A Financial CCAPM and Economic Inequalities

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ABSTRACT

This paper considers a wealth heterogeneous multi-agent financial pricing CCAPM model. It is based on the following observations: (a) A distinction between what agents are willing to pay for consumption and what they actually pay. (b) The paper assumes that each agent is in fact “Cournot-gaming” a market defined by all other agents. This results in a decomposition of an n-agents game into n games of two agents, one a specific agent and the other a synthetic agent. The paper’s essential results are two-fold. First, an agent specific martingale pricing model is defined expressing and agent’s willingness to pay for consumption (his utility price) and the unique market price—the Cournot price, that all agents pay for consumption. Second, the pricing model defined is shown to account for agents’ wealth inequalities pointing out that all agents valuations are a function of theirs and others’ wealth, the information they have about each other and other factors which are discussed in the text. In a Multi-Agent Cournot game, the paper shows that there is no wealth-price discrimination by that endowed agents have a wealth “premium” in the sense that they will pay less than what they would have been willing to pay.

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1. Introduction
An agent’s CCAPM price of consumption is based on its wealth and its utility, defining what the agent is willing to pay for current consumption and his wealth commitment for future returns and thus consumption. The price of consumption however is defined by macroeconomic (aggregate) factors such as supplies, aggregate demand and market structures that define the financial price of consumption. The price is thus implied in the exchange between agents acting in an economic and regulatory environment that define the opportunities, the rules and the constraints on such exchanges and their resultant aggregate demand of agents, each “gaming the market”. Additional factors such as agents’ information, risk attitude and power influence agents’ decisions and thereby market prices. The purpose of this paper is to provide a financial (CCAPM based) pricing model that recognizes the effects of agents’ economic inequalities as well as an approach that reconciles a mismatch of micro and macroeconomic factor in pricing models.

CCAPM pricing models have provided a financial pricing framework commensurate with the Arrow-Debreu fundamental approach to financial pricing. Publications based on theoretical and empirical research, both to validate such models and to discredit their universality abound (Lucas, 1978, Cochrane, 2001, Fama, 1996, Campbell, 2003, Mankiw et al., 1991, Hansen and Renault, 2007, Gourieroux and Monfort 2007, Gorton et al. 2013 and many others). In particular, an extensive number of issues were raised pertaining to the quality of the assumptions and the predictions underlying CCAPM pricing models. These include for example, Mehra and Prescott 1985, Hansen and Singleton, 1982 and Ferson and Constantinidis, 1991, Constantinidie and Duffie, 1996 among an extremely productive financial research in both theoretical and applied finance that has contributed to finance and continues to do so (in both discrete and continuous time, for example, using the Merton’s 1973 paper on Optimum consumption and portfolio rules in a continuous time model).

This paper, unlike previous CCAPM models is based on a representative agent, whose consumption is defined by his wealth commitment, his investment strategy for future returns and thus his risk attitudes and a random price defined by agents gaming future consumption and financial markets. Such an element was used first by Constantinidis and Duffie (1996) who pointed out to empirical difficulties encountered by using representative consumer models. They also provide a solution accounting for income heterogeneity by replacing the Euler equations of consumption in a representative-agent economy by a set of Euler equations that depend not only on the per-capita consumption growth but also on the cross-sectional variance of individual consumers’ consumption growth. Unlike the Constantinidis and Duffie’s 1996 paper, this paper considers instead heterogeneous agents expressing their consumption preferences in terms of their financial expenditures and their investment strategy. These result in a random (resulting from a multi-agent Cournot game with their future wealth and prices uncertain).

Consumption pricing is thus based on a number of assumptions that replaces the elementary decision, “how much to consume” by “how much to save for future consumption and how to allocate one’s investments?. In this context, two prices are defined: the price an agent is willing to pay for consumption and the price he pays for consumption. The former price defined by the CCAPM model, the latter and unique price of consumption is the price that results from an aggregate demand for consumption (and its supply) defined by the multi-agent Cournot game. This approach leads to a market with individual agents prices (and thus a function of their wealth and risk preferences) resulting in their bid and ask prices while the market price is a Cournot price derived from agents aggregate demand for consumption. In this
framework, the n multi-agent game is reduced to n Cournot games, each game consisting of a specific agent gaming all other agents, represented by a synthetic agent (rather than a single representative agent as commonly used many theoretical models). Such an approach leads to agents that have no wealth that can affect market prices, to be price takers, while wealth endowed agents can affect the market price in “their favor” by the decisions they make. This advantage is expressed by a “premium” defined by what they may be willing to pay for consumption and what they will pay. For poorly endowed agents, they merely adjust their consumption such that what they are willing to pay is equal to the market price (the Cournot price). Price is thus no longer the outcome of an infinite number of agents, none of which can affect appreciably the market price, but a price resulting from agents’ strategic decisions in a financial-consumption-investment game. The solutions of these agents games provide then an equilibrium that defines a common and unique financial price of consumption. Technically, this paper will show that each agent willingness to pay (the subjective agent price) and the market Cournot price, are defined by a Martingale while the unique market Cournot price, is a price that has to satisfy all agents’ Martingales. In this sense, each agent has his own pricing Martingales, intersecting with the unique multi-agent Cournot price.

The approach we use differs from other CCAPM pricing and multi-agent models in a number of ways. First, unlike utility consumption models, an agent does not, as stated above, maximizes the expected utility of consumption by setting his consumption but by setting his financial expenditure. In this sense, consumption is random, resulting from aggregate consumption, macroeconomic and supply factors that affect future prices.

A number of examples and simplifications are made to obtain tractable results and to highlight the applicability of this pricing framework to a variety of financial problems. In particular, we suggest based on marginal utility arguments that an agent pricing kernel is equivalently defined by a ratio of the price an agent he is willing to pay and the price he pays. In addition, given an agent wealth commitment for current consumption, its residual wealth (savings) is allocated to a portfolio that consists in parts of bonds and stocks, resulting in a one parameter allocation. Such a parameter is assumed within a simple model that maximizes both consumption and future assets allocation to be a proxy to an agent risk preferences—since greater investments in stocks compared to investment in bonds will imply a smaller risk aversion and thus may be considered, within an example, a proxy to risk preference.

For simplicity, we consider first a one period model to highlight the approach used and extend our results to a discrete multi-period pricing model for which each agents’ pricing Martingale’ is defined. In this sense, we maintain the fundamental (Martingale) structure of pricing models for each agent, corresponding to the price the agent is willing to pay, but faced with a single price for consumption, each agent will adjust his expenditure to levels that concord his individual martingale.

2. Pricing Consumption: The Single Period Problem
The underlying assumption of utility pricing models is that consumption has a utility that defines the price an agent is willing to pay for consumption. Given a representative agent, a unique market price model is defined. For several agents, each reaching independently its consumption decision, a Cournot game results. For example, let \( c^k_0 \) be an agent’s k decision to consume whose utility of consumption is \( u^k_0 (c^k_0) \). A
myopic utility maximizing price \( \pi^k_0 \) for consumption an agent will be willing to pay is then given by a solution of:

\[
\max_{c_0^k} u_0^k(c_0^k) - \pi^k_0 c_0^k \quad \text{and} \quad \frac{\partial u_0^k(c_0^k)}{\partial c_0^k} = \pi^k_0,
\]

A market price is defined however by the consumption of all agents and its potential supply. These lead to a consumption price \( \pi_0(C_0, S_0) \) where \( (C_0, S_0) \) denote aggregate demand and aggregate supply at time \( t=0 \) and \( c_0^k, k = 1, 2, ..., n, C_0 = \sum_{j=1}^{n} c_0^j \). A price is then the outcome of a Cournot game. When consumers are heterogeneous, each endowed with a different wealth, information and power, or when prices are regulated, or controlled, the price can be random resulting from the randomized solution of games where wealth consumption expenditure and other factors will ascribe to agents optimal consumption. Explicitly, assume that wealth distribution is heterogeneous and let an individual agent \( k \) be endowed by a wealth \( W_0^k \) fully devoted to current consumption and let the price of consumption be random \( \tilde{\pi}_0(C_0) \), a function of aggregate consumption, a-priori unknown to the agent. Consumption by the \( k \)th agent is thus random and given by \( \tilde{c}_0^k = W_0^k / \tilde{\pi}_0(C_0) \) and its expected financial utility is:

\[
E_0^k u_0^k(\tilde{c}_0^k) = E_0^k u_0^k(W_0^k / \tilde{\pi}_0(C_0)) - W_0^k, \quad \tilde{C}_0 = \sum_{j=1}^{n} \tilde{c}_0^j
\]

For example, the number of shares bought by agent \( k \) depends on the price of shares, defined by buyers demand and by sellers supply. If agent \( k \) has no prior knowledge of this demand, then of course, for agent \( k \), the number of shares he acquires in fact and their price are necessarily random. In a multi-agent pricing model over one period only, the decision to consume (however rational it may be) is thus necessarily a random variable. For \( n \) agents, buyers and sellers, a financial market price may be defined as a Cournot game, where each agent reaches independently a decision whose consequence will be (or not, depending on the size of his high wealth allocated to consumption and savings) or defined by each agent “gaming” the market against all other agents, and the market (i.e. all other agents) reacting rationally to this game. In this latter case, if each agent “games” the market then \( n \) 2-persons games result—the agent gaming all other agents and all other agents collaborating in gaming the individual agent. Since all agents face the same game, a market price equilibrium is defined when all agents, collectively and individually, recalibrate their consumption to be consistent with their consumption expenditures (resulting a-priori in random consumption levels). In other words, in a financial market, the \( k \)th agent-investor “games” all other investors defined by a synthetic financial investor including all agents except the \( k \)th agent. Setting a synthetic agent indexed by “nk”, then for an agent \( k \), \( C_0 = c_0^k + C_0^{nk} \), \( C_0 = \sum_{j=k}^{n} c_0^j \) while the market price is \( \tilde{\pi}_0 \), a function of agents’ aggregate consumption (and supplies), or \( \tilde{\pi}_0(c_0^k + C_0^{nk}) \). Current consumption of both agents \( k \) and \( nk \) are then random and given by: \( \tilde{c}_0^k = W_0^k / \tilde{\pi}_0(C_0^k + C_0^{nk}) \); \( \tilde{C}_0^{nk} = W_0^{nk} / \tilde{\pi}_0(C_0^k + C_0^{nk}) \). Over multiple periods, this will result on each agent faced by a pricing Martingale “what he is willing to pay” compared to the market price—the price he will have to pay. This latter price is unique in the sense that all agents will pay the same price and thus adjust their consumption level such that what they are willing to pay and will pay will be equal. In this process, the
individual agents utility and wealth have a role in defining the consumption price, resulting in a premium that wealthier agents will gain from their “will to pay more” for consumption and in fact pay less due to other agents willing to pay less (due to less wealth) and yet have to pay more. These elements provide then a departure from traditional CCAPM and other pricing models.

