Trading Heterogeneity, Information Transparency and Market Efficiency

Huanhuan Zheng
arwenzh@gmail.com
Department of Economics
and
Institute of Global Economics and Finance
The Chinese University of Hong Kong

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Abstract

In a market with information friction, investors extrapolate what the others are doing and decide strategically whether to do rational arbitrage or irrational speculation, depending on which one generates a higher payoff. The collective actions of all investors exert feedback on the market efficiency, which affects investors’ subsequent actions. We show that when the market is moderately inefficient, investors cluster to irrational speculation, which enlarges the market inefficiency and in extreme case generates asset price bubbles or depressions. However, when the market is sufficiently inefficient, the coordination on rational arbitrage strengthens as the market becomes more inefficient, which increases the probability of restoring market efficiency, and in extreme case leads to the correction of bubbles or depressions. Information transparency is found to magnify the size of bubbles and depressions because it discourages rational arbitrage.

Keywords: heterogeneity, global games, nonlinear dynamics, bubble, depression, financial crisis

JEL: G12, D53, D83
“A little inefficiency is necessary to give informed investors an incentive to drive prices towards efficiency.” (The Economist, 16 July 2009)

1 Introduction

Financial crisis escalates (Stiglitz, 1999; Reinhart and Rogoff, 2008) despite growing information transparency (Dincer and Eichengreen, 2007). Why does such an extreme phenomenon of market inefficiency repeat themselves while investors become more informed? This paper seeks to tackle this question in a model with investors strategically switching between rational arbitrage that is grounded on efficient market hypothesis (EMH) and irrational speculation that is built upon behavioral finance, in a market with information friction.

The backdrop of our model is a market in which investors trade based not on what the asset is worth, but rather on what they think other players will think it is worth. Although investors value the asset differently, the foundation of asset pricing can be broadly classified into two categories: EMH and behavioral finance. According to EMH, the price will revert to the fundamental when it moves away from it. Any market inefficiency thus provides an opportunity for rational arbitrage. In particular, investors can profit from buying undervalued asset and selling overvalued asset. The greater the price is away from the fundamental, that is, the more inefficient the market is, the more aggressive the rational arbitrage will be. Such actions drive the price towards its fundamental and increase market efficiency. From the behavioral perspective, investors tend to extrapolate recent trend into the future, i.e., they buy the asset when the price has increased and sell it when the price has declined. Such actions that reflect “irrational exuberance” and pay little attention to the information about fundamentals are best described as irrational speculation. Investors are open to both EMH and behavioral finance instead of adhering to a particular theory. Because all investors trade based on either theory, their actions can be classified as rational arbitrage or irrational speculation that correspond respectively to EMH and behavioral finance. This lays foundations
for investors to form expectations of actions of others. Payoff maximizing investors strategically
determine whether to perform rational arbitrage or irrational speculation, taking actions of the
others into account. Since payoffs depend on the aggregate actions of all investors, even if one
realizes the market is inefficient, they may not necessarily act on such information. Instead, they
do best riding on the trend if the market is overwhelmed by irrational speculators, and driving the
price towards the fundamental if the market is dominated by rational arbitrageurs.

In a market with information friction, investors do not have common knowledge about the
fundamental, but rather observe noisy private signals. The information friction uniquely determines
the choice of actions. The divergent opinions on the fundamental lead to different interpretations
on market efficiency and different expectations of actions of others, which result in heterogeneous
trading behavior. The trading heterogeneity arises endogenously because some investors prefer
rational arbitrage, while others are attracted to irrational speculation. Moreover, the same trading
strategy can lead to different trading orders. For example, due to the dispersion of signals, rational
arbitrageurs interpret the degree of market inefficiency differently and place diverse trading orders.
Such between- and within-group heterogeneity enriches the existing binary actions, where agents
buy a fixed number of assets or trade nothing at all\(^1\). It allows us to quantify heterogeneous trading
behavior that is not captured in the rational equilibrium model but is important for analyzing the
financial crisis (Kyle, 1985; Shleifer and Summers, 1990). The trading heterogeneity is a strategic
outcome in our setup instead of exogenously given, such as those in Brock and Hommes (1998)
and Huang, et al. (2010). The endogenous heterogeneity allows us to explain why some agents
engage in irrational speculation even if they are well informed of the fundamental.

The agents’ heterogeneous trading activities have feedback effects on the asset price. The new
asset price in turn affects agents’ subsequent actions, along with the fundamental. To account for
the dynamic interaction between agents’ actions and the asset price, we apply the market maker
framework to capture the price impact of a marginal change in the aggregate actions. In particular,
there is a market maker who passively takes in orders from all agents and adjusts the price quotation

\(^1\)See for example the setup in Ozdenoren and Yuan (2008).
up (down) if the aggregate order is positive (negative) and keeps it unchanged if the aggregate order is in balance. The greater the order imbalance, the greater the price adjustment is. The price impact function that captures the feedback effect of aggregate actions on the asset price specifies whether the price will go up or down and how much it will change. This setup accommodates multiple price outcomes that enable us to understand not just the direction but also the magnitude of future price movements.

Based on the model with heterogeneous and strategic trading strategies, we first obtain the unique equilibrium in agents’ actions, and then solve for a close-form price impact function that captures the interaction between heterogeneous trading and asset price. We show that an agent’s incentive to coordinate on rational arbitrage increases with her expected market inefficiency when the market is sufficiently inefficient. Bubbles (depressions) grow endogenously as agents consistently fail to coordinate sufficiently on rational arbitrage while clustering to speculative buying (selling). However, when market inefficiency exceeds certain thresholds, the coordination on rational arbitrage will strengthen as the market becomes more inefficient, which prevents the price from diverging infinitely from the fundamental. When the market inefficiency is sufficiently large, rational arbitrageurs may accumulate sufficient market power to burst (recover from) the bubbles (depressions).

In terms of bubble growth and crashes, coordination on rational arbitrage in our model plays a similar role as the “synchronization” (or coordinated selling at the same time) in Abreu and Brunnermeier (2003), the failure of which will sustain the market inefficiency, and the restoration of which will burst the bubble. The main difference lies in the way that trading heterogeneity is modeled. Abreu and Brunnermeier (2003) emphasize the time-series heterogeneity that arises from rational arbitrageurs’ sequential awareness of market inefficiency; we focus on the cross-sectional heterogeneity that originates from diverse opinions on the price movements. Agents in our model have diverse information on the market efficiency, but they do not necessarily act on such information in seeking high payoffs. The cross-sectional heterogeneity allows us to study not only the crash of bubbles but also the emergence of bubbles.
Understanding the emergence of bubbles and depressions helps uncover the role of information transparency during periods of financial uncertainty. Our model suggests that when there is a bubble, increasing information transparency reduces the coordination on rational arbitrage and lower average selling power by rational arbitrageurs. As a result, it reduces the aggregate selling by rational arbitrageurs and increases the aggregate buying by irrational speculators, which increases the aggregate demand and raises the price further above the fundamental. So, the size of bubble expands when the information becomes more transparent. Similarly, we show that the depression is deepened when the information becomes more transparent. Overall, information transparency is found to magnify market inefficiency during periods of financial turbulence.

**Literature Review.** Our model utilizes insights from two strands of literature, one on global games and the other on heterogeneous agent models (HAM). The global games have been widely applied in studying speculative attacks in the credit, currency, debt markets and the banking sector (Carlsson and Damme, 1993; Morris and Shin, 1998, 2002, 2004; Goldstein and Pauzner, 2005; Angeletos and Werning, 2006; Goldstein, et al., 2011; and Bebchuk and Goldstein, 2011). These applications of global games feature a static framework, in which many agents simultaneously and strategically choose action or inaction conditional on some noisy and private signals. Such strategic coordination leads to binary outcomes: the status quo is abandoned if a sufficiently large mass of agents take actions against it and is maintained otherwise. Our model extends the existing setup to accommodate multiple actions and multiple outcomes. It describes agents’ trading behavior in terms of rational arbitrage and irrational speculation, which specifies not only whether an agent will buy, sell or hold, but also how many shares of risky asset she will trade. The multiple outcomes are modeled with a price impact function, which incorporates both the direction and the magnitude of price movements. This allows us to analyze not only whether the market is efficient but also the degree of market inefficiency if there is any.

Applying global game technique to solve a model with heterogeneous actions and heterogeneous payoff functions is, however, complicated (Frankel, et al., 2003; Sákovics and Steiner, 2012; Choi, 2014). Motivated by the HAM literature, we address the difficulty by incorporating multiple
actions into two heterogeneous strategies, namely rational arbitrage and irrational speculation, and by capturing multiple outcomes in a price impact function. Such a setup enables us to solve the game with multiple actions and multiple outcomes as if it is of binary actions and binary outcomes. Comparing the payoffs of rational arbitrage and irrational speculation is sufficient to determine an agent’s optimal action.

The HAMs that account for heterogeneous trading activities of rational arbitrageurs and irrational speculators are powerful in explaining financial market phenomena that cannot be justified by EMH, i.e. bubbles and crashes (Lux, 1995; He and Westerhoff, 2005; Huang, et al., 2010). In particular, HAMs that incorporate Eq.(3) and (4) can (i) analytically explain many phenomena in the financial market, such as bubbles, crashes, and volatility clustering. (Lux, 1995; Alfarano, et al., 2008, Huang and Zheng, 2012); (ii) simulate data that match well with stylized facts (De Grauwe and Grimaldi, 2006; Huang, et al., 2013); and (iii) provide empirical specifications that outperform random walk and many existing models (Chiarella, et al., 2012; de Jong, et al., 2010).

A common feature of HAMs is that it ignores the role of information and strategic interactions. In HAM literature, the information is assumed to be complete. As a result, there exist two equilibria: a good equilibrium where all market participants are rational and the price equals its value, and a bad equilibrium where all market participants are irrational and the price can deviate infinitely away from its value. HAM steers away from the multiplicity issue by exogenously specifying whether an agent is a rational arbitrageur or an irrational speculator (Frankel and Froot, 1986, 1990; Frijns, et al., 2010; Chiarella, et al., 2012). Another constraint with HAM is that it relies heavily on the expectations of future price, which is exogenously and heuristically specified (Brock and Hommes 1998; De Grauwe and Grimaldi 2006). We borrow insight from global games to release these constraints. In particular, information friction on the fundamentals is introduced to pin down a unique equilibrium following Morris and Shin (1998). Based on the equilibrium and private signals, agents extrapolate actions of others and form price expectations strategically.

The remainder of this paper is organized as follows. Section 2 develops the model that unifies insight from global games and HAM. Section 3 analyzes the equilibrium and solves the price
impact function explicitly. Section 4 discusses the comparative statics. Section 5 studies the 
emergence and correction of bubbles and depressions. Section 6 analyzes the impact of information 
transparency on the size of bubbles and depressions. Section 7 concludes.

2 The Model

2.1 Timing and Information Structure

There is a continuum of \([0, 1]\) risk-neutral agents, e.g., fund managers, trading on one risky asset. 
The order of events are illustrated in Figure 1. At the beginning of each period \(t\), the market maker 
quotes the logarithmic price of the risky asset \(p_t\). Then the logarithmic fundamental of the risky 
asset \(\theta_t\) is realized. The value of \(\theta_t\) is not verifiable and follows the stochastic process

\[
\theta_{t+1} = \theta_t + \zeta_{t+1},
\]

where innovations \(\{\zeta_{t+1}\}\) are independently and uniformly distributed over \([-\zeta, \zeta]\). Each agent 
\(i \in [0, 1]\) observes a private signal about the fundamental, \(x^i_t = \theta_t + \epsilon_i,t\), where \(\epsilon_i,t\) is the information 
noise term that is independently and uniformly distributed over \([-\epsilon, \epsilon]\) in each period and \(\epsilon < \zeta/2\). After observing the quoted price and the private signal, each agent simultaneously and 
independently decides whether to conduct rational arbitrage or irrational speculation, depending 
on which one yields a higher payoff. The market maker takes orders from all agents and updates

Figure 1: Order of events in each period.
the price according to the price impact function proposed by Kyle (1985)

\[ R_{t+1} = p_{t+1} - p_t = \gamma (D_t + S_t), \tag{2} \]

where \( R_{t+1} \) is the log return from period \( t \) to \( t + 1 \), \( D_t \) is the aggregate demand of the risky asset by all agents, \( S_t \sim N(0, \sigma_t^2) \) is the random supply shock that is serially independent, and \( \gamma > 0 \) measures the price impact of the marginal change in the aggregate demand. Both \( D_t \) and \( S_t \) are observable to the market maker but not the agents. The subsequent price quotation, \( p_{t+1} \), goes up if \( D_t + S_t > 0 \), declines if \( D_t + S_t < 0 \), and remains unchanged otherwise. The update of the price quotation starts another round of game. We apply the market-maker mechanism instead of market-clearing mechanism to make the nonlinear model analytically tractable and solvable over multiple horizons\(^2\). As we shall show later, the price does not necessarily contain information about the fundamental, and even if it does, it is difficult to learn the information from the price due to the presence of random supply shock and nonlinear price dynamics. We therefore do not consider learning from the price.

\[ \text{2.2 Actions} \]

There are two trading strategies available: rational arbitrage and irrational speculation.

Agent \( i \)'s demand for the risky asset based on rational arbitrage is a function of the expected difference between the fundamental and the price:

\[ d_{f,t}^i = \alpha \left[ E^i \left( \theta_t | x_t^i \right) - p_t \right] = \alpha \left( x_t^i - p_t \right), \tag{3} \]

where \( \alpha > 0 \) measures the trading intensity of rational arbitrage and \( E^i \left( \theta_t | x_t^i \right) = x_t^i \) is the posterior expectation of the fundamental\(^3\). Agents engaging in rational arbitrage buy (sell) the asset when it

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\(^2\)The market-maker mechanism is also found to generate more efficient price than the market-clearing mechanism (Madhavan, 1992, Vives, 1995 and Venkataraman and Waisburd, 2007).

\(^3\)Given Eq.(1), historical signals are not informative about the expected fundamental. We can therefore ignore the past signals and interpret the dynamic game as a sequence of one-shot games.
is expected to be underpriced (overpriced), which drives the price towards revealing its fundamental and improves the market efficiency. As a result of information dispersion, agents that conduct rational arbitrage have different demand for the risky asset.

Agent $i$’s demand for the risky asset based on *irrational speculation* is independent of her private signal:

$$d_{c,t} = \beta (p_t - v),$$

where $0 < \beta < \alpha$ measures the trading intensity of irrational speculation, and $v$ is the reference price that is exogenously given. Actions based on irrational speculation capture the trading behavior that is subject to “irrational exuberance”, which pushes the price away from $v$. As both $v$ and $\beta$ are common knowledge, agents conducting irrational speculation share the same demand function.

Describing agents’ trading behavior in terms of rational arbitrage and irrational speculation specifies not only the direction but also the size of trading orders (see Eq.(3) and (4)). Such a setup also captures how agents’ trading decisions change over time and vary with price levels and fundamentals. Modeling agents’ actions based on two simple and heuristic demand functions in Eq.(3) and (4) is restricted given the numerous and complicated trading strategies in the financial market. However, HAM literature suggests that such two simple rules of thumb are representative to capture the general trading behavior and explain various financial market phenomena (Lux, 1995; De Grauwe and Grimaldi, 2006; Alfarano, et al., 2008; Huang, et al., 2013).

### 2.3 Payoffs

As documented by Hirshleifer and Thakor (1989) and Harris and Raviv (1991), fund managers that invest on others’ behalf prioritize the probability of success, i.e., earning positive profit, over the size of profit. Motivated by this finding, we define the payoff as the probability of success if rational arbitrage and irrational speculation suggest different trading directions, and as the amount

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4 Even if the reference price is time-varying, for each realization of $v$, all the propositions in this paper remain valid. One just needs to update $v$ in every period in the same way as she update the price.
of profit if both strategies suggest the same trading direction\textsuperscript{5}. The payoff differential between rational arbitrage and irrational speculation of agent $i$, denoted as $\Delta \pi_i^t$, is given by

$$\Delta \pi_i^t = I_i^t \cdot \Gamma_t,$$

where $I_i^t = 1$ if $d_{f,t}^i \geq d_{c,t}$ and $I_i^t = -1$ otherwise, and $\Gamma_t$ is the utility of buying the risky asset such that

$$\Gamma_t = \begin{cases} 
1 & \text{if } R_{t+1} > 0 \\
0 & \text{if } R_{t+1} = 0 \\
-1 & \text{if } R_{t+1} < 0 
\end{cases}.$$ 

(6)

The realization of $\Gamma_t$ depends on $R_{t+1}$, which depends on the aggregate actions of all agents. The specification of $\Gamma_t$ highlights the importance of earning positive profits: an agent's utility is 1 if she makes a profit regardless of the amount of profit, $-1$ if she makes a loss, and 0 otherwise. The definition of $\Delta \pi_i^t$ captures the attractiveness of rational arbitrage relative to irrational speculation for each agent $i \in [0, 1]$. It means that, when the two strategies result in different trading directions, the one that generates a positive profit yields a higher payoff. Suppose rational arbitrage suggests buying the risky asset, while irrational speculation suggests selling. According to Eq.(2), the price will increase (decline) when a sufficiently large number of agents conduct rational arbitrage (irrational speculation). If the price increases ($R_{t+1} > 0$), rational arbitrage is preferred over irrational speculation because it generates a positive profit, while the latter makes a loss. If the price declines ($R_{t+1} < 0$), then irrational speculation is preferred over rational arbitrage for the same reason. In this case, the strategy that is adopted by the majority is more attractive as it generates a positive profit by moving the price towards the direction that favors its trading. Eq.(5) also implies that when the two strategies suggest the same trading direction, the one that generates a higher profit yields a higher payoff. For example, if both rational arbitrage and irrational speculation result in buying the risky asset, the price will go up if $D_t > S_t$ according to Eq.(2), and both strategies

\textsuperscript{5}When one strategy suggests buying and the other suggests selling, as the return depends on the aggregate actions of all agents, only one strategy will generate positive profit. When both strategies suggest the same trading direction, they both make profit if $|D_t| > S_t$ or loss $|D_t| < S_t$. 

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generate positive profits. In this case, the strategy that yields more buying is more attractive as it generates a greater profit.

2.4 Optimal Decisions

Whether an agent will conduct rational arbitrage or irrational speculation depends on her signal as well as what she believes the others are going to do, which is captured by the conditional expected payoff differential between the two strategies:

\[
E (\Delta \pi_t | x_i^t) = I_i^t \cdot E (\Gamma_t | x_i^t).
\] (7)

Agent \(i\) favors rational arbitrage if \(E (\Delta \pi_t | x_i^t) > 0\) and irrational speculation if \(E (\Delta \pi_t | x_i^t) < 0\). As a modeling convention, we assume that agent \(i\) conducts rational arbitrage if \(E (\Delta \pi_t | x_i^t) = 0\). Therefore, her actual demand for the risky asset \(d_i^t\) is given by

\[
d_i^t = \begin{cases} 
d_{j,f,i}^t, & \text{if } E (\Delta \pi_t | x_i^t) \geq 0 \\
d_{c,f,i}^t, & \text{if } E (\Delta \pi_t | x_i^t) < 0
\end{cases}.
\] (8)

We call agent \(i\) a rational arbitrageur if \(d_i^t = d_{j,f,i}^t\) and an irrational speculator if \(d_i^t = d_{c,f,i}^t\). As \(d_i^t \geq \min(d_{j,f,i}^t, d_{c,f,i}^t)\) and \(d_i^t \leq \max(d_{j,f,i}^t, d_{c,f,i}^t)\), agent \(i\) can trade up to \(\max(d_{j,f,i}^t, d_{c,f,i}^t) (\min(d_{j,f,i}^t, d_{c,f,i}^t))\) if she expects the price to increase (decline). This suggests that agents have no incentive to adopt a mix of rational arbitrage and irrational speculation.

