Asset Pricing Implications of Learning about Long-Run Risk

Daniel Andrei∗ Michael Hasler† Alexandre Jeanneret‡

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Abstract

We develop a dynamic pure-exchange economy that nests different types of learning about expected economic growth. We find that U.S. fundamentals are more likely to reflect an environment with learning about the persistence of expected economic growth (i.e., about long-run risk) rather than about its level. In equilibrium, learning about long-run risk generates strong fluctuations in stock-market volatility, risk premium, and Sharpe ratio, which are predicted to remain constant with other types of learning. Our novel theoretical predictions, for which we provide strong empirical support, explain the conditional properties of asset returns and enrich the asset-pricing implications of existing long-run risk and learning models.
1 Introduction

The long-run risk model of Bansal and Yaron (2004) has gained increasing popularity as a potential explanation for a broad class of asset market phenomena. Its two building blocks are recursive preferences (Epstein and Zin, 1989; Weil, 1989) and the existence of a small, persistent component in expected consumption growth. One potential caveat is the assumption that investors have perfect knowledge about the existence and behavior of this small component. Arguably, neither the degree of persistence nor the level of expected consumption growth is directly observable to investors. It is important to understand whether relaxing the perfect knowledge assumption would qualitatively change the asset pricing implications of the traditional long-run risk model. This is the objective of this paper.

We propose a general equilibrium model with incomplete information in which a representative agent learns about key parameters of the consumption growth process. In particular, two dimensions are simultaneously unobservable to the agent: the level of the expected consumption growth and its persistence. While uncertainty about the level of expected consumption growth has been thoroughly analyzed in the literature, we show that learning about the second dimension—that is, learning about long-run risk—generates distinct, important implications for the dynamics of asset returns. In particular, learning about long-run risk implies significant time-series variation in stock-market volatility, risk premium, risk-free rate, and Sharpe ratio. Moreover, when uncertainty about the degree of persistence in consumption growth is present, all these market outcomes become tightly linked to current economic conditions, thereby rationalizing why stock market volatility, risk premia and the Sharpe ratio increase as economic conditions deteriorate.

Learning about the unobservable degree of persistence in consumption growth can improve our understanding of asset pricing facts that have remained difficult to explain with traditional asset pricing models. To be more precise, our model assumes away any variation in the volatility of consumption growth—an important source of risk causing a time-varying equity premium in the long-run risk model (Bansal, Kiku, and Yaron, 2012; Beeler and Campbell, 2012); or time-varying disaster risk (Gabaix, 2012; Wachter, 2013); or any utility specification that induces time-varying preferences (Campbell and Cochrane, 1999). Instead, in our model, all business-cycle fluctuations in asset pricing moments arise endogenously due to investors’ updating about the unobservable degree of long-run risk.

We start by deriving a general specification which nests a continuum of learning models. A unique learning parameter, which takes values between zero and one, specifies the type of learning that prevails in the economy. At one extreme, the only unobservable dimension is the

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level of expected consumption growth—this corresponds to the case of learning about economic growth. At the other extreme, the only unobservable dimension is the degree of persistence in consumption growth—this corresponds to the case of learning about long-run risk. In between, both types of learning are simultaneously present in the economy and the learning parameter determines their relative importance for the agents. We calibrate the model and estimate this parameter in order to understand which type of learning best explains the dynamics of economic fundamentals in the United States. A Kalman-filter Maximum Likelihood estimation (Hamilton, 1994) over the period Q4:1968 to Q4:2015 on real GDP growth and analyst forecast data indicates that learning is concentrated on the long-run risk dimension (and not on the level of expected economic growth). The estimation also shows that the agents’ perception of the persistence parameter varies significantly over time, in contrast with the traditional assumption of long-run risk models that the degree of persistence remains constant across time and economic conditions.

Based on this calibration, we focus our analysis on the special case of the model in which agents learn exclusively about long-run risk. The main prediction is that uncertainty about long-run risk greatly enhances the sensitivity of stock market volatility, the equity risk premium, and the Sharpe ratio to current economic conditions and to the perceived degree of long-run risk. The mechanism relies on how agents form expectations and react to news about expected economic growth. To grasp the intuition, let us assume that agents receive news about expected economic growth from the Survey of Professional Forecasters, as publicly available from the Federal Reserve Bank of Philadelphia. Without loss of generality, consider that the latest consensus forecast on future economic growth is lower than previously thought, which we interpret as bad news. During a recession, such negative news prompt agents to expect a longer downturn and, therefore, to perceive more long-run risk. Stock prices then drop because of both bad news and the increased long-run risk. In contrast, during an economic boom, the same piece of news induces agents to perceive less long-run risk, through an identical Bayesian updating mechanism, which dampens the initial effect of the unexpected deterioration in analyst forecasts.2 Stock returns therefore respond differently to news about future economic growth in bad versus good economic times. This learning mechanism generates significant time-series variation in all asset pricing quantities. Notably, this effect is amplified when uncertainty about long-run risk increases, yet it disappears when agents have perfect information about the degree of persistence.

Our theoretical model predicts that uncertainty about long-run risk is a key channel through which asset pricing outcomes vary with both the state of the economy and the perceived degree of long-run risk. We test this prediction with S&P 500 data and find that

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2There are other (much weaker) effects that take place. We fully describe and analyze them in the paper.
all the regression coefficients predicted by the model have the correct sign and are strongly significant. Furthermore, the model-implied price-dividend ratio, stock return volatility, risk premium, and Sharpe ratio explain a large fraction of the variation in their observed counterparts with high coefficients of determination. Our analysis thus provides strong empirical support for a model with learning about long-run risk. In comparison, the asset pricing implications differ substantially when we consider an alternative version of the model with complete information about the consumption growth process or with incomplete information about the expected consumption growth only. In particular, these specifications are unable to generate time-variation in volatility, risk premium, and Sharpe ratio, at odds with the data.

This paper contributes to the literature that aims to understand stylized asset-pricing facts in frictionless settings with rational expectations (Campbell and Cochrane, 1999; Bansal and Yaron, 2004; Gabaix, 2012; Wachter, 2013). More specifically, our theoretical approach relates to the literature on learning in financial markets.\(^3\) Most studies in this literature commonly assume that parameters are known and that the unobservable dimension is the level of expected consumption growth, with a few notable exceptions. Johannes, Lochstoer, and Mou (2016) consider different Markov switching models with both unknown parameters and unknown states and show that model and state uncertainty help rationalize the observed levels of return volatility, risk premium, and Sharpe ratio.\(^4\) They build a theoretical model in which the agent has anticipated utility (Kreps, 1998; Cogley and Sargent, 2008), and thus parameter uncertainty is not a priced risk factor. In contrast, we focus on the dynamics of asset returns in a fully rational setup with Epstein-Zin utility and show, both theoretically and empirically, that uncertainty about the degree of persistence in consumption is a key channel through which asset pricing moments depend on economic conditions. Collin-Dufresne, Johannes, and Lochstoer (2016) show that parameter uncertainty generates endogenous long-run risk and therefore implies a large equilibrium risk premium when the representative agent has a preference for early resolution of uncertainty. One key result in Collin-Dufresne et al. (2016) is that the impact of parameter uncertainty on asset prices varies substantially, depending on

\(^3\)Although a comprehensive review of this literature is beyond the scope of our paper, a few studies are particularly relevant to this paper. Veronesi (1999) shows that return volatility and the risk premium are hump-shaped functions of the state of the economy when the representative agent learns about a discrete-state expected consumption growth rate. Ai (2010) considers a production economy with learning about productivity shocks and Epstein-Zin preferences. He shows that the relation between information quality and the risk premium is negative, whereas it is positive in pure-exchange economies (Veronesi, 2000). Croce, Lettau, and Ludvigson (2015) show that, when the representative agent has a preference for early resolution of uncertainty, learning about expected consumption growth in a bounded rationality limited information model generates both a large risk-premium and a downward-sloping term structure of risk.

\(^4\)Pakoš (2013) analyzes a similar economy featuring a three-state Markov chain in which a representative agent cannot distinguish between a mild recession and a “lost decade”. This modeling choice automatically introduces a stronger response to news in bad times. In our case, the agent learns about long-run risk at all times—good or bad—and the asymmetric stock price sensitivity across economic conditions arises endogenously from the learning mechanism. See also Johnson (2001) for a model with learning about persistence.
which parameters are considered. Therefore, it is important to understand which parameters—
when unknown—matter most to investors, a question that we address in this paper. First,
we provide new empirical evidence that agents are most likely to learn about the degree of
persistence in expected consumption growth. Second, our theory offers insights into how
this learning choice generates novel interactions between the severity of long-run risk, its
uncertainty, and the state of the economy. Overall, these findings improve our understanding
of the time variation and counter-cyclicality of return volatility, risk premia, and Sharpe ratio
observed in the data.

The remainder of the paper is organized as follows. Section 2 introduces a model that
nests learning about economic growth and learning about persistence. Section 3 calibrates the
model and provides empirical evidence that investors essentially learn about the persistence
of expected consumption growth. Section 4 presents our theoretical predictions, while Section
5 tests these predictions. Section 6 offers some concluding remarks and future directions for
research.

2 Model

This section analyzes a pure-exchange economy in which the representative agent faces incom-
plete information about the dynamics of consumption. Bayesian learning allows the agent to
reduce the uncertainty about the level of economic growth, the severity of long-run risk, or a
combination of the two. We derive the equilibrium asset pricing implications based on these
different learning choices.

2.1 Environment

Consider a pure-exchange economy defined over a continuous-time horizon $[0, \infty)$ and popu-
lated by a representative agent who derives utility from consumption. The agent has stochastic
differential utility (Epstein and Zin, 1989) with subjective time preference rate $\beta$, relative risk
aversion $\gamma$, elasticity of intertemporal substitution $\psi$. The indirect utility function is given by

$$J_t = E_t \left[ \int_t^\infty h(C_s, J_s) ds \right],$$

where the aggregator $h$ is defined as in Duffie and Epstein (1992):

$$h(C, J) = \frac{\beta}{1 - 1/\psi} \left( \frac{C^{1-1/\psi}}{[(1 - \gamma)J]^{1/\phi-1}} - (1 - \gamma)J \right),$$

with $\phi \equiv \frac{1 - \gamma}{1 - 1/\psi}$. Note that if $\gamma = 1/\psi$, then $\phi = 1$ and we obtain the standard CRRA utility.
A risk-free asset is available in zero net supply. A single risky asset—the stock—is available in unit supply, and represents the claim to the aggregate consumption stream, which follows the process

$$\frac{d\delta_t}{\delta_t} = \mu_t dt + \sigma_\delta dW^\delta_t,$$

(3)

where $dW^\delta_t$ is a standard Brownian motion.

The expected consumption growth rate, $\mu_t$, is unobservable. The history of the consumption process (3) provides an incomplete signal about the expected growth rate. Additionally, we assume that the agent continuously receives a time-varying forecast of $\mu_t$, produced by a Survey of Professional Forecasters. The expected growth rate then becomes

$$\mu_t = f_t + b,$$

(4)

where $f_t$ is the observed forecast and $b$ is an unknown bias, which we assume for simplicity to be constant.\(^5\) The expected growth forecast follows

$$df_t = \Lambda(\bar{f} - f_t) dt + \sigma_f dW^f_t,$$

(5)

where the Brownians $dW^f_t$ and $dW^\delta_t$ are independent.\(^6\) The mean-reverting parameter $\Lambda$ controls the persistence of the forecast process and, thus, the persistence of the expected growth rate. We assume that the degree of persistence is unobservable and that the agent starts with a prior $\bar{\lambda}$ on $\Lambda$.

We embed the above two dimensions of uncertainty, which relate to both the level of the expected growth rate $\mu_t$ and its degree of persistence $\Lambda$, in a unified theoretical framework. To this end, we propose the following specification:

$$\frac{d\delta_t}{\delta_t} = [f_t + (1 - \theta)l] dt + \sigma_\delta dW^\delta_t,$$

$$df_t = (\theta\lambda + \bar{\lambda})(\bar{f} - f_t) dt + \sigma_f dW^f_t,$$

(6)\quad(7)

where $(1 - \theta)l$ and $\theta\lambda + \bar{\lambda}$ respectively represent the bias $b$ and the persistence $\Lambda$, which we have introduced in (4) and (5).

\(^5\)In Appendix C.1, we allow this bias to be time-varying and driven by a standard Brownian motion. We show that this richer specification, which implies an additional estimated parameter, only has a minor impact on the calibrated model and, most important, does not change our theoretical findings.

