Capital Commitment and Illiquidity Risks in Private Equity*

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Abstract

Investing in real estate, infrastructure, leveraged buyouts, or venture capital typically involves committing capital to a fund several years before the capital is actually used. Once the capital is called and invested, the asset is untradable until the fund exits the project. We present a model of capital commitment and allocation that captures this type of illiquidity along with realistic features like strategic default and liquidity cycles. With one illiquid asset, the welfare premiums associated with commitment risk are small because of the ease of adjusting liquid risky asset exposure. The presence of multiple illiquid assets allows the investor to diversify across liquidity events, increasing welfare, but generates a costly funding mismatch: some funds may call early while others return capital late. With many funds, the welfare premium associated with commitment risk actually increases and, in the limit, is equivalent to increasing investment returns by 1.63% per annum.

JEL Classification: G11, G12

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1 Introduction

Institutional investors have over $2 trillion worth of investments in private equity funds as of 2012. These funds span a wide range of investments (e.g. real estate, leveraged buyouts, venture capital) but irrespective of their focus, they always require that investors commit capital ex-ante. Fund managers can then call the capital at their discretion. The total value of committed but uncalled capital in private equity is over $1 trillion.[1] This capital commitment thus represents a sizeable amount to investors, in addition to the fact that capital distributions are at the discretion of the fund managers too. Investors do not control the timing or quantity of their entry or exit from private equity investments.

To illustrate the risk of capital commitment, consider Yale Endowment asset allocation: 18% in hedge funds, 16% in liquid equity, and about 6% in cash and fixed income. Most of the remainder of the portfolio (about 60%) has been allocated to private equity funds with various focuses.[2] Using the aggregate statistics as an indicator, Yale Endowment has capital commitments to future private equity investments worth about 30% of their portfolio – an amount greater than their liquid asset holdings. A common defense of similar arrangements is that by spreading out investments over a large number of private equity funds, the investor can diversify illiquidity risk away. Empirically, Robinson and Sensoy (2011) confirm the potential for this type of liquidity diversification. They find that while there is some cyclicality in capital calls and distributions, most of the timing risk can be diversified away by investing in many different funds.

In this paper, we develop an asset allocation model with private equity funds that allows us to evaluate commitment, risk, and diversification. A private equity fund investment goes through two stages. First, after an investor pledges capital, there is a stochastic delay before the fund issues a capital call. Second, after the investor has honored the pledge (turned over the capital to the fund) and the fund has invested the capital, there is a second stochastic delay before the fund exits and distribute the proceeds. The capital commitment mechanism generates two sources of risk: timing and quantity. Timing risk results from the fact that the pledge-to-call delay is stochastic. Quantity risk results from the fact that during the pledge-to-call delay, other sources of investors' wealth face random shocks, so investors would prefer to adjust the pledge amount. In addition, we include realistic features like the ability to default on pledges, limited secondary market sales, and multiple private equity funds.

[1] See, e.g., `https://www.preqin.com/docs/quarterly/PE/Private_Equity_Quarterly_Q3_2012.pdf`.

[2] See `https://www.preqin.com/docs/quarterly/PE/Private_Equity_Quarterly_Q3_2012.pdf`.

We solve our model with and without commitment risk – turning on and off timing certainty and the ability to adjust pledges at the time of capital calls. We find that with only one private equity fund, the welfare premium associated with commitment risk is essentially zero. However, in direct contrast to the standard diversification intuition, the welfare premium associated with commitment risk with many funds is high. It is equivalent to the welfare premium gained by increasing private equity returns by 1.63%.

Capital commitment induces investors to create an endogenous "escrow" account, setting aside the amount of the pledge in the risk free asset. This is potentially costly because they lose the opportunity to invest in the liquid risky asset (e.g. the S&P 500) or to consume out of liquid wealth. There is a similar problem after the capital call: as investors cannot sell their private equity fund investments, and because they can only consume out of liquid wealth, their consumption is much more volatile. Both situations appear to create a disaster state: investors cannot control their investment level and cannot consume out of a block of wealth.

With only one private equity fund, investors are able to avoid the "before call" disaster, even without strategic default, by controlling their allocation to the liquid risky asset. This is especially true if investors would already allocate significant wealth to the risk free asset, for example because of high risk aversion. Adding a liquidity cycle, or adding a covariance between liquidity and returns, affects the value of private equity substantially but has very little impact on the premium associated with commitment risk. The reason is the endogenous escrow account in the risk-free asset, is buffered from the private equity liquidity cycle.

In contrast, adding many different private equity investments simultaneously allows for diversification across liquidity events *but* increases the premium associated with commitment risk. The reason is that diversifying across funds, and therefore making cash flows more predictable, increases the value of private equity and increases the overall allocation. However, a large, diversified private equity allocation will require using the returns from previous funds to honor pledges made to later funds. With multiple funds, there is the chance that some funds will return capital late and others will call capital early. The result is a funding mismatch, and it can only occur because of commitment: if the investor could choose their allocation at the moment of a capital call, the problem never arises. To summarize, the obvious features of commitment risk, uncertainty over timing and quantity of investment, are unimportant only when there is only one illiquid asset. When there are multiple illiquid assets the issue grows because of the possibility of a funding shortage. The standard "illiquidity diversification" argument while present appears to be second-order.

There are three recent, related papers that add one illiquid asset to a Merton-style economy. In Ang et al. (2013), the investor cannot trade the illiquid asset during an unknown (i.e. stochastic) period of time. When the illiquid asset pays off the investor re-invests immediately an amount of her choice into the illiquid asset. This paper generalizes that model by including commitment and by allowing for multiple illiquid assets. In contemporaneous work, Buchner (2012) models the illiquid investment as an infinitely-divisible irreversible investment good with deterministic timing and commitment. We generalize this approach by considering discrete funds and random timing. In Sorensen et al. (2013) the illiquid investment is a one-shot opportunity that begins immediately and lasts for a pre-determined time. The authors are able to derive the optimal hedging portfolio and consumption policies in closed-form. Their focus is on understanding debt and option-like mechanics within the general partners's purview and also to extend the valuation of the buyout fee contract proposed by Metrick and Yasuda (2010) to a setting where buyout funds are not continuously traded. They also evaluate liquidity by varying the deterministic length of the fund. Our model focuses instead on the limited partner's problem, taking everything that happens inside the fund as given and adding commitment and stochastic timing.

Also closely related to our paper is the pioneering work in Longstaff (2001) and Longstaff (2009) on how the presence of an illiquid asset affects optimal portfolio decisions and pricing. Dai et al. (2010) investigate the case when there are periods in which the investor cannot trade and these periods are deterministic. Franzoni et al. (2012) quantify the liquidity risk premium for private equity in the four factor model of Pastor and Stambaugh (2003). Our results are consistent and complementary. Importantly, our paper introduces the notion of commitment risk.

