\lim_{\delta\to 1} q_T^H = \hat{q}.$$

$P3.2$ implies

$$\lim_{\delta\to 1} \rho_1^L = \lim_{\delta\to 1} \frac{c^H - u^L + \phi_2 - q^H(u^H - u^L)}{\alpha(1-q^H)(c^H - u^L + \phi_2)} = \frac{c^H - u^L + \alpha\bar{\phi} - q^H(u^H - u^L)}{\alpha(1-q^H)(c^H - u^L + \alpha\bar{\phi})} = \tilde{\rho}_1^L,$$

and for $1 < t < T-1$

$$\lim_{\delta\to 1} \rho_t^L = \lim_{\delta\to 1} (1-\alpha\rho_t^H) \frac{(1-\delta)\phi_{t+1}}{c^H - u^L + \phi_{t+1}} \frac{u^H - u^L}{u^H - c^H - \phi_t} = 0 = \tilde{\rho}_t^L,$$

and

$$\lim_{\delta\to 1} \rho_{T-1}^L = \lim_{\delta\to 1} (1-\alpha\rho_{T-1}^H) \frac{(1-\alpha\delta)\left(u(\hat{q}) - c^H\right)}{\alpha\hat{q}(u^H - c^H - \phi_{T-1})} = \frac{(1-\alpha)\left(u(\hat{q}) - c^H\right)}{\alpha\hat{q}(u^H - c^H - \alpha\bar{\phi})} = \tilde{\rho}_{T-1}^L.$$

Also

$$\lim_{\delta\to 1} \rho_T^L = \lim_{\delta\to 1} \left(1 - \rho_T^H\right) = 1 - \tilde{\rho}_T^H = \tilde{\rho}_T^L.$$

Thus, $\tilde{\rho}_T^H < 1$ implies $\tilde{\rho}_T^L > 0$.

As for the traders' expected utilities, we have

$$\lim_{\delta\to 1} V_1^B = \lim_{\delta\to 1} \phi_1 = \alpha\bar{\phi} = \tilde{V}_1^B,$$

and

$$\lim_{\delta\to 1} V_1^L = \lim_{\delta\to 1} \left(1 - \alpha\delta^{T-1}(1-\hat{q})\right)(u^L - c^L) = (1 - \alpha(1-\hat{q}))(u^L - c^L) = \tilde{V}_1^L.$$

Since $V_t^H = 0$, then

$$\lim_{\delta\to 1} V_t^H = 0 = \tilde{V}_t^H.$$

It is easy to check that $(\tilde{\rho}^H, \tilde{\rho}^L, \tilde{r}^H, \tilde{r}^L)$ forms an equilibrium of the market when $\delta = 1$.

Finally, we have

$$\begin{aligned}
\lim_{\delta\to 1} S^{DME} &= \lim_{\delta\to 1} \left[m^L(u^L - c^L) + m^H\delta^{T-1}\alpha(1-\hat{q})(u^L - c^L)\right] \\
&= m^L(u^L - c^L) + m^H\alpha(1-\hat{q})(u^L - c^L) \\
&= \tilde{S}^{DME}. \ \square
\end{aligned}$$

Proof of Proposition 5. If frictions are not large, then the unique DME is that given in Proposition 3. Thus, since $\lim_{T\to\infty}\phi_t = 0$ for all t, we have

$$\lim_{T\to\infty}\rho_1^H = 0 = \hat{\rho}_1^H,$$

and for $t > 1$ we have

$$\lim_{T\to\infty}\rho_t^H = (1-\delta)\frac{u^L - c^L}{\alpha\delta(c^H - u^L)} = \hat{\rho}_t^H.$$

Also

$$\lim_{T\to\infty}\rho_1^L = \frac{c^H - u^L - q^H(u^H - u^L)}{\alpha(1-q^H)(c^H - u^L)} = \frac{\bar{q} - q^H}{\alpha\bar{q}(1-q^H)} = \hat{\rho}_1^L,$$

and for $t > 1$ we have

$$\lim_{T\to\infty}\rho_t^L = 0 = \hat{\rho}_t^L.$$

Clearly $\lim_{T\to\infty} r_t^H = c^H = \hat{r}_t^H$, and $\lim_{T\to\infty} r_t^L = u^L = \hat{r}_t^L$.

We show that the strategy distribution $(\hat{\rho}^H, \hat{\rho}^L, \hat{r}^H, \hat{r}^L)$ forms a DME when $T = \infty$. Since $\alpha(1-q^H)\bar{q} > \bar{q} - q^H$, then $0 < \hat{\rho}_1^L < 1$. Since $\alpha < 1$, and $\alpha\delta(c^H - c^L) > u^L - c^L$ by $F.1$, we have

$$\alpha\delta(c^H - u^L) + \delta\left(u^L - c^L\right) > \alpha\delta(c^H - u^L) + \alpha\delta\left(u^L - c^L\right) = \alpha\delta(c^H - c^L) > u^L - c^L.$$

Hence $0 < \hat{\rho}_t^H < 1$ for all $t > 1$.

Since $\hat{r}_t^H = c^H$ and $\hat{r}_t^L = u^L$, then the (maximum) expected utility of high quality sellers is $\hat{V}_t^H = 0$ for all t. Hence $\hat{r}_t^H = c^H$ for all t satisfies $DME.H$. For $t > 1$ the expected utility of low quality sellers is

$$\hat{V}_t^L = \frac{u^L - c^L}{\delta}.$$

For $t = 1$ we have $\hat{r}_1^L = c^L + \delta\hat{V}_2^L = u^L$. Hence $\hat{r}_t^L = u^L$ for all t satisfies $DME.L$. Also

$$\hat{V}_1^L = \alpha\hat{\rho}_1^L(u^L - c^L) + (1 - \alpha\hat{\rho}_1^L)\delta\hat{V}_2^L = u^L - c^L.$$

Using $\hat{\rho}_1^H$ and $\hat{\rho}_1^L$ we have

$$q_2^H = \frac{q^H}{q^H + (1-\alpha\hat{\rho}_1^L)(1-q^H)} = \bar{q}.$$

And since $\hat{\rho}_t^L = 0$ for $t > 1$, then $q_t^H = q_2^H = \bar{q}$. Hence

$$q_t^H(u^H - c^H) + (1 - q_t^H)(u^L - c^H) = 0$$

for $t > 1$, and therefore offering the high price (c^H) leads to zero instantaneous payoff for all $t > 1$. Since $q_1^H < \bar{q}$ by assumption, then offering the high price (c^H) at $t = 1$ leads to a negative instantaneous payoff. Also since $\hat{r}_t^L = u^L$ for all t, then offering the low price (u^L) yields a zero instantaneous payoff. Thus, the buyers maximum expected utility is zero at all dates, i.e., $\hat{V}_t^B = 0$ for all t. Hence $DME.B$ is satisfied. $\square$

Proof of Corollary 3. We calculate the present value of a subsidy $\sigma^L > 0$ on low quality, which we denote for $\delta < 1$ by $PV_{\sigma^L}(\delta)$, and show that it approaches $\sigma^L m^L$ from below as δ approaches 1. We have

$$PV_{\sigma^L}(\delta) = \sigma^L \alpha \rho_1^L m_1^L + \sum_{t=2}^{\infty} \delta^{t-1} \sigma^L \alpha \rho_t^H m_t^L.$$

Since ρ_t^H is independent of t for $t > 1$ by $P5.1$, denote $\rho_t^H = \rho^H$. Also, we have $m_1^L = m^L$, and $m_t^L = (1 - \alpha\rho_1^L)(1 - \alpha\rho^H)^{t-2} m^L$ for $t > 1$. Hence

$$\begin{aligned} PV_{\sigma^L}(\delta) &= \sigma^L m^L \left(\alpha\rho_1^L + \alpha\rho^H (1 - \alpha\rho_1^L) \sum_{t=2}^{\infty} \delta^{t-1} (1 - \alpha\rho^H)^{t-2} \right) \\ &= \sigma^L m^L \left(\alpha\rho_1^L + \alpha\rho^H (1 - \alpha\rho_1^L) \sum_{t=1}^{\infty} \delta^{t} (1 - \alpha\rho^H)^{t-1} \right). \end{aligned}$$

Since

$$\begin{aligned} \sum_{t=1}^{\infty} \delta^t (1 - \alpha\rho^H)^{t-1} &= \frac{1}{(1 - \alpha\rho^H)} \sum_{t=1}^{\infty} \left(\delta(1 - \alpha\rho^H) \right)^t \\ &= \frac{1}{(1 - \alpha\rho^H)} \frac{\delta(1 - \alpha\rho^H)}{1 - \delta(1 - \alpha\rho^H)} \\ &= \frac{\delta}{1 - \delta(1 - \alpha\rho^H)}, \end{aligned}$$

then

$$PV_{\sigma^L}(\delta) = \sigma^L m^L P(\delta),$$

where

$$P(\delta) := \alpha\rho_1^L + (1 - \alpha\rho_1^L) \frac{\alpha\delta\rho^H}{\alpha\delta\rho^H + (1 - \delta)}.$$

Since $0 < \alpha\rho_1^L < 1$ and $\delta < 1$, then $P(\delta)$ is a convex combination of 1 and a number less than 1. Therefore $P(\delta) < 1$ and $PV_{\sigma^L}(\delta) < \sigma^L m^L$. Further, since $\lim_{\delta\to 1} P(\delta) = 1$, then $\lim_{\delta \to 1} PV_{\sigma^L}(\delta) = \sigma^L m^L$. $\square$

7 Appendix B: Dynamic Competitive Equilibrium

We study the market described in Section 2 when trade is *centralized*, i.e., trade is multilateral and agents are price takers. The market opens for T consecutive dates, and the traders' discount rate is $\delta \in (0,1]$.