\(^6\)The model can be extended by allowing the two Brownians in (3) and (5) to be correlated. However, we assume that they are independent. This simplifies the description of the model without changing our main message.
The parameter $\theta$, which is novel to our formulation, allows us to contrast the equilibrium implications of two types of learning. When $\theta$ is either 0 or 1, there is a unique source of uncertainty and the agent focuses her attention on one dimension only. If $\theta = 0$, the agent has perfect knowledge about the magnitude of long-run risk, which is now determined by $\bar{\lambda}$. However, she now faces uncertainty about the level of the expected economic growth rate, which in this case becomes the growth forecast $f_t$ plus an unknown bias $l$. That is, the agent views the growth forecast as imperfect and seeks to improve this estimate; we call this case learning about economic growth. Conversely, if $\theta = 1$, the agent believes that the forecast $f_t$ is a perfect estimator of expected consumption growth. However, she now faces uncertainty about the mean-reverting speed of the consumption growth rate, $\bar{\lambda} + \lambda$, and consequently aims to estimate it; we call this case learning about long-run risk.

Finally, the parameter $\theta$ can take any value between 0 and 1, in which case the agent learns simultaneously about economic growth and long-run risk. If $\theta$ approaches zero, the agent is more concerned with learning about the level of economic growth; if $\theta$ approaches one, the agent is more concerned with learning about the risk that she faces in the long term due to the slow-moving persistent component. The specification (6)-(7), therefore, conveniently embeds a continuum of learning choices in one unified framework, while preserving the linearity of the learning exercise.\footnote{The model feature that helps maintain the linearity of the learning exercise is the absence of the bias from the process (7). That is, we assume that the bias in the forecast of the expected growth rate is not following the same process as the forecast itself. While the specification (6) implies that the bias $(1 - \theta)l$ is constant, we allow for i.i.d. changes in $l$ in Appendix C.1. We show that, in this case, the structure of our model is preserved and the calibration does not change significantly.}

### 2.2 Learning dynamics

We now examine the agent’s updating of beliefs as new information arrives. The agent starts with the following priors about the unknown parameters $l$ and $\lambda$:

\[
\begin{bmatrix} l \\ \lambda \end{bmatrix} \sim \mathcal{N}\left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \nu_{l,0} & 0 \\ 0 & \nu_{\lambda,0} \end{bmatrix} \right). \tag{8}
\]

With this specification, the value of $\theta$ affects neither the agent’s prior about the long-term level of the expected growth rate, which always equals $\bar{f}$, nor the agent’s prior about the mean-reverting speed of the expected growth rate, which always equals $\bar{\lambda}$. Notice that there is no correlation between the two priors.

Define $\mathcal{F}_t$ the information set of the agent at time $t$, and denote by $\hat{l}_t \equiv \mathbb{E}[l|\mathcal{F}_t]$ the estimated parameter $l$ and its posterior variance by $\nu_{l,t} \equiv \mathbb{E}[(l - \hat{l}_t)^2|\mathcal{F}_t]$. Similarly, denote by $\hat{\lambda}_t \equiv \mathbb{E}[\lambda|\mathcal{F}_t]$ the estimated parameter $\lambda$ and its posterior variance by $\nu_{\lambda,t} \equiv \mathbb{E}[(\lambda - \hat{\lambda}_t)^2|\mathcal{F}_t]$. 
These estimates and posterior variances are such that

\[ l \sim N(\hat{l}_t, \nu_{l,t}), \quad \lambda \sim N(\hat{\lambda}_t, \nu_{\lambda,t}), \]

where \( N(m, v) \) denotes the Normal distribution with mean \( m \) and variance \( v \). It is convenient to interpret the estimates \( \hat{l} \) and \( \hat{\lambda} \) as the filters and the two posterior variances \( \nu_{l,t} \) and \( \nu_{\lambda,t} \) as the levels of uncertainty. Following Liptser and Shiryayev (1977), the filters evolve according to

\[
\begin{bmatrix}
    d\hat{l}_t \\
    d\hat{\lambda}_t
\end{bmatrix} =
\begin{bmatrix}
    (1-\theta)\nu_{l,t} & 0 \\
    0 & \theta (f_t - f) \nu_{\lambda,t}
\end{bmatrix}
\begin{bmatrix}
    d\hat{W}^\delta_t \\
    d\hat{W}^f_t
\end{bmatrix},
\]

(10)

where

\[
\begin{align*}
    d\hat{W}^\delta_t &= \frac{1}{\sigma_\delta} \left( \frac{\delta_t}{\delta_t} - [f_t + (1 - \theta)\hat{l}_t] dt \right), \\
    d\hat{W}^f_t &= \frac{1}{\sigma_f} \left( df_t - (\theta \hat{\lambda}_t + \hat{\lambda})(\bar{f} - f_t) dt \right).
\end{align*}
\]

(11)

are independent Brownian motions, resulting from the filtration of the agent. For clarity, we will hereafter use the term consumption growth shocks to refer to \( d\hat{W}^\delta \) innovations and the term expected consumption growth shocks to refer to \( d\hat{W}^f \) innovations.

One can observe from (10) that the agent does not update \( \lambda \) if \( \theta = 0 \) and thus keeps her prior \( \bar{\lambda} \) regarding the degree of long-run risk in the economy. Conversely, if \( \theta = 1 \), the agent does not learn about \( l \) and thus uses the growth forecast \( f_t \) as the predictor of future consumption growth. Finally, a value of \( \theta \) between zero and one implies that the agent updates both estimates \( \hat{l} \) and \( \hat{\lambda} \).

The estimate related to expected economic growth, \( \hat{l} \), is perfectly positively correlated with consumption \( \delta \), which implies “extrapolative expectations” (Brennan, 1998): after positive (negative) consumption shocks, the agent revises her estimate of the expected growth rate upwards (downwards). More interestingly, this extrapolative expectation formation also occurs in the case of learning about long-run risk. However, the effect now depends on the state of the economy, which is characterized by the level of the current growth rate forecast \( f_t \) relative to its unconditional mean \( \bar{f} \). In good times \((\bar{f} - f_t < 0)\), positive expected consumption growth shocks decrease the agent’s estimate of \( \lambda \); in bad times \((\bar{f} - f_t > 0)\), negative expected consumption growth shocks decrease the agent’s estimate of \( \lambda \). In both situations (i.e., positive shocks in good times or negative shocks in bad times), the agent extrapolates that expected consumption growth becomes more persistent, and that long-run risk increases. As we will show, this extrapolative expectation formation plays a critical role in the behavior of equilibrium asset prices.
The dynamics of the posterior uncertainties about \( l \) and \( \lambda \) are respectively given by

\[
d\nu_{l,t} = -(1 - \theta)^2 \nu_{l,t}^2 dt, \quad d\nu_{\lambda,t} = -\frac{\theta^2 (\bar{f} - f_t)^2 \nu_{\lambda,t}^2}{\sigma_f^2} dt, \quad (12)
\]

and indicate that both uncertainties converge to zero in the long-run, since the agent learns about constants.\(^8\) Notably, the convergence is faster for \( \nu_{\lambda} \) when \( f_t \) is far away from \( \bar{f} \). This is because news about \( \lambda \) become more informative when the economy is either in very good or very bad times.

### 2.3 Asset pricing

We turn now to equilibrium asset prices and examine the implications of learning about economic growth and/or about long-run risk.

#### 2.3.1 Equilibrium

In this environment, the equilibrium is standard and the technical details are relegated to Appendix B. Solving for the equilibrium involves writing the HJB equation for problem (1):

\[
\max_C \{h(C, J) + \mathcal{L}J\} = 0,
\]

with the differential operator \( \mathcal{L}J \) following from Itô’s lemma. We guess the following value function (Benzoni, Collin-Dufresne, and Goldstein, 2011):

\[
J(C, f, \hat{l}, \hat{\lambda}, \nu_l, \nu_{\lambda}) = C^{1-\gamma} [\beta I(x)]^\phi,
\]

where \( I(x) \) is the price-dividend ratio, and \( x \equiv [f \hat{l} \hat{\lambda} \nu_l \nu_{\lambda}]^\top \) denotes the vector of state variables, whose dynamics are given in (7), (10), and (12), respectively.

Substituting the guess (14) in the HJB equation (13) and imposing the market clearing condition, \( C = \delta \), yields the partial differential equation (45) for the log price-dividend ratio. We solve for this equation numerically using Chebyshev polynomials (Judd, 1998). The PDEs corresponding to the two polar forms of learning (i.e., \( \theta = 0 \) and \( \theta = 1 \)) are provided respectively in Equations (46) and (47) in Appendix B.

\[^8\]It is straightforward to generate positive steady-state uncertainty by assuming that learning is regenerated, i.e., that \( l \) and \( \lambda \) are not constants but move over time (we do this for the parameter \( l \) in Appendix C.1). This would unnecessarily complicate our setup without affecting our qualitative implications. Moreover, as shown by Collin-Dufresne et al. (2016), learning about a constant generates substantial effects when the representative agent has a preference for early resolution of uncertainty, as it is the case in our setup.
In order to characterize the effects of learning on equilibrium outcomes, we propose the following conjecture.

**Conjecture 1.** With preferences satisfying $\gamma > 1 > 1/\psi$, we expect:

\[
\frac{I_f}{T} > 0, \quad \frac{I_l}{T} > 0, \quad \frac{I_\lambda}{T} > 0, \quad \frac{I_{\nu}}{T} < 0, \quad \frac{I_{\nu_\lambda}}{T} < 0.
\]

This conjecture, which we will verify with a numerical exercise in Section 4, is likely to be true for a wide range of parameters. In fact, several inequalities follow directly from the definition of the value function in (14). Taking the derivative of $J$ with respect to any of the five state variables yields

\[
J_{(\cdot)} = \phi J \frac{I_{(\cdot)}}{T},
\]
with the product $\phi J$ being positive when $\gamma > 1 > 1/\psi$. Then, note that due to non-satiation, expected lifetime utility must rise as investment opportunities improve, i.e., $J_f > 0$ and $J_l > 0$. Using (16), this implies the first two inequalities of Conjecture 1. Next, being risk averse, the agent prefers less uncertainty and thus $J_{\nu} < 0$ and $J_{\nu_\lambda} < 0$, which yields the last two inequalities of Conjecture 1. The only inequality that needs numerical validation is $I_\lambda/I > 0$. Because the agent prefers early resolution of uncertainty, we expect that she prefers less persistence and thus $J_\lambda > 0$.

### 2.3.2 State-price density, risk-free rate, and market price of risk

Following Duffie and Epstein (1992), the state-price density satisfies

\[
\xi_t = e^{\int_0^t h_j(C_s, J_s) ds} h_C(C_t, J_t) = e^{\int_0^t [(\phi - 1)/I(x_s) - \beta \phi] ds} \beta \phi C_t^{-\gamma} I(x_t)^{\phi - 1},
\]
and thus the dynamics of the state-price density follow

\[
\frac{d\xi_t}{\xi_t} = -r_t dt - m_t^\top d\tilde{W}_t,
\]
where $r$ is the risk-free rate, $m$ is the 2-dimensional market price of risk, and $\tilde{W} \equiv [\tilde{W}_\delta, \tilde{W}_f]^\top$ is the 2-dimensional standard Brownian motion. Letting $\sigma_t(x) \equiv [\sigma_{I1}(x) \sigma_{I2}(x)]$ be the diffusion vector of the price-dividend ratio and applying Ito’s lemma to Equation (17) yields

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9In Appendix E, we perform a numerical evaluation of $I_\lambda/I$ for a wide range of values of the risk aversion, the intertemporal of elasticity substitution, and two state variables $f_t$ and $\hat{X}_t$. We find that $I_\lambda/I$ is positive and large in all cases. We also elaborate on parametrizations for which this term can become negative, which could happen outside the standard calibration of our model.
the risk-free rate

\[ r_t = \beta + \frac{1}{\psi^2} \left[ f_t + (1 - \theta) \hat{l}_t \right] - \frac{\gamma + \gamma \psi}{2 \psi^2} \sigma_\delta^2 - (1 - \phi) \left[ \sigma_{I1}(x_t) \sigma_\delta + \frac{1}{2} \left( \sigma_{I1}^2(x_t) + \sigma_{I2}^2(x_t) \right) \right] \] (19)

and the market price of risk

\[ m_t = \left[ \gamma \sigma_\delta + (1 - \phi) \sigma_{I1}(x_t) \right] \left( 1 - \phi \right) \sigma_{I2}(x_t) \right]^\top \] (20)

The two diffusion elements of the price-dividend ratio are given by

\[ \sigma_{I1}(x_t) = \frac{(1 - \theta) \nu_{l,t}}{\sigma_\delta} \frac{I_{\hat{l}}}{I} \] (21)
\[ \sigma_{I2}(x_t) = \frac{\sigma_f I_f}{I} + \frac{\theta (\bar{f} - f_t) \nu_{\lambda,t}}{\sigma_f} \frac{I_{\hat{\lambda}}}{I}. \] (22)

Focusing on the equilibrium risk-free rate, the second term of the expression (19) indicates that fluctuations in expected consumption growth, which result from both the mean-reverting property of the growth forecast and from learning about the level of expected consumption growth, generate a procyclical risk-free rate. Furthermore, when the investor prefers early resolution of uncertainty (i.e., \( 1 - \phi > 0 \)), the risk-free rate contains an additional term due to variations in \( f, \hat{l}, \) and \( \hat{\lambda} \). This term arises because the agent dislikes long-run risk and the uncertainty regarding the level of such risk. The resulting effect is a lower risk-free rate due to greater demand for the safe asset.