Clearly, \(I_i^t\) is known by agent \(i\), and the only uncertainty arises from \(E (\Gamma_t | x_i^t)\), which depends on the aggregate actions of all agents. So, agent \(i\) that seeks to maximize her payoff will prefer the strategy that generates a higher demand if she expects the price to increase or remain constant \((E (\Gamma_t | x_i^t) \geq 0)\), and the strategy that generates a lower demand if she expects the price to decline \((E (\Gamma_t | x_i^t) < 0)\). As we show later, \(E (\Gamma_t | x_i^t)\) increases with \(x_i^t\). So an agent with a higher signal is more likely to expect a positive return, and therefore more likely to adopt the strategy that generates more buying. Such a strategy can be either rational arbitrage or irrational speculation, depending
on the relative value of the private signal and the reference price as highlighted in Eq.(3) and (4).

Let $F_t$ be the set of agents that end up being rational arbitrageurs at period $t$:

$$F_t = \{ i | E(\Delta \pi_t|\pi_t) \geq 0 \}.$$  

The fraction of rational arbitrageurs at period $t$, denoted as $m_t \in [0, 1]$, is

$$m_t = \int_{F_t} dk.$$  \hspace{1cm} (9)

The larger the value of $m_t$, the stronger the coordination on rational arbitrage is. For this reason, we also refer to $m_t$ as the degree of coordination on rational arbitrage. Given that rational arbitrageurs and irrational speculators are complementary, the number of irrational speculators is $1 - m_t$.

The average signal of all rational arbitrageurs, denoted as $\eta_t$, is given by

$$\eta_t = \int_{F_t} x_t^k dk.$$  

Taking account of Eq.(8), the aggregate demand $D_t$ of all agents can be written as

$$D_t = \int_{F_t} (d_{I,t} - d_{c,t}) dk + d_{c,t}$$  \hspace{1cm} (10)

$$= \alpha (\eta_t - p_t) m_t + \beta (p_t - v) (1 - m_t).$$  \hspace{1cm} (11)

### 2.5 Interpretations

In this setup, rational arbitrageurs and irrational speculators can either buy or sell the risky asset with different order size (see Eq.(3) and (4)), and the returns can be either positive or negative with different magnitude (see Eq.(2)). It incorporates various buying and selling activities as well as multiple outcomes on returns. The collective actions of agents will move the price towards the fundamental if a sufficiently large number of agents act as rational arbitrageurs and drive the price away from the reference price if a sufficiently large number of agents act as irrational
speculators (see Eq.(2)). Both the convergence to the fundamental and the deviation from the reference price can be associated with either positive or negative returns. When the asset is overpriced (underpriced), the convergence towards the fundamental indicates negative (positive) returns. Similarly, when the price is above (below) the reference price, further deviation from the reference price indicates positive (negative) returns. Agents’ actions have feedback effects on the price movements, which affects their subsequent choice of trading strategies (see Eq.(7) and (8)).

To better understand the model, in the following, we first compare our model with the global game setup and then discuss its unique characteristics.

2.6 Linkage with Global Games

Our setup shares the features of global games. Agents choose their actions strategically, taking actions of others into account. The outcomes on returns depend on the aggregate actions of all agents. There are multiple equilibria when the information is complete (see Example 1). However, in the presence of information friction, the equilibrium becomes unique (see Example 2). Nonetheless, the global game is only a special case of the model. We now turn to its unique characteristics.

Example 1: Multiple Equilibria with Perfect Information. Suppose the information is perfect such that $\zeta = \varepsilon = 0$, which implies $x_i^t = \theta_i$ for any $i$. Let $p_t > v > \theta_i$ so that rational arbitrageurs sell, while irrational speculators buy the risky asset (see Eq.(3) and (4)). In this case, according to Eq.(5), the payoff is the probability of successfully generating positive profit. In the absence of supply shock, according to Eq.(2), $R_{t+1} < 0$ if there is a sufficiently large fraction of rational arbitrageurs and $R_{t+1} > 0$ if there is a sufficiently large number of irrational speculators. Conditional on the direction of price movements, the probability of success for each strategy is shown in Table 1. Clearly, agents will choose the strategy that is chosen by the others, which suggests both strategies are equilibria.

Example 2: Unique Equilibrium with Information Friction. Suppose $\varepsilon > 0$, $p_t > v > x_i^t$ and $S_t = 0$. Further suppose that $x_i^t \in [\tilde{\theta}_i - \varepsilon, \tilde{\theta}_i + \varepsilon]$, where $\tilde{\theta}_i$ is the threshold fundamental such that
Table 1: Payoffs with complete information.

<table>
<thead>
<tr>
<th></th>
<th>$R_{t+1} &gt; 0$</th>
<th>$R_{t+1} = 0$</th>
<th>$R_{t+1} &lt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rational Arbitrage</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Irrational Speculation</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 2: Payoffs with information friction.

<table>
<thead>
<tr>
<th></th>
<th>$R_{t+1} &gt; 0$</th>
<th>$R_{t+1} = 0$</th>
<th>$R_{t+1} &lt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rational Arbitrage</td>
<td>0</td>
<td>0</td>
<td>$\frac{\theta_t + \varepsilon - x'_t}{2\varepsilon}$</td>
</tr>
<tr>
<td>Irrational Speculation</td>
<td>$\frac{x'_t + \varepsilon - \theta_t}{2\varepsilon}$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

$R_{t+1} = 0$ if $\theta_t = \tilde{\theta}_t, R_{t+1} > 0$ if $\theta_t > \tilde{\theta}_t$ and $R_{t+1} < 0$ if $\theta_t < \tilde{\theta}_t$, which we prove formally latter. Note that agent $i$ expects $\theta_t$ to fall uniformly in the interval $[x'_t - \varepsilon, x'_t + \varepsilon]$. The expected probability of success for rational arbitrage is $\frac{\theta_t + \varepsilon - x'_t}{2\varepsilon}$, while that for irrational speculation is $\frac{x'_t + \varepsilon - \theta_t}{2\varepsilon}$, as shown in Table 2. Clearly, agent $i$ that seeks to maximize her probability of success will prefer rational arbitrage if $x'_t \leq \tilde{\theta}_t$ and irrational speculation otherwise.

### 2.7 Strategic Complementarities and Substitutions

To understand the game structure with trading heterogeneity, we first document the relation between the return $R_{t+1}$ and the fraction of rational arbitrageurs $m_t$, and then discuss the impact of $m_t$ on $\Delta \pi'_i$, the payoff differential between rational arbitrage and irrational speculation.

The realized return on the risky asset $R_{t+1}$ is a monotonic increasing function of the aggregate demand $D_t$ according to Eq.(2). Therefore, the qualitative relation between $R_{t+1}$ and $m_t$ is the same as that between $D_t$ and $m_t$. Based on Eq.(10), the relation between $D_t$ and $m_t$ can be analyzed in the following three scenarios.

(i) If $d^k_{f,t} < d_{c,t}$, for any $k \in [0, 1]$, that is, the demand for the risky asset based on rational arbitrage is lower than that based on irrational speculation for all agents, Eq.(10) indicates that $D_t$ declines with $m_t$. Intuitively, when $m_t$ increases, there must be some agents who switch from high-demand irrational speculation to low-demand rational arbitrage, which leads to a decline in the aggregate demand.

(ii) If $d^k_{f,t} > d_{c,t}$, for any $k \in [0, 1]$, that is, the demand based on rational arbitrage is higher than
that based on irrational speculation for all agents, then $D_t$ increases with $m_t$. This is because, as $m_t$ increases, there must be agents switching from low-demand irrational speculation to high-demand rational arbitrage, which increases the aggregate demand.

(iii) If there exists a $j \in [0, 1]$ such that $d^j_{f,t} = d_{c,t}$, the relation between $D_t$ and $m_t$ is nonlinear. Recall from Eq.(3) and that an agent’s demand based on rational arbitrage $d^k_{f,t}$ increases with her signal $x^k_t$, $d^k_{f,t} > d_{c,t}$ if $x^k_t > x^j_t$ and $d^k_{f,t} < d_{c,t}$ if $x^k_t < x^j_t$, for any $k \in [0, 1]$. It means that rational arbitrage yields a higher (lower) demand than irrational speculation for any agent that receives a signal higher (lower) than $x^j_t$. If $m_t$ increases because an agent $h$ whose signal is higher than $x^j_t$ switches from irrational speculation to rational arbitrage, then $D_t$ increases because the demand of agent $h$ increases after she switches the strategy ($d^h_{f,t} > d_{c,t}$ as $x^h_t > x^j_t$). If $m_t$ increases because an agent $l$ whose signal is lower than $x^j_t$ switches from irrational speculation to rational arbitrage, then $D_t$ declines as the demand of agent $l$ declines after she switches the strategy ($d^l_{f,t} < d_{c,t}$ as $x^l_t < x^j_t$). So, the aggregate demand $D_t$ can either increases or decreases with the fraction of rational arbitrageurs $m_t$.

The relation between the $\Delta \pi^j_t$, the payoff differential between rational arbitrage and irrational speculation, and $m_t$ can be summarized in the same three scenarios based on Eq.(5).

(i) If $d^k_{f,t} < d_{c,t}$ for any $k \in [0, 1]$, then $\Delta \pi^j_t = -\Gamma_t$ for any $i \in [0, 1]$. According to the previous analysis, as $m_t$ increases, $D_t$ and $R_{t+1}$ decrease, which decreases $\Gamma_t$ and therefore increases $\Delta \pi^j_t$. So, the incentive to conduct rational arbitrage increases with the fraction of rational arbitrageurs.

(ii) If $d^k_{f,t} > d_{c,t}$ for any $k \in [0, 1]$, then $\Delta \pi^j_t = \Gamma_t$ increases with $m_t$ (as $m_t$ increases, $D_t$ and $R_{t+1}$ increase, which leads to an increase in $\Gamma_t$). In this case, the incentive to conduct rational arbitrage increases with the fraction of rational arbitrageurs.

(iii) If there exists a $j \in [0, 1]$ such that $d^j_{f,t} = d_{c,t}$, an agent’s incentive to conduct rational arbitrage can either increase or decrease with the fraction of rational arbitrageurs. For example, if $m_t$ increases because agent $k$ that has $d^k_{f,t} > d^j_{f,t} = d_{c,t}$ switches from irrational speculation to rational arbitrage, $D_t$ increases, which leads to an increase in $\Gamma_t$. For agent $i$ that has $d^i_{f,t} > d_{c,t}$, as $\Delta \pi^j_t = \Gamma_t$, her incentive to conduct rational arbitrage increases as $m_t$ increases. However, for
agent \(i'\) that has \(d_{f,t}' < d_{c,t}\), as \(\Delta \pi_i' = -\Gamma_i\), her incentive to conduct rational arbitrage declines as \(m_t\) increases. Similarly, we can show that if \(m_t\) increases because agent \(k\) that has \(d_{f,t}^k < d_{c,t} = d_{c,t}\) switches from irrational speculation to rational arbitrage, the increase in \(m_t\) mitigates the incentive of agent \(i\) that has \(d_{f,t} > d_{c,t}\) to conduct rational arbitrage while improving the incentive of agent \(i'\) that has \(d_{f,t}' < d_{c,t}\) to conduct rational arbitrage.

In scenario (i) and (ii), there are strategic complementarities within rational arbitrageurs and within irrational speculators respectively. That is, an agent’s incentive to conduct rational arbitrage (or irrational speculation) increases with the number of other agents who employ the same strategy. In scenario (iii), there are both strategic complementarities and strategic substitutions within the group of rational arbitrageurs (or irrational speculators). In particular, strategic complementarities exist among rational arbitrageurs whose signals fall on the same side of the signal \(x_f^i\) at which \(d_{f,t} = d_{c,t}\), while strategic substitutions arise among rational arbitrageurs (or irrational speculators) whose signals fall on opposite sides of \(x_f^i\). In the case of strategic substitution, an agent’s incentive to conduct rational arbitrage (or irrational speculation) decreases with the number of other agents who employ the same strategy.

3 Equilibrium Analysis

In existing global games literature, agents follow a monotone threshold strategy by taking actions against the status quo if and only if their private signals exceed (or fall below) a certain threshold and do nothing otherwise (Morris and Shin, 1998, 2003). This property of monotone equilibrium does not hold in our nonlinear model. In particular, there is no global strategic complementarities between agents’ actions because one’s incentive to coordinate on the rational arbitrage change may either increase or decrease with her signal. Motivated by the state-dependent relationships between the payoff difference \(\Delta \pi_i\) and the fraction of rational arbitrageurs \(m_t\), for each realization of price \(p_t\), we partition the domain of fundamentals \(\theta_t\) into three states, namely the bubbly, depressed and normal state as illustrated in Figure 2.
Figure 2: Tripartite classification of the fundamentals.

The **bubbly state** is defined by the range \((-\infty, z_t - \varepsilon)\), where \(z_t = p_t - \beta (p_t - v) / \alpha\). When \(\theta_t \in (-\infty, z_t - \varepsilon)\), the demand based on rational arbitrage is always lower than that based on irrational speculation, that is, \(d_{f,t}^i < d_{c,t}\) for any \(i \in [0, 1]\). The range \((-\infty, z_t - \varepsilon)\) is obtained in the following way. As \(x_t^i \in [\theta_t - \varepsilon, \theta_t + \varepsilon]\) and \(d_{f,t}^i\) increases with \(x_t^i\), for any \(i \in [0, 1]\), there is \(d_{f,t}^i < \alpha (\theta_t + \varepsilon - p_t)\). To ensure that \(d_{f,t}^i < d_{c,t}\) for any \(i \in [0, 1]\), it must be true that \(\alpha (\theta_t + \varepsilon - p_t) < \beta (p_t - v)\), which implies \(\theta_t < z_t - \varepsilon\). We call this space of fundamentals the bubbly state because the price can rise significantly above the fundamental that characterizes asset price bubbles.

The **depressed state** is defined by the range \((z_t + \varepsilon, +\infty)\). When \(\theta_t \in (z_t + \varepsilon, +\infty)\), each agent’s demand based on rational arbitrage is always greater than that based on irrational speculation, that is, \(d_{f,t}^i > d_{c,t}\) for any \(i \in [0, 1]\). We call this space of fundamentals the depressed state because the price can fall significantly below the fundamental that characterizes asset price depressions.

The **normal state** is defined by the range \([z_t - \varepsilon, z_t + \varepsilon]\). When \(\theta_t \in [z_t - \varepsilon, z_t + \varepsilon]\), there exists a \(j \in [0, 1]\) such that \(d_{f,t}^j = d_{c,t}\). In the normal state, the asset is considered to be relatively fairly priced as the prices are relatively close to the fundamental.
The starting point of solving the equilibrium strategy is that the aggregate demand $D_t$ is an increasing function of $\theta_t$\textsuperscript{6}. We conjecture the following Claim and prove it formally in Appendix A.3.

**Claim 1** Given a realization of $p_t$, there exists a unique equilibrium threshold fundamental, $\bar{\theta}_t$, such that $D_t = 0$ if $\theta_t = \bar{\theta}_t$, $D_t > 0$ if $\theta_t > \bar{\theta}_t$, and $D_t < 0$ if $\theta_t < \bar{\theta}_t$.

As an agent’s posterior update of the fundamental is independent of her historical signals, the dynamic games can be interpreted as a sequence of identical one-shot game. Given that both the fundamental $\theta_t$ and the information noise are uniformly distributed, the posterior distribution of $\theta_t$ conditional on signal $x_i^t$ is uniformly distributed over $[x_i^t - \varepsilon, x_i^t + \varepsilon]$ and $E(\theta_t|x_i^t) = x_i^t$.

Based on Eq.(2), agent $i$’s expected return conditional on her signal has the following properties:

$$E(R_t+1|x_i^t) = 0 \text{ if } x_i^t = \bar{\theta}_t, \quad E(R_t+1|x_i^t) > 0 \text{ if } x_i^t > \bar{\theta}_t, \quad \text{and } E(R_t+1|x_i^t) < 0 \text{ if } x_i^t < \bar{\theta}_t.$$ 

It implies that $E(\Gamma_t|x_i^t) = 0$ if $x_i^t = \bar{\theta}_t$, $E(\Gamma_t|x_i^t) = 1$ if $x_i^t > \bar{\theta}_t$, and $E(\Gamma_t|x_i^t) = -1$ if $x_i^t < \bar{\theta}_t$ according to Eq.(6). In each of the three regimes, we first identify the equilibrium threshold fundamental for each realization of price $p_t$ and then solve for the price impact function explicitly.

### 3.1 Bubbly State

In the bubbly state where $\theta_t < z_t - \varepsilon$, rational arbitrage always yields a lower demand than irrational speculation, that is, $d_i^{d,r} < d_i^{c,r}$ for any $i \in [0,1]$. When $x_i^t \in [\bar{\theta}_t - \varepsilon, \bar{\theta}_t + \varepsilon]$, the probability of

$$E(\Gamma_t|x_i^t) = -1 \text{ is } \int_{x_i^t - \varepsilon}^{\bar{\theta}_t} d\theta_t = \frac{x_i^t - \bar{\theta}_t + \varepsilon}{2\varepsilon}, \quad \text{while the probability of } E(\Gamma_t|x_i^t) = 1 \text{ is } \int_{\bar{\theta}_t}^{\bar{\theta}_t + \varepsilon} d\theta_t = \frac{\bar{\theta}_t + \varepsilon - x_i^t}{2\varepsilon}. $$

So the expected payoff differential between rational arbitrage and irrational speculation in Eq.(7)

\textsuperscript{6}Intuitively, larger fundamentals increases the demand based on rational arbitrage while having no impact on the demand based on irrational speculation, which increase the aggregate demand.
can be rewritten as

\[
E(\Delta \pi^t_i|x^t_i) = -E(\Gamma^t_i|x^t_i)
\]
\[
= -\frac{1}{2\epsilon} \left[ \int_{x^t_i-\epsilon}^{\tilde{\theta}_t}(1)\,d\theta_t + \int_{\tilde{\theta}_t}^{\tilde{x}_i^t+\epsilon} 1\,d\theta_t \right]
\]
\[
= \frac{\tilde{\theta}_t - x^t_i}{\epsilon}.
\]  

Eq.(12) suggests that \(E(\Delta \pi^t_i|x^t_i)\) decreases with \(x^t_i\) when \(x^t_i \in [\tilde{\theta}_t - \epsilon, \tilde{\theta}_t + \epsilon]\). When \(x^t_i > \tilde{\theta}_t + \epsilon\), then \(E(\Delta \pi^t_i|x^t_i) = -1\) as \(E(R_{t+1}|x^t_i) > 0\) and therefore \(E(\Gamma^t_i|x^t_i) > 0\). Similarly, \(E(\Delta \pi^t_i|x^t_i) = 1\) when \(x^t_i < \tilde{\theta}_t - \epsilon\). To summarize, conditional on \(\theta_t < z_t - \epsilon\), \(E(\Delta \pi^t_i|x^t_i) = 0\) if \(x^t_i = \tilde{\theta}_t\), \(E(\Delta \pi^t_i|x^t_i) > 0\) if \(x^t_i < \tilde{\theta}_t\), and \(E(\Delta \pi^t_i|x^t_i) < 0\) if \(x^t_i > \tilde{\theta}_t\), which proves Lemma 1.