Based on these elements, we consider first simple a one period result to highlight both the approach and the basic results that will carry over to 2 and to multiple periods. The following proposition sets a price of consumption and the premium each agent pays (or profit from) as a function of the wealth allocated to consumption. Two solutions are distinguished, the usual Cournot solution and the Multi-Agent Cournot game.

**Proposition 1: A one period Cournot Game**

Let \( W_0^k \) be the financial expenditure that an agent \( k \) commits to current consumption and let \( \hat{u}^k_c (c^k_0) / \hat{c}^k_0 = \pi^k_0 \) be the marginal price the agent \( k \) is willing to pay for consumption while \( \bar{\pi}_0 (\bar{C}_0) \), \( \bar{C}_0 = \sum_{j=1}^{n} \bar{\pi}^j_0 \) is the market price of consumption (where aggregate supplies \( \bar{S}_0 \) equal aggregate demand \( \bar{C}_0 = \bar{S}_0 \)). The Cournot price is then defined by the solution of:

\[
\min \ E_0^k \left( \bar{U}_0^k (\bar{c}_0^k) \right) - W_0^k = E_0^k u_0^k \left( \frac{W_0^k}{\bar{\pi}_0 (\bar{C}_0)} \right) - W_0^k, \ k = 1, 2, ... n
\]

where expectations \( E_0^k \) are agents individual expectations regarding the market price \( \bar{\pi}_0 (\bar{C}_0) \). Let the marginal price the agent is willing to pay for consumption be \( \bar{\pi}_0^k = \frac{\partial u_0^k}{\partial c_0^k} \), then the price of consumption is unique and defined by:

\[
1 = E_0^k \left( \frac{\bar{\pi}_0^k}{\bar{\pi}_0 (\bar{C}_0)} \right), \bar{\pi}_0 (\bar{C}_0) = \bar{\pi}_0 \left( \sum_{k=1}^{n} \frac{W_0^k}{\bar{\pi}_0 (\bar{C}_0)} \right)
\]

**Proof:** The proof is a straightforward exercise and therefore neglected (see also the proof of proposition 2 which is more general.).

This result is a straightforward outcome of the utility maximization, with respect to consumption expenditure. The price however is random, a function of all other agents decisions since each reaches independently the decision to consume. Note that the price is then an implicit function. Of course, if the marginal cost of supply defines the price of supply at \( \bar{\pi}_0 (\bar{S}_0) \), \( \bar{S}_0 = \bar{C}_0 \) or if aggregate demand is rationed to \( \bar{C}_0 \), then the consumption price is defined externally by \( \bar{\pi}_0 (\bar{S}_0) \) and \( \bar{\pi}_0 (\bar{C}_0) \). Similarly, rationed prices are \( \bar{\pi}_0 \equiv \bar{\pi}_0 \). Regulation of prices and or aggregate consumption has thus a direct effect on both individual consumption and prices. In addition, consumption and price uncertainty has a direct effect on expected consumption. Assume a Taylor series expansion for an agent \( k \) random variable
\[
E^k(\tilde{c}_o^k) = \left( \frac{W_o^k}{\tilde{x}_o(\tilde{C}_0)} \right) \approx \left( \frac{W_o^k}{\tilde{x}_o(\tilde{C}_0)} \right) + \left( \frac{W_o^k}{\tilde{x}_o(\tilde{C}_0)} \right)^3 \text{var} \left( \tilde{x}_o(\tilde{C}_0) \right)
\]

Where \( \tilde{x}_o(\tilde{C}_0) = \frac{\tilde{W}_o}{\tilde{C}_0} \) and therefore, an agent’s price estimate depends on all agents consumptions and estimates of aggregate financial expenditure and aggregate demand for consumption. An approximate Taylor series expansion yields a price expectation for agent \( k \) given by the following:

\[
E^k(\tilde{x}_o(\tilde{C}_0)) = E^k \left( \frac{\tilde{W}_o}{\tilde{C}_0} \right) = \frac{\tilde{W}_o}{\tilde{C}_0} - \frac{E^k(\tilde{W}_o(\tilde{C}_0 - \tilde{C}_0))}{\tilde{C}_0^2} + \frac{E^k(\tilde{W}_o(\tilde{C}_0 - \tilde{C}_0)^2)}{\tilde{C}_0^3}
\]

which is of course a system of equations based on agents’ information and expectations of each other financial expenditures and their consumption as well as their co-variations. By the same token, setting \( \tilde{p}_0^k = \frac{W_o^k}{W_o^k + W_o^{nk}} \) to be an agent estimate of the aggregate financial expenditure, then a first order approximation yields:

\[
E^k(\tilde{p}_0^k) = \tilde{p}_0^k \approx \frac{W_o^k}{W_o^k + W_o^{nk}} \left( 1 + \frac{\text{var}_o^k(\tilde{W}_o^{nk})}{\left( W_o^k + W_o^{nk} \right)^2} \right), \quad k = 1,2,...,n
\]

Calculating the variance of \( \tilde{p}_0^k \) may also provide a two-moment Beta (discrete) probability distribution approximation to an agent financial expenditure share. For example, if the utility of agent \( k \) is logarithmic and given by \( u_k^o(\tilde{c}_o^k) = \alpha_k \ln(\tilde{c}_o^k) \), then \( \frac{W_o^k}{\alpha_k} = E^k \left( \tilde{x}_o(\tilde{C}_0) \right) \) in which case, (since \( \frac{W_o^k}{\tilde{c}_0^k} = \tilde{x}_o(\tilde{C}_0) \)) the price that agent \( k \) expects to pay \( \frac{W_o^k}{\alpha_k} \), is a function of its utility. The price variance \( \text{var}_o^k(\tilde{x}_o(\tilde{C}_0)) \) cannot of course be neglected if the kth agent estimate of aggregate financial expenditure had a large variance. The index of the Arrow-Pratt risk aversion of such an agent is then \( A_k = 1/\tilde{c}_0^k = \tilde{x}_o(\tilde{C}_0)/W_o^k \) which clearly points out to an increasing risk aversion to wealth poor agents, and vice versa for wealthier agent. In this sense, if a game was to be conceived between a very large agent (say a TBTF bank) and an individual investor, the “TBTF bank” is most likely risk neutral since its index of risk aversion will most likely be neutral. A similar conclusion can be reached if we consider an exponential utility \( u_k^o(\tilde{c}_o^k) = \frac{1}{\nu_k} (\tilde{c}_o^k)^{\nu_k} \).

Maximization of: \( \max_{\tilde{p}_0^k} E^o_k \left( \frac{W_o^k}{\tilde{x}_o(\tilde{C}_0)} \right)^{\nu_k} - W_o^k \) yields \( E^o_k \left( \frac{\tilde{x}_o^k}{\tilde{x}_o(\tilde{C}_0)} \right) \) where \( \tilde{x}_o^k = \left( \frac{W_o^k}{\tilde{x}_o(\tilde{C}_0)} \right)^{\nu_k-1} \) while the Arrow Pratt index of risk aversion is \( A_k = \frac{\partial \tilde{x}_o^k/\tilde{c}_0^k}{\tilde{x}_o(\tilde{C}_0)} = \left( 1 - \nu_k \right) \left( \frac{\tilde{x}_o(\tilde{C}_0)}{W_o^k} \right) \). The resulting Cournot market price will again be a function of all agents’ wealth and their index of implied risk aversion. A multi-agent (MA) Cournot game however, differs from the Cournot game we have considered above. Although if the number of agents is very large, none of the agents able to alter the market price, then a
Proposition 2: A Multi-Agent Cournot Game: A one period problem

Let \( \tilde{p}_0^k = W_k^0 / \tilde{W}_0^k \), \( \tilde{W}_0 = W_0^k + \tilde{W}_0^nk \) and let \( (W_0^k, \tilde{W}_0^nk) \) be the consumption expenditures of a specific agent \( k \) and all other agents, with \( \tilde{W}_0^nk = \sum_{i\neq k} \tilde{W}_0^i \). Further we let \( \tilde{\partial} c_0^k (c_0^k) / \tilde{\partial} c_0^k = \pi_0^k \) and \( \tilde{\pi}_0 (\tilde{C}_0) = \tilde{\pi}_0 (\tilde{c}_0^k + \tilde{C}_0^nk) \), \( \tilde{c}_0^nk = \sum_{i\neq k} \tilde{c}_0^i \) be the market price of consumption (where aggregate supplies \( \tilde{S}_0 \) equal aggregate demand \( \tilde{C}_0 = \tilde{S}_0 \)). The multi-agent Cournot consumption game with an agent \( k \) consumption expenditure \( W_0^k \) and its synthetic agent expenditure \( \tilde{W}_0^nk \) defines a non-zero sum game, defined by:

\[
\begin{align*}
\text{Max} \ E_0^k U_0^k (c_0^k) &= E_0^k u_0^k \left( \frac{W_0^k}{\tilde{\pi}_0 (\tilde{c}_0^k + \tilde{C}_0^nk)} \right) - W_0^k \quad \text{For agent } k \\
\text{Max} \ E_0^{nk} U_0^{nk} (C_0^{nk}) &= E_0^{nk} u_0^{nk} \left( \frac{W_0^{nk}}{\tilde{\pi}_0 (\tilde{c}_0^k + \tilde{C}_0^nk)} \right) - W_0^{nk} \quad \text{For all other agents } nk
\end{align*}
\]

where expectations \( E_0^k \) and \( E_0^{nk} \) are agents probability measures, taken with respect to both agents information regarding prices and other agents assumptions.