Empirically, Ljungqvist and Richardson (2003) were the first to model the speed of capital calls and distributions. They propose a hazard/duration regression approach and model the speed at which capital is called as a function of market conditions for a sample of US buyout funds. More recently, Robinson and Sensoy (2011) analyze a large panel of private equity funds capital commitments and net cash flows to private equity investors. They find that there is some pro cyclicality of net cash flows, but they conclude that 'illiquidity risk' is diversifiable to a large extent.

2 Institutional Details

2.1 Fund Structure

Funds set up as private partnerships with a commitment structure are typically labeled Private Equity (PE) funds, regardless of investment focus (e.g. buyout, real estate, venture capital, infrastructure, mezzanine etc.)

A private equity firm is a general partner in a private equity fund. During the fund-raising period, the firm (e.g. KKR) seeks capital from investors for its fund (e.g. KKR millennium fund). Outside investors are offered limited partnerships in the fund. If they accept, they agree to pay a certain amount of capital to the fund when the fund makes investments of its own. The capital that an investor agrees to pay to a fund is called a commitment, and the total aggregate commitment to a fund is the fund's size. No capital is paid other than in connection with the fund making a specific investment (and management fees), at which point the fund "calls" its investors for capital; such capital calls are also known as draw-downs. These draw-downs are ex-ante uncertain in terms of quantity and timing. Capital is called project-by-project and when an investment is liquidated, the resulting cash immediately goes to the limited partners and cannot be recycled to make a new investment.

Once a fund has reached its commitment target, it has its 'final close' and the year this occurs is called the fund's vintage year. After this point, it is not possible for new investors to invest in the fund, unless they purchase an interest in the fund on the secondary market from another investor (see Kleymenova et al. (2012), for a description of the secondary market). The legal length of a fund is 10 years, plus three years of possible extensions. The limited partnership agreement specifies the length of the investment period (typically, the first 5 years), during which the fund should make investments. The divestment period is flexible, spanning the entire life of the fund, including an overlap with the investment period.

2.2 Commitment Risk in Practice

Committed capital represents an option to the fund manager and removes the limited partner's control over the quantity and timing of their investment; it should therefore be associated with a return premium to the limited partner. However, pre-crisis it was often argued that the premium was negligible. For example, Fraser-Sampson (2007) writes "there is no danger of the allocation level being breached in the real world, given proper cash-flow planning... Even if this sort of unimaginable financial cataclysm gripped the investment world for a prolonged period, then in extremis existing interests could be sold as secondaries, and/or future commitment levels reduced." Similarly, Siegel (2008) writes ''Most investors have not thought very much about the liquidity of their portfolios. They have simply assumed that (...) they could meet liquidity requirements either with cash already held in the portfolio or by selling stocks and bonds."

The financial crisis generated more interest in commitment risk. As Leibowitz and Bova (2009) points out, "The horrendous declines presented liquidity problems even for many portfolio managers who were long-term oriented, had modest payment schedules, and a seemingly ample percentage of liquid assets. This perfect liquidity storm, layered on top of a perfect asset storm, resulted from a toxic combination of: 1) a need to fulfill prior commitments to private equity, venture capital, real estate, and hedge funds, 2) reduced distributions from these asset classes..." In terms used in the introduction above, the crisis generated a funding mismatch.[3] The result was a much greater practical interest in liquidity.[4]

3 The Model

3.1 Information

The information structure obeys standard continuous-time technical assumptions. There exists a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ supporting the vector of $N+2$ independent Brownian motions $Z_t = (Z_t^L, Z_t^{PE}, Z_t^1, Z_t^2, ..., Z_t^N)$ and $N+1$ independent Poisson processes $M_t = (M_t^0, M_t^1, ...M_t^N)$ or $N \geq 0$. $\mathcal{P}$ is the corresponding measure and $\mathcal{F}$ is a right-continuous

[3]From "Cash-Poor LPs Face Capital-Call Pressure", Private Equity Insider, November 5, 2008: "Brown University, Calpers and Carnegie Corp. are suddenly finding it hard to meet capital calls from private equity fund managers... A growing set of limited partners find themselves short on cash amid the financial crisis – and thus are scrambling for ways to make good on undrawn obligations to private equity vehicles. Among those in the same boat: Duke University Management, Stanford Management, University of Chicago and University of Virginia... Brown, whose $2.3 billion endowment has a 15% allocation for private equity products, is apparently thinking about redeeming capital from hedge funds to raise the money it needs to meet upcoming capital calls from private equity firms. That's similar to a strategy that University of Virginia is employing... Carnegie, a $3.1 billion charitable foundation, is also in a squeeze. Its managers have been calling on commitments faster than expected, while distributions from older funds have slowed down, creating a cash shortfall. As for Duke, the university's endowment has been named as one of the players most likely to default on private equity fund commitments. That partly explains a massive secondary-market offering that the school floated last month, as it sought to raise much-needed cash and get off the hook for undrawn obligations by unloading most of its $2 billion of holdings in the sector... Some of the bigger investors are considering tapping credit facilities to meet near-term capital calls."

[4]See e.g. Forbes, October 2009, "Did Harvard Sell At the Bottom?" and Institutional Investor, 4 November 2009, "Lessons Learned: Colleges Lose Billions in Endowments."

increasing filtration generated by $Z \times M$.

At any time t, the economy can be in one of two states $s_t = \{H, L\}$. State H corresponds to periods of high liquidity and state L to low liquidity; we will allow all asset expected returns and volatilities to vary with the liquidity state. The state of liquidity s_t follows a continuous time Markov process governed by M^0 with a transition probability matrix between t and $t + dt$ given by

$$P = \begin{pmatrix} 1 - \chi^L \, dt & \chi^L \, dt \\ \chi^H \, dt & 1 - \chi^H \, dt \end{pmatrix}. \tag{1}$$

Throughout the paper, we will use superscripts to represent states or counts and subscripts to indicate time or other labels. The notation x^s means 'the value of x in the state s', so $1/\chi^s$ is the expected duration of the state $s = (H, L)$. $x^{\sim s}$ means the value of x after the next economy transition. s_t is the state of the economy at time t.

3.2 Assets

There are $N + 2$ assets in the economy: a risk-free bond with price B_t, a liquid risky asset with price L_t, and N private equity funds. The risk-free bond appreciates at rate r^s. The price of the liquid risky asset (e.g. a public equity market index value) follows a geometric Brownian motion:

$$\frac{dL_t}{L_t} = \mu^s \, dt + \sigma^s dZ_t^L. \tag{2}$$

The first two assets are fully liquid and holdings can be rebalanced continuously.

The N private equity funds follow a private partnership structure in which they first accept investor capital pledges, then search for a project. Upon finding a project, they call the investor's capital, use it in the project, and then return the capital at the end of the project's life (e.g. an IPO). In a slight abuse of notation, we will use X_t^n as a state variable that captures the current value of capital pledged when fund n has accepted pledges but before the capital call, and we will use X_t^n again to capture the amount invested after the fund has called its pledges.

- At any time 0, an investor can pledge any positive amount to a private equity fund. This pledge is a promise to make capital available in the future, and no capital changes hands at time t. X_0^n is the amount pledged, and the Poisson clock M^n with intensity $\lambda_C^{s_0}$

is started. Because no capital is invested and the pledge cannot be changed, $dX^n_u = 0$ until the pledge is called.