The supply and demand schedules are defined as follows. Let $p = (p_1, \ldots, p_T) \in \mathbb{R}_+^T$ be a sequence of prices. The utility to a seller of quality $\tau \in \{H, L\}$ who supplies at date t is $\delta^{t-1}(p_t - c^\tau)$. Hence the maximum utility that a τ-quality seller may attain is

$$v^\tau(p) = \max_{t \in \{1,\ldots,T\}} \{0, \delta^{t-1}(p_t - c^\tau)\}.$$

The *supply of τ-quality good*, denoted by $S^\tau(p)$, is the set of sequences $s^\tau = (s_1^\tau, \ldots, s_T^\tau) \in \mathbb{R}_+^T$ satisfying:

$(S.1)$ $\sum_{t=1}^T s_t^\tau \leq m^\tau$,

$(S.2)$ $s_t^\tau > 0$ implies $\delta^{t-1}(p_t - c^\tau) = v^\tau(p)$, and

$(S.3)$ $\left(\sum_{t=1}^T s_t^\tau - m^\tau\right) v^\tau(p) = 0$.

Condition $S.1$ requires that no more of good τ than is available, m^τ, be supplied. Condition $S.2$ requires that supply be positive only at dates where it is optimal to supply. Condition $S.3$ requires that the total amount of good τ available be supplied when τ-quality sellers may attain a positive utility (i.e., when $v^\tau(p) > 0$).

Denote by $u_t \in [u^L, u^H]$ the expected value to buyers of a unit supplied at date t. Then the utility to a buyer who demands a unit of the good at date t is $\delta^{t-1}(u_t - p_t)$. If the sequence of buyers' expected values is $u = (u_1, \ldots, u_T)$, then the maximum utility a buyer may attain is

$$v^B(p, u) = \max_{t \in \{1,\ldots,T\}} \{0, \delta^{t-1}(u_t - p_t)\}.$$

The *market demand*, denoted by $D(p, u)$, is the set of sequences $d = (d_1, \ldots, d_T) \in \mathbb{R}_+^T$ satisfying:

$(D.1)$ $\sum_{t=1}^T d_t \leq m^B$,

$(D.2)$ $d_t > 0$ implies $\delta^{t-1}(u_t - p_t) = v^B(p, u)$, and

$(D.3)$ $\left(\sum_{t=1}^T d_t - m^B\right) v^B(p, u) = 0$.

Condition $D.1$ requires that the total demand not exceed the measure of buyers. Condition $D.2$ requires that the demand be positive only at dates where buying is optimal. Condition $D.3$ requires that demand be equal to the measure of buyers when buyers may attain a positive utility (i.e., when $v^B(p,u) > 0$).

We define dynamic competitive equilibrium along the lines in the literature – see e.g., Wooders (1998), and Janssen and Roy (2002).

Definition. A *dynamic competitive equilibrium (DCE)* is a profile (p, u, s^H, s^L, d) such that $s^H \in S^H(p)$, $s^L \in S^L(p)$, $d \in D(p, u)$, and for each t:

$(DCE.1)$ $s_t^H + s_t^L = d_t$, and

$(DCE.2)$ $s_t^H + s_t^L = d_t > 0$ implies $u_t = \dfrac{u^H s_t^H + u^L s_t^L}{s_t^H + s_t^L}$.

Condition $DCE.1$ requires that the market clear at each date, and condition $DCE.2$ requires that the expectations described by the vector u be correct whenever there is trade. For a market that opens for a single date (i.e., if $T = 1$), our definition reduces to Akerlof's. The surplus generated in a DCE may be calculated as

$$S^{DCE} = \sum_{\tau \in \{H,L\}} \sum_{t=1}^{T} s_t^\tau \delta^{t-1}(u^\tau - c^\tau). \tag{21}$$

In lemmas 3 and 4 we establish some properties of dynamic competitive equilibria.

Lemma 3. *In every DCE, (p, u, s^H, s^L, d), we have $\sum_{\{t|s_t^H>0\}} s_t^L < m^L$.*

Proof. Let (p, u, s^H, s^L, d) be a DCE. For all t such that $s_t^H > 0$ we have

$$\delta^{t-1}(p_t - c^H) = v^H(p) \geq 0$$

by $(S.2)$. Hence $p_t \geq c^H$. Also $d_t > 0$ by $DCE.1$, and therefore

$$v^B(p) = \delta^{t-1}(u_t - p_t) \geq 0$$

implies $0 \leq u_t - p_t \leq u_t - c^H$, i.e., $u_t \geq c^H = u(\bar{q})$. Thus

$$\frac{s_t^H}{s_t^H + s_t^L} \geq \bar{q},$$

i.e.,

$$(1 - \bar{q}) \sum_{\{t|s_t^H>0\}} s_t^H \geq \bar{q} \sum_{\{t|s_t^H>0\}} s_t^L.$$

Since $\sum_{\{t|s_t^H>0\}} s_t^H \leq m^H$, then

$$(1-\bar{q})m^H \geq (1-\bar{q}) \sum_{\{t|s_t^H>0\}} s_t^H \geq \bar{q} \sum_{\{t|s_t^H>0\}} s_t^L.$$

Since $q^H = m^H/(m^H + m^L) < \bar{q}$ by assumption, then

$$\sum_{\{t|s_t^H>0\}} s_t^L \leq \frac{1-\bar{q}}{\bar{q}} m^H < \frac{1-q^H}{q^H} m^H = \frac{\dfrac{m^L}{m^H+m^L}}{\dfrac{m^H}{m^H+m^L}} m^H = m^L. \square$$

Lemma 4 shows that low quality must trade before high quality.

Lemma 4. *Let (p, u, s^H, s^L, d) be a DCE. If $s_t^H > 0$ for some t, then there is $t' < t$ such that $s_{t'}^L > 0 = s_{t'}^H$ and $\delta^{t'-1}(u^L - c^L) \geq \delta^{t-1}(c^H - c^L)$.*

Proof. Let (p, u, s^H, s^L, d) be a DCE, and assume that $s_t^H > 0$. Then $\delta^{t-1}(p_t - c^H) = v^H(p) \geq 0$ by $S.2$, and therefore $p_t \geq c^H$. Hence $v^L(p) \geq \delta^{t-1}(p_t - c^L) \geq \delta^{t-1}(c^H - c^L) > 0$, and therefore $\sum_{k=1}^T s_k^L = m^L$ by $S.3$. Since

$$\sum_{\{k|s_k^H>0\}} s_k^L < m^L$$

by Lemma 3, then there is t' such that $s_{t'}^L > 0 = s_{t'}^H$. Hence $d_{t'} > 0$ by $DCE.1$, which implies $u_{t'} = u^L$ by $DCE.2$, and $p_{t'} \leq u^L$ by $D.2$. Also $s_{t'}^L > 0$ implies $v^L(p) = \delta^{t'-1}(p_{t'} - c^L) \geq \delta^{t-1}(p_t - c^L)$ by $S.2$. Thus

$$\delta^{t'-1}(u^L - c^L) \geq \delta^{t'-1}(p_{t'} - c^L) \geq \delta^{t-1}(p_t - c^L) \geq \delta^{t-1}(c^H - c^L).$$

Since $u^L < c^H$ this inequality implies $t' < t$. $\square$

Proposition 6 establishes that there is a DCE where all low quality units trade at date 1 at the price u^L, and none of the high quality units ever trade. Moreover, if the market opens over a sufficiently short horizon, then every DCE has these properties. Specifically, the horizon T must be less than $\overline{T}$, which is defined by the inequality

$$\delta^{\overline{T}-2}(c^H - c^L) > u^L - c^L \geq \delta^{\overline{T}-1}(c^H - c^L).$$

Since $\overline{T}$ approaches infinity as δ approaches one, for a given T the condition $T < \overline{T}$ holds when δ is near one, i.e., when traders are sufficiently patient.