(a) The One Period Pricing Model for all Agents:

\[ 1 = E_0^k \left( (1 - \tilde{p}_0^k) \frac{\tilde{c}_0^k}{\tilde{\pi}_0 (\tilde{c}_0^k + \tilde{C}_0^nk)} \right), \quad \tilde{p}_0^k = \frac{W_0^k}{W_0^k + \tilde{W}_0^nk}, \quad k = 1, \ldots, n \]

(b) For a specific agent \( k \), and is synthetic agent, assumed to be a risk neutral agents:

\[ 1 = E_0^{nk} \left( (1 - \tilde{p}_0^{nk}) \frac{\tilde{c}_0^{nk}}{\tilde{\pi}_0 (\tilde{c}_0^k + \tilde{C}_0^nk)} \right) \]

Proof: The proof is straightforward. For agent \( k \) committing \( W_0^k \) to consumption, we have:

\[ 1 = E_0^k \left( \frac{\tilde{\partial} u_0^k}{\tilde{\partial} c_0^k} \frac{\tilde{c}_0^k}{\tilde{\partial} W_0^k} \right), \quad \tilde{c}_0^k = W_0^k \left( \tilde{c}_0^k + \tilde{C}_0^nk \right) = W_0^k \left( \tilde{c}_0^k + \tilde{C}_0^nk \right) = \frac{W_0^k}{\tilde{\pi}_0}. \]

Further, \( \frac{\tilde{\partial} c_0^k}{\tilde{\partial} W_0^k} = \frac{1}{\tilde{\pi}_0} \left( \frac{\tilde{W}_0^{nk}}{W_0^k + \tilde{W}_0^{nk}} \right) = \frac{(1 - \tilde{p}_0^k)}{\tilde{\pi}_0} \) and

and \( 1 = E_0^{nk} \left( (1 - \tilde{p}_0^{nk}) \frac{\tilde{c}_0^{nk}}{\tilde{\pi}_0} \right). \) Of course, if for an agent \( k \), its synthetic agent representing “all” other agents—i.e. the consumption market excepting the kth agent, then as note above the index of such a synthetic agent will be smaller. This assumption may be used as an approximation to facilitate the equilibrium price by agent \( k \) and therefore his financial expenditure for consumption. Thus,
\[
\max E \left( \frac{W_{nk}}{\tilde{\pi}_0(\tilde{c}_0 + \tilde{C}^{nk})} \right) - W_{nk}, \text{ or } 1 = E \left( 1 - \frac{W_{nk}}{\tilde{\pi}_0(\tilde{c}_0 + \tilde{C}^{nk})} \right) = E \left( \frac{\tilde{\pi}_0(\tilde{c}_0 + \tilde{C}^{nk})}{\tilde{\pi}_0(\tilde{c}_0 + \tilde{C}^{nk})} - W_{nk} \right)
\]

However, \( \tilde{\pi}_0(\tilde{c}_0 + \tilde{C}^{nk}) = \frac{W_{nk} + \tilde{W}_{nk}}{\tilde{c}_0 + \tilde{C}^{nk}} \) and therefore,

\[
1 = E \left( 1 - \frac{W_{nk}}{\tilde{\pi}_0(\tilde{c}_0 + \tilde{C}^{nk})} \right) = E \left( \frac{\tilde{\pi}_0(\tilde{c}_0 + \tilde{C}^{nk})}{\tilde{\pi}_0(\tilde{c}_0 + \tilde{C}^{nk})} - W_{nk} \right)
\]

Q.E.D

If we were to compare our results, then,

\[
1 = E^k \left( \frac{\tilde{\pi}_0^k}{\tilde{\pi}_0^C(\tilde{C}_0)} \right) \text{ for an agent } k \text{ in a Cournot game}
\]

\[
1 = E^k \left( (1 - \tilde{p}_0^k) \frac{\tilde{\pi}_0^k}{\tilde{\pi}_0^MAC(\tilde{C}_0)} \right) \text{ for an agent } k \text{ in a multi-agent Cournot game}
\]

\[
1 = E \left( \frac{1 - p_{nk}^k}{\tilde{\pi}_0^MAC(\tilde{C}_0)} \right) \text{ for a risk neutral synthetic agent } nk \text{ in a multi-agent game}
\]

Comparing the Cournot and the Multi-Agent Cournot game we have:

\[
1 = E^k \left( (1 - \tilde{p}_0^k) \frac{\tilde{\pi}_0^k}{\tilde{\pi}_0^MAC(\tilde{C}_0)} \right) = E^k \left( \frac{\tilde{\pi}_0^k}{\tilde{\pi}_0^C(\tilde{C}_0)} \right)
\]

And therefore,

\[
E^k \left[ \frac{\tilde{\pi}_0^k - 1}{\tilde{\pi}_0^MAC(\tilde{C}_0) - \tilde{\pi}_0^MAC(\tilde{C}_0)} \right] = 0 \text{ since } \tilde{\pi}_0^k \neq 0, \tilde{\pi}_0^C(\tilde{C}_0) = \tilde{\pi}_0^MAC(\tilde{C}_0) \text{ and } (1 - \tilde{p}_0^k) < 1
\]

\( \tilde{\pi}_0^MAC(\tilde{C}_0) < \tilde{\pi}_0^C(\tilde{C}_0) \). Namely, for an agent \( k \), the price is smaller than the Cournot price \( \tilde{\pi}_0^C(\tilde{C}_0) \). The difference is smaller the greater the agent consumption expenditure. Namely a price premium for the MAC price for an agent \( k \) is \( \tilde{\pi}_0^C(\tilde{C}_0) - \tilde{\pi}_0^MAC(\tilde{C}_0) = p_0^k \tilde{\pi}_0^C(\tilde{C}_0) \), which is greater the greater the consumption expenditure. In a Cournot game, wealth is implied in the marginal utility of the agent and in the Cournot game price over which the agent has a negligible impact while in the MAC game, the premium of an agent is greater the greater his consumption expenditure. Explicitly, consider two agents, \( k \) and \( \ell \) then their relative premium are proportional to \( p_0^k / p_0^\ell \). Namely, the greater \( p_0^k \) relative to \( p_0^\ell \) the greater the premium of the \( k \)th agent. In a MAC with a large number of agents, with few with very large consumption expenditures, the latter may profit from an extremely large premium.
Further, agents’ wealth and information (and their power) also matter. For agents with partial information regarding wealth (and thus consumption of other agents), we have instead a random price of consumption for the kth agent which is a function of $\tilde{w}^k_0$—other agents financial consumption expenditures. Each agent decision will also depends on what he knows and the other know and what he does rather than just on what the other wants. For example, say that an agent knows only how much he will spend on consumption, and let other agents’ have a mean consumption expenditure $\tilde{w}^n_0$. In this case, $\tilde{p}^k_0 = \frac{W^k_0}{W^k_0 + \tilde{w}^n_0}$ and an approximate three terms Taylor series expansion about the mean, yields an expected financial share of consumption of others mean and variance financial expenditures, or:

$$E^k_0 (\tilde{p}^k_0) = p^k_0 \approx \frac{W^k_0}{W^k_0 + \tilde{w}^n_0} \left( 1 + \frac{\text{var}^k (\tilde{w}^n_0)}{(W^k_0 + \tilde{w}^n_0)^2} \right), \quad k = 1, 2, \ldots, n$$

As a result, the greater agent’s k variance of aggregate (market) wealth, the greater the spread between the price the agent k pays and the price he would be willing to pay for such consumption. Thus information augments the advantage of better informed agent (and therefore the price premium). These observations carry-over to a multi-period framework, albeit, in a more complex form.

3. The Multi Periods Consumption Problem and the CCAPM

The multi-periods Cournot as well as the MAC problems differ from the single period as the agent consumption expenditures considers both savings and their investment for future consumption. An agent whose initial wealth is $W^k_t$ and savings $\Lambda^k_t$ has a consumption expenditure $W^k_t - \Lambda^k_t$ while savings are invested in a portfolio with rate of return $\tilde{R}^k_t (a^k_0)$ (with $0 \leq a^k_0 \leq 1$ denoting his proportional investment in bonds at a known risk free rate). The next period wealth is then $W^k_{t+1} = \Lambda^k_t \left( 1 + \tilde{R}^k_{t+1} (a^k_0) \right), \quad W^k_0 > 0$ while the next period consumption is:

$$W^k_{t+1} = \Lambda^k_t \left( 1 + \tilde{R}^k_{t+1} (a^k_0) \right), \quad W^k_0 > 0$$

For a MAC game, the kth agent set invested savings $\Lambda^k_t$ for future consumption while $W^k_t - \Lambda^k_t$ is consumed in the second (last) period. As a result,

$$\hat{c}^k_t = \frac{W^k_{t+1} - \Lambda^k_{t+1}}{\tilde{\pi}^k_t \left( \hat{c}^k_T + \tilde{C}^{nk}_t \right)} \quad \text{and at a final time } T, \quad \hat{c}^k_T = \frac{\tilde{w}^k_T}{\tilde{\pi}^k_T \left( \hat{c}^k_T + \tilde{C}^{nk}_T \right)}, \quad W^k_T = 0$$

Further, since all agents pay the same price we have:

The following consumption problem for an agent k results:

$$EU^k_t (W^k_t) = \max_{\Lambda^k_t, \tilde{C}^{nk}_t} \left( \frac{W^k_t - \Lambda^k_t}{\tilde{\pi}^k_t \left( \hat{c}^k_T + \tilde{C}^{nk}_T \right)} \right) + \beta EU^k_{t+1} (W^k_{t+1}), \quad W^k_0 > 0$$

and $T^k = \inf \{ t > 0, W^k_t = 0 \}$ at which time $c^k_T = \frac{\tilde{w}^k_T}{\tilde{\pi}^k_T \left( \hat{c}^k_T + \tilde{C}^{nk}_T \right)}$. 