- When the Poisson clock M^n next hits (time τ_C), the pledge is called. At this time, the agent turns over $X^n_{\tau_C}$ in liquid wealth to the private equity fund; because of our notation, X^n_t does not immediately change. At the time the pledge is called, the Poisson clock M^n intensity becomes $\lambda^{s_{\tau_C}}_D$. Because capital is now invested in the private equity fund, we have

$$\frac{dX^n_t}{X^n_t} = \nu^s\, dt + \psi^s \left(\rho^s_L dZ^L_t + \sqrt{\rho^s_{PE} - {\rho^s_L}^2} dZ^{PE}_t + \sqrt{1-\rho_{PE}} dZ^n_t \right). \tag{3}$$

  This specification implies that each private equity fund's returns have correlation ρ^s_L with the liquid risky asset returns and correlation $\rho_{PE} \geq {\rho^s_L}^2$ with the returns of any other private equity fund.

- We assume that the investor may be able to default on a pledge: when the Poisson clock hits and the investor is to turn over $X^n_{\tau_C}$ to the private equity fund, the investor may instead default and receive an outside option equal to $V(W_t)$. We will usually take $V(W_t)$ to equal $-\infty$ or the solution to the investor's problem when N is reduced by 1. The latter represents being 'banned' from a private equity fund.

- We assume that the investor may be able to sell his or her private equity investment on a limited secondary market for a fraction α of it's value. If this option is exercised, the investor receives αX^n_t in liquid wealth and invested capital is set to zero. Without a secondary market, $\alpha = 0$.

- When the Poisson clock M^n next hits (time τ_D), any invested capital is returned to investors: the agent receives $X^n_{\tau_D}$ in liquid wealth. This payment is assumed to be net of fees, and net of general partner- or fund-level debt payments. The Poisson clock intensity and invested capital are both set to zero. The next pledge can be made immediately.

Finally, we will use $S^n_t = \{0, C, D\}$ to characterize the state of the private equity fund's pledge or call. 0 means that no capital is pledged or invested, C means that capital is pledged but not called, and D means that capital is invested but not yet returned.

This model focuses on the illiquidity of private partnership investments. Private partnerships differ from liquid investments because there are two waiting times: waiting for the pledge to be called and waiting for the invested capital to be returned. The parameters λ_C^s and λ_D^s capture the severity of the illiquidity friction. We have built in two realistic 'outs' for the investor: strategic default on pledges and sales with a haircut on a limited secondary market. Strategic default allows the investor to avoid a capital call, with the only consequence being that he or she is no longer able to invest with that fund in the future. Similarly, a limited secondary market allows the investor to sell their partnership at a discount and so to recover part of their illiquid investment. Both of these options potentially reduce the impact of illiquidity on the investor.[5] To focus on liquidity, we have abstracted from other concerns, assuming away any concavity or convexity from fees or debt payments. Our private equity capital is assumed to be the investor's residual claim.

3.3 The Investor

The investor has CRRA utility over sequences of consumption, C_t, given by

$$\mathrm{E}\left[\int_0^\infty e^{-\beta t}\frac{C^{1-\gamma}}{1-\gamma}dt\right], \tag{4}$$

where β is the subjective discount factor and $\gamma > 1$.

The agent's wealth has $N+1$ components, one liquid and N private equity funds. Liquid wealth includes the amount invested in the liquid risky asset and the risk-free asset. The evolution of the investor's liquid wealth, given by W_t, is

$$\frac{dW_t}{W_t} = \left(r^s + \left(\mu^s - r^s\right)\theta_t - c_t\right)dt + \theta_t \sigma dZ_t^L - \frac{dI_t}{W_t} \tag{5}$$

The agent invests a fraction θ of her liquid wealth into the liquid risky asset, while the remainder $(1-\theta)$ is invested in the bond. Following Dybvig and Huang (1988) and Cox and Huang (1989), we restrict the set of admissible trading strategies, θ, to those that satisfy the standard integrability conditions. All policies are appropriately adapted to $\mathcal{F}_t$. The agent consumes out of liquid wealth, so liquid wealth decreases at rate $c_t = C_t/W_t$. When a private equity fund calls capital or distributes capital, the agent transfers an amount dI_τ from her

[5] It is important, as industry practice suggests, that the investor cannot use a private equity capital as collateral on a loan. Riskless loans can be taken out (the bond position can be negative) and the liquid risky asset can be used as collateral. However, if investors could issue risky debt or write a forward contract, they would be able to avoid the illiquidity problem entirely. Similarly, we allow the investor to find buyers on a limited secondary market, but the entire investment must be sold. The discreteness is not important qualitatively.

liquid wealth to the private equity fund ($dI < 0$ for returns from the PE funds). Finally, we assume the standard discount rate restriction, that β is high enough that the investor's problem has a solution when all assets are liquid (e.g. the Merton problem).

We use dynamic programming techniques to solve the investor's problem. The agent's value function is equal to the discounted present value of her utility flow,

$$F(W_t, \{X_t^n\}, s_t, \{S_t^n\}) = \max_{\{\theta, \{X^n\}, c\}} \mathrm{E}_t \left[\int_t^{\infty} e^{-\beta(u-t)} U(C_u) du \right]. \tag{6}$$

Problem 1 (Baseline) *The investor performs the maximization in (6), subject to the intertemporal budget constraint (5) and the private equity evolution described in Section 3.2.*

To simplify further, we will use ξ_t^n to denote the fund-n-composition of the investor's wealth:

$$\xi_t^n \equiv \frac{X_t^n}{W + \sum_n X^n \mathbf{1}_{S_t^n = D}} \tag{7}$$

where the indicator is used to separate out capital pledges (which have not yet been paid) from capital invested. Notice that ξ is required to be between zero and one after capital is called, but not before. Because the utility function (4) is homothetic and the returns processes have constant moments, it must be the case that F is homogenous of degree $1-\gamma$ in total wealth:

$$F(W, \{X^n\}, s, S) = \left(W + \sum_n X^n \mathbf{1}_{S_t^n = D} \right)^{1-\gamma} H\left(\{\xi^n\}, s, S\right), \tag{8}$$

We will therefore be able to express the value function as the product of a power function of total wealth and a function of the wealth composition.

4 The Solution with One Private Equity Fund

4.1 Characterization

Since there is only one PE fund, we will suppress the fund superscript, writing for example X_t instead of X_t^1.