Proposition 6. *There are DCE in which all low quality units trade immediately at the price u^L and none of the high quality units trade, e.g., (p, u, s^H, s^L, d) given by $p_t = u_t = u^L$ for all t, $s_1^L = d_1 = m^L$, and $s_1^H = s_t^H = s_t^L = d_t = 0$ for $t > 1$ is a DCE. In these DCE the payoff to low quality sellers is $u^L - c^L$, the payoff to high quality sellers and buyers is zero, and the surplus is $\bar{S}$. Moreover, if $T < \overline{T}$, then every DCE has these properties.*[8]

Proof. The profile in Proposition 6 is clearly a DCE. We show that every DCE, (p, u, s^H, s^L, d), satisfies $p_1 = u_1 = u^L$, $s_1^L = d_1 = m^L$ and $s_1^H = s_t^H = s_t^L = d_t = 0$ for $t > 1$.

We first show that $s_t^H = 0$ for all $t \in \{1, \ldots, T\}$. Suppose that $s_t^H > 0$ for some t. Then Lemma 4 implies that there is $t' < t$ such that

$$u^L - c^L \geq \delta^{t'-1}(u^L - c^L) \geq \delta^{t-1}(c^H - c^L) \geq \delta^{T-1}(c^H - c^L),$$

which is a contradiction.

We show that $p_t \geq u^L$ for all t. If $p_t < u^L$ for some t, then

$$v^B(p, u) = \max_{t \in \{1, \ldots, T\}} \{0, \delta^{t-1}(u_t - p_t)\} > 0,$$

and therefore $\sum_{t=1}^{T} d_t = m^B = m^H + m^L$. However, $s_t^H = 0$ for all t implies

$$\sum_{t=1}^{T} (s_t^H + s_t^L) \leq m^L < m^L + m^H = \sum_{t=1}^{T} d_t,$$

which contradicts $DCE.1$.

Since $p_t \geq u^L$ for all t, then

$$v^L(p) = \max_{t \in \{1, \ldots, T\}} \{0, \delta^{t-1}(p_t - c^L)\} > 0,$$

and therefore $\sum_{t=1}^{T} s_t^L = m^L$ by $S.3$.

We show that $p_1 = u^L$ and $s_1^L = d_1 = m^L$ and $s_t^L = 0$ for $t > 1$. Let t be such that $s_t^L > 0$. Then $s_t^H = 0$ implies $u_t = u^L$. By $DCE.1$ we have $d_t = s_t^L > 0$ and thus

$$\delta^{t-1}(u_t - p_t) = \delta^{t-1}(u^L - p_t) \geq 0$$

[8] Janssen and Roy (2002)'s definition of competitive equilibrium requires additionally that the expected value to buyers of a random unit at dates when there is no trade is at least the value of the lowest quality for which there is a positive measure of unsold units. When $T < \bar{T}$ no CE with this property exists.

by $D.2$. This inequality and $p_t \geq u^L$ imply that $p_t = u^L$. Hence for all t such that $s_t^L > 0$ we have $p_t = u^L$.

Let $t > 1$ and assume that $s_t^L > 0$. Then $p_t = u^L$. Since $\delta < 1$ and as shown above $p_1 \geq u^L$, then

$$p_1 - c^L > \delta^{t-1}(u^L - c^L) = \delta^{t-1}(p_t - c^L),$$

which contradicts $S.2$. Hence $s_t^L = 0$ for $t > 1$, and therefore $\sum_{t=1}^{T} s_t^L = m^L$ implies $s_1^L = d_1 = m^L > 0$, and $p_1 = u^L$. □

The intuition for why high quality does not trade when $T < \overline{T}$ is clear: If high quality were to trade at $t \leq T$, then p_t must be at least c^H. Hence the utility to low quality sellers is at least $\delta^{t-1}(c^H - c^L)$. Since

$$\delta^{t-1}(c^H - c^L) \geq \delta^{T-1}(c^H - c^L) \geq \delta^{\overline{T}-2}(c^H - c^L) > u^L - c^L > 0,$$

then all low quality sellers trade at prices greater than u^L. But at a price $p \in (u^L, c^H)$ only low quality sellers supply, and therefore the demand is zero. Hence all trade is at prices of at least c^H. Since $u(q^H) < c^H$ by assumption, and since in equilibrium all low quality is supplied, there must be a date at which there is trade and the expected value of a random unit supplied is below c^H. This contradicts that there is demand at such a date. Thus, high quality is not supplied in a DCE. Consequently, low quality sellers capture the entire surplus, i.e., the price is u^L, as low quality sellers are the short side of the market.

By Propositions 3 the surplus realized in a decentralized market is greater than the competitive surplus, i.e., $S^{DME} > \bar{S}$, while a dynamic competitive market that opens over a finite horizon generates the competitive surplus, i.e., $S^{DCE} = \bar{S}$, by Proposition 6. Thus, *decentralized markets perform better than centralized markets when the horizon is finite.* This continues to be the case even as frictions vanish by Proposition 4.

Proposition 7 below establishes that in a centralized market that opens over a sufficiently long horizon there are dynamic competitive *separating* equilibria in which all low quality units trade immediately and all high quality units trade with delay. Specifically, the horizon T must be at least $\widetilde{T}$, which is defined by the inequality

$$\delta^{\widetilde{T}-2}(u^H - c^L) > u^L - c^L \geq \delta^{\widetilde{T}-1}(u^H - c^L).$$

Since $u^H > c^H$, then $\widetilde{T} \geq \overline{T}$.

Proposition 7. *If $T \geq \widetilde{T}$, then there are DCE in which all low quality units trade at date 1 and all high quality units trade at date $\widetilde{T}$. Such DCE yield a surplus of*

$$S^{DCE} = m^L(u^L - c^L) + m^H \delta^{\tilde{T}-1}(u^H - c^H) > \bar{S}.$$

Moreover, if $T = \infty$, then

$$\lim_{\delta \to 1} S^{DCE} = \tilde{S}^{DME}.$$

Proof. Assume that $T \geq \widetilde{T}$. We show that the profile (p, u, s^H, s^L, d) given by $p_t = u_t = u^L$ for $t < \widetilde{T}$, and $p_t = u_t = u^H$ for $t \geq \widetilde{T}$, $s_1^H = 0$, $s_1^L = m^L = d_1$, $s_{\widetilde{T}}^L = 0$, $s_{\widetilde{T}}^H = d_{\widetilde{T}} = m^H$, and $s_t^H = s_t^L = d_t = 0$ for $t \notin \{1, \widetilde{T}\}$ is a DCE.

Since $p_{\widetilde{T}} = u^H > c^H$, then $v^H(p) \geq \delta^{\widetilde{T}-1}(p_{\widetilde{T}} - c^H) > 0$. Further, since $\delta < 1$ then

$$\delta^{\widetilde{T}-1}(p_{\widetilde{T}} - c^H) = \delta^{\widetilde{T}-1}(u^H - c^H) > \delta^{t-1}(p_t - c^H)$$

for $t \neq \widetilde{T}$. Hence $s^H \in S^H(p)$. For low quality sellers, $\delta < 1$ and $u^L - c^L \geq \delta^{\widetilde{T}-1}(u^H - c^H)$ imply

$$v^L(p) = p_1 - c^L = u^L - c^L \geq \delta^{t-1}(p_t - c^H)$$

for $t > 1$. Hence $s^L \in S^L(p)$. For buyers,

$$v^B(p, u) = \delta^{t-1}(u_t - p_t) = 0$$

for all t. Hence $d \in D(p, u)$. Finally, $s_t^L + s_t^H = d_t$ for all t, and therefore $DCE.1$ is satisfied, and $u_1 = u^L$ and $u_{\widetilde{T}} = u^H$ satisfy $DCE.2$. Thus, the profile defined is a DCE. The surplus in this DCE is

$$S^{DCE} = m^L(u^L - c^L) + m^H \delta^{\widetilde{T}-1}(u^H - c^H).$$

Assume that $T = \infty$, and let $\delta < 1$. The surplus at the DCE of Proposition 7 is

$$S^{DCE}(\delta) = q^L(u^L - c^L) + q^H \delta^{\widetilde{T}(\delta)-1}(u^H - c^H).$$