The market price of risk, defined in Equation (20), contains two elements, one for each of the two Brownians in the economy. The uncertainty about the expected growth rate, \( \nu_{l,t} \), increases the first component when the agent learns about the level in expected growth (\( \theta < 1 \)). The impact of learning about long-run risk (\( \theta > 0 \)), instead, is present in the second component of the market price of risk and depends on the state of the economy, as given by the difference \( \bar{f} - f_t \). Following Conjecture 1, \( I_{\hat{\lambda}} / I > 0 \) and the market price of risk is expected to increase in bad times, when \( \bar{f} - f_t > 0 \), and to decrease in good times, when \( \bar{f} - f_t < 0 \).

We will examine in depth this theoretical prediction in Section 4.3.

2.3.3 Stock market volatility

We now determine how the level and the dynamics of stock return volatility vary with different learning choices. The diffusion of stock returns, \( \sigma \), satisfies

\[ \sigma_t = \left[ \sigma_\delta + \sigma_{I1}(x_t) \sigma_{I2}(x_t) \right], \] (23)
which, after replacing (21)-(22), can be written as

$$\sigma_t = \left[ \sigma_\delta + \frac{(1-\theta)\nu_{\lambda t}}{\sigma_\delta} \frac{L_t}{T} \right] \left[ \gamma \sigma_\delta + (1 - \phi) \frac{(1-\theta)\nu_{\lambda t}}{\sigma_\delta} \frac{L_t}{T} \right] + (1 - \phi) \left( \sigma_f \frac{L_t}{T} + \frac{\theta(f_t - f_0)\nu_{\lambda t}}{\sigma_f} \frac{L_t}{T} \right)^2. \tag{24}$$

The diffusion of stock returns indicates that learning about the level of expected consumption growth ($\theta < 1$) can generate excess volatility in stock returns. According to Conjecture 1, the term $I_f/I$ is positive, and thus uncertainty about the expected growth rate $\nu_{t,t}$ increases the magnitude of the first diffusion term.

With uncertainty and learning about long-run risk, the sensitivity of stock returns to expected consumption growth shocks now varies with the current state of the economy, as determined by $\bar{f} - f_t$. As we can see from the second diffusion term, the role of learning about long-run risk ($\theta > 0$), due to the uncertainty associated with such risk ($\nu_{\lambda} > 0$), creates an asymmetric stock market response to shocks. To understand the effect, consider a negative shock on expected consumption growth, $d\hat{W}_t f_t < 0$, although the same intuition holds for a good shock. A bad shock in bad times induces the agent to update that there is more long-run risk, as she now believes that expected consumption growth becomes more persistent. The stock price thus drops not only because of the negative shock but also because of greater long-run risk. Hence, stock returns strongly react to shocks when the economy is in a bad state. By contrast, following a bad shock in good times, the agent perceives less long-run risk, which mitigates the initial effect of the negative shock. Learning about long-run risk thus attenuates the stock price response to shocks in good times, whereas it amplifies the response in bad times.

Therefore, learning about the level of consumption growth helps generate a high, but constant, level of stock return volatility. In comparison, learning about long-run risk creates an asymmetric relation between stock returns and shocks that yields counter-cyclical stock return volatility. Our numerical analysis in Section 4.2 will illustrate and explore these theoretical predictions in greater detail.

### 2.3.4 Equity risk premium

We now examine how the equity risk premium varies with learning. The risk premium in our economy is defined as $\mu_t - r_t = \sigma_t m_t$. Using the expressions (20) and (24), we obtain

$$\mu_t - r_t = \left( \sigma_\delta + \frac{(1-\theta)\nu_{\lambda t}}{\sigma_\delta} \frac{L_t}{T} \right) \left( \gamma \sigma_\delta + (1 - \phi) \frac{(1-\theta)\nu_{\lambda t}}{\sigma_\delta} \frac{L_t}{T} \right) + (1 - \phi) \left( \sigma_f \frac{L_t}{T} + \frac{\theta(f_t - f_0)\nu_{\lambda t}}{\sigma_f} \frac{L_t}{T} \right)^2. \tag{25}$$

The equity risk premium consists of two terms. The first term is the risk premium arising from fluctuations in consumption growth. This part, which relates to the uncertainty about the level of consumption growth, becomes particularly relevant when the agent is strongly
risk-averse and prefers early resolution of uncertainty, i.e., when $\psi > 1/\gamma$.

The second term captures the risk premium arising from fluctuations in expected consumption growth. When $\gamma > 1/\psi$, this term is positive and increases with the uncertainty about the persistence in expected consumption growth because the agent dislikes both long-run risk and the uncertainty about it. Furthermore, the term in brackets is higher (lower) when the growth forecast $f_t$ is below (above) its long-term level $\bar{f}$.

As a result, the equity risk premium in our economy fluctuates over time with learning about long-run risk. However, this risk premium remains independent from the state of the economy if the agent can perfectly estimate the persistence parameter, as $\nu_{\lambda,t}$ becomes null. A similar prediction is obtained if the agent concentrates her attention to estimating the level of expected consumption growth, which means that $\theta = 0$. Various types of learning, thus, will generate different asset pricing implications. We will thoroughly examine these implications in Section 4.2.

3 Calibration

We now calibrate the model and identify the type of learning that is more likely to explain the data. To this end, we use the mean analyst forecast on 1-quarter-ahead real GDP growth as a direct measure of the growth forecast $f_t$ and the realized real GDP growth as a proxy for the growth rate of the process $\delta_t$. We consider data on output rather than on consumption to exploit time series of analyst forecasts. Data are obtained from the Federal Reserve Bank of Philadelphia and are available at quarterly frequency from Q4:1968 to Q4:2015.

We use the dynamics of the filters $\hat{l}$ and $\hat{\lambda}$ (10), the dynamics of the uncertainties about $l$ and $\lambda$ (12), and the filtered Brownian shocks (11) to generate model-implied paths of consumption and the expected growth rate. We estimate the model by Kalman-filter Maximum Likelihood (Hamilton, 1994) and determine the values of the parameters $\sigma_\delta$, $\bar{f}$, $\sigma_f$, $\bar{\lambda}$, $\theta$ that provide the closest fit to realized observations. The initial priors are $\hat{\nu}_0 = \bar{\lambda}_0 = 0$, $\nu_{l,0} = 0.02^2$, and $\nu_{\lambda,0} = 0.2^2$. The details specific to our implementation are summarized in Appendix C, while Table 1 reports the results.

The estimation suggests that an environment with learning about the persistence of economic growth is most likely, as the learning parameter is $\theta = 0.99$. While we clearly reject the null hypothesis that the learning parameter $\theta$ is equal to 0, we cannot reject the null that it is equal to 1 (p-value=0.99). Hence, the data indicate that learning is exclusively about the persistence (rather than about the level) of expected economic growth. We confirm this finding by comparing the results with two alternative specifications, which are presented in Table 1. First, we consider the specific case of $\theta = 1$ and re-estimate this constrained version
Table 1: Parameter estimates

This table reports the estimates of the model parameters. To calibrate the model, we use mean analyst forecast on the 1-quarter-ahead real GDP growth as a measure of the growth forecast and the realized real GDP growth as a proxy for consumption growth. The estimates are obtained by Maximum Likelihood for the period Q4:1968 to Q4:2015, using data from the Federal Reserve Bank of Philadelphia. The table compares the estimation results of the base case model with those of two constrained models, with $\theta = 1$ and $\theta = 0$, respectively. Standard errors are in brackets and statistical significance at the 10%, 5%, and 1% levels is denoted by *, **, and ***, respectively.

An interpretation of our finding is that learning about the unobservable degree of long-run risk offers better insights about future economic conditions than learning about the level of expected growth rate, which is a more relevant indicator of the current economic state. In addition, high quality forecasts about the level of expected growth are readily available from professional surveys, thereby reducing the marginal benefit of learning about this dimension. Our estimation thus suggests that the key uncertainty that agents are facing is less about whether the economy is in a recession or an expansion—for instance, it was pretty clear that we were in a recession during the latest financial crisis of 2007-08—but more about how persistent the current state of the economy is expected to be. Learning about the persistence component of the expected growth rate helps agents to precisely deal with this particular type of uncertainty.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Base case</th>
<th>$\theta = 1$</th>
<th>$\theta = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output growth volatility</td>
<td>$\sigma_\delta$</td>
<td>$1.42%$***</td>
<td>$1.42%$***</td>
<td>$1.45%$***</td>
</tr>
<tr>
<td></td>
<td></td>
<td>($5 \times 10^{-4}$)</td>
<td>($6 \times 10^{-4}$)</td>
<td>($6 \times 10^{-4}$)</td>
</tr>
<tr>
<td>Long-term expected growth</td>
<td>$\tilde{f}$</td>
<td>$2.59%$***</td>
<td>$2.59%$***</td>
<td>$2.61%$***</td>
</tr>
<tr>
<td></td>
<td></td>
<td>($0.005$)</td>
<td>($0.005$)</td>
<td>($0.004$)</td>
</tr>
<tr>
<td>Expected growth volatility</td>
<td>$\sigma_f$</td>
<td>$2.32%$***</td>
<td>$2.32%$***</td>
<td>$2.35%$***</td>
</tr>
<tr>
<td></td>
<td></td>
<td>($7 \times 10^{-4}$)</td>
<td>($9 \times 10^{-4}$)</td>
<td>($9 \times 10^{-4}$)</td>
</tr>
<tr>
<td>Long-term mean reversion speed</td>
<td>$\bar{\lambda}$</td>
<td>$0.78%$***</td>
<td>$0.78%$***</td>
<td>$0.95%$***</td>
</tr>
<tr>
<td></td>
<td></td>
<td>($0.259$)</td>
<td>($0.259$)</td>
<td>($0.192$)</td>
</tr>
<tr>
<td>Learning parameter</td>
<td>$\theta$</td>
<td>$0.99%$***</td>
<td>$0.99%$***</td>
<td>$0.99%$***</td>
</tr>
<tr>
<td></td>
<td></td>
<td>($0.232$)</td>
<td>($0.232$)</td>
<td>($0.232$)</td>
</tr>
<tr>
<td>Akaike information criterion</td>
<td>AIC</td>
<td>$-2,484$</td>
<td>$-2,486$</td>
<td>$-2,480$</td>
</tr>
</tbody>
</table>
To better grasp the levels and the time-variation in the main variables, Table 2 reports some descriptive statistics. The growth rate forecast $f_t$ equals 2.6%, on average, and ranges between −3.8 to 6.5%. In contrast, the estimated deviation in the forecast $\tilde{l}_t$ is close to zero over the sample period, thereby confirming the view that professional forecasts are not systematically biased. This result explains why learning about the level of the expected growth rate is not relevant when such forecasts are available. In contrast, the filter of long-run risk $\hat{\lambda}_t$ varies strongly over time, approximately between −0.12 to 0.08. As a result, the mean-reversion speed of expected consumption growth, which is equal to $\theta \hat{\lambda}_t + \bar{\lambda}$, varies approximately between 0.66 to 0.86. Finally, the uncertainty about this parameter, $\nu_{\lambda,t}$, ranges between 0.02 and 0.04. Overall, these results suggest that long-run risk clearly fluctuates over time and across economic conditions. This finding thus stands in contrast to the existing asset pricing literature, which typically considers economies with constant long-run risk.

Regarding the preference parameters, we fix the risk aversion to $\gamma = 10$, the elasticity of intertemporal substitution (EIS) to $\psi = 2$, and the subjective discount factor to $\beta = 0.03$. This choice of parameters is consistent with the asset pricing literature (e.g., Ai, Croce, Diercks, and Li, 2015). In what follows, we use this calibration to describe the asset pricing implications of our model with learning.

### 4 Numerical Illustration

This section provides a numerical analysis based on our calibration. Our structural estimation indicates that learning exclusively about long-run risk offers the best model fit to macroeconomic data. Based on this finding, we hereafter focus our analysis on the case $\theta = 1$, which implies that the model only depends on the state variables $f_t$, $\hat{\lambda}_t$, and $\nu_{\lambda,t}$.

---

**Table 2: Descriptive statistics of the main variables.**

This table reports the descriptive statistics of the main variables in the economy. The growth rate forecast $f_t$ is the state of the economy, $\tilde{l}$ reflects the estimated bias in forecast, and $\nu_l$ to its uncertainty. Then, the filter related to learning about long-run risk is given by $\hat{\lambda}$, whereas $\nu_{\lambda}$ captures the uncertainty about long-run risk.