Proposition 1 (Baseline, $N = 1$) *For Problem 1 with $N = 1$, the investor's value function can be written as in (8), where $H(\xi, s, S)$ exists and is finite, continuous, and concave for $\xi \in [0, 1)$. Whenever the investor can pledge, he selects $\xi^{s*} \equiv \arg\max_\xi H(\xi, s, S = C)$, which exists. Between private equity pledges, calls, and distributions, $H(\xi, s, S)$ is characterized by*

$$0 = \max_{c,\theta} \left[\frac{1}{1-\gamma} c^{1-\gamma} (1-\xi)^{1-\gamma} - \beta H + A(\xi, c, \theta, S) H + B(\xi, c, \theta, S) H_\xi \right. \tag{9}$$

$$\left. + C(\xi, c, \theta, S) \frac{1}{2} H_{\xi\xi} + J(H, \xi, S) \right] \tag{10}$$

where

$$\begin{aligned}
A(\xi, c, \theta, S = D) &= (1-\xi)(1-\gamma)(r^s + \theta(\mu^s - r^s) - c) + \xi(1-\gamma)\nu^s \\
&\quad + \frac{\gamma}{2}(\gamma - 1)\left(\xi^2 \psi^{s2} + \sigma^{s2}\theta^2(1-\xi)^2 + 2\xi(1-\xi)\rho_L^s \psi^s \sigma^s \theta\right) \\
B(\xi, c, \theta, S = D) &= -\xi(1-\xi)(r^s + \theta(\mu^s - r^s) - c) + \xi(1-\xi)\nu^s \\
&\quad + \gamma\left(-\psi^{s2}\xi^2(1-\xi) + \sigma^{s2}\theta^2(1-\xi)^2\xi - \xi(1-\xi)(1-2\xi)\rho_L^s \psi^s \sigma^s \theta\right) \\
C(\xi, c, \theta, S = D) &= \xi^2(1-\xi)^2\left(\psi^{s2} - 2\rho_L^s \sigma^s \theta \psi^s + \sigma^{s2}\theta^2\right) \\
J(H, \xi, S = D) &= \lambda_D^s(\max_\xi H(\xi, s, S = C) - H) + \chi^s\left(H(\xi, \sim s, S = C) - H(\xi, s, S = C)\right) \\
A(\xi, c, \theta, S = C) &= (1-\xi)(1-\gamma)(r^s + \theta(\mu^s - r^s) - c) + \frac{\gamma}{2}(\gamma - 1)\sigma^{s2}\theta^2(1-\xi)^2 \\
B(\xi, c, \theta, S = C) &= -\xi(1-\xi)(r^s + \theta(\mu^s - r^s) - c) + \gamma\sigma^{s2}\theta^2(1-\xi)^2\xi \\
C(\xi, c, \theta, S = C) &= \xi^2(1-\xi)^2\sigma^{s2}\theta^2 \\
J(H, \xi, S = C) &= \lambda_C^s\left(H(\xi, s, S = D) - H\right) + \chi^s\left(H(\xi, \sim s, S = C) - H(\xi, s, S = C)\right)
\end{aligned}$$

The proof of Proposition 1, which is omitted for brevity, closely follows a standard dynamic programming problem with verification theorem. Our methods for generating numerical result are in Appendix A.

The need to commit capital years before it is invested implies that the allocation to private equity will not be optimal at the time of the capital call. We demonstrate this in Figure 1 for a simplified economy without liquidity cycles. Panel A shows the distribution of private equity wealth to total wealth (ξ_t) at three times: the pledge (dotted vertical line), the call (dashed line), and the return (solid line). Panel B shows the agent's value function during the wait for the capital call ($H(\xi, \cdot, C)$, dashed line) and the wait for the capital return ($H(\xi, \cdot, D)$, solid line).

Figure 1 illustrates the fundamental illiquidity problem: the investor does not control his allocation to private equity. The optimal pledge is a fixed fraction of the investor's wealth, but the investor is unable to update the pledge as his wealth evolves between the pledge time and the call time. Panel A (dashed line) shows the eventual distribution of ξ_t at the time of the call, and Panel B (dashed line) shows the welfare penalty function (H) because the pledge is no longer the optimal fraction of wealth. Once the capital has been called, the investor must wait again, unable to increase or decrease the private equity allocation which evolves over time. Panel A (solid line) shows the eventual distribution of ξ_t at the time capital is returned, and Panel B (solid line) shows the welfare penalty function when the investor is away from the optimal allocation.

The investor's allocation to private equity is both more volatile *and larger* than the pledge amount, which we demonstrate in Figure 1, Panels C and D. Because private equity has a return premium ($\nu > \mu$), and because consumption is out of liquid wealth, the private equity to total wealth ratio will increase over time. Panel C shows the difference between pledged capital and average ex-post allocated capital for different values of ν. As the return to private equity increases, so does the investor's optimal allocation. Surprisingly, the investor's optimal *pledge* does not necessarily increase with returns. Instead, the investor anticipates that the invested capital will rapidly grow with high returns, and so the gap between pledge and average allocation can grow faster than average allocation, pushing the pledge down. Panel D presents the same comparison with the introduction of a secondary market. This secondary market allows the investor to sell their private equity fund investment, chopping off the right tale of the allocation distribution. The result is that ex-ante and average allocations differ by less, and the investor increases his or her initial pledge. However, both with and without the secondary market, there is a large gap between how much the investor pledges, and what fraction of wealth the investor expects to allocate to private equity ex-post.

4.2 Policies and Allocations

Figure 2 shows the investors consumption and liquid portfolio policies (c, θ) with and without the ability to default on pledges. First, we consider panels C and D, which show the policies after capital has been called. There appears to be only one line because after capital is called, strategic default has almost no effect on consumption or portfolio policies. Both consumption policy and portfolio policy show the existence of a disaster state: after the investor has turned over their capital to the private equity fund, a decline in liquid wealth causes the investor to sharply cut back on consumption and to reduce allocations to the liquid risky asset. This is the same result as in Ang et al. (2013). Figure 2, Panels A and B, appear to show a similar sort of disaster state during the pledge-to-call wait. When a pledge is large relative to wealth, and the investor cannot default (dashed line), the investor reduces both consumption and the liquid risky asset allocation. In fact, the investor's policies are very similar to the creation of an endogenous 'escrow' account, meaning that the investor puts his or her pledge entirely in the riskless bond and waits for the call. When the investor can strategically default (solid line), there is no need for such an account, and consumption and portfolio policies are kept at the higher level.

Strategic default has a large impact on policies away from the optimal pledge. However, the ability to default is almost worthless for an investor who has made the optimal pledge. In our standard calibration, the investor would only be willing to trade a fraction 0.022% (2 basis points) of his or her wealth to switch from an economy without strategic default to one with strategic default. In contrast, an investor who has just made an optimal pledge would be willing to trade 6.7% of his or her wealth to switch from an economy without a secondary market to an economy with a secondary market. This means that the ability to avoid an excessive private equity allocation is very valuable after a call, but almost useless after a pledge.

Figure 3 displays the drift and volatility for ξ_t and total wealth. After a capital call, ξ_t is both very volatile and has a positive drift. The positive drift is consistent with the results in Figure 1, Panels C and D: because private equity has a return premium and consumption is taken out of liquid wealth, the investor's wealth composition tilts toward private equity in the absence of trade. The result is that ξ_t can easily become very large; Figure 1, Panel C shows that without a secondary market, the average allocation to private equity for standard parameters is 1.6$\times$ the optimal pledge. In contrast, before capital is called, ξ_t cannot become very large: the pledge is constant and so the only thing changing is the denominator – the agent's liquid wealth. Because the agent's liquid wealth drifts up over time – asset returns exceed consumption by a small margin – ξ_t drifts down over time and is less volatile. Informally, having an excessive pledge is bad, but the probability of realizing the event starting from the optimum is very low.