9
We proceed in a similar way for a synthetic agent. Although, each agent solving his game will depend on all other’s solution, all of which depend on all others. As a result, the simultaneous solutions of these games provide an equilibrium solution.

Further, since all agents pay the same price for consumption, we have:

\[
\frac{c^k_{t+1}}{C^k_{t+1}} = \frac{\lambda^k_t (1 + \bar{R}^k_{t+1}(a^k_0))}{\Lambda^k_t (1 + \bar{R}^k_{t+1}(a^k_0))}, \quad \lambda^k_t (1 + \bar{R}^k_{t+1}(a^k_0)) = \sum_{j \neq k} \lambda^j_t (1 + \bar{R}^j_{t+1}(a^j_t))
\]

Equivalently, set \( \tilde{W}^k_t = \lambda^k_0 (1 + \bar{R}^k_0(a^k_0)) \) and \( \tilde{W}^nk_t = \lambda^nk_0 (1 + \bar{R}^nk_0(a^nk_0)) \), to be the agents’ wealth prior to consumption in the second period, then:

\[
EU^k_0(\Lambda^k_0) = \max_{\lambda^k_0} u_k \left( \frac{W^k_0 - \lambda^k_0 \tilde{C}^nk_0}{W^nk_0 - \lambda^nk_0 \tilde{C}^nk_0} \right) + \beta EU^k_1 \left( \frac{\tilde{W}^nk_1}{\tilde{W}^nk_1} \tilde{C}^nk_1 \right).
\]

\[
EU^nk_0(\lambda^nk_0) = \max_{\lambda^nk_0} u^nk_k \left( \frac{W^nk_0 - \lambda^nk_0 \tilde{C}^nk_0}{W^nk_0 - \lambda^nk_0 \tilde{C}^nk_0} \right) + \beta EU^nk_1 \left( \frac{\tilde{W}^nk_1}{\tilde{W}^nk_1} \tilde{C}^nk_1 \right).
\]

The solution of these consumption problem results in a number of multi-agent CCAPM pricing models summarized by the following proposition with proofs in the appendix. Subsequently examples are considered for brevity. The pricing results outlined below will be applied to two consecutive periods with an example of a three period problem to highlight some of the technical difficulties encountered in selecting an optimal saving policy.

**Proposition 3: The Agents Martingale**

(1) Let \( E^k_0(\cdot) \) be an agent \( k \) expectation with respect to a filtration \( \mathcal{F}_t^k \) at time \( t=0 \) and let

\[
\frac{\partial c^k_t}{\partial c^k_t} = \pi^k_t
\]

be the price agent \( k \) is willing to pay for consumption at a given time \( t \) defined by its marginal utility of consumption. Let \( \tilde{W}_t = \sum_{j=1}^n \tilde{W}_j^k \), \( \tilde{\lambda}_t = \sum_{j=1}^n \tilde{\lambda}_j^k \) while the market price is \( \pi_t = \tilde{W}_t / \tilde{C}_t \). Similarly, future wealth of an agent \( k \) at time \( t+1 \) is

\[
\tilde{W}^k_{t+1} = \lambda^k_t (1 + \bar{R}^k_{t+1}(a^k_t))
\]

while at the terminal time (in this case, period 2),

\[
\tilde{W}^k_T = \lambda^k_{t-1} (1 + \bar{R}^k_{t}(a^k_{t-1}))
\]

which is consumed entirely at a price \( \tilde{\pi}_t = \tilde{W}_t / \tilde{C}_t \). For notational convenience set \( \tilde{p}^k_t \) to be the kth agent share of aggregate wealth at \( t \) and \( q^k_t \) be an agent \( k \)’s proportional wealth saved for future consumption, or:

\[
\tilde{p}^k_t = \tilde{W}^k_t / \tilde{W}_t, \quad q^k_t = \frac{\lambda^k_t}{\tilde{p}^k_t} = \frac{\tilde{W}^k_t}{\tilde{W}^k_t}, \quad \text{while aggregate wealth at } t \text{ is:}
\]

\[
\tilde{W}_t = \left( \sum_{j=1}^n \tilde{p}^k_{t-1} (1 + \tilde{R}^k_{t}(a^k_{t-1})) \right) + \sum_{j=1}^n \tilde{q}^k_{t-1} \tilde{p}^k_{t-1} (1 + \tilde{R}^k_{t}(a^k_{t-1})) \tilde{W}_{t-1}, \quad t=1,2,....
\]
where $a^k_{t+1}$ is an agent k’s proportional investment in a risk free rate of return bond.

The following results:

(2) Each agent, has a pricing Martingale which is specific to the price he is willing to pay for consumption, the information he has regarding other agents and the investment policy he adopts. The following random variable for any agent k, define a Martingale. Explicitly, for any period of time the following holds:

$$E^k_t \left( \frac{\pi^k_t}{\pi_t(C_t)} \left( 1 + \frac{p_t^k (1-q_t^k)}{p_t^{ak}} \right) \right) = E^k_t \left( \frac{\pi_{t+1}^k}{\pi_{t+1}(C_{t+1})} \left( 1 + \tilde{R}_{t+1}^k \left( a_t^k \right) \right) \right)$$

As a result, series of random variables at any time $t<T$ define a Martingale

$$(\pi_t^k \left( \frac{1}{\pi_t(C_t)} \left( 1 + \frac{p_t^k (1-q_t^k)}{p_t^{ak}} \right) \right) \text{ and at the kth agent at } T, \left( \frac{\pi^k_t}{\pi_t(C_t)} \left( 1 + \frac{p_t^k}{p_t^{ak}} \right) \right)$$

(3) Next assume for notational convenience that $M^k_{t+1} = \pi_{t+1}^k / \pi_t^k$ is a Kernel and $1 + \tilde{\eta}_{t+1} = \pi_{t+1}(C_{t+1}) / \pi_t(C_t)$ is the inflation rate,

$$1 = E^k_t \left( \frac{M^k_{t+1} \left( 1 + \tilde{R}_{t+1}^k \left( a_t^k \right) \right)}{1 + \tilde{\eta}_{t+1}} \right)$$

And generally, at time $t=0$, we have a pricing model for time t,

$$1 = E^k_0 \left( \frac{M^k_{t+1,0} \left( 1 + \tilde{R}_{t+1}^k \left( a_0^k \right) \right)}{1 + \tilde{\eta}_{t+1,0}} \right)$$

$$1 + \tilde{\eta}_{t+1,0} = \frac{\pi_{t+1}(C_{t+1})}{\pi_0(C_0)}; \quad M^k_{t+1,0} = \frac{\pi_{t+1}^k}{\pi_0^k}$$

(4) The price of consumption $\pi_t \left( C_t \right)$ at any time t is unique defined by an equilibrium in aggregate demand for consumption. In this sense while savings for consumption and investors strategies are set individually by each agent, the price of consumption and aggregate consumption are random.

**Proof:** See Appendix 1
A number of cases arise. For example, say that an agent has negligible share of wealth, then \( p_{t+1}^k = 0 \), as a result,

\[
1 = E_t^k \left( M_{t+1}^k \left( \frac{1 + \tilde{R}_{t+1}^k (a_t^k)}{1 + \eta_{t+1}} \right) | \mathcal{F}_t \right)
\]

which is the standard kernel pricing CCAPM model with inflation adjusted rates of returns. If an agent is invested in a risk free rate TIPS, then \( a_t^k = 1 \) and therefore, \( \tilde{R}_{t+1}^k (1) = R_f^k \), the risk free rate, in which case:

\[
\left( \frac{1}{1 + R_f^{kp}} \right) = E_t^k \left( M_{t+1}^k \left| \mathcal{F}_t \right. \right)
\]

and therefore, setting a inflation adjusted probability measure, we have:

\[
M_{t+1}^{*,k} = \left( \frac{M_{t+1}^k}{E_t^k \left( M_{t+1}^k | \mathcal{F}_t \right) } \right)
\]

which defines under a probability measure the kth agent pricing kernel a multi-period pricing model.

Next consider an agent with a consequential wealth, with \( p_{t+1}^k \neq 0 \). In this case, at time t, (with \( p_{t+1}^{*,k} \) replaced by \( 1 - p_t^k \))

\[
1 = \left( \frac{1}{1 + (1 - p_t^k)(1 - q_t^k)} \right) E_t^k \left( M_{t+1}^k \left( \frac{1 + \tilde{R}_{t+1}^k (a_t^k)}{1 + \eta_{t+1}} \right) \left( 1 + \tilde{p}_{t+1}^k \left( 1 - \tilde{q}_{t+1}^k \right) \right) \left( 1 - \tilde{p}_t^k \right) \left( 1 - \tilde{q}_t^k \right) \right) \left| \mathcal{F}_t \right)
\]

Reinserting \( \tilde{p}_t^k = \frac{\tilde{W}_t^k}{W_t} \), \( q_t^k = \frac{\Lambda_t^k}{p_t^k W_t} \), and setting \( \xi_{t+1,j}^k = \left( 1 + \frac{W_{t+1}^k - \Lambda_{t+1}^k}{W_{t+1} - \Lambda^k_{t+1}} \right) \left( 1 + \frac{W_t^k - \Lambda_t^k}{W_t^k - \Lambda^k_t} \right) \) for greater notational clarity,

\[
1 = E_t^k \left( M_{t+1}^k \xi_{t+1,j}^k \left( \frac{1 + \tilde{R}_{t+1}^k (a_t^k)}{1 + \eta_{t+1}} \right) \right) \left| \mathcal{F}_t \right)
\]

On the return side, we note that there are two factors, the inflation adjusted rates of returns and \( \xi_{t+1,j}^k \) -- a ratio of relative wealth consumption expenditure growth. If an agent is increasingly wealthier compared to all other agents, then \( \xi_{t+1,j}^k > 1 \) then that ratio provides a premium to the agent since he would demand “more returns” at time t+1 for $1 invested at time t. When \( \xi_{t+1,j}^k = 1 \), we have a pricing model equivalent to that considered above. Finally, for \( \xi_{t+1,j}^k < 1 \), the agent has increasingly less disposable financial wealth for consumption expenditure and will therefore require greater returns to compensate the loss of wealth. To do so, the agent will assume more risks. In this sense, the valuation of returns is necessarily a function of variations in relative wealth. An agent spending on consumption initially less then subsequently, will seek to recuperate greater returns by paying the same $1 which means assume more risk. And vice versa, when financial consumption increases over time, the agent will be satisfied with smaller rates of returns (i.e. rebalance the portfolio to be less risky). Finally, it is worth noting that that for an agent keeping his financial consumption expenditure in line with the other agents, then \( \xi_{t+1,j}^k = 1 \) while an agent increasing and decreasing his financial
consumption expenditures will price returns differently over time. Namely increasingly assuming greater risks to augment future returns and at other times, assuming less risk.