**Lemma 1** When \(\theta_t < z_t - \epsilon\), an agent will become a rational arbitrageur if her signal is on or below \(\tilde{\theta}_t\) and an irrational speculator otherwise.

Lemma 1 suggests that an agent’s incentive to coordinate on rational arbitrage decreases with her signal in the bubbly state. The lower one’s signal is, the higher the expected probability of negative returns is, and the more attractive the rational arbitrage that generates lower demand (or more selling) will be. Lemma 1 also suggests that an agent with a lower signal expects more of the others to be rational arbitrageurs, which increases her incentive to coordinate on rational arbitrage. This property is a result of the strategic complementarity among rational arbitrageurs in the bubbly state.

Based on Lemma 1, Eq.(8) in the bubbly state can be rewritten as:

\[
d^t_i = \begin{cases} 
  d^t_{f,t} & \text{if } x^t_i \leq \tilde{\theta}_t \\
  d^t_{c,t} & \text{if } x^t_i > \tilde{\theta}_t
\end{cases}
\]  

Eq.(13) suggests that all agents follow an equilibrium threshold strategy by employing rational arbitrage if \(x^t_i \leq \tilde{\theta}_t\) and irrational speculation otherwise. Recall that agents’ signals fall uniformly in \([\theta_t - \epsilon, \theta_t + \epsilon]\). The fraction of rational arbitrageurs \(m_t\) and the average signal received by rational
arbitrageurs $\eta_t$ can be characterized by considering three scenarios. First, if $\theta_t < \bar{\theta}_t - \varepsilon$, all agents receive signals below $\bar{\theta}_t$, which leads to $m_t = 1$ and $\eta_t = \theta_t$. Second, if $\theta_t > \bar{\theta}_t + \varepsilon$, all agents receive signals above $\bar{\theta}_t$ and $m_t = 0$. Finally, if $\theta_t \in [\bar{\theta}_t - \varepsilon, \bar{\theta}_t + \varepsilon]$, the proportion of agents receiving signals not higher than $\bar{\theta}_t$ is $(\bar{\theta}_t - \theta_t + \varepsilon) / (2\varepsilon)$, and the average signal received the these rational arbitrageurs is $(\bar{\theta}_t + \theta_t - \varepsilon) / 2$. Thus, $\eta_t$ and $m_t$ are given by:

\[
\eta_t = \begin{cases} 
\theta_t & \text{if } \theta_t < \bar{\theta}_t - \varepsilon \\
(\bar{\theta}_t + \theta_t - \varepsilon) / 2 & \text{if } \theta_t \in [\bar{\theta}_t - \varepsilon, \bar{\theta}_t + \varepsilon] \\
0 & \text{if } \theta_t > \bar{\theta}_t + \varepsilon
\end{cases}
\] (14)

and

\[
m_t = \begin{cases} 
1 & \text{if } \theta_t < \bar{\theta}_t - \varepsilon \\
(\bar{\theta}_t - \theta_t + \varepsilon) / (2\varepsilon) & \text{if } \theta_t \in [\bar{\theta}_t - \varepsilon, \bar{\theta}_t + \varepsilon] \\
0 & \text{if } \theta_t > \bar{\theta}_t + \varepsilon
\end{cases}
\] (15)

Substituting Eq.(14) and (15) into Eq.(11) and rearranging yield the aggregate demand conditional on $\theta_t < z_t - \varepsilon$:

\[
D_t = \begin{cases} 
\alpha (\theta_t - p_t) & \text{if } \theta_t < \bar{\theta}_t - \varepsilon \\
\frac{\alpha (\bar{\theta}_t + \theta_t - \varepsilon - 2z_t) (\bar{\theta}_t - \theta_t + \varepsilon)}{4\varepsilon} + \beta (p_t - v) & \text{if } \theta_t \in [\bar{\theta}_t - \varepsilon, \bar{\theta}_t + \varepsilon] \\
\beta (p_t - v) & \text{if } \theta_t > \bar{\theta}_t + \varepsilon
\end{cases}
\] (16)

Eq.(16) shows that, when $\theta_t \leq \bar{\theta}_t + \varepsilon$, $D_t$ and therefore $R_{t+1}$ contains information about the fundamental, due to the presence of rational arbitrageurs. When $\theta_t > \bar{\theta}_t + \varepsilon$, the return is uninformative about the real fundamental as all agents are irrational speculators that ignore the information. In the rational expectation equilibrium model of asset prices (Grossman and Stiglitz, 1980; Angeletos and Werning, 2006; Ozdenoren and Yuan, 2008), the price will not fully reveal its value because of the supply shock. In our setup, even without supply shock, the price may not fully reveal its value due to the presence of irrational speculators. In reality, we can observe $R_{t+1}$ but not $D_t$ and $S_t$. Even if $R_{t+1}$ is informative about the fundamental, the nonlinearity of $D_t$ as a function of $\theta_t$
and the random supply shock $S_t$ make it difficult, if possible, to learn the endogenous information in $R_{t+1}$. For this reason, we do not consider the endogenous information in this paper.

Eq.(16) holds as long as $\theta_t < z_t - \varepsilon$, regardless of the value of $\bar{\theta}_t$. If $\bar{\theta}_t$ exists in the bubbly state, then it can be solved from Eq.(16). According to Claim 1 and Eq.(16), at $\theta_t = \bar{\theta}_t$,

$$D_t = \frac{\alpha}{2} \left( \frac{2\bar{\theta}_t - \varepsilon}{2} - p_t \right) + \frac{\beta}{2} (p_t - v) = 0.$$ 

Solving for $\bar{\theta}_t$ yields

$$\bar{\theta}_t = p_t - \frac{\beta (p_t - v)}{\alpha + \varepsilon/2}.$$ 

The solution is valid if and only if $\bar{\theta}_t < z_t - \varepsilon$, that is, $\beta (p_t - v)/\alpha > 3\varepsilon/4$.

### 3.2 Depressed State

In the depressed state where $\theta_t > z_t + \varepsilon$, rational arbitrage yields a higher demand than irrational speculation for all agents, that is, $d^i_{j,t} > d^c_{j,t}$ for any $i \in [0, 1]$. As the equilibrium analysis in the depressed state is symmetric to that in the bubbly state, we only describe the result briefly in the following.

When $x^i_t \in [\bar{\theta}_t - \varepsilon, \bar{\theta}_t + \varepsilon]$, Eq.(7) can be rewritten as

$$E(\Delta\pi_t^i|x^i_t) = E(\Gamma_t|x^i_t) = \frac{x^i_t - \bar{\theta}_t}{\varepsilon}. \quad (17)$$

Moreover, $E(\Delta\pi_t|x^i_t) = 1$ when $x^i_t > \bar{\theta}_t + \varepsilon$, and $E(\Delta\pi_t|x^i_t) = -1$ when $x^i_t < \bar{\theta}_t - \varepsilon$. So conditional on $\theta_t > z_t + \varepsilon$, $E(\Delta\pi_t|x^i_t) = 0$ if $x^i_t = \bar{\theta}_t$, $E(\Delta\pi_t|x^i_t) > 0$ if $x^i_t > \bar{\theta}_t$ and $E(\Delta\pi_t|x^i_t) < 0$ if $x^i_t < \bar{\theta}_t$, which proves Lemma 2.

**Lemma 2** When $\theta_t > z_t + \varepsilon$, an agent will become a rational arbitrageur if her signal is on or above $\bar{\theta}_t$ and an irrational speculator otherwise.
The results suggests that, in the depressed state, an agent’s incentive to coordinate on rational arbitrage increases with her signal. All the others being the same, the higher the signal is, the greater the expected probability of positive returns is, and the more attractive the rational arbitrage that generates a higher demand (more buying) will be.

Based on Lemma 2, Eq.(8) can be rewritten as

$$d_i^t = \begin{cases} 
  d_{f,i} & \text{if } x_i^t \geq \bar{\theta}_t \\
  d_{c,i} & \text{if } x_i^t < \bar{\theta}_t
\end{cases} \quad (18)$$

The average signal received by rational arbitrageurs, $\eta_t$, and the fraction of rational arbitrageurs, $m_t$, are computed as follows:

$$\eta_t = \begin{cases} 
  \theta_t & \text{if } \theta_t > \bar{\theta}_t + \epsilon \\
  (\theta_t + \bar{\theta}_t + \epsilon) / 2 & \text{if } \theta_t \in [\bar{\theta}_t - \epsilon, \bar{\theta}_t + \epsilon] \\
  0 & \text{if } \theta_t < \bar{\theta}_t - \epsilon
\end{cases} \quad (19)$$

and

$$m_t = \begin{cases} 
  1 & \text{if } \theta_t > \bar{\theta}_t + \epsilon \\
  (\theta_t - \bar{\theta}_t + \epsilon) / (2\epsilon) & \text{if } \theta_t \in [\bar{\theta}_t - \epsilon, \bar{\theta}_t + \epsilon] \\
  0 & \text{if } \theta_t < \bar{\theta}_t - \epsilon
\end{cases} \quad (20)$$

Substituting Eq.(19) and (20) into Eq.(11) yields the aggregate demand conditional on $\theta_t > z_t + \epsilon$:

$$D_t = \begin{cases} 
  \alpha (\theta_t - p_t) & \text{if } \theta_t > \bar{\theta}_t + \epsilon \\
  \frac{\alpha (\bar{\theta}_t + \bar{\theta}_t + \epsilon - 2z_t)}{4\epsilon} (\theta_t - \bar{\theta}_t + \epsilon) + \beta (p_t - v) & \text{if } \theta_t \in [\bar{\theta}_t - \epsilon, \bar{\theta}_t + \epsilon] \\
  \beta (p_t - v) & \text{if } \theta_t < \bar{\theta}_t - \epsilon
\end{cases} \quad (21)$$

Again, Eq.(21) holds as long as $\theta_t$ is in the depressed state ($\theta_t > z_t + \epsilon$), regardless of which state $\bar{\theta}_t$ falls into.
If $\bar{\theta}_t > z_t + \varepsilon$, it can be solved from Eq.(21). At $\theta_i = \bar{\theta}_t$, according to Claim 1 and Eq.(21), it is true that

$$D_t = \frac{\alpha}{2} \left( \frac{2\bar{\theta}_t + \varepsilon}{2} - p_t \right) + \frac{\beta}{2} (p_t - v) = 0.$$  

Solving for $\bar{\theta}_t$ yields

$$\bar{\theta}_t = p_t - \beta (p_t - v)/\alpha - \varepsilon/2.$$  

The solution is valid if and only if $\bar{\theta}_t > z_t + \varepsilon$, that is, $\beta (p_t - v)/\alpha < -3\varepsilon/4$.

### 3.3 Normal State

In the normal state where $\theta_i \in [z_t - \varepsilon, z_t + \varepsilon]$, there exists a signal $x^i_t = z_t$ such that $d^i_{f,t} = d_{c,t}$. As $d^i_{f,t}$ increases with but $d_{c,t}$ is independent of $x^i_t$, $d^i_{f,t} > d_{c,t}$ when $x^i_t > z_t$ and $d^i_{f,t} < d_{c,t}$ when $x^i_t < z_t$.

Conditional on $\theta_i \in [z_t - \varepsilon, z_t + \varepsilon]$, the expected payoff differential between rational arbitrage and irrational speculation, $E (\Delta \pi^i_t|x^i_t)$, can be evaluated in the following three scenarios.

**I** When $x^i_t \in [\bar{\theta}_t - \varepsilon, \bar{\theta}_t + \varepsilon]$, Eq.(7) can be rewritten as

$$E (\Delta \pi^i_t|x^i_t) = \left\{ \begin{array}{ll} 
\frac{x^i_t - \bar{\theta}_t}{\varepsilon} & \text{if } x^i_t \geq z_t \\
\frac{\bar{\theta}_t - x^i_t}{\varepsilon} & \text{if } x^i_t < z_t 
\end{array} \right.. \tag{22}$$

Based on Eq.(22), the sign of $E (\Delta \pi^i_t|x^i_t)$ conditional on $x^i_t \in [\bar{\theta}_t - \varepsilon, \bar{\theta}_t + \varepsilon]$ can be analyzed in five cases. (i) if $z_t < \bar{\theta}_t - \varepsilon$, then $x^i_t > z_t$ as $x^i_t \in [\bar{\theta}_t - \varepsilon, \bar{\theta}_t + \varepsilon]$. So, agent $i$ will be a rational arbitrageur if $x^i_t \geq \bar{\theta}_t$ and an irrational speculator otherwise; (ii) if $z_t \in [\bar{\theta}_t - \varepsilon, \bar{\theta}_t)$, agent $i$ will be an irrational speculator if $x^i_t \in (z_t, \bar{\theta}_t)$ and a rational arbitrageur otherwise; (iii) if $z_t = \bar{\theta}_t$, all agents are rational arbitrageurs; (iv) if $z_t \in (\bar{\theta}_t, \bar{\theta}_t^{\alpha} + \varepsilon]$, agent $i$ will be an irrational speculator if $x^i_t \in (\bar{\theta}_t, z_t)$ and a rational arbitrageur otherwise; and (v) if $z_t > \bar{\theta}_t + \varepsilon$, then $x^i_t < z_t$ as $x^i_t \in [\bar{\theta}_t - \varepsilon, \bar{\theta}_t + \varepsilon]$. So, agent $i$ will be a rational arbitrageur if $x^i_t \leq \bar{\theta}_t$ and an irrational speculator otherwise.

**II** When $x^i_t < \bar{\theta}_t - \varepsilon$, as $E (\Delta \pi^i_t|x^i_t) = -I^i_t$, agent $i$ will conduct rational arbitrage if $x^i_t \leq z_t$ and irrational speculation otherwise.
When \( \theta_t > x^t_i + \epsilon \), as \( E (\Delta \pi_i | x^t_i) = I^t_i \), agent \( i \) will conduct rational arbitrage if \( x^t_i \geq z_t \) and irrational speculation otherwise.

Summarizing scenario (I)-(III), agent \( i \) prefers irrational speculation if and only if (i) \( x^t_i \in (z_t, \tilde{\theta}_t) \) and \( z_t < \tilde{\theta}_t \), or (ii) \( x^t_i \in (\tilde{\theta}_t, z_t) \) and \( z_t > \tilde{\theta}_t \), which proves Lemma 3.

**Lemma 3** When \( \theta_t \in [z_t - \epsilon, z_t + \epsilon] \), an agent will be an irrational speculator if her signal falls into \( (\min (z_t, \tilde{\theta}_t), \max (z_t, \tilde{\theta}_t)) \) and a rational arbitrageur otherwise.

The result suggests that, in the normal state, an agent’s incentive to coordinate on rational arbitrage can either increase or decrease with her signal. Conditional on \( \theta_t \in [z_t - \epsilon, z_t + \epsilon] \), Eq.(8) can be rewritten as

\[
d^t_i = \begin{cases} 
    d^t_{f,t} & \text{if } x^t_i \notin (\min (z_t, \tilde{\theta}_t), \max (z_t, \tilde{\theta}_t)) \\
    d^t_{c,t} & \text{if } x^t_i \in (\min (z_t, \tilde{\theta}_t), \max (z_t, \tilde{\theta}_t)) 
\end{cases}
\]  

Based on Eq.(23), we derive the aggregate demand in the following three scenarios.

(I) \( z_t = \tilde{\theta}_t \)
If \( z_t = \tilde{\theta}_t \), all agents are rational arbitrageurs so that \( m_t = 1 \) and \( \eta_t = \theta_t \). Eq.(11) can be written as

\[
D_t = \alpha (\theta_t - p_t). 
\]  

If \( \tilde{\theta}_t = z_t \), recall that \( D_t = 0 \) when \( \theta_t = \tilde{\theta}_t \), which leads to

\[
\tilde{\theta}_t = p_t.
\]

For \( \tilde{\theta}_t = z_t = p_t \), it must be true that \( \beta (p_t - v) / \alpha = 0 \), that is, \( p_t = v \).

(II) \( z_t < \tilde{\theta}_t \)
If \( z_t < \tilde{\theta}_t \), recall that \( \theta_t \in [z_t - \epsilon, z_t + \epsilon] \), the average signal received by rational arbitrageurs \( \eta_t \)
and the fraction of rational arbitrageurs $m_t$ are

$$
\eta_t = \begin{cases} 
(\theta_t + z_t - \varepsilon) / 2 & \text{if } \theta_t < \tilde{\theta}_t - \varepsilon \\
\frac{z_t^2 + 4\varepsilon \theta_t - \tilde{\theta}_t^2}{2(2\varepsilon - \tilde{\theta}_t + z_t)} & \text{if } \theta_t \geq \tilde{\theta}_t - \varepsilon 
\end{cases}
$$

and

$$
m_t = \begin{cases} 
(\varepsilon - \theta_t + z_t) / (2\varepsilon) & \text{if } \theta_t < \tilde{\theta}_t - \varepsilon \\
(2\varepsilon - \tilde{\theta}_t + z_t) / (2\varepsilon) & \text{if } \theta_t \geq \tilde{\theta}_t - \varepsilon 
\end{cases}
$$

Substituting Eq. (25) and (26) into Eq. (11) and rearranging yield the aggregate demand conditional on $z_t < \tilde{\theta}_t$ and $\theta_t \in [z_t - \varepsilon, z_t + \varepsilon]$:

$$
D_t = \begin{cases} 
\frac{-\alpha (\theta_t - z_t - \varepsilon)^2 + \beta (p_t - v)}{4\varepsilon} & \text{if } \theta_t < \tilde{\theta}_t - \varepsilon \\
\alpha (\theta_t - z_t) - \frac{\alpha (\tilde{\theta}_t - z_t)^2}{4\varepsilon} + \beta (p_t - v) & \text{if } \theta_t \geq \tilde{\theta}_t - \varepsilon 
\end{cases}
$$

If $\theta_t = \tilde{\theta}_t$ exists given $z_t < \tilde{\theta}_t$ and $\tilde{\theta}_t \in [z_t - \varepsilon, z_t + \varepsilon]$, then at $\theta_t = \tilde{\theta}_t$,

$$
D_t = \alpha (\tilde{\theta}_t - z_t) - \frac{\alpha (\tilde{\theta}_t - z_t)^2}{4\varepsilon} + \beta (p_t - v) = 0.
$$

Solving for $\tilde{\theta}_t$ yields

$$
\tilde{\theta}_t = z_t + 2\varepsilon \left(1 + \sqrt{1 + \beta (p_t - v) / (\alpha \varepsilon)} \right).
$$

Clearly, $\tilde{\theta}_t = z_t + 2\varepsilon \left(1 + \sqrt{1 + \beta (p_t - v) / (\alpha \varepsilon)} \right) > z_t + 2\varepsilon$, which contradicts $\tilde{\theta}_t = \theta_t \leq z_t + \varepsilon$. The solution $\tilde{\theta}_t = z_t + 2\varepsilon \left(1 - \sqrt{1 + \beta (p_t - v) / (\alpha \varepsilon)} \right)$ exists if and only if $\tilde{\theta}_t > z_t$ and $\tilde{\theta}_t \in [z_t - \varepsilon, z_t + \varepsilon]$, that is, $\beta (p_t - v) / \alpha \in [-3\varepsilon / 4, 0)$.