By definition $\widetilde{T}(\delta)$ satisfies

$$\delta^{\widetilde{T}(\delta)-1}(u^H - c^L) \leq u^L - c^L < \delta^{\widetilde{T}(\delta)-2}(u^H - c^L).$$

i.e.,

$$\delta < \frac{u^H - c^L}{u^L - c^L}\delta^{\widetilde{T}(\delta)-1} \leq 1$$

Hence

$$\lim_{\delta\to 1}\delta = \frac{u^H - c^L}{u^L - c^L}\lim_{\delta\to 1}\delta^{\widetilde{T}(\delta)-1} = 1,$$

i.e.,

$$\lim_{\delta\to 1}\delta^{\widetilde{T}(\delta)-1} = \frac{u^L - c^L}{u^H - c^L} = (1-\hat{q})\frac{u^L - c^L}{u^H - c^H}.$$

Substituting, we have

$$\lim_{\delta\to 1}\hat{S}^{DCE}(\delta) = \left[m^L + m^H(1-\hat{q})\right](u^L - c^L) = \tilde{S}^{DME}. \square$$

Centralized markets that open over a sufficiently long horizon eventually *recover* from adverse selection, i.e., have equilibria in which high quality trades and the surplus is above the competitive surplus. Consequently, *when the horizon is infinite, centralized markets may outperform decentralized markets* – which by Proposition 5 yield the competitive surplus.[9]

In the proof of Proposition 7 we show that

$$\lim_{\delta\to 1}\delta^{\widetilde{T}-1} = \frac{u^L - c^L}{u^H - c^L},$$

and therefore that the surplus realized from trading high quality in this equilibrium approaches

$$m^H\frac{u^L - c^L}{u^H - c^L}(u^H - c^H) = m^H(1-\hat{q})(u^L - c^L).$$

Thus, as δ approaches one, the surplus approaches $\tilde{S}^{DME}$, which is also the surplus realized in the DME when $T < \infty$ as α and δ approach one – see Proposition 4. This result reveals that the same incentive constraints are at play in both centralized and decentralized markets: In a separating DCE, high quality trades with a sufficiently long delay that low quality sellers prefer trading immediately at a low price to waiting and trading at a high price. Likewise, in a DME, high price offers are made with sufficiently low probability that low quality sellers accept a low price offer.

[9]When $\overline{T} \leq T < \widetilde{T}$ there are no separating CE, but there are *partially pooling* CE in which high quality trades. In the most efficient of these CE, in which some low quality trades at date 1 while the remaining low quality and all the high quality trade at date T, the surplus is greater than $\bar{S}$.

Policy Intervention and Liquidity

As noted earlier, the effect of a subsidy or tax is akin to that of a change of the value of the good, i.e., of u^L or u^H. Marginal changes in these values do not affect the value of $\overline{T}$ or $\widetilde{T}$ generically, and hence do not affect the net surplus in a centralized market. If $T < \infty$ and δ is near one, then subsidies have no impact on net surplus. If $T = \infty$, a subsidy on low quality or tax on high quality that reduces $\widetilde{T}$ increases net surplus in the separating DCE since high quality trades earlier.

When $T < \overline{T}$, low quality is liquid as it trades immediately, while high quality is illiquid as it never trades. When $T = \infty$ all units trade in the separating DCE, but high quality trades with delay, and therefore is less liquid than low quality, which trades immediately.

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8 Online Appendix

Proof of Lemma 1: Let $t \in \{1, \ldots, T\}$. We prove $L1.1$. Write $\bar{p} = \max\{r_t^H, r_t^L\}$, and suppose that $\lambda_t(\bar{p}) < 1$. Then there is $\hat{p} > \bar{p}$ in the support of λ_t. Since $I(\bar{p}, r_t^\tau) = I(\hat{p}, r_t^\tau) = 1$ for $\tau \in \{H, L\}$, we have

$$\begin{aligned}
V_t^B &\geq \alpha \sum_{\tau \in \{H,L\}} q_t^\tau I(\bar{p}, r_t^\tau)(u^\tau - \bar{p}) + \left[1 - \alpha \sum_{\tau \in \{H,L\}} q_t^\tau I(\bar{p}, r_t^\tau)\right] \delta V_{t+1}^B \\
&= \alpha \sum_{\tau \in \{H,L\}} q_t^\tau \left(u^\tau - \bar{p}\right) + (1-\alpha)\, \delta V_{t+1}^B \\
&> \alpha \sum_{\tau \in \{H,L\}} q_t^\tau \left(u^\tau - \hat{p}\right) + (1-\alpha)\, \delta V_{t+1}^B \\
&= \alpha \sum_{\tau \in \{H,L\}} q_t^\tau I(\hat{p}, r_t^\tau)(u^\tau - \hat{p}) + \left[1 - \alpha \sum_{\tau \in \{H,L\}} q_t^\tau I(\hat{p}, r_t^\tau)\right] \delta V_{t+1}^B,
\end{aligned}$$

which contradicts $DME.B$.

We prove $L1.2$ by induction. Because $V_{T+1}^\tau = 0$ for $\tau \in \{B, H, L\}$, then $DME.H$ and $DME.L$ imply

$$r_T^H = c^H + \delta V_{T+1}^H = c^H > c^L = r_T^L = c^L + \delta V_{T+1}^L.$$

Hence $\lambda_T(c^H) = 1$ by $L1.1$, and therefore $V_T^H = 0$ and $V_T^L \leq c^H - c^L$. Also, if $q_T^H > \bar{q}$, then offering the high price $r_T^H = c^H$ yields a payoff $u(q_T^H) - c^H > u(\bar{q}) - c^H > 0$, and if $q_T^H \leq \bar{q}$, then $q_T^L > 0$, and therefore offering the low price $r_T^L = c^L$ yields a payoff $q_T^L \left(u^L - c^L\right) > 0$. Hence in either case $V_T^B > 0$. Let $k \leq T$, and assume that $L1.2$ holds for $t \in \{k, \ldots, T\}$; we show that it holds for $k-1$. Since $V_k^H = 0$, $DME.H$ implies $r_{k-1}^H = c^H + \delta V_k^H = c^H$. Since $V_k^L \leq c^H - c^L$ and $\delta < 1$, then $DME.L$ implies $r_{k-1}^L = c^L + \delta V_k^L \leq (1-\delta)c^L + \delta c^H < c^H$. Hence $\lambda_k(c^H) = 1$ by $L1.1$, and therefore $V_{k-1}^H = 0$. Also since $\lambda_t(c^H) = 1$ for $t \geq k-1$, then $V_{k-1}^L \leq c^H - c^L$. Finally, $V_{k-1}^B \geq \delta V_k^B > 0$.

In order to prove $L1.3$, note that $L1.2$ implies $\lambda_t^H \leq \lambda_t^L$. Hence

$$q_{t+1}^H = \frac{m_{t+1}^H}{m_{t+1}^H + m_{t+1}^L} = \frac{(1-\alpha\lambda_t^H)m_t^H}{(1-\alpha\lambda_t^H)m_t^H + (1-\alpha\lambda_t^L)m_t^L} \geq \frac{m_t^H}{m_t^H + m_t^L} = q_t^H.$$

As for $L1.4$, it is a direct implication of $L1.1$ and $L1.2$.

We prove $L1.5$. Suppose that $\lambda_t(p) > \lambda_t(r_t^L)$ for some $p \in (r_t^L, r_t^H)$. Then there is $\hat{p}$ in the support of λ_t such that $r_t^L < \hat{p} < r_t^H$. Since $I(\hat{p}, r_t^L) = 1$ and $I(\hat{p}, r_t^H) = 0$, then

$$\begin{aligned} V_t^B &\geq \alpha \sum_{\tau \in \{H,L\}} q_t^\tau I(r_t^L, r_t^\tau)(u^\tau - r_t^L) + \left[1 - \alpha \sum_{\tau \in \{H,L\}} q_t^\tau I(r_t^L, r_t^\tau)\right] \delta V_{t+1}^B \\ &= \alpha q_t^L (u^L - r_t^L) + (1 - \alpha q_t^L)\delta V_{t+1}^B \\ &> \alpha q_t^L (u^L - \hat{p}) + (1 - \alpha q_t^L)\delta V_{t+1}^B \\ &= \alpha \sum_{\tau \in \{H,L\}} q_t^\tau I(\hat{p}, r_t^\tau)(u^\tau - \hat{p}) + \left[1 - \alpha \sum_{\tau \in \{H,L\}} q_t^\tau I(\hat{p}, r_t^\tau)\right] \delta V_{t+1}^B, \end{aligned}$$

which contradicts $DME.B$. $\square$

Proof of Lemma 2: We prove $L2.1$. Assume by way of contradiction that the claim does not hold, and let $\bar{t}$ be the first date such that $\rho_{\bar{t}}^H = 1$. By $P2.2$, $\bar{t} > 1$. We show that $\rho_{\bar{t}-1}^H = 1$, which contradicts that $\bar{t}$ is the first date for which $\rho_{\bar{t}}^H = 1$. Since $\rho_{\bar{t}}^H = 1$ and $V_t^L \geq 0$ for all t, we have

$$V_{\bar{t}}^L = \alpha(c^H - c^L) + (1-\alpha)\,\delta V_{\bar{t}+1}^L \geq \alpha(c^H - c^L).$$