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Standard deviation</th>
<th>Minimum</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f$</td>
<td>2.61%</td>
<td>1.70%</td>
<td>−3.75%</td>
<td>6.45%</td>
</tr>
<tr>
<td>$\tilde{l}$</td>
<td>$1.83 \times 10^{-4}$</td>
<td>$1.73 \times 10^{-4}$</td>
<td>$-1.36 \times 10^{-4}$</td>
<td>$5.14 \times 10^{-4}$</td>
</tr>
<tr>
<td>$\nu_l$</td>
<td>$4.00 \times 10^{-4}$</td>
<td>$2.14 \times 10^{-8}$</td>
<td>$4.00 \times 10^{-4}$</td>
<td>$4.00 \times 10^{-4}$</td>
</tr>
<tr>
<td>$\hat{\lambda}$</td>
<td>0.037</td>
<td>0.037</td>
<td>−0.121</td>
<td>0.083</td>
</tr>
<tr>
<td>$\nu_{\lambda}$</td>
<td>0.024</td>
<td>0.006</td>
<td>0.020</td>
<td>0.040</td>
</tr>
</tbody>
</table>

---

10Because the estimated values of $\tilde{l}$ and $1 - \theta$ are close to 0, the uncertainty associated with the filter $\nu_l$ becomes almost constant over time (see Equation 12).
We first discuss the behavior of the price-dividend ratio in subsection 4.1. Then, subsections 4.2-4.3 analyze the implications of learning about long-run risk on the volatility of asset returns, the risk premium in the economy, the equilibrium risk-free rate, and the risk-return trade-off. To clearly highlight the properties of a model with learning about long-run risk, we compare our predictions with those coming from a model without uncertainty about long-run risk, i.e., when $\nu_\lambda$ is set to zero.\(^\text{11}\)

\[4.1 \text{ Log price-dividend ratio}\]

The starting point of our analysis is the behavior of the log price-dividend ratio with respect to the main state variables, which we illustrate in Figure 1. The left panel shows that the log price-dividend ratio increases, almost linearly, with economic conditions, as captured by the growth forecast $f_t$. Hence, $I_f/I > 0$. The right panel demonstrates that the log price-dividend ratio increases with the filter of the persistence parameter $\hat{\lambda}_t$, which implies $I_{\hat{\lambda}}/I > 0$, and thus decreases with the agent’s estimate of long-run risk. Both panels suggest a decreasing relationship between the price-dividend ratio and uncertainty about long-run risk $\nu_\lambda$, which yields $I_{\nu_\lambda}/I < 0$.

Hence, the price-dividend ratio increases with expected consumption growth, decreases with uncertainty about long-run risk, and decreases with the persistence of expected consumpt-

\[^{11}\text{Note that the results of the model with learning about the level of expected consumption growth only (i.e., } \theta = 0\text{) are discussed in Appendix D. They are qualitatively the same as those of the traditional long-run risk model.}\]
tion growth. These findings, which we will also validate empirically in Section 5, numerically confirm our Conjecture 1. Eventually, Figure 1 suggests that uncertainty and learning about long-run risk should be important drivers of asset prices, as they exert a strong influence on the log price-dividend ratio.

4.2 Stock return volatility and equity risk premium

We turn now to the analysis of the volatility of asset returns and the equity risk premium. Following Equation (24) in the case $\theta = 1$, stock market volatility is given by

$$
\sigma_t = \sqrt{\sigma_\delta^2 + \left(\sigma_f \frac{I_f}{I} + \frac{(\bar{f} - f_t)\nu_{\lambda,t} I_{\hat{\lambda}}}{\sigma_f} \right)^2},
$$

(26)

while the equity risk premium, as determined by Equation (25), becomes

$$
\mu_t - r_t = \gamma \sigma_\delta^2 + (1 - \phi) \left(\sigma_f \frac{I_f}{I} + \frac{(\bar{f} - f_t)\nu_{\lambda,t} I_{\hat{\lambda}}}{\sigma_f} \right)^2.
$$

(27)

The first component of stock return volatility captures the uncertainty about consumption growth, while the first term of the equity risk premium represents the compensation for this risk, which is also present in the CRRA case. When the agent prefers early resolution of uncertainty, we have $\phi < 1$ and the equity risk premium now incorporates an additional part, which consists of the same quadratic term as the one driving stock return volatility in (26). This term captures, on one hand, the compensation for long-run risk in the economy (Bansal and Yaron, 2004) and, on the other hand, the uncertainty regarding the degree of long-run risk. This last component, specific to our model, directly associates with learning about long-run risk.

We study the properties of stock return volatility and the equity risk premium in the top and bottom panels of Figure 2, respectively. The left panels depict the relations of these moments with the growth forecast $f_t$, setting the filter $\hat{\lambda}_t$ at zero. The right panels depict the relations of these moments with the filter of the mean-reversion speed $\hat{\lambda}_t$, setting the growth forecast $f_t$ at its long-run level $\bar{f}$. All panels report values for various levels of uncertainty about long-run risk.

When the growth forecast is at its long-term level ($f_t = \bar{f}$), the model demonstrates that uncertainty about long-run risk magnifies stock return volatility and increases the equity risk premium. The reason is that incomplete information about the persistence in expected growth represents a new source of risk to investors. Thus, uncertainty about long-run risk affects asset prices even though agents are not currently learning about this risk. Indeed, when $f_t = \bar{f}$,
Figure 2: Stock return volatility and equity risk premium with learning about long-run risk. For the left plots, we fix $\hat{\lambda}_t = 0$. For the right plots, we fix $\tilde{f}_t = \bar{f}$. The dashed vertical line in the left plots corresponds to $f_t = \bar{f}$. Unless otherwise specified, the calibration used is provided in Table 1 and Section 3.

consumption growth shocks remain uninformative for estimating the mean-reverting speed in expected consumption growth (see Equation 10), which implies that the second term in brackets in (26) and (27) becomes null.

Departing now from the special case $f_t = \bar{f}$, we find that stock return volatility and the equity risk premium vary across economic conditions in presence of uncertainty about long-run risk. The left panels of Figure 2 show that stock return volatility and the equity risk premium become negatively related with the growth forecast. In comparison, without uncertainty about long-run risk ($\nu_\lambda = 0$), these moments remain constant, as the second term in brackets in (26) and (27) vanishes. This is the result of the long-run risk model (without stochastic
volatility), which features complete information about the persistence parameter (Bansal and Yaron, 2004). Hence, learning about long-run risk induces key asset pricing moments to be time-varying and counter-cyclical. Our main theoretical prediction is that the sensitivity of stock return volatility and equity risk premium to economic conditions crucially depends on the degree of uncertainty about long-run risk. Positive uncertainty activates the second term in brackets in (26) and (27), which in turn alters the response of stock returns to expected consumption growth shocks.

The intuition behind this mechanism is as follows. In bad economic times \((f_t < \bar{f})\), the agent perceives positive (negative) shocks as very good (bad) news. Focusing on positive shocks, not only these shocks signal higher consumption growth in the future, but the agent now also expects less long-run risk and a shorter recession. Similarly, negative shocks during bad times imply lower future consumption growth, stronger long-run risk, and a recession that will last longer than previously expected. Overall, learning about long-run risk amplifies the sensitivity of asset returns to economic shocks during bad times, which increases stock return volatility and the equity risk premium.

By contrast, learning about long-run risk has the opposite impact in good economic times. The presence of uncertainty about long-run risk reduces stock return volatility and equity risk premium during periods of high expected consumption growth \((f_t > \bar{f})\), because positive (negative) expected growth shocks become moderately good (bad) news only. To see that, positive shocks always signal higher consumption growth in the future and a longer expansion but, at the same time, induce the agent to believe that there is more long-run risk. Similarly, negative expected growth shocks signal lower consumption growth in the future and a shorter expansion, but also reduces the degree of long-run risk. Agent’s updating of the long-run risk parameter thus partially offsets the effect of the initial shock when \(f_t > \bar{f}\), reducing stock return volatility and the equity risk premium in good times.

We confirm this intuition by showing that stock return volatility and the equity risk premium strongly and negatively depend on the estimated persistence parameter \(\hat{\lambda}_t\), as displayed

\[12\text{The only indirect channel through which equity volatility and risk premium could potentially depend on } f_t \text{ is through the term } I_f/I. \text{ With our calibration, this effect is negligible. In fact, this effect is zero with a standard log-linear approximation (Bansal and Yaron, 2004).}\]

\[13\text{The quadratic functions in (26) and (27) reach a minimum at } f_t = \bar{f} + \frac{\sigma^2}{\nu_{f,t} I_f} I_f. \text{ According to Conjecture 1, this minimum point is situated at the right of } \bar{f}. \text{ Hence the decreasing relationship depicted in the left panels of Figure 2.}\]

\[14\text{It is reasonable to expect that positive shocks during good times also signal a longer economic boom than previously expected, whereas negative shocks also signal a shorter economic boom. These expectations partially offset the effect of learning about long-run risk in good times. However, we find that the magnitude of this short-term effect is negligible when compared with the effect of changes in the persistence of consumption growth. Consequently, investors always dislike an increase in the persistence of consumption growth, because it is associated with greater long-run risk, and ultimately the term } I_{\hat{\lambda}/I} \text{ remains positive at all times. Figure 4 in Appendix E clarifies this point.}\]
in the right panels of Figure 2. Stock return volatility and the equity risk premium are thus higher when expected consumption growth is more persistent, which is in line with Bansal and Yaron (2004). Furthermore, Figure 2 highlights an interaction between the level and the uncertainty of the persistence parameter driving long-run risk. Specifically, the positive relation between stock return volatility (and the equity risk premium) and the perceived level of long-run risk strengthens when uncertainty about long-run risk is higher. Therefore, learning about long-run risk enriches and amplifies the results of the traditional long-run risk model.

Overall, our model generates a set of new predictions. First, uncertainty about long-run risk increases stock market volatility and the equity risk premium, on average. Second, this uncertainty induces stock market volatility and the equity risk premium to fluctuate over time and, in particular, to increase in bad economic times. Third, this uncertainty amplifies the sensitivity of asset pricing moments to the level of long-run risk in the economy. Consequently, incomplete information about the mean-reverting speed in consumption growth, which dictates the severity of long-run risk, has fundamental, new implications for the dynamics of asset prices. We will evaluate these predictions empirically in Section 5.

### 4.3 Risk-free rate and the risk-return tradeoff

When agents learn about long-run risk only (i.e. \( \theta = 1 \)), the risk-free rate and the market price of risk satisfy

\[
 r_t = \beta + \frac{1}{\psi} f_t - \frac{\gamma(1 + 1/\psi)}{2} \sigma^2 - \frac{1}{2} (1 - \phi) \left( \frac{I_f}{I} + \frac{(\tilde{f} - f_t)\nu_{\lambda, t}}{\sigma_f} \right)^2
\]

and

\[
 m_t = \left[ \gamma \sigma_\delta \left( 1 - \phi \right) \left( \frac{I_f}{I} + \frac{(\tilde{f} - f_t)\nu_{\lambda, t}}{\sigma_f} \right) \right]^\top.
\]

The upper panels of Figure 3 depict the behavior of the equilibrium risk-free rate in an economy with incomplete information about long-run risk. The risk-free rate increases with growth forecast \( f_t \) (left panel) and decreases with long-run risk, which is negatively related to the filter \( \hat{\lambda}_t \) (right panel). Furthermore, uncertainty about long-run risk drives the risk-free rate in two ways. First, it decreases the level of the risk-free rate, on average, as risky assets become more uncertain. Second, it amplifies the procyclicality of the risk-free rate, particularly in bad times, as the risk-free rate becomes more sensitive to changes in economic conditions. This effect can be seen through the last term of (28), which is magnified when \( f_t < \tilde{f} \). Hence, learning about long-run risk helps explain the low levels of risk-free rate observed during economic recessions.
Figure 3: Risk-free rate and Sharpe ratio with learning about long-run risk. For the left plots, we fix \( \hat{\lambda}_t = 0 \). For the right plots, we fix \( f_t = \bar{f} \). The dashed vertical line in the left plots corresponds to \( f_t = \bar{f} \). Unless otherwise specified, the calibration used is provided in Table 1 and Section 3.

The lower panels of Figure 3 illustrate the model predictions regarding the Sharpe ratio. This ratio is driven by the interaction between uncertainty about long-run risk, the degree of long-run risk, and the growth forecast, following the mechanism discussed previously in the case of stock return volatility and the equity risk premium. The Sharpe ratio increases when economic conditions deteriorate, particularly when long-run risk is highly uncertain, and increases with the magnitude of long-run risk. These effects result from the market price of risk component associated with expected consumption growth shocks, i.e., the second term in (29). This term magnifies the compensation per unit of risk required by the agent during bad times when she faces incomplete information about long-run risk. Uncertainty and learning about
long-run risk, thus, appear to be necessary ingredients for our understanding of the negative relation between the risk-return tradeoff and economic conditions.