5 Liquidity Premiums with One Private Equity Fund

The description of private equity funds in Section 3.2 creates two separate sources of risk from private equity pledges: timing and quantity. Timing risk results from the fact that the wait between pledge and call is uncertain and so the investor does not know if he will be able to make his allocation soon or not. This uncertainty creates consumption volatility as in any model where investment opportunities are uncertain.

Quantity risk results from the fact that the pledge cannot be changed over time. The investor would like to invest a constant fraction of his wealth in a PE fund, but his wealth level changes between the time of the pledge and the time of the call. This requires only a deterministic wait between pledge and call, but one might expect timing risk to exacerbate the welfare cost of quantity risk.

Throughout this section, we will make reference to welfare premiums. To assign a premium, we will ask "What fraction of his wealth would the investor be willing to pay to switch from economy A to economy B?". We measure this at time 0, just before the investor makes an optimal pledge. Thus, the premium is ζ^s such that

$$H^A(\xi^{sA*}, s, S = C) = (1 - \zeta^s)^{1-\gamma} H^B(\xi^{sB*}, s, S = C) \tag{11}$$

In general, this premium will be different in the liquid and illiquid states of the economy, so we measure them separately.

5.1 Timing Risk

We assess the welfare cost of 'timing risk' – the fact that the arrival of the call and distribution times are uncertain. To do this, we will go through two constructions. The first is to change the working of the Poisson clock. In the baseline economy there is one clock with intensity λ_C, and so the time between pledge and call has an exponential distribution with parameter λ_C. The mean wait is $\frac{1}{\lambda_C}$ and the variance is $\frac{1}{\lambda_C^2}$. Our goal is to go through a sequence of economies in which the exponential distribution shrinks to a point, keeping the mean constant but reducing the variance to zero. To do so, we will use the fact that the sum of M exponentially distributed variables with parameter $M\lambda_C$ has a distribution $\Gamma(M, M\lambda_C)$. The mean time for all M clocks to hit in series, each M times as intense as the single clock, is $\frac{1}{\lambda_C}$, the same as the mean for the single clock. However the variance of the sum of times is $\frac{1}{M\lambda_C^2}$, which goes to zero as M becomes large. Thus, by increasing the value of M, we can see the effect of changing the variance of the timing of the capital call, without changing its mean. We can do the same thing with the timing of the capital return to investors:

Problem 2 (M-clock Call) *The investor solves the Baseline problem, except that M Poisson clocks, each with intensity $M\lambda_C^s$ must hit in series before the pledge is called.*

Problem 3 (M-clock Return) *The investor solves the Baseline problem, except that M Poisson clocks, each with intensity $M\lambda_D^s$ must hit in series before capital is returned.*

Figure 4 presents the resulting premiums for the M-clock economies, where each economy is compared to the baseline ($M = 1$) economy. The premium for timing risk on the eventual return of capital is large and positive (as in Ang et al. (2013)), but the premium associated with timing risk for the capital call is very small and *negative*.[6]

To proceed, we will use a second, more direct construction: we will simulate the investor's realized utility in the baseline model, conditional on the first call or return time being realized at a particular value. We start with the solution to the standard problem with stochastic time, and we assign the investor 1 unit of liquid wealth. We then simulate the economy, using the optimal policies from Proposition 1 and assuming that the first capital call is made at time T. We assign that simulation run a utility value of

$$\int_0^T e^{-\beta t} \frac{1}{1-\gamma} C_t^{1-\gamma} dt + e^{-\beta T} (W_T + X_T)^{1-\gamma} H(\xi_T, \cdot, D)$$

We repeat the simulation 100,000 times to find the average realized utility in which the first call occurs at time T; then we vary T. We repeat the procedure for capital returns, starting at the time of a call and assigning a pledge-to-wealth ratio drawn from the appropriate distribution (Figure 1, Panel A, lashed line).

We present the results of these simulations in Figure 4, Panels B and C. In Panel B, we see that over six years ($\frac{1}{\lambda_C} \times 3$), conditional expected utility from the pledge-to-call wait is a decreasing and almost linear function of time.[7] As a result, there is little premium associated with call uncertainty.

[6]The benefit of examining the M-clock economies is that the change is simple, but the cost is that uncertainty is resolved in M different steps, and so the agent can update portfolio and consumption policies at each step. In our model (in un-reported results), we verify that this effect is very small, but it is of the same order of magnitude as the (very small) negative timing premium.

[7]One might ask what this graph would look like if we made T large enough. Our belief is that the conditional utility would continue to decline asymptotically to the welfare value of an economy without private equity. Thus, our expectation is that this graph is on average *convex*. A similar result will hold for the far right tail of the call-to-return plot in Panel C.

In contrast, Panel C shows that conditional expected utility from waiting for capital to be returned is concave over 12 years ($\frac{1}{\lambda_D} \times 3$). This result is intuitive: if the delay is zero, there is no value to this round of private equity because there is no time to accumulate returns. Similarly, if the delay is very large, then there is a risk of entering the high-ξ disaster state. The best realizations of the Poisson clock are moderate – and the curve is concave in the important region.

5.2 Quantity Risk

Next, we assess the welfare cost of 'quantity risk' – the fact that even with a constant pledge, the pledge-to-wealth ratio will change between pledge and call times. To proceed, we compare the baseline economy to one in which the investor can change his pledge at the time of the capital call:

Problem 4 (Change Pledge) *The investor solves the Baseline problem, except that the investor can change the pledge amount X_t^n at the time of the capital call.*

We can continue by assessing the premium associated with eliminating the pledge process entirely

Problem 5 (No Pledge Time) *The investor solves the Baseline problem, except that all pledges are called immediately.*

One of the costs of the pledge-to-call wait that has nothing to do with risk is the fact that the investor does not have any way to diversify his asset holdings during this period. In our standard calibration, the average wait time between pledge and call is $\frac{1}{\lambda_C} = 2$ and between call and return is $\frac{1}{\lambda_D} = 4$. Thus, the investor is constrained to have a zero allocation to private equity over 1/3 of his life. Since we have already concluded that timing risk is relatively unimportant, we can compare the 'Change Pledge' economy to the 'No Pledge Time' economy to determine the welfare cost of the expected wait time – the constrained zero allocation.