Since frictions are small, then $\alpha\delta(c^H - c^L) > u^L - c^L$, and therefore

$$r_{\bar{t}-1}^L = c^L + \delta V_{\bar{t}}^L \geq c^L + \alpha\delta(c^H - c^L) > c^L + u^L - c^L = u^L.$$

Hence offering $r_{\bar{t}-1}^L$ at date $\bar{t}-1$ is suboptimal, i.e., $\rho_{\bar{t}-1}^L = 0$. Moreover, $q_{\bar{t}-1}^H = q_{\bar{t}}^H$. Since offering $r_{\bar{t}}^H$ at date $\bar{t}$ is optimal we have

$$V_{\bar{t}}^B = \alpha(u(q_{\bar{t}}^H) - c^H) + (1-\alpha)\,\delta V_{\bar{t}+1}^B,$$

and $u(q_{\bar{t}}^H) - c^H \geq \delta V_{\bar{t}+1}^B > 0$ (by $L1.2$). Thus, offering $r_{\bar{t}-1}^H = c^H$ ($L1.2$) at date $\bar{t}-1$ yields

$$\alpha(u(q_{\bar{t}-1}^H) - c^H) + (1-\alpha)\,\delta V_{\bar{t}}^B = \alpha(u(q_{\bar{t}}^H) - c^H)\,(1 + (1-\alpha)\delta) + (1-\alpha)^2\,\delta^2 V_{\bar{t}+1}^B.$$

Then we have

$$\begin{aligned} \alpha(u(q_{\bar{t}-1}^H) - c^H) + (1-\alpha)\,\delta V_{\bar{t}}^B - \delta V_{\bar{t}}^B &= \alpha(u(q_{\bar{t}}^H) - c^H)\,(1 - \alpha\delta) - (1-\alpha)\,\delta^2 \alpha V_{\bar{t}+1}^B \\ &\geq \alpha(u(q_{\bar{t}}^H) - c^H)\,(1-\delta)\,(1 + \delta(1-\alpha)) \\ &> 0. \end{aligned}$$

Hence offering a negligible price at date $\bar{t}-1$ is suboptimal, i.e., $1-\rho_{\bar{t}-1}^{L}-\rho_{\bar{t}-1}^{H}=0$. Since $\rho_{\bar{t}-1}^{L}=0$, then $\rho_{\bar{t}-1}^{H}=1$.

We prove $L2.2$. We first show that $\rho_t^L<1$ for $t<T$. Assume by way of contradiction that $\rho_t^L=1$ for some $t<T$. Then $L1.3$ and$\bar{\rho}/\alpha\delta<1$ by the inequality $F.2$ imply

$$q_T^H \geq q_{t+1}^H = g(q_t^H, 0) > g(q^H, \bar{\rho}/\alpha\delta) > \hat{q}.$$

Hence

$$q_T^H u^H + q_T^L u^L - c^H > \hat{q}u^H + (1-\hat{q})\, u^L - c^H = (1-\hat{q})\,(u^L - c^L) > q_T^L(u^L - c^L),$$

i.e., offering $r_T^L = c^L$ at date T is suboptimal, and therefore $\rho_T^L=0$. Thus, $\rho_T^H=1$ by $P2.3$, which contradicts $L2.1$.

We show that $\rho_T^L<1$. Assume that $\rho_T^L=1$. Then $q_T^H\leq\hat{q}$ (since otherwise an offer of r_T^L is suboptimal), $V_T^L=0$ and $V_T^B=\alpha q_T^L(u^L-c^L)$. Hence $r_{T-1}^L=c^L$ by $DME.L$, and

$$\begin{aligned}
q_{T-1}^L(u^L - r_{T-1}^L) + q_{T-1}^H\delta V_T^B &= q_{T-1}^L(u^L-c^L) + (1-q_{T-1}^L)\delta V_T^B \\
&> q_{T-1}^L\delta V_T^B + (1-q_{T-1}^L)\delta V_T^B \\
&= \delta V_T^B,
\end{aligned}$$

i.e., the payoff to offering r_{T-1}^L at date $T-1$ is greater than that of offering a negligible price. Therefore $\rho_{T-1}^L+\rho_{T-1}^H=1$. Since $q_{T-1}^H\leq q_T^H$ by $L1.3$ and $q_T^H\leq\hat{q}$, then the payoff to offering $r_{T-1}^H=c^H$ at $T-1$ is

$$\begin{aligned}
q_{T-1}^H u^H + q_{T-1}^L u^L - c^H &\leq q_T^H u^H + q_T^L u^L - c^H \\
&\leq q_T^L(u^L-c^L) \\
&\leq q_{T-1}^L(u^L-c^L) \\
&< q_{T-1}^L(u^L-c^L) + q_{T-1}^H\delta V_T^B,
\end{aligned}$$

where the last term is the payoff to offering $r_{T-1}^L=c^L$ at $T-1$. Hence $\rho_{T-1}^H=0$, and therefore $\rho_{T-1}^L=1$, which contradicts that $\rho_t^L<1$ for all $t<T$ as shown above. Hence $\rho_T^L<1$.

We prove $L2.3$. By $P2.3$, $L2.1$ and $L2.2$, we have $\rho_T^H>0$ and $\rho_T^L>0$. Since both high price offers and low price offers are optimal at date T, and reservation prices are

$r_T^H = c^H$ and $r_T^L = c^L$, we have

$$q_T^H u^H + q_T^L u^L - c^H = q_T^L(u^L - c^L).$$

Thus, using $q_T^L = 1 - q_T^H$ and solving for q_T^H yields

$$q_T^H = \frac{c^H - c^L}{u^H - c^L} = \hat{q}.$$

We prove $L2.4$ by induction. By $L2.3$, $V_T^L = \alpha\rho_T^H\left(c^H - c^L\right) > 0$. Since $V_t^L \geq \delta V_{t+1}^L$ for all $t \leq T$, then $V_t^L \geq \delta^{T-t}V_T^L > 0$.

We prove $L2.5$. Suppose by way of contradiction that $\rho_t^L = 0$ for some t. Since $\rho_T^L > 0$ by $L2.3$, then $t < T$. Also $\rho_t^L = 0$ implies $\rho_t^H > 0$ by $P2.1$. Since $\rho_t^H < 1$ by $L2.1$, then buyers are indifferent at date t between offering c^H or a negligible price, i.e.,

$$q_t^H u^H + q_t^L u^L - c^H = \delta V_{t+1}^B.$$

We show that $\rho_{t+1}^H = 0$. Suppose that $\rho_{t+1}^H > 0$; then

$$V_{t+1}^B = \alpha(q_{t+1}^H u^H + q_{t+1}^L u^L - c^H) + (1-\alpha)\delta V_{t+2}^B.$$

Hence $\delta < 1$ and $V_{t+1}^B > 0$ by $L1.2$ imply

$$q_t^H u^H + q_t^L u^L - c^H = \delta V_{t+1}^B < V_{t+1}^B = \alpha(q_{t+1}^H u^H + q_{t+1}^L u^L - c^H) + (1-\alpha)\delta V_{t+2}^B,$$

But $\rho_t^L = 0$ implies that $q_{t+1}^H = q_t^H$, and therefore

$$q_{t+1}^H u^H + q_{t+1}^L u^L - c^H < \delta V_{t+2}^B,$$

i.e., offering c^H at date $t+1$ yields a payoff smaller than offering a negligible price, which contradicts that $\rho_{t+1}^H > 0$.

Since $\rho_{t+1}^H = 0$, then $DME.L$ implies

$$V_{t+1}^L = \alpha\rho_{t+1}^L(r_{t+1}^L - c^L) + (1 - \alpha\rho_{t+1}^L)\delta V_{t+2}^L = \delta V_{t+2}^L.$$

Since $V_{t+1}^L > 0$ by $L2.4$, then $V_{t+2}^L > 0$, and therefore $DME.L$ and $\delta < 1$ imply

$$r_t^L = c^L + \delta V_{t+1}^L = c^L + \delta^2 V_{t+2}^L < c^L + \delta V_{t+2}^L = r_{t+1}^L.$$

i.e., $r_t^L < r_{t+1}^L$. We show that this inequality cannot hold, which leads to a contradiction.