Essential questions we may be concerned with are of course, to what extent wealth inequalities increase, or the rich getting richer. First, these depend on the growth rate \( \Delta \tilde{P}_t \). Second, they depend, on the effects of savings on agents’ wealth? Is the propensity to save \( \tilde{q}_t^k \) greater the richer an agent? The financial consumption model considered above has essentially pointed out to the effect of financial consumption changes over time and not to wealth (albeit, both depend). Third, is there a relationship between an agent’s wealth, his propensity to save and the agent’s investment strategy \( \tilde{a}_t^k \)? A tentative answer is as noted above, embedded in the rate of growth in financial consumption expenditure. These are complex issues that are function of numerous factors but their integration in a financial pricing model may provide some insights on these fundamental economic, financial and social issues. Due to the complexity of these game pricing models, a solution is necessarily numerical.

Further, in the financial CCAPM model, the ratio \( \left( \frac{\tilde{M}_t^k}{1 + \tilde{\eta}_t (\tilde{C}_t)} \right) \) may be interpreted as the relative measure of an agent willingness to pay to the price it will pay for consumption. Such ratio depends on the agent’s disposable wealth which requires that CCAPM and other models be far more aware of the effects of wealth in such models. A price for future returns will then be a function of such an agent want in addition to the effects of financial expenditures growth, or:

\[
1 = E_{\tilde{a}^k_{t-1}, \tilde{p}^k_{t-1}} \left( \frac{\tilde{M}_t^k}{1 + \tilde{\eta}_t (\tilde{C}_t)} \right) \tilde{R}_t^k \left( \tilde{a}_t^k \right) \tilde{C}_t^k, k = 1, 2, \ldots, n
\]

An agent financial strategy is of course an important factor to account for in a CCAPM model. If an investor is invested long, then \( \tilde{a}_t^k = \tilde{a}_t^k \) while for active investor, this investment parameter will change continuously as new information (and perceived new opportunities regarding future returns) are identified. Changing one’s portfolio is costly however and thus, will dampen variations of a portfolio composition. By the same token, an agent proportional wealth \( \tilde{p}_t^k \) will vary over time, defined as a function of his savings, prices and his investment strategy. As a result, in a multi agent framework, pricing is far more difficult to assess. If the rich get richer and the poor becomes poorer, such a process in fact ought to accelerate with a poorer agent, reaching a terminal wealth-less state much faster than linear rates of economic inequalities might indicate. Further, our interpretation of the pricing kernel is that of a ratio of what an agent will be willing to pay for consumption and what he will pay for it, may not be in fact a discount pricing probability measure. Rather, it is a predictor of financial trades. If an agent would be willing to pay more than another agent for a future market price, he may in fact be willing to buy that future from the agent. And vice versa if the agent would be willing to pay less than the market price another agent will be willing to pay, the agent might be a seller. This approach concurs with a Cournot price as economics theory suggests and resulting from what agents do rather than what their expectations may be. For example, since consumption is a function of an agent’s wealth, the poorer that agent becomes, the greater his needs and the more he will be willing to pay, accelerating further he rate at which
economic inequalities increase. At the final time, say T, 
\( q^T_i = 0, q^T_i = 0 \) and thus the agent will join a
growing population that consumes their current earnings (in case they have such earnings or consume
transfer payments). In other words, such agents are no longer “financial agents” and the number of wealth–based consuming agents is reduced. Thus, the number of agents, in a multi-agent financial pricing model
is necessarily dynamic, changing over time as agents gradually depart from the circle of wealth-endowed
agents.

**Example: A two period problem**

Assume two times \( t=0 \) and the terminal time \( t=1 \), then:

\[
E^k_0 \left( \frac{\pi^k_0}{\pi_0 (C_0)} \right) \mathfrak{S}^k_0 = E^k_0 \left( \frac{\pi^k_i}{\pi_1 (C_1)} \right) \left( 1 + \tilde{R}^k_i (a^k_i) \right) \left( 1 + \sum_{i=k}^{n} \Lambda^k_i \left( 1 + \tilde{R}^k_i (a^k_i) \right) \right) \left( \frac{1 + \sum_{i=k}^{n} \Lambda^k_i \left( 1 + \tilde{R}^k_i (a^k_i) \right)}{\frac{1}{1 + \sum_{i=k}^{n} (W^k_i - \Lambda^k_i)} \right) \]  

Or

\[
1 = E^k_0 \left( M^k_1 \left( 1 + \tilde{R}^k_i (a^k_i) \right) \right) \left( 1 + \sum_{i=k}^{n} \Lambda^k_i \left( 1 + \tilde{R}^k_i (a^k_i) \right) \right) \left( \frac{1 + \sum_{i=k}^{n} \Lambda^k_i \left( 1 + \tilde{R}^k_i (a^k_i) \right)}{\frac{1}{1 + \sum_{i=k}^{n} (W^k_i - \Lambda^k_i)} \right) \]  

If we invest in TIPS with rate of return \( R^\text{TIPS}_f \) then:

\[
1 = \frac{\left( 1 + R^\text{TIPS}_f \right) \sum_{i=k}^{n} (W^k_i - \Lambda^k_i)}{\sum_{i=k}^{n} (W^k_i - \Lambda^k_i)} E^k_0 \left( M^k_1 \left( 1 + \sum_{i=k}^{n} \Lambda^k_i \left( 1 + R^\text{TIPS}_f \right) \right) \right) \left( \frac{1 + \sum_{i=k}^{n} \Lambda^k_i \left( 1 + \tilde{R}^k_i (a^k_i) \right)}{\frac{1}{1 + \sum_{i=k}^{n} (W^k_i - \Lambda^k_i)} \right) \]  

which is a function of all other agents’ investment strategy. A pricing model will all agents investing in
TIPS can however be constructed, in which case,

\[
\frac{\sum_{i=k}^{n} (W^k_i - \Lambda^k_i) \sum_{i=k}^{n} \Lambda^k_i}{\left( 1 + R^\text{TIPS}_f \right) \sum_{i=k}^{n} (W^k_i - \Lambda^k_i) \sum_{i=k}^{n} \Lambda^k_i} = E^k_0 \left( M^k_1 \right) \left( \mathfrak{S}^k_0 \right) \]  

and therefore

\[
1 = \frac{\sum_{i=k}^{n} \Lambda^k_i}{\left( 1 + R^\text{TIPS}_f \right) \sum_{i=k}^{n} \Lambda^k_i} E^M^k_0 \left( 1 + \tilde{R}^k_i (a^k_i) \right) \left( \frac{1 + \sum_{i=k}^{n} \Lambda^k_i \left( 1 + \tilde{R}^k_i (a^k_i) \right)}{\frac{1}{1 + \sum_{i=k}^{n} (W^k_i - \Lambda^k_i)} \right) \left( \mathfrak{S}^k_0 \right) \]  

\[14\]
Note that such an equation provides also a solution to an agent proportional investment, say $\theta^k_0 = \Lambda^k_0 / \sum_{i=k} \Lambda^i_0$, since:

$$\frac{1}{1 - \theta^k_0} = \frac{1}{(1 + R^\text{i}_{j_{\text{inv}}})} E_{0}^{M^j_{0}} \left( \frac{1 + \tilde{R}^i_k (a^i_k)}{1 + \eta_i} \right) \left( 1 + \frac{\theta^k_0 (1 + \tilde{R}^i_k (a^i_k))}{\sum_{i=k} \theta^i_0 (1 + \tilde{R}^i_k (a^i_k))} \right) [\tilde{\xi}^k_0]$$

In this case, the effects of wealth are implied in probability measure of the agent while the market price of future returns is a function of what the agent invests proportionately to other agents and of course their investment strategy. Explicitly, say that all agent invest in a common market providing a common rate of return $\tilde{r}_j$, then for an agent $k$, $1 + \tilde{R}^i_k (a^i_k) = a^i_k (1 + \tilde{r}_j) + (1 - a^i_k) (1 + \tilde{r}_i) = 1 - a^i_k (\tilde{r}_i - \tilde{r}_j)$ where $\tilde{\lambda}_i = (\tilde{r}_i - \tilde{r}_j)$ is a risk premium. In this case

$$\frac{1}{1 - \theta^k_0} = \frac{1}{(1 + R^\text{i}_{j_{\text{inv}}})} E_{0}^{M^j_{0}} \left( \frac{1 - a^i_k \tilde{\lambda}_i}{1 + \eta_i} \right) \left( 1 + \frac{\theta^k_0 - a^i_k \theta^k_0 \tilde{\lambda}_i}{(1 - \theta^k_0) - \tilde{\lambda}_i \sum_{i=k} \theta^i_0 a^i_0} \right) [\tilde{\xi}^k_0]$$

Considering the term in the parenthesis $E_{0}^{M^j_{0}} (\cdot)$, we have:

$$\frac{\partial}{\partial a^i_0} E_{0}^{M^j_{0}} (\cdot) = -E_{0}^{M^j_{0}} \left( \frac{\tilde{\lambda}_i}{1 + \eta_i} \right) \left( 1 + \frac{2 \theta^k_0 (1 - a^i_k \tilde{\lambda}_i)}{(1 - \theta^k_0) - \tilde{\lambda}_i \sum_{i=k} \theta^i_0 a^i_0} \right) [\tilde{\xi}^k_0]$$

In other words, there is a loss in return premium when augmenting investments in risk free assets if the following equation holds: $1 + \theta^k_0 (1 - 2 a^i_k \tilde{\lambda}_i) > \tilde{\lambda}_i \sum_{i=k} \theta^i_0 a^i_0$. For example, if $a^i_0 = 1$ (an all bonds investment), then $1 + \theta^k_0 (1 - 2 \tilde{\lambda}_i) > \tilde{\lambda}_i \sum_{i=k} \theta^i_0 a^i_0$ while if the agent $k$ pursues an all equity investment, we have:

$1 + \theta^k_0 > \tilde{\lambda}_i \sum_{i=k} \theta^i_0 a^i_0$. Dependence resulting from all agents investing in a common equity index for example was not considered as it is likely to change the price and therefore the rate of returns of equity and bond prices in a far more complicated manner. Such an issue is not in the scope of the paper and may at best be assessed through an extensive simulation of assets prices in a dependent environment.