(III) $z_t > \tilde{\theta}_t$

If $z_t > \tilde{\theta}_t$, $\eta_t$ and $m_t$ can be written as
speculator if and only if $x_i \leq \tilde{\theta}_i + \epsilon$ in the following. (i) When $\theta_t < \tilde{\theta}_t - \epsilon$, $x_i^t < \theta_t + \epsilon < z_t$ for any $i \in [0, 1]$. According to Lemma 1, an agent becomes a rational arbitrageur if and only if $x_i^t \leq \min(z_t, \tilde{\theta}_t)$ and an irrational speculator if and only if $x_i^t \in (\tilde{\theta}_t, z_t)$ and $\tilde{\theta}_t < z_t$. (ii) When $\theta_t > z_t + \epsilon$, $x_i^t > \theta_t - \epsilon > z_t$ for

$$\eta_t = \begin{cases} 
(\theta_t + z_t + \epsilon) / 2 & \text{if } \theta_t > \tilde{\theta}_t + \epsilon \\
\frac{\tilde{\theta}_t^2 + 4\epsilon \theta_t - z_t^2}{2(2\epsilon + \tilde{\theta}_t - z_t)} & \text{if } \theta_t \leq \tilde{\theta}_t + \epsilon
\end{cases} \tag{28}$$

and

$$m_t = \begin{cases} 
(\epsilon + \theta_t - z_t) / (2\epsilon) & \text{if } \theta_t > \tilde{\theta}_t + \epsilon \\
(2\epsilon + \tilde{\theta}_t - z_t) / (2\epsilon) & \text{if } \theta_t \leq \tilde{\theta}_t + \epsilon
\end{cases} \tag{29}$$

Substituting Eq.(28) and (29) into Eq.(11) and rearranging yield the aggregate demand conditional on $z_t > \tilde{\theta}_t$ and $\theta_t \in [z_t - \epsilon, z_t + \epsilon]$:

$$D_t = \begin{cases} 
\frac{\alpha (\epsilon + \theta_t - z_t)^2}{4\epsilon} + \beta (p_t - v) & \text{if } \theta_t > \tilde{\theta}_t + \epsilon \\
\frac{\alpha (\tilde{\theta}_t - z_t)^2}{4\epsilon} + \alpha (\theta_t - z_t) + \beta (p_t - v) & \text{if } \theta_t \leq \tilde{\theta}_t + \epsilon
\end{cases} \tag{30}$$

If $\tilde{\theta}_t \in [z_t - \epsilon, z_t + \epsilon]$ and $z_t > \tilde{\theta}_t$, at $\theta_t = \tilde{\theta}_t$,

$$D_t = \frac{\alpha (\tilde{\theta}_t - z_t) (\tilde{\theta}_t - z_t + 4\epsilon)}{4\epsilon} + \beta (p_t - v) = 0.$$

Solving for $\tilde{\theta}_t$ yields

$$\tilde{\theta}_t = z_t + 2\epsilon \left( \pm \sqrt{1 - \beta (p_t - v_t) / (\alpha \epsilon)} - 1 \right).$$

Clearly, $\tilde{\theta}_t = z_t + 2\epsilon \left( -\sqrt{1 - \beta (p_t - v_t) / (\alpha \epsilon)} - 1 \right) < z_t - \epsilon$, which contradicts $\theta_t = \tilde{\theta}_t \geq z_t - \epsilon$. The solution $\tilde{\theta}_t = z_t + 2\epsilon \left( \sqrt{1 - \beta (p_t - v_t) / (\alpha \epsilon)} - 1 \right)$ exists in the normal regime if and only if $\tilde{\theta}_t < z_t$ and $\tilde{\theta}_t \in [z_t - \epsilon, z_t + \epsilon]$, that is, $\beta (p_t - v_t) / \alpha \in (0, 3\epsilon / 4]$.

### 3.4 All States

To explore the common features of agents’ strategic behavior across states, we reinterpret Lemma 1-3 in the following. (i) When $\theta_t < z_t - \epsilon$, $x_i^t < \theta_t + \epsilon < z_t$ for any $i \in [0, 1]$. According to Lemma 1, an agent becomes a rational arbitrageur if and only if $x_i^t \leq \min(z_t, \tilde{\theta}_t)$ and an irrational speculator if and only if $x_i^t \in (\tilde{\theta}_t, z_t)$ and $\tilde{\theta}_t < z_t$. (ii) When $\theta_t > z_t + \epsilon$, $x_i^t > \theta_t - \epsilon > z_t$ for
any \( i \in [0,1] \). According to Lemma 2, an agent becomes a rational arbitrageur if and only if 
\[ x_i' \geq \max (z_i, \bar{\theta}_t) \] and an irrational speculator if and only if 
\[ x_i' \in (z_i, \bar{\theta}_t) \] and \( \bar{\theta}_t > z_t \). (iii) When 
\( \theta_t \in [z_t - \varepsilon, z_t + \varepsilon] \), according to Lemma 3, an agent will be an irrational speculator if her signal falls into \((z_t, \bar{\theta}_t)\) or \((\bar{\theta}_t, z_t)\) and a rational arbitrageur otherwise. To summarize, regardless of which state the fundamental resides, an agent acts as an irrational speculator if her private signal falls into \((\min (z_t, \bar{\theta}_t), \max (z_t, \bar{\theta}_t))\) and a rational arbitrageur otherwise. Therefore, Lemma 1-3 can be compressed into Proposition 1.

**Proposition 1** Agent \( i \) will be an irrational speculator if \( x_i' \in \left(\min (z_t, \bar{\theta}_t), \max (z_t, \bar{\theta}_t)\right) \) and a rational arbitrageur otherwise, where

\[
\bar{\theta}_t = \begin{cases} 
p_t - \mu_t + \varepsilon/2 & \text{if } \mu_t \in (3\varepsilon/4, +\infty) 
z_t + 2\varepsilon \left(\sqrt{1 - \mu_t/\varepsilon} - 1\right) & \text{if } \mu_t \in (0, 3\varepsilon/4] 
p_t & \text{if } \mu_t \in \{0\} 
z_t + 2\varepsilon \left(1 - \sqrt{1 + \mu_t/\varepsilon}\right) & \text{if } \mu_t \in [-3\varepsilon/4, 0) 
p_t - \mu_t - \varepsilon/2 & \text{if } \mu_t \in (+\infty, -3\varepsilon/4) 
\end{cases} \tag{31}
\]

and

\[
\mu_t = \beta \left( p_t - v \right) / \alpha.
\]

Based on Proposition 1 and Claim 1, we show in Appendix A.1 how agents’ actions can be mapped into various combinations of “buy, sell and hold” and how the various outcomes of the aggregate actions are jointly determined by \( p_t \) and \( \theta_t \). This highlights the comprehensiveness of modeling heterogeneous trading in terms of rational arbitrage and irrational speculation.

Note that \( \bar{\theta}_t < z_t - \varepsilon \) if \( \mu_t \in (3\varepsilon/4, +\infty) \), \( \bar{\theta}_t \in [z_t - \varepsilon, z_t] \) if \( \mu_t \in (0, 3\varepsilon/4] \), \( \bar{\theta}_t = z_t \) if \( \mu_t \in \{0\} \), \( \bar{\theta}_t \in (z_t, z_t + \varepsilon] \) if \( \mu_t \in [-3\varepsilon/4, 0) \), and \( \bar{\theta}_t > z_t + \varepsilon \) if \( \mu_t \in (+\infty, -3\varepsilon/4) \). So, \( \min (z_t, \bar{\theta}_t) = \bar{\theta}_t \) and \( \max (z_t, \bar{\theta}_t) = z_t \) if \( p_t \geq v \), and \( \min (z_t, \bar{\theta}_t) = z_t \) and \( \max (z_t, \bar{\theta}_t) = \bar{\theta}_t \) if \( p_t < v \). When \( p_t = v, z_t = \bar{\theta}_t \).

For any \( p_t \neq v \), we can show that \( \min (z_t, \bar{\theta}_t) < p_t \) and \( \max (z_t, \bar{\theta}_t) > p_t \). According to Proposition 1, an agent \( i \) will become a rational arbitrageur if her signal \( x_i' \) falls out of \((\min (z_t, \bar{\theta}_t), \max (z_t, \bar{\theta}_t))\),
that is, $x^i_t \leq \min (z_t, \bar{\theta}_t) < p_t$ or $x^i_t \geq \max (z_t, \bar{\theta}_t) > p_t$, which leads to Corollary 1.

**Corollary 1** When $p_t \neq v$, an agent will be a rational arbitrageur if and only if she expects the risky asset to be sufficiently overpriced or underpriced.

Corollary 1 means that, except when $p_t = v$, rational arbitrage is preferred over irrational speculation if and only if the market is sufficiently inefficient. The result is intuitive if we recall that agents seek to maximize the probabilities of successfully generating more positive profit. Consider an example with $p_t - v > 0$ so that irrational speculators are buying the risky asset. When $x^i_t \leq p_t$, because the two strategies suggest different trading directions, agent $i$ will seek to maximize her probability of success. If rational arbitrage were to have a higher probability of success, it must generate sufficient selling power in order to move the price towards its favorable direction, which is possible if and only if the asset is sufficiently overpriced. When $x^i_t > p_t$, as both strategies suggest buying the asset, agent $i$ will seek to maximize her profit. If rational arbitrage were to have a higher profit, it must generate greater buying power than the irrational speculation, which is possible if the asset is sufficiently underpriced.

We measure the degree of overpricing and underpricing with the absolute value of the difference between the fundamental and the price, $|\theta_t - p_t|$, which we refer to as market inefficiency and use interchangeably with the degree of overpricing and (or) underpricing in the following. Clearly, the market is most efficient if $\theta_t = p_t$, and the market inefficiency increases with $|\theta_t - p_t|$. Following this definition, agent $i$’s expected market inefficiency is $|x^i_t - p_t|$.

What degree of market inefficiency is sufficient to motivate rational arbitrage? Note that the upper bound of the rational arbitrageur’s signal is $\min (z_t, \bar{\theta}_t)$ when the asset is expected to be overpriced, and the lower bound of the rational arbitrageur’s signal is $\max (z_t, \bar{\theta}_t)$ when the asset is expected to be underpriced. So when $p_t \neq v$, the minimum degree of expected overpricing and underpricing are $p_t - \min (z_t, \bar{\theta}_t)$ and $\max (z_t, \bar{\theta}_t) - p_t$, respectively. The differences between $\min (z_t, \bar{\theta}_t)$ and $p_t$ can be written as
\[ p_t - \min (z_t, \bar{\theta}_t) = \begin{cases} 
\mu_t - \varepsilon / 2 & \text{if } \mu_t \in (3\varepsilon/4, +\infty) \\
-\mu_t + 2\varepsilon \left( 1 - \sqrt{1 - \mu_t / \varepsilon} \right) & \text{if } \mu_t \in (0, 3\varepsilon/4] \\
0 & \text{if } \mu_t \in \{0\} \\
\mu_t + 2\varepsilon \left( \sqrt{1 + \mu_t / \varepsilon} - 1 \right) & \text{if } \mu_t \in [-3\varepsilon/4, 0) \\
-\mu_t + \varepsilon / 2 & \text{if } \mu_t \in (+\infty, -3\varepsilon/4) 
\end{cases} \]

It suggests that, for any realization of \( p_t \), \( p_t - \min (z_t, \bar{\theta}_t) \) increases with \( |\mu_t| \). Similarly, we can show that, for any \( p_t \), \( \max (z_t, \bar{\theta}_t) - p_t \) increases with \( |\mu_t| \). This leads to Corollary 2.

**Corollary 2** When \( p_t \neq v \), \( p_t - \min (z_t, \bar{\theta}_t) \) and \( \max (z_t, \bar{\theta}_t) - p_t \), the minimum expected market inefficiency that is sufficient to trigger rational arbitrage, increases with \( |\mu_t| \).

When \( |\mu_t| \) increases, the buying or selling power of each irrational speculator increases. Intuitively, for rational arbitrage to generate a higher probability of success or greater profit, the buying or selling power based on such a strategy must increase accordingly, which is possible if and only if the expected market inefficiency enlarges. The result suggests that, for a given \( \theta_t \), agents will switch from rational arbitrage to irrational speculation if \( |\mu_t| \) increases. Consider a case with \( \mu_t > 0 \) and \( x_i' = \bar{\theta}_t < p_t \) so that agent \( i \) expects the rational arbitrage to generate the same probability of success as irrational speculation. All the others being the same, if \( \mu_t \) increases, \( \bar{\theta}_t \) increases above \( x_i' \), and agent \( i \) will switch from rational arbitrage to irrational speculation according to Proposition 1.

### 4 Comparative Statics

#### 4.1 Changes in the Information Transparency

Information transparency plays a subtle role in shaping the coordination on rational arbitrage. When \( |\mu_t| > 3\varepsilon/4 \), Proposition 2 suggests that information transparency increases the coordination on rational arbitrage if the market is sufficiently inefficient and the effect is opposite otherwise.
**Proposition 2** When $|\mu_t| > 3\varepsilon / 4$, $\partial m_t / \partial \varepsilon < 0$ if $|\theta_t - p_t| > |\mu_t|$ and $\partial m_t / \partial \varepsilon \geq 0$ if $|\theta_t - p_t| \leq |\mu_t|$.

**Proof.** see Appendix A.6. ■

When the market is sufficiently inefficient such that $|\theta_t - p_t| > |\mu_t| > 3\varepsilon / 4$, the coordination on rational arbitrage strengthens as information becomes more transparent (that is, $\varepsilon$ becomes smaller). The intuition is as follows. Recall from Proposition 1 that an agent will become a rational arbitrageur if she expects the market to be sufficiently inefficient, that is, $x_t^i \leq \min (z_t, \bar{\theta}_t)$ or $x_t^i \geq \max (z_t, \bar{\theta}_t)$, which implies $|x_t^i - p_t| > |\mu_t| - \varepsilon / 2$ or $|x_t^i - p_t| > |\mu_t|$ when $|\mu_t| > 3\varepsilon / 4$.

Given $|\theta_t - p_t| > |\mu_t| > 3\varepsilon / 4$, if an agent observes the perfect signal such that $x_t^i = \theta_t$, she will expect the market to be sufficiently inefficient so that rational arbitrage yields a higher payoff than irrational speculation. However, as a result of information friction, not all agents realize that rational arbitrage is more preferable. As the information becomes more transparent, agents’ signals approach the real fundamental. Therefore, more agents realize the market is more inefficient than what is required to trigger rational arbitrage and become rational arbitrageurs, which strengthens the coordination on rational arbitrage.

However, when $|\theta_t - p_t| \leq |\mu_t|$ and $|\mu_t| > 3\varepsilon / 4$, that is, the market is moderately inefficient, greater information transparency leads to weaker coordination on rational arbitrage. This is because information transparency reduces the fraction of agents who mistakenly expect the market to be sufficiently inefficient when it is not. The intuition is gained by considering the marginal rational arbitrageur who observes a signal $x_t^i = \bar{\theta}_t$ in an example with $\mu_t > 3\varepsilon / 4$ and $\bar{\theta}_t < \theta_t < z_t - \varepsilon$. All the others being the same, when the information noise $\varepsilon$ decreases to $\varepsilon'$, $\bar{\theta}_t$ decreases to $\bar{\theta}_t'$. As $x_t^i = \bar{\theta}_t > \bar{\theta}_t'$, agent $i$ will no longer be a rational arbitrageur according to Lemma 1. This is because the expected market inefficiency falls below the minimum value that is required for rational arbitrage as a result of increasing information transparency.

When $|\mu_t| \leq 3\varepsilon / 4$, we show in appendix A.6 that $\partial m_t / \partial \varepsilon < 0$ if (i) $|\theta_t - p_t| > \beta |p_t - v| / \alpha$ and $\theta_t \in [z_t - \varepsilon, z_t + \varepsilon]$ or (ii) $|\theta_t - p_t| > -|\mu_t| + \max (\varepsilon, \tau)$ and $\theta_t \notin [z_t - \varepsilon, z_t + \varepsilon]$, where $\tau = \frac{|\mu_t|}{\sqrt{1 - |\mu_t|}}$, and $\partial m_t / \partial \varepsilon \geq 0$ otherwise. Again, information transparency increases the coordination on rational arbitrage when the market is sufficiently inefficient, but what degree of inefficiency is sufficient
varies across fundamentals.

### 4.2 Changes in the Market Efficiency

An agent’s incentive to coordinate on rational arbitrage varies with her expected market efficiency. According to Eq.(12), (17) and (22), an agent $i$’s expected payoff difference between rational arbitrage and irrational speculation, $E (\Delta \pi_i^t | x_i^t)$, increases with $\tilde{\theta}_t$ and therefore $p_t$ when $x_i^t < z_t$ and decreases with $p_t$ when $x_i^t \geq z_t$, which leads to Proposition 3.

**Proposition 3** $E (\Delta \pi_i^t | x_i^t)$ increases with $|x_i^t - p_t|$ when $x_i^t \notin [\min (p_t, z_t), \max (p_t, z_t))$ and decreases with $|x_i^t - p_t|$ otherwise.

**Proof.** see Appendix A.8.

Proposition 3 suggests that one’s incentive to coordinate on rational arbitrage increases with her expected market inefficiency when the market is expected to be sufficiently inefficient, that is, $x_i^t \notin [\min (p_t, z_t), \max (p_t, z_t))$. The intuition is as follows. Recall from Eq.(3) that, the greater the expected market inefficiency, the more aggressively a rational arbitrageur trades. When $x_i^t < \min (p_t, z_t)$, rational arbitrage yields lower demand than irrational speculation. If $|x_i^t - p_t|$ increases because $x_i^t$ becomes lower or $p_t$ becomes higher, the aggregate demand of rational arbitrageurs decreases, or equivalently, the aggregate selling of rational arbitrageurs increases. As a result, the price is more likely to drop, and the rational arbitrage is more likely to generate more positive profit. When $x_i^t > \max (p_t, z_t)$, with similar reasoning, we can show that an agent’s incentive to coordinate on rational arbitrage increases with her expected market inefficiency.

The degree of coordination on rational arbitrage $m_t$ also responds to the changes in market efficiency. As observed from Eq.(31), $\tilde{\theta}_t$ is an increasing function of $p_t$ because $\alpha > \beta$. When $\theta_t < z_t - \epsilon$, according to Eq.(26), $m_t$ increases with $p_t$ while decreasing with $\theta_t$. When $\theta_t > z_t + \epsilon$, according to Eq.(20), $m_t$ decreases with $p_t$ while increasing with $\theta_t$. As changes in prices and fundamentals can be ascribed to changes in market efficiency, the state-dependent impact of prices and fundamentals on $m_t$ can be summarized into Proposition 4.
Proposition 4 \( m_t \) increases with \(|\theta_t - p_t|\) if \( \theta_t < \min(p_t, z_t - \varepsilon) \) or \( \theta_t > \max(p_t, z_t + \varepsilon) \).

Proof. see Appendix A.7.

Proposition 4 suggests that market inefficiency enhances the coordination on rational arbitrage when it exceeds certain thresholds, i.e., \( \theta_t < \min(p_t, z_t - \varepsilon) \) or \( \theta_t > \max(p_t, z_t + \varepsilon) \). The intuition is gained by considering \( \theta_t < \min(p_t, z_t - \varepsilon) \). With the asset being overpriced, an increase in the price enlarges the market inefficiency \(|\theta_t - p_t|\). Moreover, as the price increases, both \( z_t \) and \( \bar{\theta}_t \) increases, which increases the fraction of signals falling into \((-\infty, \min(z_t, \bar{\theta}_t))\) and therefore \( m_t \) according to Lemma 1. The impact of increasing market inefficiency that originates from a decline in \( \theta_t \) can be analyzed similarly.

5 Asset Price Bubbles and Depressions

Once agents’ choices of trading strategies are determined, the price dynamic functions for any given price and fundamental can be written explicitly in close-form. The change in the price, if any, will lead to a change in the state-dependent threshold fundamental \( \bar{\theta}_t \), which, together with the fundamental \( \theta_t \), reshuffles agents’ choices of trading strategies according to Proposition 1 and feedback on the subsequent price. The previous section analyzes agents’ trading behavior for a given realization of price. In this section, we first study the price dynamics for a given realization of fundamental and then discuss the multi-step price dynamics, taking account of the dynamic interaction between asset price and strategic coordination.