Since $\rho_t^H < 1$ by $L2.1$, then $\rho_t^L = 0$ implies $1 - \rho_t^H - \rho_t^L > 0$; i.e., negligible price offers are optimal at date t. Hence at date t the payoff to offering r_t^L must be less than or equal to the payoff to offering a negligible price, i.e.,

$$q_t^H \delta V_{t+1}^B + q_t^L(u^L - r_t^L) \leq \delta V_{t+1}^B.$$

Using $q_t^H = 1 - q_t^L$ we may write this inequality as

$$u^L - r_t^L \leq \delta V_{t+1}^B.$$

Likewise, $\rho_{t+1}^H = 0$ implies $0 < \rho_{t+1}^L < 1$ by $P2.1$ and $L2.2$, and therefore $1 - \rho_{t+1}^H - \rho_{t+1}^L > 0$. Hence low and negligible price offers are both optimal at date $t+1$, and therefore

$$V_{t+1}^B = \alpha q_{t+1}^L(u^L - r_{t+1}^L) + (1 - \alpha q_{t+1}^L)\delta V_{t+2}^B = \delta V_{t+2}^B.$$

Hence

$$V_{t+1}^B = u^L - r_{t+1}^L.$$

Thus, $\delta < 1$ and $V_{t+1}^B > 0$ by $L1.2$ imply

$$u^L - r_t^L \leq \delta V_{t+1}^B < V_{t+1}^B = u^L - r_{t+1}^L.$$

Therefore $r_t^L > r_{t+1}^L$, which contradicts $r_t^L < r_{t+1}^L$.

We prove $L2.6$. For $t \in \{1, \ldots, T\}$, since $V_t^L \geq 0$, and $r_t^L - c^L = \delta V_{t+1}^L$ by $DME.L$, we have

$$\begin{aligned} V_t^L &= \alpha\left(\rho_t^H(c^H - c^L) + \rho_t^L(r_t^L - c^L)\right) + \left(1 - \alpha(\rho_t^H + \rho_t^L)\right)\delta V_{t+1}^L \\ &\geq \alpha\rho_t^H(c^H - c^L). \end{aligned}$$

By $P2.2$, we have $\rho_1^H = 0 < \bar{\rho}/\alpha\delta$. For $1 < t \leq T$, since $\rho_{t-1}^L > 0$ by $L2.5$ (i.e., low price offers are optimal at date $t-1$) and $V_{t-1}^B > 0$ by $L1.2$, then $u^L > r_{t-1}^L$. Hence

$$u^L - c^L > r_{t-1}^L - c^L = \delta V_t^L \geq \alpha\delta\rho_t^H\left(c^H - c^L\right),$$

and therefore

$$\rho_t^H < \frac{u^L - c^L}{\alpha\delta\left(c^H - c^L\right)} = \bar{\rho}/\alpha\delta.$$

Finally, we prove ($L2.7$). Let $t \in \{1, \ldots, T-1\}$. We proceed by showing that (i) $\rho_t^H > 0$ implies $\rho_t^H + \rho_t^L < 1$, and (ii) $\rho_t^H + \rho_t^L < 1$ implies $\rho_{t+1}^H > 0$. Then $L2.7$ follows by induction: Since $\rho_1^H = 0$ by $P2.2$ and $\rho_1^L < 1$ by $L2.2$, then $\rho_1^H + \rho_1^L < 1$, and therefore $\rho_2^H > 0$ by (ii). Assume that $\rho_k^H + \rho_k^L < 1$ and $\rho_{k+1}^H > 0$ holds for some $1 \leq k < T-1$; we show that $\rho_{k+1}^H + \rho_{k+1}^L < 1$ and $\rho_{k+2}^H > 0$. Since $\rho_{k+1}^H > 0$, then $\rho_{k+1}^H + \rho_{k+1}^L < 1$ by (i), and therefore $\rho_{k+2}^H > 0$ by (ii).

We establish (i), i.e., $\rho_t^H > 0$ implies $\rho_t^H + \rho_t^L < 1$. Suppose not; let $t < T$ be the first date such that $\rho_t^H > 0$ and $\rho_t^H + \rho_t^L = 1$. Since $q_t^H \geq q_1^H = q^H$ by $L1.3$, and $\rho_t^H < \bar{\rho}/\alpha\delta$ by $L2.6$, then $g(q^H, \bar{\rho}/\alpha\delta) > \hat{q}$ (by $F.2$) and $L2.3$ imply

$$q_{t+1}^H = g(q_t^H, \rho_t^H) > g(q^H, \bar{\rho}/\alpha\delta) > \hat{q} = q_T^H,$$

which contradicts $L1.3$.

Next we prove (ii), i.e., $\rho_t^H + \rho_t^L < 1$ implies $\rho_{t+1}^H > 0$. Suppose by way of contradiction that $\rho_t^H + \rho_t^L < 1$ and $\rho_{t+1}^H = 0$ for some $t < T$. Since $\rho_t^L > 0$ by $L2.5$, then low and negligible offers are optimal at date t. Hence

$$u^L - r_t^L = \delta V_{t+1}^B.$$

Since $\rho_{t+1}^H = 0$, then

$$V_{t+1}^L = \delta V_{t+2}^L.$$

Since $V_{t+1}^L > 0$ by $L2.4$ and $\delta < 1$, we have

$$r_{t+1}^L = c^L + \delta V_{t+2}^L = c^L + V_{t+1}^L > c^L + \delta V_{t+1}^L = r_t^L.$$

Since $0 < \rho_{t+1}^L < 1$ by $L2.2$ and $L2.5$ and $\rho_{t+1}^H = 0$, then $1 - \rho_{t+1}^H - \rho_{t+1}^L > 0$; i.e., low and negligible offers are optimal at $t+1$. Therefore

$$u^L - r_{t+1}^L = \delta V_{t+2}^B.$$

Thus, $V_{t+1}^B > 0$ by $L1.2$ and $\delta < 1$ imply

$$u^L - r_t^L = \delta V_{t+1}^B < V_{t+1}^B = \delta V_{t+2}^B = u^L - r_{t+1}^L,$$

i.e., $r_t^L > r_{t+1}^L$, which contradicts the inequality above. $\square$

The Public-Private Investment Program for Legacy Assets

As noted in Section 4, the introduction of a small PPIP subsidy $s > 0$ in a market where $1 < T < \infty$ affects the equilibrium sequences of probabilities of high price offers ρ^H and the reservation prices of low quality sellers r^L, as well as the traders' payoffs and surplus, via its impact on $\hat{q}(s)$, where

$$\hat{q}(s) = \frac{c^H - c^L - s}{u^H - c^L - s},$$

and hence via the functions $\bar{\phi}(s) = (1 - \hat{q}(s))\,(u^L - c^L)$, and $\phi_t(s) = \alpha\delta^{T-t}\bar{\phi}(s)$. The formulae describing the sequence of probabilities of low price offers ρ^L is

$$\rho_1^L(s) = \frac{\phi_2(s) - (u(q^H) - c^H) - (1 - q^H)s}{\alpha(1 - q^H)(c^H - u^L - s + \phi_2(s))},$$

and $\rho_T^L = 1 - \rho_T^H$. If $T > 2$, then

$$\rho_t^L(s) = (1 - \alpha\rho_t^H(s))\frac{(1 - \delta)\phi_{t+1}(s)}{\alpha(c^H - u^L - s + \phi_{t+1}(s))}\frac{u^H - u^L - s}{u^H - c^H - \phi_t(s)}$$

for all $1 < t < T - 1$, and

$$\rho_{T-1}^L(s) = (1 - \alpha\rho_{T-1}^H(s))\frac{u(\hat{q}(s)) - c^H + (1 - \hat{q}(s))s - \phi_{T-1}(s)}{\alpha\hat{q}(s)(u^H - c^H - \phi_{T-1}(s))}.$$

As δ approaches one, the high price is offerred with positive probability only at date T. Hence the cost of the subsidy $C(s)$ is

$$\begin{aligned}
C(s) &= s\alpha\rho_T^H(s)m_T^L(s) \\
&= s\frac{u^L - c^L - \alpha\bar{\phi}(s)}{c^H - c^L}m^H\frac{(1 - \hat{q}(s))}{\hat{q}(s)} \\
&= sm^H\frac{u^L - c^L - \alpha\bar{\phi}(s)}{c^H - c^L}\frac{u^H - c^H}{c^H - c^L - s}.
\end{aligned}$$

The net surplus, $NS(s)$, is

$$\begin{aligned}
NS(s) &= [\tilde{S}^{DME}(s) - C(s)] - \tilde{S}^{DME}(0) \\
&= m^H\alpha(u^L - c^L)\,(\hat{q}(0) - \hat{q}(s)) - sm^H\frac{u^L - c^L - \alpha\bar{\phi}(s)}{c^H - c^L}\frac{u^H - c^H}{c^H - c^L - s} \\
&= \frac{sm^H\left(u^H - c^H\right)\left(u^L - c^L\right)}{u^H - c^L - s}N(\alpha),
\end{aligned}$$

where

$$N(\alpha) := \frac{\alpha}{u^H - c^L} - \frac{u^H - c^L - s - \alpha\left(u^H - c^H\right)}{\left(c^H - c^L - s\right)\left(c^H - c^L\right)}.$$

Since

$$N(1) = -\frac{1}{c^H - c^L}\frac{u^H - c^H}{u^H - c^L} < 0,$$

and $dN(\alpha)/d\alpha > 0$, then $N(\alpha) < 0$. Therefore $NS(s) < 0$ for $s > 0$.