**Example: Debt, lending and the price of credit**

An extension to include borrowing and lending among agents is well defined in terms of an initial financial transaction with its current and future consequences for both agents. In this case, let there be two agents, one borrower (with index B) and the other a lender (with index L). Let $b_{L,B}$ is the amount lent by agent L to the borrowing agent B. Based on our previous results, the following two periods consumption model results:
Where current and future consumption for both the borrowing and lending agents are defined by their initial transaction and its terms resulting in the following rates of returns:

\[
1 + \tilde{R}_i^B(a_B) = \frac{\Lambda_0^B}{\Lambda_0^L + \frac{\Lambda_0^B}{1 + W^L_0 - \Lambda_0^L}} \left( 1 + R_i^B + (1 - a_B^B) \left( 1 + 9R_i \right) - b_0^{LB} \left( 1 + R_i^{LB} \right) \right), \quad W^B_0 = w^B_0 + b_0^{LB}
\]

\[
1 + \tilde{R}_i^L(a_L) = \frac{\Lambda_0^L}{\Lambda_0^B + \frac{\Lambda_0^L}{1 + W^B_0 - \Lambda_0^B}} \left( 1 + R_i^L + (1 - a_L^L) \left( 1 + 9R_i \right) + b_0^{LB} \left( 1 + R_i^{LB} \right) \right), \quad W^{nk,B} = W^{nk} - b_0^{LB}
\]

And therefore,

\[
1 = E^B_0 \left( \frac{1 + \tilde{R}_i^B(a_B)}{1 + \eta_i} \right) \left( 1 + \frac{\Lambda_0^B}{\Lambda_0^L + \frac{\Lambda_0^B}{1 + W^L_0 - \Lambda_0^L}} \left( 1 + R_i^B + (1 - a_B^B) \left( 1 + 9R_i \right) - b_0^{LB} \left( 1 + R_i^{LB} \right) \right) \right)
\]

\[
1 = E^L_0 \left( \frac{1 + \tilde{R}_i^L(a_L)}{1 + \eta_i} \right) \left( 1 + \frac{\Lambda_0^L}{\Lambda_0^B + \frac{\Lambda_0^L}{1 + W^B_0 - \Lambda_0^B}} \left( 1 + R_i^L + (1 - a_L^L) \left( 1 + 9R_i \right) + b_0^{LB} \left( 1 + R_i^{LB} \right) \right) \right)
\]

A fair price for debt is thus defined by a solution for $b_0^{LB}$, the amount borrowed and the interest $R_i^{LB}$ that the borrower pays to the lender, both implied in the borrower and in the lender pricing model, each valuing the returns from such a transaction based on its own information and analysis (their filtration) and
the pricing kernel embedding their risk preferences. However if we assume that risk preferences are in fact embedded in the portfolio selection each adopts. First note that by definition, rates of returns are:

$$1 + \tilde{R}^b_t \left( a_o^b \right) = \Lambda^b_o \left( a_o^b \left( 1 + R^f_t \right) + (1 - a_o^b) \left( 1 + \eta_t \right) \right) - b_{0B}^L \left( 1 + R^l_t \right)$$  
$$W_0^B = w_0^B + b_{0B}^L$$

$$1 + \tilde{R}^l_t \left( a_o^l \right) = \Lambda^l_o \left( a_o^l \left( 1 + R^f_t \right) + (1 - a_o^l) \left( 1 + \eta_t \right) \right) + b_{0B}^L \left( 1 + R^l_t \right)$$  
$$W_0^L = w_0^L - b_{0B}^L$$

And for simplicity say that the agents willingness to pay for consumption at times t=0 and times t=1 are the same, in this case, $\tilde{\pi}^B_0 = \tilde{\pi}^L_0 = 1$ and thus,

$$\begin{align*}
1 + \frac{W_0^B - \Lambda^B_0}{W_0^B - \Lambda^B_0} &= E \left[ \left( 1 + \tilde{R}^B_t \left( a_o^b \right) \right) \left( 1 + \frac{\Lambda^B_o \left( 1 + \tilde{R}^B_t \left( a_o^b \right) \right)}{\Lambda^B_0 \left( 1 + \tilde{R}^B_t \left( a_o^b \right) \right)} \right) \right] \\
1 + \frac{W_0^L - \Lambda^L_0}{W_0^B - \Lambda^B_0} &= E \left[ \left( 1 + \tilde{R}^L_t \left( a_o^l \right) \right) \left( 1 + \frac{\Lambda^L_o \left( 1 + \tilde{R}^L_t \left( a_o^l \right) \right)}{\Lambda^L_0 \left( 1 + \tilde{R}^L_t \left( a_o^l \right) \right)} \right) \right]
\end{align*}$$

Inserting $(\tilde{R}^B_t \left( a_o^b \right), \tilde{R}^L_t \left( a_o^l \right))$ as defined above, we obtain a system of two equations in the two credit risk trade parameters $(b_{0B}^L, R^l_t)$. Setting $W_0^B = w_0^B + b_{0B}^L$, $W_0^L = w_0^L - b_{0B}^L$ we obtain the following system of equations:

$$\begin{align*}
1 + \frac{w_0^B + b_{0B}^L - \Lambda^B_0}{w_0^B - b_{0B}^L - \Lambda^B_0} &= E \left[ \left( 1 + \tilde{R}^B_t \left( a_o^b \right) \right) \left( 1 + \frac{\Lambda^B_o \left( 1 + \tilde{R}^B_t \left( a_o^b \right) \right)}{\Lambda^B_0 \left( 1 + \tilde{R}^B_t \left( a_o^b \right) \right)} \right) \right] \\
1 + \frac{w_0^L - b_{0B}^L - \Lambda^L_0}{w_0^B + b_{0B}^L - \Lambda^B_0} &= E \left[ \left( 1 + \tilde{R}^L_t \left( a_o^l \right) \right) \left( 1 + \frac{\Lambda^L_o \left( 1 + \tilde{R}^L_t \left( a_o^l \right) \right)}{\Lambda^L_0 \left( 1 + \tilde{R}^L_t \left( a_o^l \right) \right)} \right) \right]
\end{align*}$$

where $\eta_t$ is the rate of return on common equity market:

$$1 + \tilde{R}^B_t \left( a_o^b \right) = \Lambda^B_o \left( a_o^b \left( 1 + R^f_t \right) + (1 - a_o^b) \left( 1 + \eta_t \right) \right) - b_{0B}^L \left( 1 + R^l_t \right)$$

$$1 + \tilde{R}^L_t \left( a_o^l \right) = \Lambda^L_o \left( a_o^l \left( 1 + R^f_t \right) + (1 - a_o^l) \left( 1 + \eta_t \right) \right) + b_{0B}^L \left( 1 + R^l_t \right)$$

The solution for the credit parameters are then clearly a system of cubic equations which may be solved numerically or by simulation of a given random distribution of rates of returns. Let,

$$A^B = \frac{\Lambda^B_o \left( a_o^b \left( 1 + R^f_t \right) + (1 - a_o^b) \left( 1 + \eta_t \right) \right)}{\Lambda^B_0}$$  
$$A^L = \frac{\Lambda^L_o \left( a_o^l \left( 1 + R^f_t \right) + (1 - a_o^l) \left( 1 + \eta_t \right) \right)}{\Lambda^L_0}$$

As well as

$$1 + \tilde{R}^B_t \left( a_o^b \right) = A^B - \frac{b_{0B}^L \left( 1 + R^l_t \right)}{\Lambda^B_0}$$  
$$1 + \tilde{R}^L_t \left( a_o^l \right) = A^L + \frac{b_{0B}^L \left( 1 + R^l_t \right)}{\Lambda^L_0}$$
With $\tilde{A}^b$ and $\tilde{A}^l$ normal random variables if equity rates of returns are normally distributed. Neglecting for simplicity the inflation rate, this is reduced to:

$$
\left(1 + \frac{w^b_0 + b^b_0 - \Lambda^b_0}{w^b_0} \right) = -\frac{b^b_0 (1 + R^b_1)}{\Lambda^b_0} + E(\tilde{A}^b) + \frac{\Lambda^b_0}{\Lambda^b_0} E\left( \frac{\tilde{A}^b - b^b_0 (1 + R^b_1)}{\Lambda^b_0} \right)
$$

$$
\left(1 + \frac{w^l_0 - b^l_0 - \Lambda^l_0}{w^l_0} \right) = \frac{b^l_0 (1 + R^l_1)}{\Lambda^l_0} + E(\tilde{A}^l) + \frac{\Lambda^l_0}{\Lambda^l_0} E\left( \frac{\tilde{A}^l - b^l_0 (1 + R^l_1)}{\Lambda^l_0} \right)
$$