5.1 Equilibrium Price

According to Claim 1, \( D_t = 0 \) if and only if \( \theta_t = \bar{\theta}_t \). So for each realization of \( \theta_t \), we can solve for the equilibrium price \( \bar{p}_t \) from \( \theta_t = \bar{\theta}_t \), which leads to Proposition 5.

---

7When \( p_t \leq \theta_t < z_t - \varepsilon \) or \( z_t + \varepsilon < \theta_t \leq p_t \), the market is moderately inefficient, we show in the appendix that \( m_t \) decreases with \(|\theta_t - p_t|\) in this case. When \( \theta_t \in [z_t - \varepsilon, z_t + \varepsilon] \), we show in Appendix A.7 that \( m_t \) increases with \(|\theta_t - p_t|\) if (i) \( \mu_t > 0 \) and \( \theta_t \in [p_t, z_t + \varepsilon] \), or (ii) \( \mu_t < 0 \) and \( \theta_t \in [z_t - \varepsilon, p_t] \), and \( m_t \) decreases with \(|\theta_t - p_t|\) otherwise.
Proposition 5 For each realization of $\theta_t$, there exists a unique equilibrium price $\bar{p}_t$ such that $D_t = 0$ when $p_t = \bar{p}_t$, $D_t > 0$ when $p_t < \bar{p}_t$, and $D_t < 0$ when $p_t > \bar{p}_t$, where

$$
\bar{p}_t = \begin{cases} 
\frac{\alpha (\theta_t - \varepsilon/2) - \beta v}{\alpha - \beta} & \text{if } \theta_t > v + \kappa \\
\theta_t & \text{if } \theta_t = v \\
\frac{-2\alpha^2 \varepsilon^2 + \alpha^2 \theta_t + \alpha \beta (v + \theta_t) - 2\alpha \sqrt{\alpha^2 \varepsilon^2 - \beta \varepsilon (\alpha + \beta) (\theta_t - v)}}{(\alpha + \beta)^2} & \text{if } \theta_t \in (v, v + \kappa] \\
\frac{-2\alpha^2 \varepsilon^2 + \alpha^2 \theta_t + \alpha \beta (v + \theta_t) + 2\alpha \sqrt{\alpha^2 \varepsilon^2 + \beta \varepsilon (\alpha + \beta) (\theta_t - v)}}{(\alpha + \beta)^2} & \text{if } \theta_t \in [v - \kappa, v) \\
\frac{\alpha (\theta_t + \varepsilon/2) - \beta v}{\alpha - \beta} & \text{if } \theta_t < v - \kappa
\end{cases}
$$

and $\kappa = \frac{3\alpha - \beta}{4\beta} \varepsilon$.

Proof. see Appendix A.4.

The result suggests that there always exists a unique equilibrium price for any realization of $\theta_t$. If $\theta_t \in [v - \kappa, v + \kappa]$, the difference between $\theta_t$ and $\bar{p}_t$ is bounded. If $\theta_t > v + \kappa$, then $\bar{p}_t > \theta_t + 3\varepsilon/4$, which suggests that the asset is overpriced at the equilibrium, and the degree of overpricing is not bounded. Similarly, if $\theta_t < v - \kappa$, then $\bar{p}_t < \theta_t - 3\varepsilon/4$, which suggests that the asset is underpriced at the equilibrium, and the degree of underpricing is not bounded. In conventional wisdom, it is said to be a bubble (depression) if the asset is sufficiently overpriced (underpriced). Regardless of what degree of market inefficiency is considered to be sufficient, the equilibrium price conditional on $\theta_t > v + \kappa$ ($\theta_t < v - \kappa$) can always meet the criteria given proper parameter values. We therefore define $\bar{p}_t$ conditional on $\theta_t > v + \kappa$ and $\theta_t < v - \kappa$ as bubbly equilibrium $\bar{p}_t^b$ and depressed equilibrium $\bar{p}_t^d$, respectively:

$$
\bar{p}_t^b = \frac{\alpha (\theta_t - \varepsilon/2) - \beta v}{\alpha - \beta} \text{ if } \theta_t > v + \kappa,
$$

$$
\bar{p}_t^d = \frac{\alpha (\theta_t + \varepsilon/2) - \beta v}{\alpha - \beta} \text{ if } \theta_t > v + \kappa.
$$

To understand the emergence and correction of asset price bubbles and depressions, we study how the price dynamics evolve to and escape away from $\bar{p}_t^b$ and $\bar{p}_t^d$. We first consider a benchmark
case in which $\theta_t \equiv \theta$ and $S_t \equiv 0$ for any $t$. In this case, the model is reduced to a one-dimensional dynamics of the price. Given $\theta$, the equilibrium price is also constant such that $\bar{p}_t \equiv \bar{p}$. Moreover, as $S_t \equiv 0$, according to Proposition 5 and Eq.(2), $R_{t+1} = 0$ when $p_t = \bar{p}$, $R_{t+1} > 0$ when $p_t < \bar{p}$, and $R_{t+1} < 0$ when $p_t > \bar{p}$.

5.2 Boom and Burst of Bubbles

The necessary condition for the existence of a bubbly equilibrium is $\theta > v + \kappa$. Conditional on $\theta > v + \kappa$, for any initial price $p_0 < \bar{p}^b$, the price will increase as long as $p_t < \bar{p}^b$, remain unchanged if $p_t = \bar{p}^b$, and decline if $p_t > \bar{p}^b$, for any $t > 1$. If the bubbly equilibrium is stable, the price dynamics will converge to $\bar{p}^b$ and remain at $\bar{p}^b$ afterwards. Otherwise, it will diverge from $\bar{p}^b$. The following Corollary provides the stability conditions of $\bar{p}^b$.

**Corollary 3** There exists a stable bubbly equilibrium $\bar{p}^b$ if $\alpha \in (\beta, \beta + 2/\gamma)$ and $\theta \in (v + \kappa, v + \bar{\kappa})$, where

$$\bar{\kappa} = \frac{(8/\gamma + 3\beta - \alpha) \varepsilon}{4\beta}.$$  

**Proof.** see Appendix A.5. 

Figure 3 illustrates the emergence of bubbles when $\theta > v + \kappa$. The standard parameters used for demonstration are as follows: $\alpha = 0.6, \beta = 0.5, \gamma = 2, \varepsilon = 0.5, v = 7.5$ and $p_0 = 5.2$. The left panels plot the phase diagram of the price dynamics when $\theta = 8.2$ (top panel), $\theta = 8.68$ (middle panel), and $\theta = 9.8$ (bottom panel). The $x$-axis represents the current price $p_t$ and the $y$-axis represents the price in the next period $p_{t+1}$. The 45 degree line represents the scenario when $p_t = p_{t+1}$. The arrow line captures the motion of price movements. The solid curve plots the phase line that describes the relation between $p_{t+1}$ and $p_t$. The left panels of Figure 3 plot the time path of the prices corresponding to the right panels.

In the top left panel, the phase line is positively inclined with a slope less than unity when evaluated at the bubbly equilibrium, which suggests that the bubbly equilibrium is stable. Starting from the initial price $p_0$, the price moves up consistently to $\bar{p}^b$ in a steady time path. This is because
when the market is moderately inefficient, agents are more attracted to irrational speculation that pushes the price higher and higher above the fundamental. However as the market inefficiency becomes more and more pronounced, the coordination on rational arbitrage enhances, so does the aggregate selling power of rational arbitrageurs. The price will continue to go up until it reaches $\bar{p}^b$, at which the selling power of rational arbitrageurs increases to an extent that breaks even with the buying power of irrational speculators. The price cannot go up further as there is no incentive for more agents to coordinate on rational arbitrage. As a result, the price will stabilize at $\bar{p}^b$, which allows the market inefficiency to persist.

In the middle left panel, the phase line is negatively inclined with a slope less than unity in its absolute value when evaluating at $\bar{p}^b$. In this case, the price converges to $\bar{p}^b$ in an oscillating time path. When $p_t > \bar{p}^b$, rational arbitrageurs accumulate sufficiently strong selling power that significantly dominates the buying power of the irrational speculators, which drives the price down below $\bar{p}^b$ such that $p_{t+1} < \bar{p}^b$. When $p_{t+1} < \bar{p}^b$, the selling power of rational arbitrageurs is dominated by the buying power of irrational speculators, which drives the price up and leads to $p_{t+2} > \bar{p}^b$. Such an oscillation process continues as long as $p_{t+n} \neq \bar{p}^b$, for $n > 1$. The price will move closer and closer to $\bar{p}^b$ and eventually stabilize at $\bar{p}^b$, at which the selling power of rational arbitrageurs is completely offset by the buying power of irrational speculators.

The bottom left panel plots the phase diagram when $\bar{p}^b$ is not stable, with the phase line negatively inclined with a slope greater than unity in its absolute value when evaluating at $\bar{p}^b$. It is observed that the price diverges from $\bar{p}^b$ whenever it moves close to it, which suggests that $\bar{p}^b$ is stable. Such price dynamics lead to frequent booms and busts of bubbles even without any external shock.

The bubbly equilibrium no longer exists if $\theta \leq v + \kappa$. To understand how the bubble burst, consider an example with $p_0 = \bar{p}^b$. All the others being the same, if $\theta$ declines to such an extent that $\theta \leq v + \kappa$, the bubble will burst, as illustrated in Figure 4. If $\theta = 8.2$ drops to $\theta' = 7.2 \in [v - \kappa, v + \kappa]$, the price will crash immediately and then gradually converge to a new equilibrium price $\bar{p}'$ that is relatively close to the fundamental but lower than $\bar{p}^b$. The dash-dotted phase lines
Figure 3: The emergence of bubbles.
in the left panel of Figure 4 illustrate the steps of price dynamics that lead to the burst of bubbles, while that in the right panel plot the price trajectory after the change of the fundamental. If $\theta$ drops to $\theta'' = 6.7 < v - \kappa$, the price will first crash and then decline consistently until it reaches the new equilibrium $\bar{p}'$, as described by the dashed lines in Figure 4. Similarly, the bubble will burst if the reference price $v$ increases to such an extent that $\theta \leq v + \kappa$. The increase in $v$ plays the same role as the decline in $\theta$. The burst of bubble can also be triggered by the increases in rational arbitrageurs’ trading intensity $\alpha$ or information noise $\epsilon$, or the decline of irrational speculators’ trading intensity $\beta$ as long as it leads to $\theta \leq v + \kappa$. However, changing $\alpha$, $\beta$, or $\epsilon$ will not transmit the bubbly equilibrium to the depressed one.

5.3 Emergence and Recovery of Depressions

Since the emergence and recovery of depressions can be analyzed symmetrically, we describe the results briefly. The necessary condition for the existence of a depressed equilibrium is $\theta < v - \kappa$. The stability conditions of the depressed equilibrium $\bar{p}^d$ are given in Corollary 4.
Corollary 4  There exists a stable depressed equilibrium $\bar{p}^d$ if $\alpha \in (\beta, \beta + 2/\gamma)$ and $\theta \in (v - \kappa, v - \kappa)$.

Proof. see Appendix A.5. ■

Figure 5 illustrates the emergence of depression when $\theta < v - \kappa$. The parameters used are the same with the standard parameter set except for $p_0 = 8.5$. For $\theta = 6.7$ and $\theta = 5.2$, Figure 5 shows that the price converges to $\bar{p}^d$ in a steady and an oscillating path respectively. When $\theta = 6.2$, $\bar{p}^d$ is no longer stable so that the price tends to diverge from $\bar{p}^d$ whenever it gets close to it.

The asset price will recover from the depression if $\theta_t \geq v - \kappa$, as shown in Figure 6. The dash-dotted lines in Figure 6 plot the steps of price dynamics (left panel) and the price trajectory (right panel) when $\theta = 6.7$ increases to $\theta' = 7.2 \in [v - \kappa, v + \kappa]$. The dotted lines plot the similar statistics when $\theta$ increases to $\theta = 8.2 > v + \kappa$. Similarly, the price dynamics will recover from the depression if $v$ declines to such an extent that $\theta_t \geq v - \kappa$. The increase in $\alpha$ or $\varepsilon$, or the decline in $\beta$, will also trigger the recovery of the depression if it results in $\theta_t \geq v - \kappa$, but they can not transmit the depressed equilibrium into the bubbly equilibrium.

5.4 Price Dynamics with Stochastic Fundamental

When $S_t \equiv 0$, for each realization of $\theta_t$, Corollaries 3 and 4 hold by replacing $\theta$ with $\theta_t$. The problem is that, before the price reaches the equilibrium, the value of $\theta_t$ that follows a stochastic process changes, which may lead to a new equilibrium. Allowing for the presence of supply shock complicates the process further. Incorporating the stochastic $\theta_t$ and $S_t$ into the endogenous price dynamics makes the price intractable, but does not affect the model’s general characteristics. Figure 7 illustrates the emergence and reversal of asset bubbles and depressions as well as their transitions when $\theta_t$ and $S_t$ are time-varying. The parameters used for the simulation are as follows: $\sigma^2 = 0.5, \zeta = 1, p_0 = 5.2, \theta_0 = 8.2$ and all the other parameters are defined in the standard parameter set.
Figure 5: The emergence of depressions.
Figure 6: The recovery from depressions.

Figure 7: The emergence of bubbles, depressions and their transitions.
6 Information Transparency and Market Efficiency

We measure the size of bubbles (depressions) by the absolute difference between the bubbly (depressed) equilibrium and the fundamental:

\[
|\theta_t - \bar{p}_b^t| = \frac{\beta (\theta_t - v) - \alpha \varepsilon / 2}{\alpha - \beta} \tag{32}
\]

\[
|\theta_t - \bar{p}_d^t| = \frac{\beta (v - \theta_t) - \alpha \varepsilon / 2}{\alpha - \beta}. \tag{33}
\]

Conditional respectively on the existence of \(\bar{p}_b^t\) and \(\bar{p}_d^t\), Eq.(32) and (33) suggest that (i) greater information transparency, that is, smaller information noise \(\varepsilon\), is associated with larger bubbles and depressions; (ii) a stronger fundamental \(\theta\) expands the bubble but mitigates the depression; (iii) the larger the trading intensity of rational arbitrageurs \(\alpha\), the smaller the size of bubbles and depressions; (iv) the larger the trading intensity of irrational speculator \(\beta\), the greater the size of bubbles and depressions.

To understand why information transparency increases the magnitude of bubbles and depressions, consider the following two scenarios. Suppose the price is initially at the bubbly equilibrium \(\bar{p}_b^t\) at which \(\theta_t = \bar{\theta}\). Note that the threshold equilibrium fundamental in the bubbly state, \(\bar{\theta}\), is an increasing function of the information noise \(\varepsilon\) when \(p_t = \bar{p}_b^t\) according to Eq.(31). All the others being the same, when the information becomes more transparent (that is, \(\varepsilon\) declines), \(\bar{\theta}\) will drop below the actual fundamental \(\theta_t\), leading to a price increment in the next period according to Claim 1. As \(p_t\) increases, \(\bar{\theta}\) also increases. As long as \(\bar{\theta} < \theta\), the price will continue to increase. Only when the price increases to such an extent that offsets the impact of \(\varepsilon\) on \(\bar{\theta}\) will \(\bar{\theta} = \theta\), at which the price dynamics reach the new bubbly equilibrium that is higher than the initial one. Similarly, if the price is initially at the depressed equilibrium \(\bar{p}_d^t\), the decline in \(\varepsilon\) decreases \(\bar{\theta}\), which leads to price decline in the next period. The new depressed equilibrium will only be restored if and only if the price drops to an extent that offsets the impact of \(\varepsilon\) on \(\bar{\theta}\). The result suggests that information transparency can magnifies the market inefficiency.
The relation between information transparency and market efficiency can also be analyzed based on Proposition 2. Note that $|\mu_t| > 3\epsilon/4$ and $|\theta_t - p_t| < |\mu_t|$ when $p_t = \bar{p}^b$, the coordination on rational arbitrage declines as the information becomes more transparent according to Proposition 2. Moreover, conditional on $p_t = \bar{p}^b$, the average signal received by rational arbitrageurs increases as the information becomes more transparent according to Eq.(14), which reduces the average selling of rational arbitrageurs. As a result, the aggregate selling of rational arbitrageurs decreases while the aggregate buying of irrational speculators increase. According to Eq.(16), this raises the aggregate demand from zero to positive, which increases the subsequent asset price and therefore market inefficiency. Similarly, we can show that more information transparency results in greater market inefficiency conditional on $p_t = \bar{p}^d$.

7 Conclusion

We develop a model with elements of heterogeneous trading, strategic interactions and information friction. The model captures not only the direction but also the size of trading orders and price movements, accommodating multiple actions and multiple outcomes in a game setup. We solve the model with modified global game technique and obtain a close-form price impact function that captures the dynamic interaction between agents’ strategic trading and the asset price. The price impact function varies across the states of prices and fundamentals, which generates rich dynamic patterns that enable us to analyze the emergence and correction of asset price bubbles and depressions as well as their implications for market efficiency.

Although market inefficiency provides an opportunity for rational arbitrage according to the efficient market hypothesis, agents may not necessarily act on such information because whether the market will restore its efficiency depends on the aggregate actions of all agents. Agents utilize both efficient market hypothesis and behavioral finance to extrapolate actions of others and trade strategically by switching between rational arbitrage and irrational speculation to maximize their payoffs. When the market is moderately inefficient, the dynamic interaction between heterogeneous
trading and market inefficiency may lead to consistent clustering to irrational speculation, which persists the market inefficiency and in extreme case generates asset bubbles or depressions. However, market inefficiency enhances agents’ coordination on rational arbitrage when it exceeds certain thresholds. As the market becomes sufficiently inefficient, more and more agents are motivated to coordinate on rational arbitrage because the probability of returning the price towards its fundamental increases. In some cases, this may lead to the correction of bubbles or depressions and even the restoration of market efficiency. Greater information transparency does not necessarily lead to stronger coordination on rational arbitrage. In fact, at the bubbly or depressed equilibrium, information transparency discourages the coordination on rational arbitrage and enlarges the market inefficiency. The results have important implications for regulators and investors. It suggests that when imposing stricter requirement on information disclosure, it is important to account for the financial market environment. It also implies that riding on the trend is rewarded if the market is moderately inefficient, but it can become risky once the market inefficiency is sufficiently large.
References


Appendix Table 1: State-dependent actions. This table summarizes the composition of rational arbitrageurs (RA) and irrational speculators (IS) in different states of prices and fundamentals. Conditional on the existence of RA or IS, the table reports whether RA or IS buy, sell or hold the risky asset for a price specified in the first column and a fundamental specified in the first row. If RAs or ISs do not exist, we input ‘-’ in the cell corresponding cell.