The Effect of a Subsidy Conditional on Trading at a Low Price

With a subsidy $s > 0$ to either buyers or sellers who trade the good at a low price $p < c^H$ the fraction of high quality in the market at the last date solves the equation

$$q_T^H u^H + \left(1 - q_T^H\right) u^L - c^H = \left(1 - q_T^H\right) \left(u^L - c^L + s\right).$$

Solving for q_T^H yields

$$q_T^H = \check{q}(s) = \frac{c^H - c^L + s}{u^H - c^L + s}.$$

Hence

$$\frac{d\check{q}(s)}{ds} = \frac{u^H - c^H}{\left(u^H - c^L + s\right)^2} > 0.$$

Also the role played by the functions $\bar{\phi}$ and ϕ_t in Proposition 3, is played by the functions $\check{\phi}(s) := (1 - \check{q}(s))\,(u^L - c^L + s)$ and $\check{\phi}_t(s) := \alpha\delta^{T-t}\check{\phi}(s)$. Hence

$$\frac{d\check{\phi}(s)}{ds} = -(u^L - c^L + s)\frac{d\check{q}(s)}{ds} + 1 - \check{q}(s) = \frac{\left(u^H - c^H\right)\left(u^H - u^L\right)}{\left(u^H - c^L + s\right)^2} \in (0, 1),$$

and

$$\frac{d\check{\phi}_t(s)}{ds} = \alpha\delta^{T-t}\frac{d\check{\phi}(s)}{ds} \in (0, 1).$$

The formulae for the probabilities of low price offers at each date are obtained by replacing $\hat{q}$, $\bar{\phi}$ and ϕ_t in the formulae given in Proposition 3 with $\check{q}(s)$, $\check{\phi}(s)$ and $\check{\phi}_t(s)$, respectively. However, the formulae describing the sequence of probabilities of high price offers and the traders' payoffs and surplus are as follows:

High Price Offers: $\rho_1^H = 0$,

$$\rho_t^H(s) = \frac{1 - \delta}{\alpha\delta}\frac{u^L - c^L + s}{c^H - u^L - s + \check{\phi}_t(s)},$$

for all $1 < t < T$, and

$$\rho_T^H(s) = \frac{u^L - c^L + s - \check{\phi}_{T-1}(s)}{\alpha\delta\left(c^H - c^L\right)}.$$

Payoffs and Surplus: $V_1^B(s) = \check{\phi}_1(s)$, $V_1^L(s) = u^L - c^L + s - \check{\phi}_1(s)$, and

$$S^{DME}(s) = m^L(u^L - c^L) + m^H\alpha\delta^{T-1}\check{\phi}(s) + sm^L.$$

Reservation prices: $r_t^L(s) = u^L + s - \check{\phi}_t(s)$ for all $t < T$ and $r_T^L(s) = c^L$ if the subsidy is given to buyers, and $r_t^L(s) = u^L - \check{\phi}_t(s)$ and $r_T^L(s) = c^L - s$ if it is given to sellers.

Corollary 8 follows readily by differentiating these formulae. We have

$$\frac{d\rho_T^H(s)}{ds} = \frac{1 - \frac{d\check{\phi}_{T-1}(s)}{ds}}{\alpha\delta\left(c^H - c^L\right)} > 0,$$

and

$$\begin{aligned}\frac{d\rho_1^L(s)}{ds} &= \frac{1}{\alpha(1-q^H)(c^H - u^L + \check{\phi}_2(s))}\left(1 - \frac{\check{\phi}_2(s) + c^H - u(q^H)}{c^H - u^L + \check{\phi}_2(s)}\right)\frac{d\check{\phi}_2(s)}{ds} \\ &= \frac{u(q^H) - u^L}{\alpha(1-q^H)(c^H - u^L + \check{\phi}_2(s))^2}\frac{d\check{\phi}_2(s)}{ds} \\ &> 0.\end{aligned}$$

Also $dV_1^B(s)/ds = d\check{\phi}_1(s)/ds > 0$ and $dV_1^L(s)/ds = 1 - d\check{\phi}_1(s)/ds > 0$. The effect on the net surplus is positive, since the cost of the subsidy is at most sm^L, while the subsidy increases the surplus by $m^H\alpha\delta^{T-1}(\check{\phi}(s) - \bar{\phi}) + sm^L > sm^L$.

If $T = \infty$, then

$$\hat{\rho}_1^L(s) = \frac{c^H - u(q^H)}{\alpha(1-q^H)(c^H - u^L)},$$

and $\hat{\rho}_t^L(s) = 0$ for $t > 1$. Also $\hat{\rho}_1^H = 0$, and

$$\hat{\rho}_t^H(s) = \frac{1-\delta}{\alpha\delta}\frac{u^L - c^L + s}{c^H - u^L - s}.$$

for $t > 1$. Thus, the subsidy increases the liquidity of both qualities. Moreover, the surplus is

$$\hat{S}^{DME}(s) = m^L(u^L - c^L) + sm^L,$$

the cost of the subsidy is $\alpha\hat{\rho}_1^L sm^L$, and hence the net surplus increases by $\left(1 - \alpha\hat{\rho}_1^L(s)\right) sm^L > 0$.

The Effect of a Subsidy Conditional on Trading at the High Price

With a subsidy $s > 0$ to either buyers or sellers who trade at the high price c^H the fraction of high quality in the market at the last date solves the equation

$$q_T^H u^H + \left(1 - q_T^H\right)u^L - c^H + s = \left(1 - q_T^H\right)\left(u^L - c^L\right).$$

Solving for q_T^H yields

$$q_T^H = \check{q}(s) = \frac{c^H - c^L - s}{u^H - c^L}.$$

Hence

$$\frac{d\check{q}(s)}{ds} = -\frac{1}{u^H - c^L} < 0.$$

The role played by the functions $\bar{\phi}$ and ϕ_t in Proposition 3 is now played by $\breve{\phi}(s) := (1 - \breve{q}(s))(u^L - c^L)$ and $\breve{\phi}_t(s) := \alpha\delta^{T-t}\breve{\phi}(s)$, respectively. Hence

$$\frac{d\breve{\phi}(s)}{ds} = -(u^L - c^L)\frac{d\breve{q}(s)}{ds} = \frac{u^L - c^L}{u^H - c^L} \in (0, 1),$$

and

$$\frac{d\breve{\phi}_t(s)}{ds} = \alpha\delta^{T-t}\frac{d\breve{\phi}(s)}{ds} \in (0, 1).$$

The formulae for the probabilities of high price offers at each date, and the traders' payoffs and surplus are obtained by replacing $\hat{q}$, $\bar{\phi}$ and ϕ_t in the formulae given in Proposition 3 with $\breve{q}(s)$, $\breve{\phi}(s)$ and $\breve{\phi}_t(s)$, respectively. However, the formulae describing the sequence of probabilities of low price offers are as follows:

$$\rho_1^L(s) = \frac{c^H - u(q^H) - s + \breve{\phi}_2(s)}{\alpha(1 - q^H)(c^H - u^L + \breve{\phi}_2(s))},$$

and $\rho_T^L(s) = 1 - \rho_T^H(s)$. If $T > 2$, then

$$\rho_t^L(s) = (1 - \alpha\rho_t^H(s))\frac{(1-\delta)\breve{\phi}_{t+1}(s)}{\alpha(c^H - u^L - s + \breve{\phi}_{t+1}(s))}\frac{u^H - u^L}{u^H - c^H + s - \breve{\phi}_t(s)}$$

for $t \in \{2, \dots, T-2\}$, and

$$\rho_{T-1}^L(s) = (1 - \alpha\rho_{T-1}^H(s))\frac{(1 - \alpha\delta)\,\breve{\phi}(s)}{\alpha\breve{q}(s)(u^H - c^H + s - \breve{\phi}_{T-1}(s))}.$$