**Example: A Three Periods Problem**

A three period problem is defined by initial conditions, as well the final condition, where all wealth is expended on consumption. This example will emphasize some of the difficulties when we move from a two period problem to more than 2 periods. We consider therefore, three period, $t=0,1,2$. The Martingale property of Proposition 3 yields:

$$
\left( \frac{\pi^k}{\sigma^k} \right) \left( 1 + \left( \frac{W^k_0 - \Lambda^k_0}{W^k_0} \right) \right) = E_0 \left( \frac{\pi^k}{\pi^k} \left( \frac{C_1}{C_1} \right) \left( 1 + \tilde{R}_1^k (a^k_0) \right) \left( 1 + \tilde{W}_1^k - \Lambda^k_1 \right) \right) = E_0 \left( \frac{\pi^k}{\pi^k} \left( \frac{C_2}{C_2} \right) \left( 1 + \tilde{R}_2^k (a^k_0) \right) \left( 1 + \tilde{W}_2^k \right) \right)
$$

Where for convenience we assume that initially, the price is known and the decision to save are set at time $t=0$ and at time $t=1$ while at time 2, the agent retires and therefore spends all his outstanding wealth on consumption. At the retirement time 2,

$$
E_0 \left( \frac{\pi^k}{\pi^k} \left( \frac{C_1}{C_1} \right) \left( 1 + \tilde{R}_1^k (a^k_0) \right) \left( 1 + \tilde{W}_1^k - \Lambda^k_1 \right) - \frac{\pi^k}{\pi^k} \left( \frac{C_2}{C_2} \right) \left( 1 + \tilde{R}_2^k (a^k_0) \right) \left( 1 + \tilde{W}_2^k \right) \right) = 0
$$

And therefore, with $p^k_2$ denoting the relative disposable wealth of the agent (where $W^{nk}_i, \Lambda^{nk}_i$ and the rate of return at time 1 is known):

$$
\Lambda^{nk}_i = W^{nk}_i - (W^{nk}_i - \Lambda^{nk}_i) E_k \left( \frac{M^k_2}{1 + \eta_2} \left( 1 + \tilde{R}_2^k (a^k_0) \right) \left( 1 + \frac{p^k_2}{1 - p^k_2} \right) + (W^{nk}_i - \Lambda^{nk}_i) \left( 1 + R^k_1 (a^k_0) \right) \right)
$$

Similarly, at the initial time:

$$
1 + \left( \frac{W^0_0 - \Lambda^0_0}{W^{nk}_0 - \Lambda^{nk}_0} \right) = E_0 \left( \frac{M^k_1}{1 + \eta_1} \left( 1 + \tilde{R}_1^k (a^k_0) \right) \left( 1 + \tilde{W}_1^k - \Lambda^{nk}_i \right) \right)
$$
Where $\Lambda^t$ is currently a random estimate of future savings which depends on the current investment policy and its returns, as well as all other agents’ current states and future decisions.

**Conclusion**

Caveats of the complete financial market theory for assets pricing underlying both rational expectations and CCAPM models (Arrow 1951a, 1951b, 1963, and Arrow and Debreu, 1954; Debreu, 1959, 1960; Lucas, 1978; Constantinides and Duffie, 1996; Cochrane 2001; Fama, 1996) have been profusely researched, pointing out that its assumptions do not always hold. Yet, there are few general and meaningful explanatory theories to replace these models. The utility based approach, CCAPM, to consumption rationality (Lucas, 1978) in an exchange economy led to the Euler equation for prices, $1 = E(\tilde{M}_{t+1}(1 + \tilde{R}_{t+1})|\mathcal{F}_t)$ with $\tilde{M}_{t+1}$, a pricing kernel, $1 + \tilde{R}_{t+1}$ the return on an initial investment of one dollar and $\mathcal{F}_t$, a filtration, summarizing all the relevant and common information at time $t$. A transformation to a stochastic discount factor by pricing relative to a risk free rate, say $R_f$, led to a probability measure defined in terms of $\tilde{M}_{t+1}/E(\tilde{M}_{t+1})$ and therefore to a pricing model for future returns $1 + \tilde{R}_{t+1}$, given by: $1 = \frac{1}{1+R_f}E^M(\tilde{M}_{t+1}(1 + \tilde{R}_{t+1})|\mathcal{F}_t)$. Rational expectations models, unlike CCAPM pricing models, are based on Walras equilibrium theory extended by the Arrow-Debreu framework. In this framework a general equilibrium in competitive markets with complete state preferences is extended to future prices, say $\tilde{\pi}_{t+1}$, trading around a rational expectation $E(\tilde{\pi}_{t+1}|\mathcal{F}_t)$. In this case, $\tilde{\pi}_{t+1} - E(\tilde{\pi}_{t+1}|\mathcal{F}_t) = \tilde{\epsilon}_{t+1}$ where $\tilde{\epsilon}_{t+1}$ are random variables that are independently and identically (iid) distributed. These mathematical properties lead to defining a unique Martingale that underlies such price processes. Both approaches support “the law of one price” which is a weak form of the principle of no arbitrage. Formally, no-arbitrage implies that non-negative payoffs that are positive with positive conditional probability have a strictly positive price (Hansen 1957). Achievements of these pricing models have blurred however the fact that in a real world risks may have multiple and interactive causes; be both exogenous and strategic—resulting from what we and others want to pay. It also fails to recognize the economic effects of financial inequalities and a mis-match of micro and macroeconomic factors. In such an environment, financial pricing models are necessarily and in fact incomplete.

Numerous papers have attempted to circumvent these failings. Mehra and Prescott, 1985, have pointed out that utility-consumption based models fare poorly in explaining security prices. They pointed out that these models predict a mean equity premium that is too low and a mean interest rate that is too high—given the observable low variability of aggregate growth (see Constantinidis and Duffie, 1996 for an early study on the effects of heterogeneous consumer models). Many other authors have thus questioned the validity of these models.

Alternatively, economic theories based on Cournot (and Bertrand) games have pointed out that the price is reached by what others and by what we do. Explicitly, the price of a stock or consumption depends on the financial commitment of what the agent as well as others are willing to commit. In this case, buyers and sellers of such stocks obtain a quantity of stocks based on the resulting equilibrium (random) price. In a simple Cournot game, with no agent able to sway the market price, a pricing model was defined similar
to that of the CCAPM model. When some agents are able by their financial commitment to sway the market price, the resultant prices are much more complex, albeit, this paper has defined agents’ pricing Martingale where their willingness to pay is measured relative to that of the resulting market Cournot price. What agents do then, is to alter their decision to comply to the market Cournot price. In this case, agents’ endowment (wealth, savings for future consumption, investment strategies, as well as power and information) provide for endowed agents a price which they may be willing to pay which is higher than the market price. The effects of economic inequalities are then expressed by the relative price that poorly endowed and richly endowed agents are paying relative to what they would be willing to pay. The endowed agent paying less, while the poorly endowed agent paying more. In this sense there no price discrimination due to economic inequalities.

The Multi-Agent financial CCAPM considered in this paper may also account for both exogenous and endogenous (strategic) factors. This approach is in the spirit of Adam Smith statement about prices, defined by both “needs and exchange” with the utility of consumption defining an agent need while prices defined by the exchange that occurs between agents, which we define by the equilibrium Cournot price. When the number of agents is immense, none of which has any sufficient endowment allowing him to influence market prices, or when consumption decisions are independent of market prices, the multi-agent CCAPM model is reduced to its standard Euler equation as indicated in the paper and the price is a Cournot price in the sense that it is defined by agents aggregate decision. When this is not the case, namely, some agents may be extremely powerful (such as regulatory agencies, extremely large financial institutions, Sovereign States interventions, etc.), or significantly endowed (compared to consumers and other agents), then unlike most models that assume a single and representative agent, we have reduced the multi agent problem to that of n two-agents games, each consisting of an individual agent and a synthetic agent representing all remaining agents. Such an assumption can be reduced however by considering a single agent whose decision is a function of all others. Since all others do the same, based on their own rationality and endowment, the result is a set of equations that are all dependent and therefore can be solved numerically.

Multi-agent (MA) pricing problems underlie a great variety of problems that seek to reconcile “individual and collective” behaviors of financial agents (for example, Le Baron, 2000, Hommes, 2006, Brian Arthur et al. 1997, Levy, Levy and Solomon, 2000, Lux and Marchesi, 1999, Boswijk et al. 2007. Some problems are based on assumptions seeking to reconcile individual traders and investors decisions’ and financial markets prices and trends. However, modeling and techniques are diverse, based on simulations, learning dynamics, private and common-shared information, and their like. In addition, agents endowments (their wealth, private information, predictions—whether chartists or fundamentalists, strategic positioning in a financial market—such as TBTF firms versus small investors) and their attitudes led to broad number of assumptions that are needed to resolve some of the complex problems that interacting and multi-agent raises. For example, Aoki, 1998, Chapter 5 suggests a (physics) Mean Field, approach based on averaging, economic agents (see also Aoki, 2000). Aoki and Hawkins 2010 expand on this approach and points out that equilibrium models based on self-averaging lead to strong market violations, reinforcing our contention that simple averaging is misleading. It is not clear however that such an approach leads to reasonable results characterizing macro-economic factors. Duffie and Sun, 2007, Duffie, Garleanu and Pedersen, 2005, Duffie and Manso, 2007 consider heterogeneous traders and under specific assumptions, provide aggregate responses. Anufriev and Dindo, 2010, study the co-
evolution of asset prices and individual wealth in a financial market with an arbitrary number of heterogeneous bounded rational investors. Their models have similar elements to this paper but also points out to the importance of income inequalities as relevant to asset pricing as this paper has highlighted. In particular, and similarly to Anufriev and Dindo, 2010, this paper included wealth effects on prices—found to lead to incomplete pricing. As in Boswijk, et al. 2007, all agents are assumed to access to two assets—a risk free asset and a stock market index (although, we can also consider more elaborate portfolios). As a result, every agent investment’s is defined by its portfolio with preferred risk-returns tailored to his wealth and his risk attitudes. In fact, an agent investment portfolio provides information for the agent’s risk preferences and therefore the implied probability distribution used to price future returns. Future returns by all agents are however necessarily statistically dependent, as all investments are made in common market indices (or their components).