<table>
<thead>
<tr>
<th>$\mu_t \in$</th>
<th>$\theta_t &lt; z_t - \varepsilon$</th>
<th>$\theta_t \in [z_t - \varepsilon, z_t)$</th>
<th>$\theta_t = z_t$</th>
<th>$\theta_t \in (z_t, z_t + \varepsilon]$</th>
<th>$\theta_t &gt; z_t + \varepsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>RA</td>
<td>IS</td>
<td>RA</td>
<td>IS</td>
<td>RA</td>
<td>IS</td>
</tr>
<tr>
<td>(3\varepsilon/4, +\infty)</td>
<td>Sell</td>
<td>Buy</td>
<td>Buy</td>
<td>Buy</td>
<td>-</td>
</tr>
<tr>
<td>(0, 3\varepsilon/4]</td>
<td>Sell</td>
<td>Buy</td>
<td>Buy/Sell</td>
<td>Buy</td>
<td>Buy</td>
</tr>
<tr>
<td>{0}</td>
<td>Sell</td>
<td>-</td>
<td>Buy/hold</td>
<td>-</td>
<td>Buy/hold</td>
</tr>
<tr>
<td>[-3\varepsilon/4, 0]</td>
<td>Sell</td>
<td>-</td>
<td>Buy/Sell</td>
<td>Sell</td>
<td>Sell</td>
</tr>
<tr>
<td>(+\infty, -3\varepsilon/4)</td>
<td>Sell</td>
<td>-</td>
<td>Sell</td>
<td>Sell</td>
<td>-</td>
</tr>
</tbody>
</table>

Appendix

A.1 The State-Dependent Actions and Outcomes

To illustrate how agents’ actions vary with the state of prices and fundamentals, we summarize in Appendix Table 1 the trading behavior of rational arbitrageurs (RA) and irrational speculators (IR) in terms of buying and selling based on Proposition 1. Clearly, the composition of rational arbitrageurs and irrational speculators varies with the price and the fundamental. Agents that receive the same signals may switch their strategies between rational arbitrage and irrational speculation because the price or the fundamental has changed. Moreover, agents that practice the same strategy may switch among buying, selling, and holding. Whether rational arbitrageurs will buy, sell, or hold the risky asset depends on the range of the price as well as the fundamental. They always sell when $\theta_t < z_t - \varepsilon$, buy when $\theta_t > z_t + \varepsilon$, and can either buy, sell, or hold when $\theta_t \in [z_t - \varepsilon, z_t + \varepsilon]$ depending on the range of $\beta \left( p_t - v \right)/\alpha$. It means that all rational arbitrageurs expect the asset to be overpriced (underpriced) when the fundamental falls into the bubbly (depressed) state. Whether irrational speculators, if exist, will buy or sell depends on the price (captured by $\beta \left( p_t - v \right)$) but not on the fundamental. Regardless of the fundamental, irrational speculators buy the risky asset if $\beta \left( p_t - v \right) > 0$ and sell if $\beta \left( p_t - v \right) < 0$\(^8\). The result highlights the importance of describing agents’ actions from the perspective of rational arbitrage and irrational speculation that accommodates various possibilities of state-dependent trading behavior.

We now turn to how the outcomes of the aggregate actions of all agents vary with the prices and fundamentals. Appendix Table 2 presents the sign of $D_t$ that is jointly determined by $p_t$ and $\theta_t$, based on Claim 1 and Eq.(31). For a price specified in the first column and a fundamental specified in the first row, we input ‘+’ in the corresponding cell if $D_t > 0$, ‘-’ if $D_t > 0$, and ‘→’ if $D_t = 0$. For a given $p_t$, the greater $\theta_t$ is, the more likely $D_t$ is positive. The result also follows directly from the fact that $D_t$ is an increasing function of $\theta_t$. For a given $\theta_t$, the greater $p_t$ is, the more likely $D_t$ is positive. This follows from Claim 1 and the observation that $\bar{\theta}$ is an increasing function of $p_t$. In the absence of supply shock, a positive $D_t$ leads to an increase in the subsequent price while a negative $D_t$ leads to a decline. The change in the price then affects agents’ subsequent

\(^8\)There is no irrational speculator when $p_t = v$. 

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Appendix Table 2: The sign of aggregate demand. This table shows how the sign of the aggregate demand, which reflects the outcome of all agents’ collective actions, vary with the states of prices and fundamentals. For a price specified in the first column and a fundamental specified in the first row, we input ‘+’ in the corresponding cell if \( D_t > 0 \), ‘−’ if \( D_t < 0 \), and ‘→’ if \( D_t = 0 \).

<table>
<thead>
<tr>
<th>( \mu_t )</th>
<th>( \theta_t &lt; z_t - \varepsilon )</th>
<th>( \theta_t \in [z_t - \varepsilon, z_t) )</th>
<th>( \theta_t = z_t )</th>
<th>( \theta_t \in (z_t, z_t + \varepsilon) )</th>
<th>( \theta_t &gt; z_t + \varepsilon )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((3\varepsilon/4, +\infty))</td>
<td>+−→</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>((0,3\varepsilon/4))</td>
<td>−</td>
<td>−+→</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>({0})</td>
<td>−</td>
<td>−</td>
<td>→</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>([-3\varepsilon/4, 0))</td>
<td>−</td>
<td>−</td>
<td>−</td>
<td>−+→</td>
<td>+</td>
</tr>
<tr>
<td>((+\infty, -3\varepsilon/4))</td>
<td>−</td>
<td>−</td>
<td>−</td>
<td>−</td>
<td>−+→</td>
</tr>
</tbody>
</table>

actions, which has a feedback effect on the price. Such a loop continues, which leads to dynamic interaction between agents’ strategic actions and the asset price.

### A.2 Alternative Specifications

In the previous analysis, we restrict \( \alpha > \beta \) so that rational arbitrageurs are more responsive to the price than irrational speculators. If \( \alpha \leq \beta \), Lemma 1–3 remains. But Proposition 5 and Corollaries 3 and 4 no longer hold. If \( \alpha = \beta \), the equilibrium price exists if and only if \( \theta \in [\nu - \varepsilon, \nu + \varepsilon] \). If \( \alpha < \beta \), then \( \bar{\theta} \) is decreasing with \( p_t \). Repeat the calculation for Proposition 5, we can show that if \( \beta > 3\alpha \), the equilibrium price \( \bar{p}_t \) is unique:

\[
\bar{p}_t = \left\{ \begin{array}{ll}
\frac{\alpha (\theta_t - \varepsilon / 2) - \beta \nu}{\alpha - \beta} & \text{if } \theta_t < \nu + \kappa \\
\frac{2\alpha^2 \varepsilon + \beta^2 \nu + \alpha^2 \theta_t + \alpha \beta (\nu + \theta_t) - 2\alpha \sqrt{\alpha^2 \varepsilon^2 - \beta \varepsilon (\alpha + \beta) (\theta_t - \nu)}}{(\alpha + \beta)^2} & \text{if } \theta_t \in [\nu + \kappa, \nu) \\
\frac{\theta_t}{(\alpha + \beta)^2} & \text{if } \theta_t = \nu \\
\frac{-2\alpha^2 \varepsilon^2 + \beta^2 \varepsilon + \alpha^2 \theta_t + \alpha \beta (\nu + \theta_t) - 2\alpha \sqrt{\alpha^2 \varepsilon^2 + \beta \varepsilon (\alpha + \beta) (\theta_t - \nu)}}{(\alpha + \beta)^2} & \text{if } \theta_t \in (\nu, \nu - \kappa] \\
\frac{\alpha (\theta_t + \varepsilon / 2) - \beta \nu}{\alpha - \beta} & \text{if } \theta_t > \nu - \kappa
\end{array} \right.
\]

If \( \alpha < \beta < 3\alpha \), there exist multiple price equilibria and \( \bar{p}_t \) is given by:

\[
\bar{p}_t = \left\{ \begin{array}{ll}
\frac{\alpha (\theta_t - \varepsilon / 2) - \beta \nu}{\alpha - \beta} & \text{if } \theta_t < \nu + \kappa \\
\frac{2\alpha^2 \varepsilon + \beta^2 \nu + \alpha^2 \theta_t + \alpha \beta (\nu + \theta_t) + 2\alpha \sqrt{\alpha^2 \varepsilon^2 - \beta \varepsilon (\alpha + \beta) (\theta_t - \nu)}}{(\alpha + \beta)^2} & \text{if } \theta_t \in [\nu + \kappa, \nu) \\
\frac{\theta_t}{(\alpha + \beta)^2} & \text{if } \theta_t = \nu \\
\frac{-2\alpha^2 \varepsilon^2 + \beta^2 \nu + \alpha^2 \theta_t + \alpha \beta (\nu + \theta_t) + 2\alpha \sqrt{\alpha^2 \varepsilon^2 + \beta \varepsilon (\alpha + \beta) (\theta_t - \nu)}}{(\alpha + \beta)^2} & \text{if } \theta_t \in (\nu, \nu - \kappa] \\
\frac{\alpha (\theta_t + \varepsilon / 2) - \beta \nu}{\alpha - \beta} & \text{if } \theta_t > \nu - \kappa
\end{array} \right.
\]

When \( \alpha < \beta \), \( D_t = 0 \) when \( p_t = \bar{p}_t \), \( D_t > 0 \) when \( p_t > \bar{p}_t \), and \( D_t < 0 \) when \( p_t < \bar{p}_t \).

Moreover, the bubbly or depressed equilibrium, if exist, are always unstable. To see why,
consider the following example. Suppose $\theta > \bar{\theta}$, the price in the next period will increase so that $p_1 > p_0$. Because $\bar{\theta}$ is decreasing with $p_t$, when $\alpha < \beta$, there is $\bar{\theta}_t < \bar{\theta}_0 < \theta$, which increases the price in the next period. As such a process continues, the price will increase infinitely. Similarly, we can show that the price will decline infinitely in the depression regime if $\alpha < \beta$. So, it is impossible for the price to stabilize at either $\bar{p}^b$ or $\bar{p}^d$.

A.3 Existence of Equilibrium Threshold Fundamental

**Proof.** Step 1: For a given realization of $p_t, D_t$ is an increasing function of $\theta_t$.

(I) When $\theta_t < z_t - \varepsilon$, based on Eq.(16), differentiating $D_t$ with respect to $\theta_t$ results in

$$\frac{\partial D_t}{\partial \theta_t} = \left\{ \begin{array}{ll} \frac{\alpha}{2\varepsilon} & \text{if } \theta_t < \bar{\theta}_t - \varepsilon \\ \frac{\alpha z_t + \varepsilon - \theta_t}{2\varepsilon} & \text{if } \theta_t \in [\bar{\theta}_t - \varepsilon, \bar{\theta}_t + \varepsilon] \\ 0 & \text{if } \theta_t > \bar{\theta}_t + \varepsilon \end{array} \right.$$

Note that $\frac{\alpha z_t + \varepsilon - \theta_t}{2\varepsilon} > 0$ because $\theta_t < z_t - \varepsilon$. So $\frac{\partial D_t}{\partial \theta_t} > 0$ when $\theta_t < z_t - \varepsilon$.

(II) When $\theta_t > z_t + \varepsilon$, based on Eq.(21), differentiating $D_t$ with respect to $\theta_t$ results in

$$\frac{\partial D_t}{\partial \theta_t} = \left\{ \begin{array}{ll} \frac{\theta_t + \varepsilon - z_t}{2\varepsilon} & \text{if } \theta_t \in [\bar{\theta}_t - \varepsilon, \bar{\theta}_t + \varepsilon] \\ \frac{\alpha}{2} & \text{if } \theta_t > \bar{\theta}_t + \varepsilon \end{array} \right.$$

Note that $\frac{\theta_t + \varepsilon - z_t}{2\varepsilon} > 0$ because $\theta_t > z_t + \varepsilon$. So $\frac{\partial D_t}{\partial \theta_t} > 0$ when $\theta_t > z_t + \varepsilon$.

(III) When $\bar{\theta}_t \in [z_t - \varepsilon, z_t + \varepsilon]$, we discuss the relation between $D_t$ and $\theta_t$ based on the aggregate demand function in Eq.(24), (27) and (30), respectively.

If $z_t = \bar{\theta}_t$, Eq.(24) suggests that $D_t$ increases with $\theta_t$.

If $z_t < \bar{\theta}_t$, according to Eq.(27), differentiating $D_t$ with respect to $\theta_t$ yields

$$\frac{\partial D_t}{\partial \theta_t} = \left\{ \begin{array}{ll} \frac{\alpha z_t + \varepsilon - \theta_t}{4\varepsilon} & > 0 \text{ if } \theta_t < \bar{\theta}_t - \varepsilon \\ \frac{\alpha}{2\varepsilon} & > 0 \text{ if } \theta_t > \bar{\theta}_t - \varepsilon \end{array} \right.$$

where the inequality for the first line is obtained because $\theta_t \in [z_t - \varepsilon, z_t + \varepsilon]$ and that for the second line is because $\bar{\theta}_t < \theta_t + \varepsilon < z_t + 2\varepsilon$.

If $z_t > \bar{\theta}_t$, according to Eq.(30), differentiating $D_t$ with respect to $\theta_t$ yields

$$\frac{\partial D_t}{\partial \theta_t} = \left\{ \begin{array}{ll} \frac{\alpha \theta_t + \varepsilon - z_t}{4\varepsilon} & > 0 \text{ if } \theta_t > \bar{\theta}_t + \varepsilon \\ \frac{\alpha}{2\varepsilon} & > 0 \text{ if } \theta_t < \bar{\theta}_t + \varepsilon \end{array} \right.$$

where the inequality for the first line is obtained because $\theta_t \in [z_t - \varepsilon, z_t + \varepsilon]$ and that for the second line is because $\bar{\theta}_t > \theta_t - \varepsilon > z_t - 2\varepsilon$.
So $D_t$ increases with $\theta_t$ when $\theta_t \in [z_t - \varepsilon, z_t + \varepsilon]$.

Note that when $\theta_t = z_t - \varepsilon$, $D_t$ can also be evaluated by Eq.(16), which yields the same result as it is evaluated by Eq.(27) or (30). Similarly, when $\theta_t = z_t + \varepsilon$, evaluating $D_t$ with Eq.(16) yields the same result as with Eq.(27) or (30).

So, regardless of the range of $\theta_t$, $\partial D_t / \partial \theta_t > 0$. Moreover, $\partial D_t / \partial \theta_t > 0$ when $\theta_t \in [z_t - \varepsilon, z_t + \varepsilon]$ or $\theta_t \in \left[ \bar{\theta}_t - \varepsilon, \bar{\theta}_t + \varepsilon \right]$.

**Step 2:** For each realization of $p_t$, there exists a unique $\bar{\theta}_t$ such that $D_t = 0$ if $\theta_t = \bar{\theta}_t$, $D_t > 0$ if $\theta_t > \bar{\theta}_t$ and $D_t < 0$ if $\theta_t < \bar{\theta}_t$.

(i) If $\mu_t > 3\varepsilon/4$, then $\bar{\theta}_t = p_t - \mu_t + \varepsilon/2 < z_t - \varepsilon$. In this case, it is only possible for $\theta_t = \bar{\theta}_t$ if and only if $\theta_t < z_t - \varepsilon$.

When $\theta_t < z_t - \varepsilon$, according to Eq.(16), evaluating $D_t$ at $\theta_t = \bar{\theta}_t$ yields $D_t|_{\theta_t = \bar{\theta}_t} = 0$. Recall from the first step that $D_t$ increases with $\theta_t$ when $\theta_t \in \left[ \bar{\theta}_t - \varepsilon, \bar{\theta}_t + \varepsilon \right]$, then $D_t < 0$ if $\theta_t \in \left[ \bar{\theta}_t - \varepsilon, \bar{\theta}_t \right)$ and $D_t > 0$ if $\theta_t \in (\bar{\theta}_t, \bar{\theta}_t + \varepsilon]$. In particular, $D_t|_{\bar{\theta}_t - \varepsilon} < 0$ and $D_t|_{\bar{\theta}_t + \varepsilon} > 0$. As $D_t$ is a nondecreasing function of $\theta_t$,

$$D_t|_{\theta_t < \bar{\theta}_t - \varepsilon} \leq D_t|_{\theta_t = \bar{\theta}_t - \varepsilon} < 0$$

and

$$D_t|_{\theta_t > \bar{\theta}_t + \varepsilon} > D_t|_{\theta_t = \bar{\theta}_t + \varepsilon} > 0.$$

When $\theta_t > z_t + \varepsilon$, then $\theta_t > z_t + \varepsilon > \bar{\theta}_t + 2\varepsilon > \bar{\theta}_t + \varepsilon$, which suggests

$$D_t|_{\theta_t > \bar{\theta}_t + \varepsilon} > D_t|_{\theta_t = \bar{\theta}_t + \varepsilon} > 0.$$

When $\theta_t \in [z_t - \varepsilon, z_t + \varepsilon]$, then $\theta_t > z_t - \varepsilon > \bar{\theta}_t$. Recall from the first step that $D_t$ increases with $\theta_t$, there is

$$D_t|_{\theta_t \in [z_t - \varepsilon, z_t + \varepsilon]} > D_t|_{\theta_t = \bar{\theta}_t} = 0.$$

So when $\mu_t > 3\varepsilon/4$, $D_t = 0$ if $\theta_t = \bar{\theta}_t$, $D_t > 0$ if $\theta_t > \bar{\theta}_t$ and $D_t < 0$ if $\theta_t < \bar{\theta}_t$.

(ii) If $\mu_t \in (0, 3\varepsilon/4]$, then $\bar{\theta}_t = z_t + 2\varepsilon \left( \sqrt{1 - \frac{\mu_t}{4\varepsilon}} - 1 \right) \in [z_t - \varepsilon, z_t]$. In this case, it is only possible for $\theta_t = \bar{\theta}_t$ if and only if $\theta_t \in [z_t - \varepsilon, z_t)$. When $\theta_t \in [z_t - \varepsilon, z_t)$, evaluating $D_t$ at $\theta_t = \bar{\theta}_t$ yields $D_t|_{\theta_t = \bar{\theta}_t} = 0$.

As $D_t$ increases with $\theta_t$ when $\theta_t \in [z_t - \varepsilon, z_t + \varepsilon]$,

$$D_t|_{\theta_t \in [z_t - \varepsilon, \bar{\theta}_t]} < D_t|_{\theta_t = \bar{\theta}_t} = 0$$

and

$$D_t|_{\theta_t \in (\bar{\theta}_t, z_t + \varepsilon]} > D_t|_{\theta_t = \bar{\theta}_t} = 0.$$

As $D_t$ is a nondecreasing function of $\theta_t$ when $\theta_t < z_t - \varepsilon$ or $\theta_t > z_t + \varepsilon$,

$$D_t|_{\theta_t < z_t - \varepsilon} < D_t|_{\theta_t = z_t - \varepsilon} < 0$$

and

$$D_t|_{\theta_t > z_t + \varepsilon} > D_t|_{\theta_t = z_t + \varepsilon} > 0.$$

So when $\mu_t \in (0, 3\varepsilon/4]$, $D_t = 0$ if $\theta_t = \bar{\theta}_t$, $D_t > 0$ if $\theta_t > \bar{\theta}_t$ and $D_t < 0$ if $\theta_t < \bar{\theta}_t$.

(iii) When $p_t = v$, $\bar{\theta}_t = p_t = z_t$. In this case, $\theta_t = \bar{\theta}_t$ if and only if $\theta_t = z_t$. Evaluating $D_t$ at
\( \theta_t = \bar{\theta}_t = z_t \) yields \( D_t|_{\theta_t=\bar{\theta}_t}=0 \). With similar analysis in (II), we can show that \( D_t = 0 \) if \( \theta_t = \bar{\theta}_t \), \( D_t > 0 \) if \( \theta_t > \bar{\theta}_t \) and \( D_t < 0 \) if \( \theta_t < \bar{\theta}_t \).