Overall, learning about long-run risk induces all asset pricing quantities—stock return volatility, equity risk premium, risk-free rate, and the Sharpe ratio—to be tightly linked to the state of the economy. Specifically, incomplete information about long-run risk implies that stock market volatility, risk premia and the risk-return trade-off increase as economic conditions deteriorate. This arises without assuming any behavioral biases, time-varying risk-aversion, stochastic consumption volatility, or time-varying disaster intensity. We turn now to empirical tests of these theoretical predictions.

5 Empirical Tests

In this section, we empirically evaluate our model outlined in Section 4. As a preliminary analysis, we verify that the price-dividend ratio increases with the growth forecast, but decreases with long-run risk and the uncertainty about its level. Then, we propose a direct empirical assessment of the model. To this end, we examine how the model-implied price-dividend ratio, stock return volatility, risk premium, Sharpe ratio, and excess stock returns help explain their empirical counterparts. Furthermore, our theory suggests that learning about long-run risk generates three testable theoretical predictions, which we aim to validate empirically. First, stock return volatility, the equity risk premium, and the Sharpe ratio are more sensitive to economic conditions when there is greater uncertainty about long-run risk. This prediction suggests that the interaction between \((f_t - \bar{f})\) and \(\nu_{\lambda,t}\) not only drives the response of asset returns to expected consumption growth shocks but has also a negative impact on the aforementioned moments. Second, uncertainty about long-run risk amplifies the response of these moments to variations in long-run risk. More precisely, the interaction between \(\hat{\lambda}_t\) and \(\nu_{\lambda,t}\) has a negative impact on these moments. Third, the model predicts that uncertainty about long-run risk \(\nu_{\lambda,t}\) amplifies stock return volatility and increases both the equity risk premium and the Sharpe ratio. We test these three theoretical predictions in the data.

Our empirical analysis uses quarterly data over the period Q4:1968–Q4:2015. Following the calibration, we consider the mean analyst forecast on 1-quarter-ahead real GDP growth as a measure of the growth forecast \(f_t\) and use the observed real GDP growth to proxy the realized growth rate of \(\delta_t\). The estimation performed in Section 3 provides time series of the long-run risk filter \(\hat{\lambda}_t\) and of its uncertainty \(\nu_{\lambda,t}\). These state variables allows constructing model-implied time series for the price-dividend ratio \(PD^{MI}\), stock return volatility \(Vol^{MI}\), equity risk premium \(RP^{MI}\), and the Sharpe ratio \(SR^{MI}\).

The observed counterparts of these moments, which we denote by the superscript \(\text{Obs}\),
are computed using data on the S&P 500 index. Our proxy for the equity risk premium on
the S&P 500 follows the return predictability literature (Cochrane, 2008; van Binsbergen and
Koijen, 2010). Specifically, we compute the fitted values obtained by regressing future S&P 500
excess returns on current dividend yields. We also use the residuals of this regression, which
abstract from the predictive components, to compute the return volatility on the S&P 500.
Our estimation is based on the EGARCH(1,1,1) to account for potential asymmetric volatility
responses across good and bad news.\textsuperscript{15}

As a first test of the model, we compare the dynamics of the theoretical price-dividend ratio,
stock return volatility, and equity risk premium with their observed counterparts. Table 3
shows that the model-implied moments, which we derive in equilibrium, successfully explain
the variations in the moments observed in the data. The slopes are all positive, as expected,
and highly statistically significant. The coefficients of determination are also surprisingly high.
Moreover, we find that the model-implied excess stock returns, $r^{MI}$, can explain the excess
returns on the S&P 500 index with a slope and a constant that are not statistically different
from one and zero, respectively. These tests thus confirm that the dynamics of stock returns
obtained with our model align well with what is observed in the data.

We now investigate the relation between the log price-dividend ratio and the state variables.
Table 4 reports the results within the model (left columns) and in the data (right columns).
Both the model-implied and the observed log price-dividend ratios increase with the growth
forecast and the degree of persistence (i.e., decreases with long-run risk), but decreases with
uncertainty about long-run risk. Importantly, the level of long-run risk and its uncertainty
appear to explain a large fraction of the variations in the price-dividend ratio. Both sources
of risk largely outperform the role of economic conditions in terms of $R^2$.

Finally, Table 5 presents the model-implied (left columns) and observed (right columns)
relations between the main asset pricing moments and the state variables of interest. The
results for stock return volatility, the equity risk premium, and the Sharpe ratio are presented
in panels A, B, and C, respectively. The data confirm the theoretical predictions that these
moments i) increase with uncertainty about long-run risk; ii) decrease with the interaction be-
tween the growth forecast and long-run risk uncertainty; and iii) decrease with the interaction
between the long-run risk estimate and its uncertainty. Furthermore, the interaction between
the growth forecast and uncertainty about long-run risk is the most important driver of return
volatility in both the model and the data, as it generates the largest $R^2$. Consequently, these
results confirm our main prediction that stock return volatility, the equity risk premium, and

\textsuperscript{15}This choice of specification is motivated by the leverage (or asymmetric) effect documented by Black
(1976), French, Schwert, and Stambaugh (1987), Schwert (1989), Nelson (1991), and Glosten, Jagannathan,
and Runkle (1993), among others. Note that using a model-free proxy for volatility such as the absolute return
or the square return does not qualitatively alter our results.
Table 3: Observed vs. model-implied asset prices
This table reports the relations between key asset pricing moments and their model-implied counterparts. Regression using the price-dividend ratio is reported in the first column, whereas the second column reports the results for the stock return volatility, the third column for the equity risk premium, the fourth column for the Sharpe ratio, and the last column for excess stock returns. \( N \) is the number of observations. Newey and West (1987) standard errors are in brackets and statistical significance at the 10%, 5%, and 1% levels is denoted by *, **, and ***, respectively.

<table>
<thead>
<tr>
<th></th>
<th>( \log PD^{Obs} )</th>
<th>( Vol^{Obs} )</th>
<th>( RP^{Obs} )</th>
<th>( SR^{Obs} )</th>
<th>( r^{Obs} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Const.</td>
<td>-23.503***</td>
<td>0.101***</td>
<td>-0.013***</td>
<td>-0.116***</td>
<td>-0.005</td>
</tr>
<tr>
<td></td>
<td>(2.992)</td>
<td>(0.013)</td>
<td>(0.000)</td>
<td>(0.005)</td>
<td>(0.006)</td>
</tr>
<tr>
<td>( \log PD^{MI} )</td>
<td>6.962***</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.769)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( Vol^{MI} )</td>
<td></td>
<td>2.124***</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.600)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( RP^{MI} )</td>
<td></td>
<td></td>
<td>0.097***</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(0.013)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( SR^{MI} )</td>
<td></td>
<td></td>
<td></td>
<td>0.077***</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(0.010)</td>
<td></td>
</tr>
<tr>
<td>( r^{MI} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.738**</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(0.359)</td>
</tr>
<tr>
<td>( R^2 )</td>
<td>0.374</td>
<td>0.143</td>
<td>0.169</td>
<td>0.207</td>
<td>0.029</td>
</tr>
<tr>
<td>( N )</td>
<td>189</td>
<td>189</td>
<td>189</td>
<td>189</td>
<td>189</td>
</tr>
</tbody>
</table>

Table 4: Price-dividend ratio with learning about long-run risk
This table reports the relations between the \( \log PD^{MI} \) and the state variables related with learning about long-run risk. \( N \) is the number of observations. Newey and West (1987) standard errors are in brackets and statistical significance at the 10%, 5%, and 1% levels is denoted by *, **, and ***, respectively.

<table>
<thead>
<tr>
<th></th>
<th>( \log PD^{MI} )</th>
<th>( \log PD^{Obs} )</th>
<th>( \log PD^{MI} )</th>
<th>( \log PD^{Obs} )</th>
<th>( \log PD^{MI} )</th>
<th>( \log PD^{Obs} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Const.</td>
<td>3.864***</td>
<td>3.497***</td>
<td>3.861***</td>
<td>3.293***</td>
<td>4.010***</td>
<td>3.464***</td>
</tr>
<tr>
<td></td>
<td>(0.006)</td>
<td>(0.070)</td>
<td>(0.003)</td>
<td>(0.045)</td>
<td>(0.008)</td>
<td>(0.143)</td>
</tr>
<tr>
<td>( f )</td>
<td>1.130***</td>
<td>4.140*</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.197)</td>
<td>(2.307)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \lambda )</td>
<td>0.868***</td>
<td>8.359***</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.044)</td>
<td>(0.844)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \nu_{\lambda} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( R^2 )</td>
<td>0.263</td>
<td>0.027</td>
<td>0.750</td>
<td>0.536</td>
<td>0.526</td>
<td>0.172</td>
</tr>
<tr>
<td>( N )</td>
<td>189</td>
<td>189</td>
<td>189</td>
<td>189</td>
<td>189</td>
<td>189</td>
</tr>
</tbody>
</table>

---

16One potential concern is that the the level of uncertainty \( \nu_{\lambda} \) gradually decreases over time (as shown in Equation 12). Our empirical results could therefore be driven by a trend in \( \nu_{\lambda} \). In Appendix F, we consider a stationarized time-series of \( \nu_{\lambda} \) (using the method of Savitzky and Golay, 1964) and reproduce our empirical
### Panel A: Stock return volatility

<table>
<thead>
<tr>
<th></th>
<th>$V_{ol}^{MT}$</th>
<th>$V_{ol}^{Obs}$</th>
<th>$V_{ol}^{MT}$</th>
<th>$V_{ol}^{Obs}$</th>
<th>$V_{ol}^{MT}$</th>
<th>$V_{ol}^{Obs}$</th>
<th>$V_{ol}^{MT}$</th>
<th>$V_{ol}^{Obs}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Const.</td>
<td>0.025***</td>
<td>0.153***</td>
<td>0.029***</td>
<td>0.162***</td>
<td>0.015***</td>
<td>0.101***</td>
<td>0.016***</td>
<td>0.083***</td>
</tr>
<tr>
<td>$(f - f)\nu_A$</td>
<td>-15.944***</td>
<td>-31.769***</td>
<td>-0.065</td>
<td>-11.770**</td>
<td>-0.003</td>
<td>-0.016***</td>
<td>-0.002</td>
<td>-0.011***</td>
</tr>
<tr>
<td>$\lambda_\nu_A$</td>
<td>-6.06***</td>
<td>-11.770**</td>
<td>(1.049)</td>
<td>(4.935)</td>
<td>0.389***</td>
<td>2.143***</td>
<td>(0.144)</td>
<td>(0.810)</td>
</tr>
<tr>
<td>$\nu_A$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.582***</td>
<td>0.178***</td>
<td>(0.177)</td>
<td>(0.038)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.807</td>
<td>0.101</td>
<td>0.367</td>
<td>0.044</td>
<td>0.065</td>
<td>0.062</td>
<td>0.975</td>
<td>0.190</td>
</tr>
<tr>
<td>$N$</td>
<td>189</td>
<td>189</td>
<td>189</td>
<td>189</td>
<td>189</td>
<td>189</td>
<td>189</td>
<td>189</td>
</tr>
</tbody>
</table>

### Panel B: Equity risk premium

<table>
<thead>
<tr>
<th></th>
<th>$RP_{MT}$</th>
<th>$RP_{Obs}$</th>
<th>$RP_{MT}$</th>
<th>$RP_{Obs}$</th>
<th>$RP_{MT}$</th>
<th>$RP_{Obs}$</th>
<th>$RP_{MT}$</th>
<th>$RP_{Obs}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Const.</td>
<td>0.011***</td>
<td>-0.012***</td>
<td>0.017***</td>
<td>-0.011***</td>
<td>-0.003</td>
<td>-0.016***</td>
<td>0.001</td>
<td>-0.011***</td>
</tr>
<tr>
<td>$(f - f)\nu_A$</td>
<td>-19.346***</td>
<td>-0.982*</td>
<td>(2.220)</td>
<td>(0.525)</td>
<td>-8.381***</td>
<td>-2.092***</td>
<td>-5.313***</td>
<td>-2.001***</td>
</tr>
<tr>
<td>$\lambda_\nu_A$</td>
<td>-1.525***</td>
<td>-2.172**</td>
<td>(3.075)</td>
<td>(1.624)</td>
<td>-1.525***</td>
<td>-2.172**</td>
<td>(3.075)</td>
<td>(1.624)</td>
</tr>
<tr>
<td>$\nu_A$</td>
<td>0.582***</td>
<td>0.178***</td>
<td>(0.177)</td>
<td>(0.038)</td>
<td>0.582***</td>
<td>0.178***</td>
<td>(0.177)</td>
<td>(0.038)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.692</td>
<td>0.032</td>
<td>0.409</td>
<td>0.455</td>
<td>0.085</td>
<td>0.141</td>
<td>0.900</td>
<td>0.460</td>
</tr>
<tr>
<td>$N$</td>
<td>189</td>
<td>189</td>
<td>189</td>
<td>189</td>
<td>189</td>
<td>189</td>
<td>189</td>
<td>189</td>
</tr>
</tbody>
</table>