Finally, we will compare our illiquid economies to ones in which all investments, including private equity, are fully liquid (e.g. the Merton two-risky-asset model):

Problem 6 (Fully Liquid) *Private equity is fully liquid and can be adjusted continuously.*

Table 2 and 3 present the welfare premium results. The columns are meant to be understood "step-by-step": the premium associated with the 'Baseline Model' is the amount of wealth the investor would give up to switch from an economy with no private equity to our baseline model. The premium associated with 'Change Pledge' is the amount of wealth the investor would give up to switch from the baseline economy to one in which pledges could be changed at the time of the call, etc. For the standard parameters, we conclude that quantity risk is relatively unimportant: The investor will give up half a percent of his wealth to switch to an economy in which pledges can be changed. This compares to a 9.5% premium to acquire access to private equity, and a 5.5% premium to eliminate the pledge process entirely. Alternatively, the investor would also pay half a percent of his wealth to switch to an economy in which the expected return on private equity (ν) increases by 0.37%. Thus we conclude that the welfare cost of quantity risk is small in our standard calibration.

To test the robustness of this result, we change parameters and institutional details: allowing default on pledges, increasing call and return waits, changing the risk free-rate, and removing intermediate consumption[8] all have little effect on the quantity risk premium.

Two changes that significantly increase the premium are lowering the investor's risk aversion and adding a secondary market. Both of these changes increase the premium by making the endogenous escrow account more expensive. In the baseline economy, the need for an endogenous escrow account has a small welfare cost because the change in the agent's portfolio policy is small. If there were no private equity, the agent would invest a fraction $\frac{\mu-r}{\gamma\sigma^2} = .89$ of his wealth in the liquid risky asset, leaving 0.11 in the riskless bond. The investor's optimal pledge to private equity is a .167 fraction of wealth, meaning that the investor is very close to the necessary escrow account without making any changes. Additionally, the 0.89 liquid asset allocation is the fully liquid optimal policy, and by standard first-order condition arguments, the cost to reducing the liquid-risky allocation by a small amount will be second-order.

In contrast, with a secondary market available, the investor is willing to pledge much more – there is no longer a disaster state in which he cannot consume his illiquid wealth. The pledge amount rises to 0.364, and so the investor must significantly change his liquid risky asset allocation in order to ensure he can honor the pledge. Another way to think about this result is that different aspects of liquidity are complements: increasing secondary market liquidity increases the quantity of investment, which makes increasing primary market liquidity more valuable.

[8]For no-intermediate consumption economies, the agent is assumed to maximize $\mathrm{E}\left[(W_\tau + X_\tau 1_{S=D})^{1-\gamma}\right]$, where τ is exponentially distributed with parameter β.

The impact of reducing risk aversion works through the opposite channel. When $\gamma = 2$ instead of 4, the investor's desired liquid risky asset allocation is 1.78 times wealth. Reducing this enough to ensure that the investor can honor the pledge is very costly, and this effect is numerically more important that the more risk averse investor's tolerance for consumption volatility.

We find that the optimal pledge declines as we move from the 'Change Pledge' economy to the 'No Pledge Time' economy. This is a mechanical effect: when there is a pledge-to-call delay, the investor's liquid wealth grows during the wait, and so the pledge is a smaller fraction of wealth on average at the time of the call. In fact, in unreported results, the difference in the average realized allocation across the two economies is less than 1% (see Appendix A for our numerical methods).

Adding a private equity cycle does not change the basic liquidity premium result. Whether liquidity is positively or negatively correlated with private equity returns, public equity returns, or public-private correlations, there is very little effect on the premium associated with quantity risk. This does not mean that the cycle is does not impact the value of private equity – Table 3 shows that the premium the investor is willing to pay to switch from an economy with only the liquid risky asset to an economy with private equity is variable.

Our conclusion is that *neither* timing nor quantity risk is important when $N = 1$. In fact, timing risk may actually be a small benefit. Instead, the cost of commitment is almost entirely due to the *expected* wait. We will now introduce multiple private equity funds and show this result no longer holds when there is an interaction between different funding needs.

6 The Funding Mismatch

We now turn to an analysis of the economy where there is more than one private equity fund. We will look explicitly at economies with $N = 2$ and $N = \infty$. Our analysis will focus on the gains or losses from *liquidity diversification*, and so we set $\rho_{PE} = 1$. This assumption means that all funds have perfectly correlated returns, and so any welfare changes from changing N must be the result of diversifying liquidity. To simplify, we will remove private equity cycles from the analysis.

The important point is that there is an interaction effect between the funding needs of multiple funds: when an investor has allocations to two funds, he runs the risk that the second fund will call his pledge before the first fund returns his capital. This effect is not possible with $N = 1$ because the fund cannot call for new pledges until after returning capital to its investors. The investor would like to take advantage of liquidity diversification – by spreading out his allocation across different funds, the investor can smooth his capital inflows and outflows. As with all risk-averse investors, smoothing wealth flows results in higher risky asset allocation; in this case the sum of allocations to two funds is greater than the allocation to in the $N = 1$ problem. However, there is always the possibility that both funds will call capital before either returns capital, requiring a larger endogenous 'escrow' account and resulting in a total allocation to PE that is higher than desired. As a result, the investor is more willing to pay a premium to avoid quantity risk.

In contrast to the standard diversification intuition, the funding interaction becomes more acute as we increase the number of funds the investor has access to. At N grows, the timing of calls and returns within the population becomes more predictable. As a result, the investor increases his allocation to private equity as a whole, while reducing it to any individual fund. However, the investor's total investment in private equity is still uncertain because he cannot freely liquidate his positions. The frequency of fund call arrival (continuous with $N = \infty$) means that being able to change pledge amounts grant much more control over the total amount invested. Notice that the intuition is slightly different than for the $N = 2$ case: the investor wants to control his allocation overall, and he values controlling the pledge process because he cannot control the return timing process.

6.1 Two Funds

Our analysis of the $N = 2$ case proceeds similarly to the proceeding $N = 1$ case. The primary difference is that the value function now admits two state variables and two fund states: $H(\xi^1, \xi^2, S^1, S^2)$. Pledge policy is now a function $\Xi^1(\xi^2, S^2) = \arg\max_{\xi^1} H(\xi^1, \xi^2, S^1 = C, S^2)$, and similarly for Ξ^2. At time zero, the investor will pledge Ξ^* in both funds, where Ξ^* solves the fixed point problems $\Xi^* = \Xi^1(\Xi^2(\Xi^*, C), C)$ and by symmetry $\Xi^* = \Xi^2(\Xi^1(\Xi^*, C), C)$.

Figure 5 presents value function and the optimal consumption and allocation policies. We present the investor's pledge policy in Panel A, where the dashed line plots $\Xi^2(\xi^1, S^1 = C)$ (before call) and the solid line plots $\Xi^2(\xi^1, S^1 = D)$ (after call). The vertical dashed line is Ξ^*, and the horizontal dashed line is the optimal pledge in the $N = 1$ economy. Pledges to individual funds are smaller than in the $N = 1$ economy, but the aggregate private equity pledge is higher. As with $N = 1$, the value function is concave (Panel B), but now we must examine three different penalty functions. The dashed line corresponds to the value of H

as ξ^2 changes when neither fund has yet called capital and ξ^1 is held constant at Ξ^*. The solid line corresponds to the value of H after fund 1 has called its pledge. The dotted line corresponds to the value of H after fund 1 and 2 have both called their pledges. The key point is that the penalty to having too high a pledge to fund 2 is higher *after* fund 1 is called than before, consistent with the funding interaction mentioned earlier.