(iv) When \( \mu_t \in [-3\varepsilon/4, 0) \), \( \bar{\theta}_t = z_t + 2\varepsilon \left( \sqrt{1 - \mu_t/\varepsilon - 1} \right) \in (z_t, z_t + \varepsilon] \). In this case, \( \theta_t = \bar{\theta}_t \) if and only if \( \theta_t \in (z_t, z_t + \varepsilon] \). Evaluating \( D_t \) at \( \theta_t = \bar{\theta}_t \) conditional on \( \theta_t \in (z_t, z_t + \varepsilon] \) yields \( D_t|_{\theta_t=\bar{\theta}_t}=0 \). With similar analysis in (II), we can show that \( D_t = 0 \) if \( \theta_t = \bar{\theta}_t \), \( D_t > 0 \) if \( \theta_t > \bar{\theta}_t \) and \( D_t < 0 \) if \( \theta_t < \bar{\theta}_t \).

(v) When \( \mu_t < -3\varepsilon/4 \), \( \bar{\theta}_t = p_t - \mu_t - \varepsilon/2 > z_t + \varepsilon \). In this case, it is possible for \( \theta_t = \bar{\theta}_t \) if and only if \( \theta_t > z_t + \varepsilon \).

When \( \theta_t > z_t + \varepsilon \), evaluating \( D_t \) at \( \theta_t = \bar{\theta}_t \) yields \( D_t|_{\theta_t=\bar{\theta}_t}=0 \). As \( D_t \) increases with \( \theta_t \) when \( \theta_t \in [\bar{\theta}_t - \varepsilon, \bar{\theta}_t + \varepsilon] \),

\[
D_t|_{\theta_t \in [\bar{\theta}_t - \varepsilon, \bar{\theta}_t]} < D_t|_{\theta_t = \bar{\theta}_t} = 0
\]

and

\[
D_t|_{\theta_t \in (\bar{\theta}_t, \bar{\theta}_t + \varepsilon]} > D_t|_{\theta_t = \bar{\theta}_t} = 0.
\]

As \( D_t \) is a nondecreasing function of \( \theta_t \),

\[
D_t|_{\theta_t < \bar{\theta}_t - \varepsilon} \leq D_t|_{\theta_t = \bar{\theta}_t - \varepsilon} < 0
\]

and

\[
D_t|_{\theta_t > \bar{\theta}_t + \varepsilon} \geq D_t|_{\theta_t = \bar{\theta}_t + \varepsilon} > 0.
\]

When \( \theta_t < z_t - \varepsilon \), the condition \( \mu_t < -3\varepsilon/4 \) implies that \( \theta_t < \bar{\theta}_t - \varepsilon \), which leads to

\[
D_t|_{\theta_t < z_t - \varepsilon} \leq D_t|_{\theta_t = \bar{\theta}_t - \varepsilon} < 0.
\]

When \( \theta_t \in [z_t - \varepsilon, z_t + \varepsilon] \), as \( D_t \) increases with \( \theta_t \) and \( \theta_t < z_t + \varepsilon < \bar{\theta}_t \),

\[
D_t|_{\theta_t \in [z_t - \varepsilon, z_t + \varepsilon]} < D_t|_{\theta_t = \bar{\theta}_t} = 0.
\]

So when \( \mu_t < -3\varepsilon/4 \), \( D_t = 0 \) if \( \theta_t = \bar{\theta}_t \), \( D_t > 0 \) if \( \theta_t > \bar{\theta}_t \) and \( D_t < 0 \) if \( \theta_t < \bar{\theta}_t \).

To summarize, regardless of the value of \( \mu_t \), there always exists a unique \( \bar{\theta}_t \).

**A.4 Existence of Equilibrium Price**

**Proof.** For each realization of \( \theta_t \), we solve \( \bar{\theta}_t = \theta_t \) for the equilibrium price \( \bar{p}_t \).

(i) When \( \mu_t > 3\varepsilon/4 \), solving from

\[
\theta_t = \bar{\theta}_t = \bar{p}_t - \beta (\bar{p} - v) / \alpha + \varepsilon/2
\]

yields

\[
\bar{p}_t = \frac{\alpha (\theta_t - \varepsilon/2) - \beta v}{\alpha - \beta}.
\]

Note that \( \beta (\bar{p}_t - v) / \alpha > 3\varepsilon/4 \) implies that \( \theta_t > v + \frac{3\alpha - \beta}{4\beta} \varepsilon \).
(ii) When \( \mu_t \in (0, 3\epsilon/4] \), solving from

\[
\theta_t = \hat{\theta}_t = \bar{p}_t + \beta (\bar{p}_t - v) / \alpha + 2\epsilon \left( \sqrt{1 - \beta} (\bar{p}_t - v) / (\alpha \epsilon) - 1 \right)
\]
yields

\[
\bar{p}_t = \frac{2\alpha^2 \epsilon + \beta^2 v + \alpha^2 \theta_t + \alpha \beta (v + \theta_t) + 2\alpha \sqrt{\alpha^2 \epsilon^2 - \beta \epsilon (\alpha + \beta) (\theta_t - v)}}{(\alpha + \beta)^2}
\]
where \( \alpha^2 \epsilon^2 - \beta \epsilon (\alpha + \beta) (\theta - v) \geq 0 \), that is, \( \theta_t \leq v + \frac{\alpha^2 \epsilon}{\beta (\alpha + \beta)} \). As \( \beta (\bar{p}_t - v) / \alpha \in (0, 3\epsilon/4] \), it must be true that \( \theta_t \in \left[ v, v + \frac{3\alpha - \beta}{4\beta} \epsilon \right] \). We can verify that there is only one valid solution when \( \theta_t \in \left[ v, v + \frac{3\alpha - \beta}{4\beta} \epsilon \right] \), that is,

\[
\bar{p}_t = \frac{2\alpha^2 \epsilon + \beta^2 v + \alpha^2 \theta_t + \alpha \beta (v + \theta_t) - 2\alpha \sqrt{\alpha^2 \epsilon^2 - \beta \epsilon (\alpha + \beta) (\theta_t - v)}}{(\alpha + \beta)^2}.
\]

(iii) When \( \mu_t = 0 \), at the equilibrium, \( \theta_t = \hat{\theta}_t = p_t = v \).

(iv) When \( \mu_t \in [-3\epsilon/4, 0) \), solving from

\[
\theta_t = \hat{\theta}_t = \bar{p}_t + \beta (\bar{p}_t - v) / \alpha + 2\epsilon \left( 1 - \sqrt{1 + \beta} (\bar{p}_t - v) / (\alpha \epsilon) \right)
\]
yields

\[
\bar{p}_t = \frac{-2\alpha^2 \epsilon + \beta^2 v + \alpha^2 \theta_t + \alpha \beta (v + \theta_t) + 2\alpha \sqrt{\alpha^2 \epsilon^2 + \beta \epsilon (\alpha + \beta) (\theta_t - v)}}{(\alpha + \beta)^2},
\]
where \( \theta \geq v - \frac{\alpha^2 \epsilon}{\beta (\alpha + \beta)} \). As \( \beta (\bar{p}_t - v) / \alpha \in [-3\epsilon/4, 0) \), \( \theta_t \in \left[ v - \frac{3\alpha - \beta}{4\beta} \epsilon, v \right] \). We can verify that there is only a valid solution when \( \theta \in \left[ v - \frac{3\alpha - \beta}{4\beta} \epsilon, v \right] \), that is,

\[
\bar{p}_t = \frac{-2\alpha^2 \epsilon + \beta^2 v + \alpha^2 \theta + \alpha \beta (v + \theta) - 2\alpha \sqrt{\alpha^2 \epsilon^2 + \beta \epsilon (\alpha + \beta) (\theta - v)}}{(\alpha + \beta)^2}.
\]

(v) When \( \mu_t < -3\epsilon/4 \), solving from

\[
\theta_t = \hat{\theta}_t = \bar{p}_t - \beta (\bar{p}_t - v) / \alpha - \epsilon / 2
\]
yields

\[
\bar{p}_t = \frac{\alpha (\theta_t + \epsilon/2) - \beta v}{\alpha - \beta}.
\]

Note that \( \beta (\bar{p}_t - v) / \alpha < -3\epsilon/4 \) implies that \( \theta_t < v - \frac{3\alpha - \beta}{4\beta} \epsilon \).

To summarize, regardless of the range of fundamental, there always exists a unique equilibrium.
price $\bar{p}_t$ such that

$$\bar{p}_t = \begin{cases} \\
\frac{\alpha (\theta_t - \varepsilon / 2) - \beta \varepsilon}{\alpha - \beta} & \text{if } \theta_t > v + \kappa \\
\frac{2\alpha^2 \varepsilon + \beta^2 + \alpha^2 \theta_t + \alpha \beta (v + \theta_t) - 2\alpha \sqrt{\alpha^2 \varepsilon^2 - \beta \varepsilon (\alpha + \beta)(\theta_t - v)}}{(\alpha + \beta)^2} & \text{if } \theta_t \in (v, v + \kappa] \\
\theta_t & \text{if } \theta_t = v \\
\frac{-2\alpha^2 \varepsilon + \beta^2 + \alpha^2 \theta_t + \alpha \beta (v + \theta_t) - 2\alpha \sqrt{\alpha^2 \varepsilon^2 + \beta \varepsilon (\alpha + \beta)(\theta_t - v)}}{(\alpha + \beta)^2} & \text{if } \theta_t \in [v - \kappa, v) \\
\frac{\alpha (\theta_t + \varepsilon / 2) - \beta \varepsilon}{\alpha - \beta} & \text{if } \theta_t < v - \kappa \\
\end{cases}$$

where $\kappa = \frac{3\alpha - \beta}{4\beta} \varepsilon$.

As $\tilde{\theta}_t$ is strictly increasing with $\bar{p}_t$, $\tilde{\theta}_t = \theta_t$ if $p_t = \bar{p}_t$, $\tilde{\theta}_t > \theta_t$ if $p_t > \bar{p}_t$, and $\tilde{\theta}_t > \theta_t$ if $p_t < \bar{p}_t$. According to Claim 1, this suggests $D_t = 0$ if $p_t = \bar{p}_t$, $D_t < 0$ if $p_t > \bar{p}_t$, and $D_t > 0$ if $p_t < \bar{p}_t$. $\blacksquare$

### A.5 Stability of the Equilibrium Price

**Proof.** The equilibrium price $\bar{p}$ is stable if $\left. \frac{\partial p_{t+1}}{\partial p_t} \right|_{p_t=\bar{p}} < 1$, or, $-2 < \left. \frac{\partial R_{t+1}}{\partial p_t} \right|_{p_t=\bar{p}} < 0$. Note that at $p_t = \bar{p}$, it is always true that $\theta = \tilde{\theta}_t$.

(i) When $\mu_t > 3\varepsilon / 4$, $\tilde{\theta}_t = p_t - \mu_t + \varepsilon / 2 < z_t - \varepsilon$. In this case, it is only possible for $\theta_t = \tilde{\theta}_t$ if and only if $\theta_t < z_t - \varepsilon$. Based on Eq. (16), differentiating $R_{t+1}$ with respect to $p_t$ conditional on $\theta_t \in [\tilde{\theta}_t - \varepsilon, \tilde{\theta}_t + \varepsilon]$ yields

$$\left. \frac{\partial R_{t+1}}{\partial p_t} \right|_{p_t=\bar{p}} = \gamma \left[ \frac{\frac{\partial \tilde{\theta}_t}{\partial p_t} - 2 \frac{\alpha + \beta}{\alpha} \frac{\alpha (\tilde{\theta}_t - \theta + \varepsilon)}{4\varepsilon}}{\tilde{\theta}_t} + \frac{\frac{\partial \tilde{\theta}_t}{\partial p_t} \left( \tilde{\theta}_t + \theta - \varepsilon - 2z_t \right)}{4\varepsilon} + \beta \right]$$

$$= \gamma \left( \frac{\alpha - \beta}{4\varepsilon} \theta_t - \varepsilon - \frac{2\varepsilon}{4\varepsilon} \right)$$

$$= \gamma \frac{4\beta (\theta_t - \varepsilon) + (\alpha - 3\beta) \varepsilon}{4\varepsilon},$$

where the second and third equality is obtained by substituting $\theta = \tilde{\theta}_t$, $\theta_t / \partial p_t = (\alpha + \beta) / \alpha$, and $\theta_t / \partial p_t = (\alpha + \beta) / \alpha$. As $\alpha > \beta$ and $\theta_t < z_t + \varepsilon$, clearly $\left. \frac{\partial R_{t+1}}{\partial p_t} \right|_{p_t=\bar{p}} < 0$. In this case, $\bar{p}^b$ is stable if $\frac{\partial \Delta p_{t+1}}{\partial p_t} \bigg|_{p_t=\bar{p}} > -2$, that is, $\theta < v + \tilde{k}$, where $\tilde{k} = \frac{(8\gamma + 3\beta - \alpha) \varepsilon}{4\beta}$. Therefore, the necessary conditions for $\bar{p}^b = \frac{\alpha (\theta - \varepsilon / 2) - \beta \varepsilon}{\alpha - \beta}$ to exist and stabilize are $(\alpha - \beta) \in (0, 2 / \gamma)$, and $(\theta_t - v) \in (\kappa, \tilde{k})$. It can be further proved that (i) the necessary conditions for $0 < \left. \frac{\partial p_{t+1}}{\partial p_t} \right|_{p_t=\bar{p}} < 1$ (steady convergence) are $(\alpha - \beta) \in (0, 2 / \gamma)$, and $(\theta_t - v) \in \left( \kappa, \tilde{k} - \frac{\varepsilon}{\beta \gamma} \right)$; and (ii) the necessary
conditions for \(-1 < \frac{\partial p_{t+1}}{\partial p_t} |_{p_t = \tilde{p}^b} < 0\) (oscillating convergence) is \((\alpha - \beta) \in (0, 2/\gamma)\) and \((\theta_t - \nu) \in \left(\kappa - \frac{\nu}{\beta \gamma}, \kappa\right)\).

It is straightforward from the second equality that \(\frac{\partial \Delta p_{t+1}}{\partial p_t} |_{p_t = \tilde{p}^b} > 0\) if \(\alpha < \beta\), which suggests that \(\tilde{p}^b\) is unstable if \(\alpha < \beta\).

(ii) When \(\mu_t < -3\epsilon/4\), \(\bar{\theta}_t = p_t - \mu_t - \epsilon/2 > z_t + \epsilon\). Based on Eq.(18), differentiating \(R_{t+1}\) with respect to \(p_t\) conditional on \(\theta \in [\bar{\theta}_t - \epsilon, \bar{\theta}_t + \epsilon]\) yields

\[
\frac{\partial R_{t+1}}{\partial p_t} |_{p_t = \tilde{p}^d} = \gamma \left(\frac{\partial \tilde{\theta}_t}{\partial p_t} - \frac{2\alpha + \beta}{\alpha} \frac{\alpha (\bar{\theta}_t - \bar{\theta}_t + \epsilon)}{4\epsilon} \right.
\]
\[
- \frac{\alpha (\bar{\theta}_t + \theta_t + \epsilon - 2z_t)}{4\epsilon} \left. \frac{\partial \bar{\theta}_t}{\partial p_t} + \beta\right)
\]
\[
= -\gamma \frac{(\alpha - \beta) (\theta + \epsilon - z_t)}{2\epsilon}
\]
\[
= -\gamma \frac{-4\beta (\theta - \nu) + (\alpha - 3\beta) \epsilon}{4\epsilon},
\]

where the second equality is obtained by substituting \(\theta = \bar{\theta}_t\), \(\partial \tilde{\theta}_t/\partial p_t = (\alpha - \beta)/\alpha\), and \(\partial z_t/\partial p_t = (\alpha + \beta)/\alpha\); and the third equality is obtained by substituting \(z_t|_{p_t = \tilde{p}^d} = \frac{(8/\gamma + 3\beta - \alpha) \epsilon}{\alpha - \beta}\). Therefore, the necessary condition for \(\tilde{p}^d = \frac{\alpha (\theta - \epsilon/2) - \beta \nu}{\alpha - \beta}\) to exist and stabilize are \((\alpha - \beta) \in (0, 2/\gamma)\), and \(\theta \in (\nu - \kappa, \nu - \kappa)\). It can be further proved that (i) the necessary conditions for \(0 < \frac{\partial p_{t+1}}{\partial p_t} |_{p_t = \tilde{p}^d} < 1\) (steady convergence) are \((\alpha - \beta) \in (0, 2/\gamma)\), and \((\theta_t - \nu) \in \left(\kappa, \kappa - \frac{\nu}{\beta \gamma}\right)\); and (ii) the necessary conditions for \(-1 < \frac{\partial p_{t+1}}{\partial p_t} |_{p_t = \tilde{p}^d} < 0\) (oscillating convergence) are \((\alpha - \beta) \in (0, 2/\gamma)\), and \((\theta_t - \nu) \in \left(\kappa - \frac{\epsilon}{\beta \gamma}, \kappa\right)\).

**A.6 The Relation between \(m_t\) and \(\epsilon\)**

**Proof.** (I) If \(p_t > \nu\), \(m_t = 1\) if \(\theta_t > z_t + \epsilon\). In this case, \(m_t\) is associated with \(\epsilon\) when (i) \(\theta_t < z_t - \epsilon\) or (ii) \(\theta_t \in [z_t - \epsilon, z_t + \epsilon]\) and \(z_t > \bar{\theta}_t\). We consider the relation between \(m_t\) and \(\epsilon\) conditional on \(\mu_t \in (3\epsilon/4, +\infty)\) and \(\mu_t \in (0, 3\epsilon/4)\) respectively.

(a) \(\mu_t \in (3\epsilon/4, +\infty)\)

When \(\theta_t < z_t - \epsilon\), \(m_t\) is associated with \(\epsilon\) if and only if \(\theta_t \in [\bar{\theta}_t - \epsilon, \bar{\theta}_t + \epsilon]\). Based on Eq.(15),
differentiating \( m_t \) with respect to \( \varepsilon \) yields

\[
\frac{\partial m_t}{\partial \varepsilon} = \frac{\partial (\frac{\tilde{\theta}_t - \theta_t + \varepsilon}{2\varepsilon})}{\partial \varepsilon} = \frac{\theta_t - \bar{\theta}_t + \varepsilon / 2}{2\varepsilon^2},
\]

where the second equality is obtained by substituting \( \frac{\partial \bar{\theta}_t}{\partial \varepsilon} = 1/2 \). Clearly, \( \frac{\partial m_t}{\partial \varepsilon} < 0 \) if \( \theta_t < \bar{\theta}_t - \varepsilon / 2 \), which implies \( \theta_t - p_t < \mu_t < -3\varepsilon / 4 \) and \( \frac{\partial m_t}{\partial \varepsilon} \geq 0 \) if \( \theta_t \geq \bar{\theta}_t - \varepsilon / 2 \).

When \( \theta_t \in [z_t - \varepsilon, z_t + \varepsilon] \) and \( z_t > \bar{\theta}_t \), differentiating \( m_t \) in Eq.(29) with respect to \( \varepsilon \) yields

\[
\frac{\partial m_t}{\partial \varepsilon} = \begin{cases} \frac{z_t - \theta_t}{\varepsilon} & \text{if } \theta_t > \bar{\theta}_t + \varepsilon \\ \frac{z_t - \theta_t + \varepsilon / 2}{2\varepsilon^2} & \text{if } \theta_t \leq \bar{\theta}_t + \varepsilon. \end{cases}
\]

As \( \bar{\theta}_t < z_t - \varepsilon \) conditional on \( \mu_t \in (3\varepsilon / 4, +\infty) \), \( \frac{\partial m_t}{\partial \varepsilon} > 0 \) when \( \theta_t \leq \bar{\theta}_t + \varepsilon \). When \( \theta_t > \bar{\theta}_t + \varepsilon \), \( \frac{\partial m_t}{\partial \varepsilon} \geq 0 \) if \( \theta_t \leq z_t \) and \( \frac{\partial m_t}{\partial \varepsilon} < 0 \) if \( \theta_t > z_t \), or equivalently, \( \theta_t - p_t > \mu_t > 3\varepsilon / 4 \).