### Panel C: Sharpe ratio

<table>
<thead>
<tr>
<th></th>
<th>$SR_{MT}$</th>
<th>$SR_{Obs}$</th>
<th>$SR_{MT}$</th>
<th>$SR_{Obs}$</th>
<th>$SR_{MT}$</th>
<th>$SR_{Obs}$</th>
<th>$SR_{MT}$</th>
<th>$SR_{Obs}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Const.</td>
<td>0.390***</td>
<td>-0.087***</td>
<td>0.483***</td>
<td>-0.072***</td>
<td>0.204***</td>
<td>-0.140***</td>
<td>0.220***</td>
<td>-0.112***</td>
</tr>
<tr>
<td>$\lambda_\nu_A$</td>
<td>-7.549**</td>
<td>2.243***</td>
<td>(3.099)</td>
<td>(0.454)</td>
<td>8.415***</td>
<td>1.483***</td>
<td>(3.099)</td>
<td>(0.454)</td>
</tr>
<tr>
<td>$\nu_A$</td>
<td>0.834</td>
<td>0.078</td>
<td>0.349</td>
<td>0.297</td>
<td>0.055</td>
<td>0.168</td>
<td>0.984</td>
<td>0.361</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.834</td>
<td>0.078</td>
<td>0.349</td>
<td>0.297</td>
<td>0.055</td>
<td>0.168</td>
<td>0.984</td>
<td>0.361</td>
</tr>
<tr>
<td>$N$</td>
<td>189</td>
<td>189</td>
<td>189</td>
<td>189</td>
<td>189</td>
<td>189</td>
<td>189</td>
<td>189</td>
</tr>
</tbody>
</table>

**Table 5: Asset pricing with learning about long-run risk**

This table reports the relations between key asset pricing moments and the state variables obtained with learning about long-run risk. Panel A reports the results for the stock return volatility, Panel B for the equity risk premium, and Panel C for the Sharpe ratio. $N$ is the number of observations. Newey and West (1987) standard errors are in brackets and statistical significance at the 10%, 5%, and 1% levels is denoted by *, **, and ***, respectively.
Overall, the data lend strong support to the predictions offered by a model with learning about long-run risk. Moreover, our theory successfully pins down key determinants of the dynamics of the observed price-dividend ratio, stock return volatility, risk premia, and the risk-return tradeoff. Our results therefore highlight the importance of considering learning and uncertainty about long-run risk in understanding the dynamics of asset prices.

6 Conclusion

An important result of our paper is that a model of learning about the persistence of expected economic growth is more likely to prevail in the U.S. economy than a model of learning about the level of expected economic growth. A plausible interpretation of this finding is that relatively accurate analyst forecasts on economic growth are readily available to investors from professional surveys of forecasters, whereas accurate forecasts of the persistence of the growth rate itself are clearly lacking. Yet, we show that learning about persistence (or about the magnitude of long-run risk—a central ingredient for asset pricing theory) is not without consequences for financial markets. In particular, this learning type generates negative dynamic relations between key asset-pricing moments (e.g., equity return volatility, equity risk premium, and the Sharpe ratio) and economic conditions, which is not the case with learning about the level of expected economic growth. In addition, the level and the conditional properties of these moments increase with the uncertainty about long-run risk, theoretical predictions for which we find strong support in the data.

Our analysis offers several extensions for future research. First, a theory in which agents endogenously choose which learning type to perform would be useful to investigate whether indeed learning about long-run risk can be a rational response of investors, and also to compute the welfare gains of a publicly available accurate indicator of the persistence of economic growth. Second, extending our model to an economy with multiple risky assets would allow to investigate the implications of uncertainty and learning about long-run risk for the cross-section of asset returns. Third, since the persistence of economic growth is unobservable and can only be accurately estimated using a long history of data, agents are very likely to disagree about it. Future work could then construct a measure of disagreement about the persistence of economic growth by exploiting the cross-section of analyst forecasts on future economic growth at different horizons. This measure of disagreement about persistence might help forecast future market returns and their volatility more accurately than existing measures of disagreement about the level of economic growth itself.

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analysis with this new variable. We show that our results remain robust to this change in specification.
A Learning

Theorem 1. (Liptser and Shiryaev, 1977) Consider an unobservable process \( u_t \) and an observable process \( s_t \) with dynamics given by

\[
\begin{align*}
\dot{u}_t &= [a_0(t, s_t) + a_1(t, s_t)u_t]dt + b_1(t, s_t)dz^u_t + b_2(t, s_t)dz^s_t, \\
\dot{s}_t &= [A_0(t, s_t) + A_1(t, s_t)u_t]dt + B_1(t, s_t)dz^u_t + B_2(t, s_t)dz^s_t.
\end{align*}
\]

All the parameters can be functions of time and of the observable process. Liptser and Shiryaev (1977) show that the filter evolves according to (we drop the dependence of coefficients on \( t \) and \( s_t \) for notational convenience):

\[
\begin{align*}
\dot{\hat{u}}_t &= (a_0 + a_1\hat{u}_t)dt + [(b \circ B) + \nu_tA_1^1](B \circ B)^{-1}[\dot{s}_t - (A_0 + A_1\hat{u}_t)dt] \\
\dot{\nu}_t &= a_1\nu_t + \nu_1a_1^\top + (b \circ b) + [(b \circ B) + \nu_tA_1^1](B \circ B)^{-1}[(b \circ B) + \nu_tA_1^\top],
\end{align*}
\]

where

\[
\begin{align*}
b \circ b &= b_1b_1^\top + b_2b_2^\top, \\
B \circ B &= B_1B_1^\top + B_2B_2^\top, \\
b \circ B &= b_1B_1^\top + b_2B_2^\top.
\end{align*}
\]

Write the dynamics of the observable variables:

\[
\begin{bmatrix} d\log\delta_t \\ df_t \end{bmatrix} = \begin{bmatrix} f_t - \frac{1}{2}\sigma_\delta^2 \\ \lambda(f - f_t) \end{bmatrix} + \begin{bmatrix} 1 - \theta \\ \theta(f - f_t) \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \sigma_f \end{bmatrix} \begin{bmatrix} dW^\delta_t \\ dW^f_t \end{bmatrix}
\]

All the other matrices in Theorem 1 are equal to zero. Then,

\[
\begin{bmatrix} d\hat{\lambda} \\ d\hat{\nu}_t \\ d\hat{\nu}_\lambda \\ d\hat{\nu}_s \\ d\hat{\nu}_f \\ d\hat{\nu}_{\lambda,t} \\ d\hat{\nu}_{\lambda,f} \\ d\hat{\nu}_{\delta,t} \\ d\hat{\nu}_{\delta,f} \\ d\hat{\nu}_{\lambda,s} \\ d\hat{\nu}_{\lambda,f} \\ d\hat{\nu}_{\delta,t} \\ d\hat{\nu}_{\delta,f} \end{bmatrix} = \begin{bmatrix} (1-\theta)\nu_{\lambda,t} \\ \theta(f - f_t)\nu_{\lambda,t} \\ \theta(f - f_t)\nu_{\lambda,t} \\ \theta(f - f_t)\nu_{\lambda,t} \\ \theta(f - f_t)\nu_{\lambda,t} \\ \theta(f - f_t)\nu_{\lambda,t} \end{bmatrix} \begin{bmatrix} d\hat{W}_t^\delta \\ d\hat{W}_t^f \\ d\hat{W}_t^{\delta,t} \\ d\hat{W}_t^{\delta,s} \\ d\hat{W}_t^{\delta,f} \\ d\hat{W}_t^{\lambda,t} \\ d\hat{W}_t^{\lambda,f} \\ d\hat{W}_t^{\lambda,s} \\ d\hat{W}_t^{\lambda,f} \end{bmatrix},
\]

where the independent Brownian motions \( \hat{W}_t^\delta \) and \( \hat{W}_t^f \) are such that

\[
\begin{align*}
\frac{d\delta_t}{\delta_t} &= [f_t + (1 - \theta)\hat{\nu}_t]dt + \sigma_\delta d\hat{W}_t^\delta, \\
\frac{df_t}{df_t} &= (\theta\hat{\lambda}_t + \hat{\lambda})(\hat{f} - f_t)dt + \sigma_f d\hat{W}_t^f.
\end{align*}
\]

If \( \theta = 0 \), there is no way to update \( \lambda \) and thus the agent sticks with the prior \( \lambda \) as mean-reverting speed. If \( \theta = 1 \), there is no way to learn about \( l \) and thus the agent sticks with the prior \( f \) as long-term growth of the economy.

The posterior uncertainties about \( l \) and \( \lambda \) evolve according to

\[
\begin{align*}
d\nu_{l,t} &= -\frac{(1-\theta)^2\nu_{l,t}^2}{\sigma_l^2}dt, \\
d\nu_{\lambda,t} &= -\frac{\theta^2(\hat{f} - f_l)^2\nu_{\lambda,t}^2}{\sigma_{\lambda}^2}dt.
\end{align*}
\]
and thus the uncertainty about $l$ has an easy solution

$$\nu_{l,t} = \frac{1}{(1-\theta)^2 \sigma^2 + \frac{1}{\nu_{l,0}}}$$

(43)

but this is not the case for the uncertainty about $\lambda$, because of the $(\bar{f} - f_t)$ term.

**B Equilibrium**

The dynamics of the vector of state variables are

$$\begin{bmatrix}
    d\delta_t \\
    df_t \\
    d\bar{l}_t \\
    d\lambda_t \\
    d\nu_{l,t} \\
    d\nu_{\lambda,t}
\end{bmatrix} =
\begin{bmatrix}
    \frac{\delta_t[f_t + (1 - \theta)\bar{l}_t]}{(\theta \lambda_t + \hat{\lambda})(\bar{f} - f_t)} \\
    0 \\
    0 \\
    -(1-\theta)^2 \nu_{l,t} \\
    -\theta^2 (\bar{f} - f_t)^2 \nu_{l,t} \\
    -\theta^2 (\bar{f} - f_t)^2 \nu_{\lambda,t}
\end{bmatrix} dt +
\begin{bmatrix}
    \delta_t \sigma_\delta \\
    0 \\
    0 \\
    \frac{\sigma_\delta}{\sigma_f} \\
    \frac{\sigma_\delta}{\sigma_f} \\
    \frac{\sigma_\delta}{\sigma_f}
\end{bmatrix}
\begin{bmatrix}
    d\bar{W}_t^\delta \\
    d\bar{W}_t^f
\end{bmatrix}.
$$

(44)

Substituting the guess (14) in the HJB equation (13) and imposing the market clearing condition, $C = \delta$, yields the following PDE for the log price-dividend ratio $i \equiv \log I$

$$0 = \left(\gamma - 1\right) \left[-f - (1 - \theta)l + \frac{1}{2} \gamma \sigma_\delta^2\right] - \beta \phi + \phi e^{-i}$$

$$+ \phi (\bar{f} - f)[\theta \hat{\lambda} + \lambda]i_f - \phi \nu_{\lambda}^2 (\theta - 1)^2 \nu_{l} - \phi \frac{(\bar{f} - f)^2}{\sigma_f^2} \nu_{\lambda}^2 \hat{\lambda} + \phi^2 (1 - 1/\psi) \nu_{l}(\theta - 1)i_{\bar{l}}$$

$$+ \phi \frac{\sigma_f^2}{2} i_{ff} + \phi \frac{(1 - \theta)^2 \nu_{l}^2}{2\sigma_\delta^2} i_{\bar{l}l} + \phi \frac{\sigma_\lambda^2}{2\sigma_f^2} i_{\bar{l}\lambda} + \phi \theta \nu_{\lambda} (\bar{f} - f) i_{f\bar{l}}$$

$$+ \phi^2 \frac{\sigma_\lambda^2}{2} i_{\bar{l}\lambda}$$

(45)

Setting $\theta = 0$ in Equation (45) yields

$$0 = \left(\gamma - 1\right) \left[-f - l + \frac{1}{2} \gamma \sigma_\delta^2\right] - \beta \phi + \phi e^{-i}$$

$$+ \phi (\bar{f} - f) \hat{\lambda} i_f - \phi \nu_{\lambda}^2 i_{\bar{l}l} - \phi^2 \nu_{l}(1 - 1/\psi) i_{\bar{l}}$$

$$+ \phi \frac{\sigma_f^2}{2} i_{ff} + \phi \frac{\nu_{l}^2}{2\sigma_\delta^2} i_{\bar{l}l}$$

$$+ \phi^2 \frac{\sigma_\delta^2}{2} i_{\bar{l}\lambda}.$$

(46)
Setting \( \theta = 1 \) in Equation (45) yields

\[
0 = -(\gamma - 1)f + \gamma(\gamma - 1)\frac{\sigma^2_c}{2} - \beta \phi + \phi e^{-i} + \phi \left[ (\hat{\lambda} + \check{\lambda})(f - \bar{f})i_f - \frac{(\bar{f} - f)^2 \nu^2_\lambda}{\sigma^2_f} i_{\nu_\lambda} \right] + \phi \left[ \frac{\sigma^2_f}{2} i_{ff} + \frac{(f - \bar{f})^2 \nu^2_\lambda}{2\sigma^2_f} i_{\nu_\lambda} + \nu_\lambda (f - \bar{f})i_{\lambda} \right] + \phi^2 \left[ \frac{\sigma^2_f}{2} i^2_f + \frac{(f - \bar{f})^2 \nu^2_\lambda}{2\sigma^2_f} i^2_{\lambda} + \nu_\lambda (f - \bar{f})i_{\lambda} \right].
\]  