Corollary 9 readily follows by differentiating these formulae. Differentiating ρ_t^H for $t \in \{2, \dots, T-1\}$ yields

$$\frac{d\rho_t^H(s)}{ds} = -\frac{1-\delta}{\alpha\delta}\frac{u^L - c^L}{(c^H - u^L + \breve{\phi}_t(s))^2}\frac{d\breve{\phi}_t(s)}{ds} < 0.$$

Also

$$\frac{d\rho_T^H(s)}{ds} = \bar{\rho}\frac{d\breve{q}(s)}{ds} < 0.$$

For low price offers,

$$\frac{d\rho_1^L(s)}{ds} = -\frac{1}{\alpha(1 - q^H)(c^H - u^L + \breve{\phi}_2(s))}\left(1 - \frac{d\breve{\phi}_2(s)}{ds} + \frac{c^H - u(q^H) - s + \breve{\phi}_2(s)}{c^H - u^L + \breve{\phi}_2(s)}\frac{d\breve{\phi}_2(s)}{ds}\right) < 0.$$

Finally, $dV_1^B(s)/ds = d\breve{\phi}_1(s)/ds > 0$ and $dV_1^L(s)/ds = -d\breve{\phi}_1(s)/ds < 0$, and

$$\frac{dS^{DME}(s)}{ds} = \alpha\delta^{T-1}m^H\frac{d\breve{\phi}(s)}{ds} > 0.$$

Thus, for $T = \infty$ the subsidy has no impact on either the payoffs or the surplus, and is purely wasteful.

Government Purchases

Assume that at the market open the government offers to buy β units of the good, e.g., via a uniform price auction. In equilibrium, the government acquires β units of low quality at a price equal to the reservation price of low quality sellers in the market that follows, i.e., r_1^L. In this market, after the government purchase, the measure of buyers exceeds the measure of sellers by β. We assume that the probability that a buyer is matched at date t is $\alpha\theta_t$, where

$$\theta_t = \frac{m_t^H + m_t^L}{m_t^H + m_t^L + \beta}$$

is the market tightness at date t.

Let us consider a market that opens over two dates, i.e., $T = 2$. A small government intervention does not affect the basic structure of the DME; specifically, at date 1 buyers only offer low and negligible prices with positive probability, and at date 2 only offer high and low prices with positive probability.

Since at date 2 buyers are indifferent between low and high price offers, then

$$q_2^H u^H + (1 - q_t^H)u^L - c^H = (1 - q_2^H)(u^L - c^L).$$

Thus, in equilibrium $q_2^H = \hat{q}$. At date 1, buyers are indifferent between offering low and negligible prices, i.e.,

$$u^L - r_1^L = \delta V_2^B = \delta\alpha\theta_2\bar{\phi},$$

which implies

$$r_1^L = u^L - \delta\alpha\theta_2\bar{\phi}.$$

Also by $DME.L$ the reservation price of low quality sellers satisfies

$$r_1^L = c^L + \delta V_2^L = c^L + \delta\alpha\rho_2^H(c^H - c^L).$$

Solving for ρ_2^H in the system of equations involving r_1^L yields

$$\rho_2^H = \frac{u^L - c^L - \delta\alpha\theta_2\bar{\phi}}{\delta\alpha(c^H - c^L)}.$$

Since the high price is offered with probability zero at date 1, then $m_2^H = m_1^H = m^H$. Also $m_2^L = (1-\alpha\rho_1^L)m_1^L$ and $m_1^L = m^L - \beta$. Hence

$$\frac{m_2^H}{m_2^H + m_2^L} = \frac{m^H}{m^H + (1-\alpha\rho_1^L)(m^L-\beta)} = \hat{q},$$

and therefore

$$m_2^L = (1-\alpha\rho_1^L)(m^L-\beta) = \frac{1-\hat{q}}{\hat{q}} m^H,$$

and

$$\theta_2 = \frac{m_2^H + m_2^L}{m_2^H + m_2^L + \beta} = \frac{m^H + \frac{1-\hat{q}}{\hat{q}} m^H}{m^H + \frac{1-\hat{q}}{\hat{q}} m^H + \beta} = \frac{m^H}{m^H + \hat{q}\beta}.$$

(We assume $\beta \le m^L - \frac{1-\hat{q}}{\hat{q}} m^H$ to ensure that $\rho_1^L \ge 0$.) Note that m_2^L, and therefore the measure of low quality sellers that trades at date 1, is independent of β. Since all low quality sellers matched at date 2 trade, then the liquidity of low quality and the volume of trade of low quality are also independent of β.

Substituting the expression for m_2^L into the expression for ρ_2^H gives

$$\rho_2^H = \frac{u^L - c^L - \delta\alpha \frac{m^H + m_2^L}{m^H + m_2^L + \beta}\bar{\phi}}{\delta\alpha(c^H - c^L)} = \frac{u^L - c^L - \delta\alpha \frac{m^H}{m^H + \hat{q}\beta}\bar{\phi}}{\delta\alpha(c^H - c^L)}.$$

Payoffs are

$$V_1^L = u^L - c^L - \delta\alpha\theta_2\bar{\phi},$$

and

$$V_1^B = \delta\alpha\theta_2\bar{\phi}.$$

Let ε be the amount by which the government values low quality less than buyers. The net surplus is

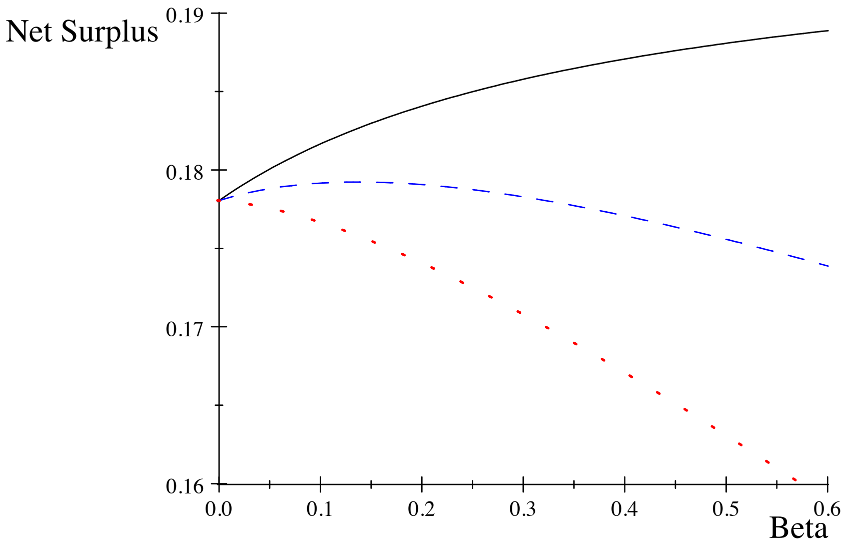
$$(m^H + m^L)V_1^B + m^L V_1^L + \beta(u^L - \varepsilon - r_1^L) = m^L(u^L - c^L) + (m^H + \beta)\delta\alpha \frac{m^H}{m^H + \hat{q}\beta}\bar{\phi} - \beta\varepsilon.$$

Differentiating this expression with respect to β and setting $\beta = 0$ yields $(1-\hat{q})\alpha\delta\bar{\phi} - \varepsilon$. Hence net surplus is increasing in β at $\beta = 0$ so long as $(1-\hat{q})\alpha\delta\bar{\phi} > \varepsilon$.

As an example, consider the market of Example 1 with $T = 2$ and $\alpha = \delta = .95$. Note that β cannot exceed $m^L - \frac{1-\hat{q}}{\hat{q}} m^H = .8 - .2 = 0.6$. Net surplus is increasing at $\beta = 0$ so long as

$$\alpha\delta(1-\hat{q})\bar{\phi} = (.95)^2(.5)(.1) = .045125 > \varepsilon.$$

Figure 1 below shows net surplus as a function of β for $\varepsilon = 0$ (solid line), $\varepsilon = .025$ (dashed line), and $\varepsilon = .05$ (dotted line).


Net Surplus
0.19
0.18
0.17
0.16
0.0
0.1
0.2
0.3
0.4
0.5
0.6
Beta