6.2 Many Funds

We next examine an economy in which $N = \infty$. To do so, we will make an assumption analogous to a 'law of large numbers' result: if funds have a pledge-to-call wait that is exponentially distributed with parameter λ_C, then a fraction $\lambda_C dt$ of funds call capital over the interval dt. We will make a similar assumption for the time between a capital call and return for parameter λ_D. In addition, we assume there are an infinity of funds calling at any given time, so the investor can fully diversify his holdings, taking no idiosyncratic risk across funds. Since all funds returns are perfectly correlated ($\rho_{PE} = 1$), we need only keep track of the stock of capital invested in all funds (X_t^∞) rather than each individual fund separately. Similarly, we need only to keep track of the aggregate amount pledged (P_t^∞). These assumptions imply

$$\frac{dP_t^\infty}{P_t^\infty} = \frac{dI_t}{P_t^\infty} - \lambda_C dt \tag{12}$$

$$\frac{dX_t^\infty}{X_t^\infty} = \lambda_C \frac{P_t^\infty}{X_t^\infty} dt - \lambda_D dt + \nu dt + \psi \left(\rho_L dZ_t^L + \sqrt{1 - {\rho_L}^2} dZ_t^{PE} \right) \tag{13}$$

$$\frac{dW_t}{W_t} = \left(r + (\mu - r)\,\theta_t - c_t\right) dt + \theta_t \sigma dZ_t^L - \lambda_C \frac{P_t^\infty}{X_t^\infty} dt + \lambda_D \frac{X_t^\infty}{W_t} dt \tag{14}$$

As in the case with finite N, we will define composition variables: $\pi_t \equiv \frac{P_t^\infty}{X_t^\infty + W_t}$ and $\xi_t \equiv \frac{X_t^\infty}{X_t^\infty + W_t}$ to denote the fraction of wealth pledged and invested.

Problem 7 **($N = \infty$)** *The investor's problem is to maximize 4 subject to the budget constraints (12-14) and the constraint that I_t is non-decreasing.*

Proposition 2 **($N = \infty$)** *For Problem 7, the investor's value function can be written as $(W + X)^{1-\gamma} H(\pi, \xi)$, where $H(\pi, \xi)$ exists and is finite, continuous, and concave for $\xi \in [0, 1)$ and $\pi \in [0, 1 - \xi)$.*

The investor chooses $dI_t = \max(0, \pi^*(\xi_t) - \pi_t)$, where $\pi^*(\xi)$ is characterized by the value matching and super-contact conditions $H_\pi(\pi^*, \xi) = H_{\pi\pi}(\pi^*, \xi) = 0$.

On $\pi \in [\pi^*(\xi), 1-\xi)$ and $\xi \in [0,1)$, $H(\pi,\xi)$ is characterized by

$$0 = \max_{c,\theta} \left[\frac{1}{1-\gamma} c^{1-\gamma} (1-\xi)^{1-\gamma} - \beta H + A(\pi,\xi,c,\theta) H + B(\pi,\xi,c,\theta) H_{\xi} \right. \tag{15}$$
$$\left. + C(\pi,\xi,c,\theta)\frac{1}{2} H_{\xi\xi} + D(\pi,\xi,c,\theta) H_{\pi} + E(\pi,\xi,c,\theta) \frac{1}{2} H_{\pi\pi} + F(\pi,\xi,c,\theta) H_{\xi\pi} \right]$$

where

$$\begin{aligned}
A(\pi,\xi,c,\theta) &= (1-\gamma)\left((1-\xi)(r+\theta(\mu-r)-c) + \xi\lambda_D - \pi\lambda_C\right) + (1-\gamma)\left(\xi\nu + \pi\lambda_C - \xi\lambda_D\right) \\
&\quad + \frac{\gamma}{2}(\gamma-1)\left(\xi^2\psi^2 + \sigma^2\theta^2(1-\xi)^2 + 2\xi(1-\xi)\rho_L\psi\sigma\theta\right) \\
B(\pi,\xi,c,\theta) &= -\xi(1-\xi)(r+\theta(\mu-r)-c) + (1-\xi)\left(\xi\nu + \pi\lambda_C - \xi\lambda_D\right) \\
&\quad + \gamma\left(-\psi^2\xi^2(1-\xi) + \sigma^2\theta^2(1-\xi)^2\xi - \xi(1-\xi)(1-2\xi)\rho_L\psi\sigma\theta\right) \\
C(\pi,\xi,c,\theta) &= \xi^2(1-\xi)^2\left(\psi^2 - 2\rho_L\sigma\theta\psi + \sigma^2\theta^2\right) \\
D(\pi,\xi,c,\theta) &= -\pi\left((1-\xi)(r+\theta(\mu-r)-c) - \xi\lambda_D + \pi\lambda_C\right) - \pi\left(\lambda_C + \pi\lambda_C - \xi\lambda_D + \xi\nu\right) \\
&\quad + \frac{\gamma}{2}\left(\xi^2\psi^2 + \sigma^2\theta^2(1-\xi)^2 + 2\xi(1-\xi)\rho_L\psi\sigma\theta\right) \\
E(\pi,\xi,c,\theta) &= \pi^2\xi^2\psi^2 + 2\xi\pi^2(1-\xi)\rho_L\sigma\theta\psi + \pi^2(1-\xi)^2\sigma^2\theta^2 \\
F(\pi,\xi,c,\theta) &= -\xi(1-\xi)\pi\left(\xi\psi^2 + (1-2\xi)\rho_L\sigma\theta\psi - (1-\xi)\sigma^2\theta^2\right)
\end{aligned}$$

We also examine the impact of changes in liquidity on welfare, particulary the investor's ability to change his pledge. With $N = \infty$, there are an infinity of funds calling capital at any given time, o the ability to change pledges implies that the investor can free adjust upward his investment in private equity. Thus the investor's problem becomes

Problem 8 ($N = \infty$, Change Pledge) *The investor's problem is to maximize 4 subject to the budget constraints*

$$\begin{aligned}
\frac{dX_t^{\infty}}{X_t^{\infty}} &= \frac{dI_t}{X_t^{\infty}} - \lambda_D dt + \nu dt + \psi\left(\rho_L dZ_t^L + \sqrt{1-{\rho_L}^2} dZ_t^{PE}\right) \\
\frac{dW_t}{W_t} &= \left(r + (\mu - r)\theta_t - c_t\right) dt + \theta_t \sigma dZ_t^L - \frac{dI_t}{W_t} + \lambda_D \frac{X_t^{\infty}}{W_t} dt
\end{aligned}$$

and the constraint that I_t is non-decreasing.

Proposition 3 ($N = \infty$, Change Pledge) *For Problem 8, the investor's value function can be written as $(W+X)^{1-\gamma}H(\xi)$, where $H(\xi)$ exists and is finite, continuous, and concave* *for* $\xi \in [0, 1)$.