(47)

C Estimation procedure

To fit our continuous-time model to the data, we first discretize the filtered dynamics in (10), (11), and (12) using the following approximations

\[
\log(\delta_{t+\Delta}/\delta_t) = \left( f_t + (1 - \theta)\check{\iota}_t - \frac{1}{2}\frac{\sigma^2_c}{\sigma_\delta} \right) \Delta + \sigma_\delta \sqrt{\Delta} \epsilon_{1,t+\Delta},
\]

(48)

\[
f_{t+\Delta} = e^{-\lambda^f_t \Delta} f_t + \left( 1 - e^{-\lambda^f_t \Delta} \right) \bar{f} + \sigma_f \sqrt{\frac{1 - e^{-2\lambda^f_t \Delta}}{2\lambda^f_t}} \epsilon_{2,t+\Delta},
\]

(49)

\[
\check{\iota}_{t+\Delta} = \check{\iota}_t + \frac{(1 - \theta)\nu \iota_t}{\sigma_\delta} \log(\Delta) \epsilon_{1,t+\Delta},
\]

(50)

\[
\hat{\lambda}_{t+\Delta} = \hat{\lambda}_t + \frac{\theta(\bar{f} - f_t)\nu \lambda_t}{\sigma_f} \log(\Delta) \epsilon_{2,t+\Delta},
\]

(51)

\[
\nu_{\lambda,t+\Delta} = \nu_{\lambda,t} - \left( \frac{\theta(\bar{f} - f_t)\nu \lambda_t}{\sigma_f} \right)^2 \Delta,
\]

(52)

\[
\nu_{t,t+\Delta} = \nu_{t,0}(t + \Delta)(1 - \theta)^2 + \sigma^2_\delta,
\]

(53)

where \( \lambda^f_t \equiv \theta \hat{\lambda}_t + \bar{\lambda} \) and \( \epsilon_{1,t}, \epsilon_{2,t} \) are independent normally distributed random variables with mean 0 and variance 1. Note that the uncertainty \( \nu_\iota_t \) solves an ODE that has a closed form solution provided in (53).

We use the mean analyst forecast on the 1-quarter-ahead real GDP growth as a proxy for the expected growth rate \( f_t \) and the realized real GDP growth as a proxy for the output growth \( \log(\delta_{t+\Delta}/\delta_t) \). That is, the time interval is \( \Delta = 1/4 \). Note that the system above shows that, conditional on knowing the parameters of the model and the priors \( (\tilde{\iota}_0, \bar{\lambda}_0, \nu_{t,0}, \nu_{\lambda,0}) \), the time series of the GDP growth forecast and realized GDP growth allow us to sequentially back out the time series of the posteriors \( (\tilde{\iota}_t, \hat{\lambda}_t, \nu_{t,t}, \nu_{\lambda,t}) \) and the noises \( (\epsilon_{1,t+\Delta}, \epsilon_{1,t+\Delta}) \) for \( t = \Delta, 2\Delta, 3\Delta \ldots \).

To estimate the vector of parameters \( \Theta \equiv (\sigma_\delta, f, \sigma_f, \bar{\lambda}, \theta)^\top \), we maximize the following log-likelihood function \( L \)

\[
L(\Theta; u_\Delta, \ldots, u_{N\Delta}) = \sum_{i=1}^N \log \left( \frac{1}{2\pi \sqrt{\Sigma(i-1)\Delta}} \right) - \frac{1}{2} u_{i\Delta}^\top \Sigma_{i\Delta}^{-1} u_{i\Delta},
\]

(54)
where $N$ is the number of observations, $\mathcal{T}$ is the transpose operator, and $|.|$ the determinant operator. The 2-dimensional vector $u$ satisfies

$$
u_{t+\Delta} = \begin{pmatrix} u_{1,t+\Delta} \\ u_{2,t+\Delta} \end{pmatrix} = \left( \begin{array}{c} \log(\delta_{t+\Delta}/\delta_t) - \left( f_t + (1 - \theta)\hat{l}_t - \frac{1}{2}\sigma_f^2 \right) \Delta \\ f_{t+\Delta} - e^{-\lambda^t_{t+\Delta} f_t} - (1 - e^{-\lambda^t_{t+\Delta}}) \hat{f} \end{array} \right).$$

(55)

Therefore, the conditional expectation and conditional variance-covariance matrix of $u_{t+\Delta}$ are

$$E_t(u_{t+\Delta}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{Var}_t(u_{t+\Delta}) = \Sigma_t = \begin{pmatrix} \sigma^2_{\delta} \Delta & 0 \\ 0 & \sigma^2_f \frac{1 - e^{-2\lambda^t_{t+\Delta}}}{2\lambda^t_{t+\Delta}} \end{pmatrix}. \quad (56)$$

### C.1 The case of i.i.d. changes in $l_t$

In this section, we extend our framework and assume that $l$ is an i.i.d. process instead of a constant. That is, the dynamics of $l$ satisfy

$$dl_t = \sigma_l dW^l_t, \quad (57)$$

where $W^l_t$ is a Brownian motion independent from $W^\delta_t$ and $W^f_t$. The dynamics of aggregate consumption $\delta$ and the growth forecast $f$ are

$$d\delta_t = [f_t + (1 - \theta\hat{l}_t)] dt + \sigma_d dW^\delta_t, \quad df_t = (\theta \lambda + \hat{\lambda})(\hat{f} - f_t) dt + \sigma_f dW^f_t. \quad (58)$$

In this case, solving the investor’s learning problem yields the following dynamics

$$\frac{d\delta_t}{\delta_t} = \left[ f_t + (1 - \theta \hat{l}_t) \right] dt + \sigma_d d\hat{W}^\delta_t, \quad df_t = (\theta \lambda + \hat{\lambda})(\hat{f} - f_t) dt + \sigma_f d\hat{W}^f_t, \quad (59)$$

$$d\lambda_t = \frac{\theta \lambda_t - \lambda_t}{\sigma_f} d\hat{W}^f_t, \quad \hat{\lambda}_t = \frac{\theta (\hat{f} - f_t) + \lambda_t}{\sigma_f} d\hat{W}^f_t, \quad (60)$$

$$d\nu_{t,t} = \left( \frac{-1}{2} \frac{\nu^2_{t,t}}{\sigma^2_{\delta}} + \sigma^2_f \right) dt, \quad d\nu_{\lambda,t} = \frac{-\theta^2 (\hat{f} - f_t)^2 \nu^2_{\lambda,t}}{\sigma^2_f} dt. \quad (61)$$

Using the dynamics in (59), (60), and (61), the estimation procedure described in Section 3 and Appendix C yields the set of parameters provided in Table 6. Interestingly, the parameter $\sigma_l$ is only weekly significant, meaning that assuming a constant level $l$ is realistic. Similar to what we obtain under the assumption that $l$ is constant, the most likely model given observed data is one that considers learning about the degree of persistence $\lambda$ and not learning about the level $l$. Indeed, the estimation shows that $\theta$ is close to one even when $l$ is assumed to be stochastic. Furthermore, the model with the largest Akaike information criterion is the one assuming that $\theta = 1$.

Comparing the moments in Table 2 (where $l$ is assumed constant) to those in Table 7 (where $l$ is i.i.d.) shows that the filtered degree of persistence $\hat{\lambda}$ and the uncertainty about long-run risk $\nu_{\lambda}$ are independent of our modeling assumption. The moments of the filtered level $\hat{l}$ and uncertainty about the level $\nu_l$, however, differ significantly. Indeed, both the level and the variation of $\hat{l}$ and $\nu_l$ are significantly larger when $l$ is assumed to be i.i.d. than when it is assumed constant. This reason is that the value of the volatility parameter $\sigma_l$ is relatively large when $l$ is assumed to be i.i.d.

To summarize, assuming an i.i.d. process for the level $l$ yields exactly the same conclusion as when the level is assumed to be constant. Indeed, in both cases the most likely model given observed
Table 6: Parameter estimates

This table reports the estimates of the model parameters. To calibrate the model, we use mean analyst forecast on the 1-quarter-ahead real GDP growth as a measure of the growth forecast and the realized real GDP growth as a proxy for consumption growth. The estimates are obtained by Maximum Likelihood for the period Q4:1968 to Q4:2015, using data from the Federal Reserve Bank of Philadelphia. The table compares the estimation results of the base case model with those of two constrained models, with $\theta = 1$ and $\theta = 0$, respectively. Standard errors are in brackets and statistical significance at the 10%, 5%, and 1% levels is denoted by *, **, and ***, respectively.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Base case</th>
<th>$\theta = 1$</th>
<th>$\theta = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output growth volatility</td>
<td>$\sigma_\delta$</td>
<td>1.42%***</td>
<td>1.42%***</td>
<td>1.43%***</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(5 x 10^{-4})</td>
<td>(6 x 10^{-4})</td>
<td>(6 x 10^{-4})</td>
</tr>
<tr>
<td>Long-term expected growth</td>
<td>$\hat{f}$</td>
<td>2.59%***</td>
<td>2.59%***</td>
<td>2.61%***</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.005)</td>
<td>(0.005)</td>
<td>(0.004)</td>
</tr>
<tr>
<td>Expected growth volatility</td>
<td>$\sigma_f$</td>
<td>2.32%***</td>
<td>2.32%***</td>
<td>2.35%***</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(8 x 10^{-4})</td>
<td>(9 x 10^{-4})</td>
<td>(0.001)</td>
</tr>
<tr>
<td>Long-term mean reversion speed</td>
<td>$\bar{\lambda}$</td>
<td>0.79***</td>
<td>0.78***</td>
<td>0.95***</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.259)</td>
<td>(0.255)</td>
<td>(0.193)</td>
</tr>
<tr>
<td>Learning parameter</td>
<td>$\theta$</td>
<td>0.99***</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.015)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Volatility of level $l_t$</td>
<td>$\sigma_l$</td>
<td>0.216</td>
<td></td>
<td>0.003***</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.135)</td>
<td></td>
<td>(0.046)</td>
</tr>
<tr>
<td>Akaike information criterion</td>
<td>AIC</td>
<td>-2,484</td>
<td></td>
<td>-2,486</td>
</tr>
</tbody>
</table>

Table 7: Statistics of variables related to learning about long-risk.

This table reports the descriptive statistics of the state of the economy, measured by the growth forecast $f_t$, the estimate of long-run risk, $\hat{\lambda}$, the uncertainty about long-run risk, captured by $\nu_\lambda$, the estimate of the level, $\hat{l}$, and the uncertainty about the level, $\nu_l$.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>Standard deviation</th>
<th>Minimum</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f$</td>
<td>2.61%</td>
<td>1.70%</td>
<td>-3.75%</td>
<td>6.45%</td>
</tr>
<tr>
<td>$\hat{\lambda}$</td>
<td>0.037</td>
<td>0.037</td>
<td>-0.121</td>
<td>0.082</td>
</tr>
<tr>
<td>$\nu_\lambda$</td>
<td>0.024</td>
<td>0.006</td>
<td>0.020</td>
<td>0.040</td>
</tr>
<tr>
<td>$\hat{l}$</td>
<td>0.215</td>
<td>0.515</td>
<td>-0.822</td>
<td>1.274</td>
</tr>
<tr>
<td>$\nu_l$</td>
<td>0.292</td>
<td>0.072</td>
<td>4.00 x 10^{-4}</td>
<td>0.326</td>
</tr>
</tbody>
</table>

data is one considering learning about the degree of persistence and not learning about the level. Therefore, the asset pricing implications remain unchanged under the assumption of an i.i.d. process for $l$. 31
D The case of learning about expected growth

When the agents learn about the expected consumption growth ($\theta = 0$), the volatility of stock returns becomes:

$$\|\sigma_t\| = \sqrt{\left(\sigma_\delta + \frac{\nu_{t,} I_f}{\sigma_\delta} \hat{I}_f\right)^2 + \left(\sigma_f I_f\right)^2}. \quad (62)$$

While it is indeed the case that uncertainty about the level of economic growth increases stock market volatility, it is easy to see from (62) that learning about economic growth does not generate significant variation in volatility. In fact, a standard log-linearization of the price-dividend ratio $I$ would imply constant volatility. A similar result arises for the equity risk premium, which, for $\theta = 0$, becomes,

$$\mu_t - r_t = \left(\sigma_\delta + \frac{\nu_{t,} I_f}{\sigma_\delta}\right) \gamma_\delta + (1 - \phi) \left(\sigma_f I_f\right)^2. \quad (63)$$

and therefore, although it increase with the uncertainty about expected consumption growth, it does not vary over the business cycle. Overall, these results show that the two types of learning have different implications for the behavior of asset prices. In particular, uncertainty and learning about long-run risk generates significant time-variation in all asset pricing quantities.