(b) \( \mu_t \in (0, 3\varepsilon / 4] \)

When \( \theta_t < z_t - \varepsilon \), given \( \theta_t \in [z_t - \varepsilon, \bar{\theta}_t + \varepsilon] \), differentiating \( m_t \) in Eq.(15) with respect to \( \varepsilon \) yields

\[
\frac{\partial m_t}{\partial \varepsilon} = \frac{\theta_t - z_t + \tau}{2\varepsilon^2},
\]

where \( \tau = \frac{\mu_t}{\sqrt{1 - \mu_t / \varepsilon}} \in (0, 3\varepsilon / 2] \). Clearly, \( \frac{\partial m_t}{\partial \varepsilon} \geq 0 \) if \( \theta_t \geq z_t - \tau \) and \( \tau > \varepsilon \), and \( \frac{\partial m_t}{\partial \varepsilon} < 0 \) if \( \theta_t < z_t - \max(\tau, \varepsilon) < z_t \), or equivalently, \( \theta_t - p_t < \mu_t - \max(\tau, \varepsilon) < 0 \).

When \( \theta_t \in [z_t - \varepsilon, z_t + \varepsilon] \) and \( z_t > \bar{\theta}_t \), differentiating \( m_t \) in Eq.(29) with respect to \( \varepsilon \) yields

\[
\frac{\partial m_t}{\partial \varepsilon} = \begin{cases} \frac{z_t - \theta_t}{\varepsilon} & \text{if } \theta_t > \bar{\theta}_t + \varepsilon \\ \frac{z_t - \theta_t + \varepsilon / 2}{2\varepsilon^2} & \text{if } \theta_t \leq \bar{\theta}_t + \varepsilon. \end{cases}
\]

When \( \theta_t \leq \bar{\theta}_t + \varepsilon \), \( \frac{\partial m_t}{\partial \varepsilon} > 0 \). When \( \theta_t > \bar{\theta}_t + \varepsilon \), \( \frac{\partial m_t}{\partial \varepsilon} \geq 0 \) if \( \theta_t \leq z_t \) and \( \frac{\partial m_t}{\partial \varepsilon} < 0 \) if \( \theta_t > z_t \), or equivalently, \( \theta_t - p_t > \beta (p_t - \nu) / \alpha \).

(II) If \( p_t < \nu \), then \( m_t = 1 \) if \( \theta_t < z_t - \varepsilon \). In this case, \( m_t \) is associated with \( \varepsilon \) when \( \theta_t > z_t + \varepsilon \) or when \( \theta_t \in [z_t - \varepsilon, z_t + \varepsilon] \) and \( z_t < \bar{\theta}_t \). We consider the relation between \( m_t \) and \( \varepsilon \) conditional on \( \mu_t \in (+\infty, -3\varepsilon / 4) \) and \( \mu_t \in [-3\varepsilon / 4, 0) \) respectively.

(c) \( \mu_t \in (+\infty, -3\varepsilon / 4) \)

When \( \theta_t < z_t + \varepsilon \), \( m_t \) is associated with \( \varepsilon \) if and only if \( \theta_t \in [\bar{\theta}_t - \varepsilon, \bar{\theta}_t + \varepsilon] \). Differentiating \( m_t \) in Eq.(20) with respect to \( \varepsilon \) yields

\[
\frac{\partial m_t}{\partial \varepsilon} = \frac{\bar{\theta}_t - \theta_t + \varepsilon / 2}{2\varepsilon^2}.
\]

So \( \frac{\partial m_t}{\partial \varepsilon} \geq 0 \) if \( \theta_t \leq \bar{\theta}_t + \varepsilon / 2 \), and \( \frac{\partial m_t}{\partial \varepsilon} < 0 \) if \( \theta_t > \bar{\theta}_t + \varepsilon / 2 \), or equivalently, \( \theta_t - p_t > -\mu_t > 3\varepsilon / 4 \).

When \( \theta_t \in [z_t - \varepsilon, z_t + \varepsilon] \) and \( z_t < \bar{\theta}_t \), differentiating \( m_t \) in Eq.(26) with respect to \( \varepsilon \) yields

\[
\frac{\partial m_t}{\partial \varepsilon} = \begin{cases} \frac{\theta_t - z_t}{\varepsilon} & \text{if } \theta_t < \bar{\theta}_t - \varepsilon \\ \frac{\bar{\theta}_t + \varepsilon / 2 - z_t}{2\varepsilon^2} & \text{if } \theta_t \geq \bar{\theta}_t - \varepsilon. \end{cases}
\]

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Appendix Table 3: The necessary conditions for information transparency to enhance the coordination on rational arbitrage.

Panel I: \( p_t > v \)

<table>
<thead>
<tr>
<th>( \mu_t )</th>
<th>( \theta_t &lt; z_t - \varepsilon )</th>
<th>( \theta_t \in [z_t - \varepsilon, z_t + \varepsilon] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (3\varepsilon/4, +\infty) )</td>
<td>( \theta_t - p_t &lt; -\mu_t )</td>
<td>( \theta_t &gt; \bar{\theta}_t )</td>
</tr>
<tr>
<td>( (0, 3\varepsilon/4) )</td>
<td>( \theta_t - p_t &gt; \mu_t )</td>
<td>( \theta_t &gt; \bar{\theta}_t )</td>
</tr>
</tbody>
</table>

Panel II: \( p_t < v \)

<table>
<thead>
<tr>
<th>( \mu_t )</th>
<th>( \theta_t &lt; z_t - \varepsilon )</th>
<th>( \theta_t \in [z_t - \varepsilon, z_t + \varepsilon] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( [-3\varepsilon/4, 0) )</td>
<td>( \theta_t - p_t &gt; \mu_t )</td>
<td>( \theta_t &gt; \bar{\theta}_t )</td>
</tr>
<tr>
<td>( (+\infty, -3\varepsilon/4) )</td>
<td>( \theta_t - p_t &lt; \mu_t )</td>
<td>( \theta_t &gt; \bar{\theta}_t )</td>
</tr>
</tbody>
</table>

As \( \bar{\theta}_t > z_t + \varepsilon \) when \( \mu_t \in (+\infty, -3\varepsilon/4) \), \( \partial m_t / \partial \varepsilon > 0 \) if \( \theta_t > \bar{\theta}_t - \varepsilon > z_t \). If \( \theta_t < \bar{\theta}_t - \varepsilon \), \( \partial m_t / \partial \varepsilon > 0 \) if \( z_t < \theta_t < \bar{\theta}_t - \varepsilon \), and \( \partial m_t / \partial \varepsilon < 0 \) if \( \theta_t < z_t \), or equivalently \( \theta_t - p_t < \mu_t < -3\varepsilon/4 \).

(c) \( \mu_t \in [-3\varepsilon/4, 0) \)

When \( \bar{\theta}_t > z_t + \varepsilon \), \( m_t \) is associated with \( \varepsilon \) if and only if \( \theta_t \in [\bar{\theta}_t - \varepsilon, \bar{\theta}_t + \varepsilon] \). Differentiating \( m_t \) in Eq.(20) with respect to \( \varepsilon \) yields

\[
\frac{\partial m_t}{\partial \varepsilon} = -\frac{\theta_t + z_t + \varepsilon'}{2\varepsilon^2},
\]

where \( \varepsilon' = \frac{-\mu_t}{\sqrt{1 + \mu_t/\varepsilon}} \in (0, 3\varepsilon/2] \). Clearly, \( \partial m_t / \partial \varepsilon > 0 \) if \( \theta_t \leq z_t + \varepsilon' \) and \( \varepsilon' > \varepsilon \), and \( \partial m_t / \partial \varepsilon < 0 \) if \( \theta_t > z_t + \max(\epsilon, \varepsilon') > \theta_t \), that is, \( \theta_t - p_t > \mu_t + \max(\epsilon, \varepsilon') > 0 \).

When \( \theta_t \in [z_t - \varepsilon, z_t + \varepsilon] \) and \( z_t < \theta_t \), differentiating \( m_t \) in Eq.(26) with respect to \( \varepsilon \) yields

\[
\frac{\partial m_t}{\partial \varepsilon} = \begin{cases} \frac{\theta_t - z_t}{2\varepsilon'} & \text{if } \theta_t < \bar{\theta}_t - \varepsilon \\ \frac{\theta_t - z_t}{2\varepsilon} & \text{if } \theta_t > \bar{\theta}_t - \varepsilon \end{cases}
\]

As \( \varepsilon' \in (0, 3\varepsilon/2] \), \( \partial m_t / \partial \varepsilon > 0 \) if \( \theta_t > \bar{\theta}_t - \varepsilon \). Note that \( z_t < \bar{\theta}_t \leq z_t + \varepsilon \) when \( \mu_t \in [-3\varepsilon/4, 0) \), \( \partial m_t / \partial \varepsilon < 0 \) if \( \theta_t < \bar{\theta}_t - \varepsilon < z_t \), which implies \( \theta_t - p_t < \mu_t \).

Table 3 summarizes the necessary conditions for \( \partial m_t / \partial \varepsilon < 0 \). It suggests that information transparency increases the coordination on rational arbitrage if the market is sufficiently inefficient such that \( |\theta_t - p_t| > |\mu_t| \) when \( |\mu_t| > 3\varepsilon/4 \), which proves Proposition 2. Note that \( \tau = \frac{\varepsilon'}{\sqrt{1 - \varepsilon/\mu_t}} \).

when \( \mu_t \in (0, 3\varepsilon/4) \) and \( \varepsilon' = \frac{|\mu_t|}{\sqrt{1 - |\mu_t|/\varepsilon}} \) when \( \mu_t \in [-3\varepsilon/4, 0) \). Moreover, \( \mu_t - \max(\tau, \varepsilon) < 0 \) when \( \mu_t \in (0, 3\varepsilon/4) \) and \( \mu_t + \max(\tau, \varepsilon) > 0 \) when \( \mu_t \in [-3\varepsilon/4, 0) \). So when \( |\mu_t| \leq 3\varepsilon/4 \), \( \partial m_t / \partial \varepsilon < 0 \) if (i) \( |\theta_t - p_t| > |\mu_t| \) and \( \theta_t \in [z_t - \varepsilon, z_t + \varepsilon] \) or (ii) \( |\theta_t - p_t| > -|\mu_t| + \max(\varepsilon, \tau) \) and \( \theta_t > z_t + \varepsilon \).
### A.7 The Relation between $m_t$ and $|\theta_t - p_t|$  

**Proof.** When $\theta_t < z_t - \varepsilon$, based on Eq.(15),

\[
\frac{\partial m_t}{\partial p_t} \geq 0 \\
\frac{\partial m_t}{\partial \theta_t} \leq 0.
\]

When $\theta_t \in [z_t - \varepsilon, z_t + \varepsilon]$ and $z_t > \bar{\theta}$, based on Eq.(29),

\[
\frac{\partial m_t}{\partial p_t} \leq 0 \\
\frac{\partial m_t}{\partial \theta_t} \geq 0.
\]

When $\theta_t \in [z_t - \varepsilon, z_t + \varepsilon]$ and $z_t < \bar{\theta}$, based on Eq.(26),

\[
\frac{\partial m_t}{\partial p_t} \geq 0 \\
\frac{\partial m_t}{\partial \theta_t} \leq 0.
\]

When $\theta_t > z_t + \varepsilon$, based on Eq.(20),

\[
\frac{\partial m_t}{\partial p_t} \leq 0 \\
\frac{\partial m_t}{\partial \theta_t} \geq 0.
\]

We consider the implications of these results for the relation between $m_t$ and $|\theta_t - p_t|$ in the following four scenarios.

(a) When $\mu_t > 3\varepsilon/4$, $\bar{\theta} = p_t - \mu_t + \varepsilon/2 < z_t - \varepsilon$.

When $\theta_t < z_t - \varepsilon$, that is, $\theta_t - p_t < \mu_t - \varepsilon$, we consider two scenarios. (i) If $\theta_t - p_t < 0$ so that the asset is overpriced, an increase in $p_t$ or a decline in $\theta_t$ leads to higher $|\theta_t - p_t|$. As $\frac{\partial m_t}{\partial p_t} \geq 0$ and $\frac{\partial m_t}{\partial \theta_t} \leq 0$ when $\theta_t < z_t - \varepsilon$ and $\theta_t - p_t < 0$, it implies $\frac{\partial m_t}{\partial |\theta_t - p_t|} \geq 0$. (ii) If, alternatively, $\theta_t - p_t \geq 0$ and $\theta_t - p_t < \mu_t - \varepsilon$, which exists if and only if $\mu_t > \varepsilon$, then $\frac{\partial m_t}{\partial |\theta_t - p_t|} \leq 0$.

When $\theta_t \in [z_t - \varepsilon, z_t + \varepsilon]$ and $z_t > \bar{\theta}$, there are two scenarios. (i) If $\theta_t \in [z_t - \varepsilon, p_t]$ and $\mu_t \in (3\varepsilon/4, \varepsilon)$, then the asset is overpriced. So $\frac{\partial m_t}{\partial |\theta_t - p_t|} \leq 0$. (ii) If $\theta_t \in [p_t, z_t + \varepsilon]$, then $\frac{\partial m_t}{\partial |\theta_t - p_t|} \geq 0$.

When $\theta_t > z_t + \varepsilon$, then $m_t = 1$, which results in $\frac{\partial m_t}{\partial |\theta_t - p_t|} = 0$.

(b) When $\mu_t \in (0, 3\varepsilon/4]$, then $\bar{\theta} < z_t$.

When $\theta_t < z_t - \varepsilon$, it implies $\theta_t - p_t < \mu_t - \varepsilon < 0$. So $\frac{\partial m_t}{\partial |\theta_t - p_t|} \geq 0$.

When $\theta_t \in [z_t - \varepsilon, z_t + \varepsilon]$ and $z_t > \bar{\theta}$, $\frac{\partial m_t}{\partial |\theta_t - p_t|} \leq 0$ if $\theta_t \in [z_t - \varepsilon, p_t]$ and $\frac{\partial m_t}{\partial |\theta_t - p_t|} \geq 0$ if $\theta_t \in [p_t, z_t + \varepsilon]$.

When $\theta_t > z_t + \varepsilon$, $\frac{\partial m_t}{\partial |\theta_t - p_t|} = 0$.

(c) When $\mu_t \in [-3\varepsilon/4, 0)$, $\bar{\theta} > z_t$.

If $\theta_t < z_t - \varepsilon$, $\frac{\partial m_t}{\partial |\theta_t - p_t|} = 0$.

If $\theta_t \in [z_t - \varepsilon, z_t + \varepsilon]$ and $z_t < \bar{\theta}$, $\frac{\partial m_t}{\partial |\theta_t - p_t|} \geq 0$ if $\theta_t < p_t$ and $\frac{\partial m_t}{\partial |\theta_t - p_t|} \leq 0$ if $\theta_t \geq p_t$.

If $\theta_t > z_t + \varepsilon$, that is, $\theta_t - p_t > \mu_t + \varepsilon$, we consider two scenarios. (i) If $\theta_t - p_t > 0$ so that the asset is underpriced, an increase in $p_t$ or an increase in $\theta_t$ leads to higher $|\theta_t - p_t|$ and therefore $\frac{\partial m_t}{\partial |\theta_t - p_t|} \geq 0$. (ii) If $\theta_t - p_t \leq 0$ and $\theta_t - p_t > \mu_t + \varepsilon$, which exists if and only if $\mu_t < -\varepsilon$, then $\frac{\partial m_t}{\partial |\theta_t - p_t|} \leq 0$. 

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A.8 The Relation between $\mathbb{E}_t$ and $\pi_t$

**Proof.** According to Eq. (12), (17) and (22), $\mathbb{E}_t$ increases with $\hat{\theta}$ and therefore $p_t$ when $x_t < z_t$ and decreases with $p_t$ when $x_t \geq z_t$. We discuss the implication of this result in four cases.

First, if $p_t < v$, for $x_t < z_t$, that is, $x_t - p_t < \mu_t < 0$, a higher price indicates greater expected overpricing $|x_t - p_t|$, which suggests $\mathbb{E}_t$ increases with $|x_t - p_t|$. 

Second, if $p_t > v$, for $x_t \geq z_t$, that is, $x_t - p_t \geq \mu_t > 0$, a lower price indicates greater expected underpricing $|x_t - p_t|$. In this case, our result suggests $\mathbb{E}_t$ increases with $|x_t - p_t|$. 

Third, if $p_t < v$ and $x_t \geq z_t$, note that $p_t > z_t$ when $p_t < v$, an increase in the price is associated with greater expected overpricing if $p_t > x_t \geq z_t$ and smaller expected underpricing if $x_t \geq p_t$. In this case, the result means that $\mathbb{E}_t$ increases with $|x_t - p_t|$ if $x_t \geq p_t$ and decreases with $|x_t - p_t|$ if $p_t > x_t \geq z_t$.

Finally, if $p_t > v$ and $x_t < z_t$, with similar analysis, we can show that $\mathbb{E}_t$ increases with $|x_t - p_t|$ if $x_t < p_t$ and decreases with $|x_t - p_t|$ if $p_t \leq x_t < z_t$.

Similarly, we can analyze the impact of $|x_t - p_t|$ that originates from changes in $x_t$ on $\mathbb{E}_t$.

To summarize, $\mathbb{E}_t$ decreases with $|x_t - p_t|$ if $x_t \in [\min(p_t, z_t), \max(p_t, z_t))$ and increases with $|x_t - p_t|$ otherwise. 

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(d) When $\mu_t < -3\varepsilon/4$, $\hat{\theta} = p_t - \mu_t - \varepsilon/2 > z_t + \varepsilon$. When $\theta_t < z_t - \varepsilon$, $m_t = 1$, which results in $\partial m_t / \partial |\theta_t - p_t| = 0$. When $\theta_t \in [z_t - \varepsilon, z_t + \varepsilon]$ and $z_t < \hat{\theta}$, $\partial m_t / \partial |\theta_t - p_t| \geq 0$ if $\theta_t \in [z_t - \varepsilon, p_t]$ and $\partial m_t / \partial |\theta_t - p_t| \leq 0$ if $\theta_t \in [p_t, z_t + \varepsilon]$. When $\theta_t > z_t + \varepsilon$, that is, $\theta_t - p_t > \mu_t + \varepsilon$, we consider two scenarios. (i) If $\theta_t - p_t > 0$ so that the asset is underpriced, an increase in $p_t$ or an increase in $\theta_t$ leads to higher $|\theta_t - p_t|$ and therefore $\partial m_t / \partial |\theta_t - p_t| \geq 0$. (ii) If, alternatively, $\theta_t - p_t \leq 0$ and $\theta_t - p_t > \mu_t + \varepsilon$, which exists if and only if $\mu_t < -\varepsilon$, then $\partial m_t / \partial |\theta_t - p_t| \leq 0$.

Similarly, we can show that the results are the same if $|\theta_t - p_t|$ changes because of $\theta_t$. To summarize, $\partial m_t / \partial |\theta_t - p_t| \geq 0$ if (i) $\theta_t < \min(p_t, z_t - \varepsilon)$, (ii) $\theta_t > \max(p_t, z_t + \varepsilon)$, (iii) $\mu_t > 0$ and $\theta_t \in [p_t, z_t + \varepsilon]$, or (iv) $\mu_t < 0$ and $\theta_t \in [z_t - \varepsilon, p_t]$.

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A.8 The Relation between $\mathbb{E}_t (\Delta \pi_t^i \mid x_t^i)$ and $|x_t^i - p_t|$