The multi-agent Financial CCAPM approach is particularly relevant to markets where there are dominant agents such as extremely large financial institutions and extreme economic inequalities where “few families own or control a sizable share of trading volume and thus act as dominant factors contributing to price formation and markets incompleteness. For example, the current expansion of financial controls over trades and prices, may have a significant impact on financial market prices which ought to be accounted for. The Multi-Agent CCAPM provides an avenue of research to assess the effects of both regulation and controls. These situations are shown to lead to trade disadvantages to individual and small financial trading agents and is therefore relevant to regulatory agencies who seek to control the excessive power some institutions have (due to their size) and who may also harbor systemic risks (see also Tapiero 2013a, 2013b). The MA framework is also relevant to High Frequency Trading (HFT) increasing accounting for a large part of financial volumes traded, and with a relatively few automatic traders “making decisions” using algorithmic trading to both reach financial decisions, initiate orders, cancel them and profit by seeking a temporary advantage they may be able to profit from.

This paper differs also from an extensive literature in Behavioral Finance which seeks to reconcile micro and macro-economic decisions by rejecting consumers’ utility maximization (Stracca, 2004). Rather, a utility maximization of consumers in a multi-agent strategic CCAPM framework is maintained with consumption prices endogenous and heterogeneous agents-consumers wealth, financial returns performance, portfolios investments’ etc. Thus whether investors and traders are chartists or fundamentalist—the one investing based on signals generated from current and costless information or investing based on equilibrium mean reverting models to define an expected future fundamental price (e.g. Chiarella, He and Wang, 2006), is an issue that the paper avoids (although it can be accounted for by agents ex-ante predictions of future prices relative to their ex-post realizations). Further, the multi-person game approach (see Aumann, 1964) implied in multi agent games was reduced to an n persons game, providing a solution based on a collaboration of the market of all other agents, each of which is confronted in fact by a similar pronle—one against all.

Numerous extensions of this paper including debt and credit, the pricing of insurance contracts, MA binomial pricing models, the effects of supply shortage, consumption rationing, etc. provide an opportunity for further study that are, for brevity sake, are not considered in this paper (for some elements pertaining to these problems, see Tapiero, 2013a). The MA-CCAPM, although technically challenging, thus provides ample opportunities for further research, currently being pursued.
References


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Appendix 1: Proof of Proposition 3

For convenience we consider first a two periods problem for an agent $k$, sat time $t=0$ and time $t=1$, with time $t=1$ is a terminal retirement period where the agent has consumed all is available wealth for consumption. Subsequently, the problem is generalized to $T$ periods. The agent’ optimization problem is:

$$EU_0^k \left( \Lambda_0^k \right) = \max_{\Lambda_0^k} \left\{ EU_k \left( \frac{W_0^k - \Lambda_0^k}{\pi_0 (\bar{C}_0)} \right) + \beta EU_1^k \left( \frac{\Lambda_1^k \left( 1 + \tilde{R}_1^k (a_0^k) \right)}{\pi_1 (\bar{C}_1)} \right) \right\}$$

And therefore, the necessary condition for optimal savings $\Lambda_0^k$, with $W_0^k - \Lambda_0^k$ the agent investment for current consumption is:

$$\frac{\partial c_0^k}{\partial \Lambda_0^k} = 0$$

where:

$$\frac{\partial c_0^k}{\partial \Lambda_0^k} = \frac{1}{\pi_0 (\bar{C}_0)} \left\{ 1 - \left( W_0^k - \Lambda_0^k \right) \frac{\partial \ln \pi_0}{\partial \bar{C}_0} \left( \frac{\partial c_0^k}{\partial \Lambda_0^k} + \frac{\partial c_0^k}{\partial \Lambda_0^k} \right) \right\}$$

Proposition

$$\frac{\partial c_{0k}}{\partial \Lambda_0^k} = \frac{\partial \tilde{C}_{0k}}{\partial \Lambda_0^k} = 0$$

Proof:

Consider $\tilde{c}_0^k = \frac{W_0^k - \Lambda_0^k}{W_0^k - \Lambda_0^k} \tilde{C}_0^k$ and therefore, $\frac{\partial \tilde{c}_0^k}{\partial \Lambda_0^k} = \frac{1}{W_0^k - \Lambda_0^k} \left( \frac{\partial c_0^k}{\partial \Lambda_0^k} \right)$

Thus

$$\frac{\partial \tilde{c}_{0k}}{\partial \Lambda_0^k} = \frac{W_0^k - \Lambda_0^k}{W_0^k - \Lambda_0^k} \left( \frac{\partial \tilde{c}_0^k}{\partial \Lambda_0^k} \right)$$

As a result,

$$\frac{\partial c_0^k}{\partial \Lambda_0^k} = \frac{1}{\pi_0 (\bar{C}_0)} \left\{ 1 - \left( W_0^k - \Lambda_0^k \right) \frac{\partial \ln \pi_0}{\partial \bar{C}_0} \left( \frac{\partial c_0^k}{\partial \Lambda_0^k} \right) \right\}$$

and therefore,
\[
\frac{\partial c^k}{\partial \lambda^k_0} = -\frac{1}{\pi_0(\bar{C}_0)} \left( 1 - \left( W^k_0 - \lambda^k_0 \right) \right) \frac{1}{\pi^2_0(\bar{C}_0)} \frac{\partial \pi_0}{\partial \bar{C}_0}^{-1}
\]

\[
\frac{\partial c^k_i}{\partial \lambda^k_0} = \left( \frac{1 + \bar{R}_i^k(a^k)}{\pi_1(\bar{C}_1)} \right) \left[ 1 - \lambda^k_i \left( 1 + \bar{R}_i^k(a^k) \right) \right] \frac{1}{\pi^2_1(\bar{C}_1)} \frac{\partial \pi_1}{\partial \bar{C}_1}^{-1}
\]

Since \( \pi_0 = \frac{\sum_{i=k, n^k} (W^i_0 - \lambda^i_0)}{C_0} \), and \( \frac{1}{\pi^2_0(\bar{C}_0)} \frac{\partial \pi_0}{\partial \bar{C}_0} = -\left( \sum_{i=k, n^k} (W^i_0 - \lambda^i_0) \right)^{-1} \) we have

\[
\frac{\partial c^k}{\partial \lambda^k_0} = -\frac{1}{\pi_0(\bar{C}_0)} \left[ \frac{\sum_{i=k, n^k} (W^i_0 - \lambda^i_0)}{W^nk_0 - \lambda^nk_0} \right] \left[ 1 + \frac{W^k_0 - \lambda^k_0}{W^nk_0 - \lambda^nk_0} \right]
\]

By the same token,

\[
\frac{\partial c^k_i}{\partial \lambda^k_0} = \left( \frac{\pi_1(\bar{C}_1)}{1 + \bar{R}_i^k(a^k)} \right) \left[ 1 - \lambda^k_i \left( 1 + \bar{R}_i^k(a^k) \right) \right] \frac{\partial \ln \pi_1}{\partial \bar{C}_1^{-1}} \]

And therefore, at the initial time and conditional on a filtration \( \mathcal{F}_0 \), we have

\[
E^0 \left[ \frac{\partial u_k}{\partial c^k / \pi_0(\bar{C}_0)} \left( 1 + \frac{W^k_0 - \lambda^k_0}{W^nk_0 - \lambda^nk_0} \right) \mid \mathcal{F}_0 \right] = E^0 \left[ \frac{\partial u_k}{\partial c^k_i / \pi_1(\bar{C}_1)} \left( 1 + \frac{W^k_0 - \lambda^k_0}{W^nk_0 - \lambda^nk_0} \right) \mid \mathcal{F}_{T-1} \right]
\]

Or at time T-1, one period before the final time T, we have:

\[
1 = \frac{1}{\left( 1 + \frac{W^k_{T-1} - \lambda^k_{T-1}}{W^nk_{T-1} - \lambda^nk_{T-1}} \right)} E^T \left[ \frac{\partial u_k}{\partial c^k / \pi_1(\bar{C}_T)} \left( 1 + \frac{\bar{R}_i^k(a^k)}{\pi_1(\bar{C}_T)} \right) \mid \mathcal{F}_{T-1} \right]
\]

Let \( M_{T-1} = \frac{\partial u_k / \partial c^k_i}{\partial u_k / \partial c^k_{T-1}} \) be a kernel and let \( 1 + \eta_T = \frac{\pi_1(\bar{C}_{T-1})}{\pi_T(\bar{C}_{T-1})} \) with \( \eta_T \) its inflation rate, thus:

\[
1 = \frac{1}{\left( 1 + \frac{W^k_{T-1} - \lambda^k_{T-1}}{W^nk_{T-1} - \lambda^nk_{T-1}} \right)} E^T \left[ M_{T-1} \left( 1 + \frac{\bar{R}_i^k(a^k)}{\pi_1(\bar{C}_T)} \right) \right]
\]

Finally, over any two consecutive periods, we have:

\[
E^T \left[ \frac{\pi_i^k}{\pi_T(\bar{C}_T)} \left( 1 + p_i^k \left( 1 - q_i^k \right) \right) \right] \mid \mathcal{F}_0 = E^T \left[ \frac{\pi_i^k}{\pi_T(\bar{C}_T)} \left( 1 + \tilde{p}_i^k \left( 1 - \tilde{q}_i^k \right) \right) \right] \mid \mathcal{F}_T
\]

As a result, the series of random variables at any time \( t<T \) defines a Martingale

\[
\left( \frac{\pi_i^k}{\pi_T(\bar{C}_T)} \left( 1 + p_i^k \left( 1 - q_i^k \right) \right) \right)
\]
while at time $T = T^k$, the kth retirement time of an agent $k$, it is defined by:

$$
\left( \frac{\pi^k_T}{\pi_T(C_T)} \left( 1 + \frac{p^k_T}{p^k_T (1 - q^k_T)} \right) \right)
$$

Q.E.D