E Numerical evaluation of $I_{\hat{\lambda}}/I$

We evaluate numerically the coefficient $I_{\hat{\lambda}}/I$. According to our discussion of Conjecture 1, it is not clear whether the sign of this coefficient is always positive. In particular, the coefficient $I_{\hat{\lambda}}/I$ depends not only on the utility parameters $\gamma$ and $\psi$, but also on the value of the state variables $f_t$ and $\hat{\lambda}_t$.

<table>
<thead>
<tr>
<th>(a) $f_t = -1%$, $\hat{\lambda}_t = 0$</th>
<th>(c) $f_t = 2.6%$, $\hat{\lambda}_t = 0$</th>
<th>(e) $f_t = 6.2%$, $\hat{\lambda}_t = 0$</th>
<th>(b) $f_t = 2.6%$, $\hat{\lambda}_t = -0.1$</th>
<th>(d) $f_t = 2.6%$, $\hat{\lambda}_t = 0$</th>
<th>(f) $f_t = 2.6%$, $\hat{\lambda}_t = 0.1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma/\psi$</td>
<td>1.5</td>
<td>2</td>
<td>2.5</td>
<td>$\gamma/\psi$</td>
<td>1.5</td>
</tr>
<tr>
<td>3</td>
<td>2.3</td>
<td>5.0</td>
<td>7.8</td>
<td>3</td>
<td>1.5</td>
</tr>
<tr>
<td>10</td>
<td>7.2</td>
<td>15.0</td>
<td>22.7</td>
<td>10</td>
<td>6.4</td>
</tr>
<tr>
<td>15</td>
<td>11.3</td>
<td>22.7</td>
<td>33.1</td>
<td>15</td>
<td>10.5</td>
</tr>
<tr>
<td>(b) $f_t = 2.6%$, $\hat{\lambda}_t = -0.1$</td>
<td>(d) $f_t = 2.6%$, $\hat{\lambda}_t = 0$</td>
<td>(f) $f_t = 2.6%$, $\hat{\lambda}_t = 0.1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\gamma/\psi$</td>
<td>1.5</td>
<td>2</td>
<td>2.5</td>
<td>$\gamma/\psi$</td>
<td>1.5</td>
</tr>
<tr>
<td>3</td>
<td>2.2</td>
<td>5.1</td>
<td>8.3</td>
<td>3</td>
<td>1.5</td>
</tr>
<tr>
<td>10</td>
<td>9.7</td>
<td>20.3</td>
<td>30.7</td>
<td>10</td>
<td>6.4</td>
</tr>
<tr>
<td>15</td>
<td>17.4</td>
<td>34.2</td>
<td>48.7</td>
<td>15</td>
<td>10.5</td>
</tr>
</tbody>
</table>

Table 8: The value of the coefficient $I_{\hat{\lambda}}/I$

This table shows the coefficient $I_{\hat{\lambda}}/I$ for different values of the risk aversion, $\gamma \in \{3, 10, 15\}$ and different values of the elasticity of intertemporal substitution, $\psi \in \{1.5, 2, 2.5\}$. There are six sub-tables. The upper row of three tables keeps $\hat{\lambda}_t = 0$ and varies $f_t$. The lower row of three tables keeps $f_t = \bar{f} = 2.6\%$ and varies $\hat{\lambda}_t$. For all tables, the uncertainty $\nu_\lambda$ is fixed at its prior $\nu_\lambda = \nu_{\lambda,0} = 0.04$ (the effect of changes in uncertainty on $I_{\hat{\lambda}}/I$ is relatively smaller).
In Table 8, we generate values of the coefficient $I_{\hat{\lambda}}/I$ in several situations. Subtables correspond to various levels of the growth forecast $f_t$ and of the estimate $\hat{\lambda}$. Within each subtable, we tabulate the value of $I_{\hat{\lambda}}/I$ for different preferences parameters $\gamma$ and $\psi$. Our calculations show that all values are positive, confirming numerically our Conjecture 1.

We note that the coefficient $I_{\hat{\lambda}}/I$ tends to increase in recessions (for lower $f_t$) and when there is more long-run risk (for lower $\hat{\lambda}_t$). The coefficient gets smaller when the risk aversion is smaller and when the elasticity of intertemporal substitution is smaller. However, it remains positive in all cases. It is possible that the coefficient becomes negative, especially when the risk aversion gets close to one, which is far away from the standard long-run risk calibration.

The fact that $I_{\hat{\lambda}}/I$ becomes smaller in good times is related to the effect discussed in Section 4. More precisely, positive shocks in good times do not only signal more persistence (which is bad for the agent), but also a longer economic boom (which is good). However, because the term $I_{\hat{\lambda}}/I$ remains positive even in good times, this suggests that the second effect is smaller in magnitude. To confirm this intuition, we plot in the left panel of Figure 4 the long-price dividend ratio as a function of $\hat{\lambda}_t$ in good, normal, and bad economic times. The log-price dividend ratio clearly remains an increasing function of $\hat{\lambda}_t$ at all times. As a result, the decreasing relationship between volatility/risk-premium and $\hat{\lambda}_t$ is preserved in all cases, as shown in panels (b) and (c) of Figure 4.

![Figure 4: Log price-dividend ratio, stock return volatility, and risk premium as functions of the estimated long-run risk parameter $\hat{\lambda}_t$.](image)

**Figure 4**: Log price-dividend ratio, stock return volatility, and risk premium as functions of the estimated long-run risk parameter $\hat{\lambda}_t$. Each plot has three lines, corresponding to three levels of expected consumption growth: normal times (solid line), bad times (dashed line), and good times (dotted line). The left panel shows that the price dividend ratio increases in $\hat{\lambda}_t$, irrespective of the level of economic growth. The middle and right panels show that stock return volatility and the risk premium decrease with $\hat{\lambda}_t$, irrespective of the level of economic growth.

### F Robustness empirical analysis

#### F.1 Stationarized uncertainty about long-run risk

Learning about long-run risk gradually reduces the level of uncertainty $\nu_\lambda$ over time, as seen in (12). One potential concern, therefore, is that learning plays a greater role and yields stronger asset-pricing implications at the beginning of the sample period than at the end. We show in this section that our empirical results remain robust when we address this concern.
To verify the validity of our empirical findings, we now consider a stationarized time-series of \( \nu_\lambda \) and reproduce our empirical analysis with this variable. Given that the trend in uncertainty is not a deterministic function of time, we use the filter of Savitzky and Golay (1964) to detrend \( \nu_\lambda \).\(^{17}\) More precisely, our detrended measure of \( \nu_\lambda \), which we denote by \( \nu^d_{\lambda,t} \), satisfies

\[
\nu^d_{\lambda,t} = (\nu_{\lambda,t} - m_t) + |\min(\nu_{\lambda,t} - m_t)|,
\]

(64)

where \( m_t \) is the value of Savitzky and Golay (1964)'s filter at time \( t \). We add to the demeaned uncertainty, \( (\nu_{\lambda,t} - m_t) \), the absolute value of its minimum to insure that our detrended measure \( \nu^d_{\lambda,t} \) remains always positive.

Table 9 reports our empirical results when we substitute the uncertainty about long-run risk \( \nu_\lambda \) by its stationary counterpart \( \nu^d_\lambda \). A comparison of Table 5 with Table 9 shows that our empirical results regarding the asset-pricing moments remain qualitatively the same with and without the stationarization. Hence, the decrease over time in uncertainty is not the driver of our findings.\(^{18}\)

\(^{17}\)Detrending \( \nu_\lambda \) using other non-parametric methods such as “Loess” or “Lowess” (Cleveland, 1979, 1981) yields similar results.

\(^{18}\)Unsurprisingly, the statistical significance of the new regression coefficients tends to be slightly lower compared to the case with decaying uncertainty, as \( \nu^d_\lambda \) is noisier than \( \nu_\lambda \).
Table 9: Robustness analysis – stationary uncertainty about long-run risk
This table reproduces Table 5 but using a stationary uncertainty about long-run risk. This time-series is detrended using the approach of Savitzky and Golay (1964). Panel A reports the results for the stock return volatility, Panel B for the equity risk premium, and Panel C for the Sharpe ratio. \( N \) is the number of observations. Newey and West (1987) standard errors are in brackets and statistical significance at the 10%, 5%, and 1% levels is denoted by *, **, and ***, respectively.

<table>
<thead>
<tr>
<th>Panel A: Stock return volatility</th>
<th>( V_{\text{obs}} )</th>
<th>( V_{\text{obs}} )</th>
<th>( V_{\text{obs}} )</th>
<th>( V_{\text{obs}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Const.</td>
<td>0.152*** (0.007)</td>
<td>0.165*** (0.007)</td>
<td>0.137*** (0.012)</td>
<td>0.155*** (0.008)</td>
</tr>
<tr>
<td>( (f - \bar{f}) \nu^d_\lambda )</td>
<td>-899.316** (407.167)</td>
<td></td>
<td></td>
<td>-767.525 (480.974)</td>
</tr>
<tr>
<td>( \hat{\nu}^d_\lambda )</td>
<td>-346.012** (144.219)</td>
<td></td>
<td></td>
<td>-177.473 (175.242)</td>
</tr>
<tr>
<td>( \nu^d_\lambda )</td>
<td></td>
<td>17.515* (10.423)</td>
<td></td>
<td>3.224 (11.040)</td>
</tr>
<tr>
<td>( R^2 )</td>
<td>0.122</td>
<td>0.056</td>
<td>0.008</td>
<td>0.134</td>
</tr>
<tr>
<td>( N )</td>
<td>189</td>
<td>189</td>
<td>189</td>
<td>189</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: Equity risk premium</th>
<th>( R^P_{\text{obs}} )</th>
<th>( R^P_{\text{obs}} )</th>
<th>( R^P_{\text{obs}} )</th>
<th>( R^P_{\text{obs}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Const.</td>
<td>-0.012*** (0.001)</td>
<td>-0.010*** (0.001)</td>
<td>-0.013*** (0.001)</td>
<td>-0.011*** (0.000)</td>
</tr>
<tr>
<td>( (f - \bar{f}) \nu^d_\lambda )</td>
<td>-33.292*** (8.865)</td>
<td></td>
<td></td>
<td>16.774 (12.918)</td>
</tr>
<tr>
<td>( \hat{\nu}^d_\lambda )</td>
<td>-61.894*** (10.402)</td>
<td></td>
<td></td>
<td>-65.900*** (11.260)</td>
</tr>
<tr>
<td>( \nu^d_\lambda )</td>
<td></td>
<td>0.967** (0.470)</td>
<td></td>
<td>1.523*** (4.777)</td>
</tr>
<tr>
<td>( R^2 )</td>
<td>0.055</td>
<td>0.587</td>
<td>0.008</td>
<td>0.610</td>
</tr>
<tr>
<td>( N )</td>
<td>189</td>
<td>189</td>
<td>189</td>
<td>189</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel C: Sharpe ratio</th>
<th>( S^R_{\text{obs}} )</th>
<th>( S^R_{\text{obs}} )</th>
<th>( S^R_{\text{obs}} )</th>
<th>( S^R_{\text{obs}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Const.</td>
<td>-0.087*** (0.010)</td>
<td>-0.067*** (0.003)</td>
<td>-0.094*** (0.012)</td>
<td>-0.075*** (0.006)</td>
</tr>
<tr>
<td>( (f - \bar{f}) \nu^d_\lambda )</td>
<td>-535.507*** (155.686)</td>
<td></td>
<td></td>
<td>-135.902 (234.194)</td>
</tr>
<tr>
<td>( \hat{\nu}^d_\lambda )</td>
<td>-576.954*** (60.484)</td>
<td></td>
<td></td>
<td>-548.558*** (77.631)</td>
</tr>
<tr>
<td>( \nu^d_\lambda )</td>
<td></td>
<td>8.451* (4.529)</td>
<td></td>
<td>7.727** (3.326)</td>
</tr>
<tr>
<td>( R^2 )</td>
<td>0.107</td>
<td>0.382</td>
<td>0.005</td>
<td>0.395</td>
</tr>
<tr>
<td>( N )</td>
<td>189</td>
<td>189</td>
<td>189</td>
<td>189</td>
</tr>
</tbody>
</table>
References