The investor chooses $dI_t = \max(0, \xi^* - \xi_t)$, *where* ξ^* *is characterized by the value matching and super-contact conditions* $H_\xi(\xi^*) = H_{\xi\xi}(\xi^*) = 0$.

On $\xi \in [\xi^*, 1)$, $H(\xi)$ *is characterized by*

$$0 = \max_{c,\theta} \left[\frac{1}{1-\gamma} c^{1-\gamma}(1-\xi)^{1-\gamma} - \beta H + A(0, \xi, c, \theta)H + B(0, \xi, c, \theta)H_\xi + C(0, \xi, c, \theta)\frac{1}{2}H_{\xi\xi} \right]$$

where A, B, *and* C, *are defined in Proposition 2.*

We present the optimal pledging policies and the resulting distribution of allocations in Figure 6. In Panel A, the solid line shows the value of $\pi^*(\xi)$ when the investor cannot adjust pledges, while the dashed vertical line depicts ξ^* when the investor can adjust pledges. Panel B shows the distribution of allocations for both Problems 7 and 8. The solid line shows that the distribution of allocations is much less variable when pledges can be adjusted. Importantly, the ability to adjust pledges reduces the right tail as well as the left because there is no stock of un-called pledges to contribute to private equity when the allocation is already too high.

6.3 Implied Return Premium

Finally, we calculate welfare and an implied return premium, and we present the results in Table 4. The welfare premiums are reported in the same 'step-by-step' method as in Tables 2 and 3. The implied return premium answers the question 'By how much do private equity returns (ν) have to increase in the Baseline Model to generate the same investor welfare as allowing the investor to change his or her pledge amounts?'. For a well diversified private equity investor, the implied premium is 1.37%.

7 Conclusion

In this paper, we present a model to investigate the effects of capital commitment and commitment risk on asset allocation and welfare. We calibrate our model with and without commitment risk – certain timing and the ability to adjust pledges at the time of capital calls. We find that with only one private equity fund, the welfare premium associated with commitment risk is essentially zero. However, the welfare premium associated with commitment risk with many funds is equivalent to the welfare premium gained by increasing private equity returns by 1.63%. The standard intuition that illiquidity can be diversified away is only partially correct: individual shocks be diversified away, but this induces the investor to use the proceeds from previous funds to honor pledges made to later funds. This policy creates a possible funding mismatch that an investor will pay to avoid.

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A Numerical Methods

We use the numerical methods of Kushner and Dupuis (2001) to evaluate the Hamilton-Jacobi-Bellman linked-ODE and linked-PDE equations (linked by state of the economy). For the $N = 1$ case, we use logged, scaled variables (e.g. $x = \log(X/W)$) on a grid from -10 to 5 with intervals of $dx = \frac{1}{100}$. We stop the procedure when the sum of the absolute value of all innovations was below 10^{-6}: $\sum_j |H_{i+1}(x_j) - Hi(x_j) < 10^{-6}$. For $N = 2$ and $N\infty$ we used grid intervals of $\frac{1}{50}$ and a total tolerance of 10^{-3}. We note that because $N = 2$ and $N = \infty$ require two state variables, we are solving linked-PDEs on $751 \times 751 = 564,001$ points, rather than linked-ODEs on 1501 points.

For probability distributions we use Monte Carlo methods: we use $dt = \frac{1}{100}$ ($dt = \frac{1}{50}$ for $N = 2, \infty$) and simulate wealth shocks using $dZ_t = \epsilon \sim N(0, \sqrt{dt})$. We create a single times series lasting for $1,000,000$ years, taking the evolution of wealth from the budget equations and optimal allocations and consumption from the HJB equation.

Our standard parameters are shown in Table 1.

Table 1: Standard Numerical Parameters

Parameter	Symbol	Value	Source
Private equity expected returns	ν	.15	Mean US buyout fund IRR, Preqin
Private equity volatility	ψ	.15	Std dev. US buyout fund IRR, Preqin
Liquid asset expected returns	μ	.11	Mean CRSP value-weighted index
Liquid asset volatility	σ	.15	Std dev. CRSP value-weighted index
Risk-free rate	r	.03	Mean 3-month T-bill rate
Liquid, private equity return correlation	ρ_L	.5	Arbitrary
Inter-private-equity fund return correlation	ρ_{PE}	1	Arbitrary
Pledge-to-call intensity	λ_C	.5	Private equity fund cash flow, Calpers
Call-to-return intensity	λ_D	.25	Lopez-de Silanes et al. (2013)
Investor time discounting	β	.1	Standard / arbitrary
Investor risk aversion	γ	4	Standard / arbitrary
Secondary market value	α	0	Arbitrary
Pledge default welfare		$-\infty$	Arbitrary
With cycle pledge-to-call intensity	λ_C^s	$\{\frac{1}{3}, \frac{2}{3}\}$	Arbitrary
With cycle call-to-Return intensity	λ_D^s	$\{\frac{1}{6}, \frac{2}{6}\}$	Arbitrary
Cycle H to L Poisson intensity	χ^H	.33	Arbitrary
Cycle L to H Poisson intensity	χ^L	.33	Arbitrary

Table 2: Liquidity Premiums and Pledges Without Cycles

Panel A: Premiums

	Standard	Default on Pledge	Secondary Market	No Int. Consumption	Longer Call Wait	Longer Distribution Wait	Lower Risk Aversion	Lower Risk-Free Rate
No Private Equity	-	-	-	-	-	-	-	-
Baseline Model	0.095	0.095	0.168	0.045	0.066	0.060	0.068	0.100
Change Pledge	0.005	0.005	0.014	0.002	0.009	0.002	0.012	0.009
No Pledge Time	0.055	0.055	0.108	0.053	0.081	0.022	0.052	0.060
Fully Liquid	0.303	0.303	0.189	0.028	0.303	0.354	0.471	0.366

Panel B: Optimal Pledges

	Standard	Default on Pledge	Secondary Market	No Int. Consumption	Longer Call Wait	Longer Distribution Wait	Lower Risk Aversion	Lower Risk-Free Rate
Baseline Model	0.167	0.169	0.364	0.705	0.150	0.082	0.157	0.153
Change Pledge	0.194	0.194	0.454	0.756	0.202	0.093	0.210	0.192
No Pledge Time	0.179	0.179	0.423	0.756	0.179	0.089	0.202	0.176
Fully Liquid	1.185	1.185	1.185	1.185	1.185	1.185	2.370	1.333

Panel A reports welfare premiums (defined in Equation 11) while Panel B reports optimal pledge amounts. Each row is references an economy with a different illiquidity fraction. 'No Private Equity' is a version of the model in which private equity does not exist (e.g. it is the classic Merton one-risky-asset model). 'Baseline Model' is the model of private equity described in Section 3 and Problem 1. 'Change Pledge' corresponds to Problem 4; 'No Pledge Time' corresponds to Problem 5; 'Fully Liquid' corresponds to Problem 6. Each column represents a change in a parameter or other economic condition. 'Standard' is our standard calibration, and the reference point for all other changes. 'Default on Pledge' enables strategic default by the investor on pledges (see Section 3.2). 'Secondary Market' allows for $\alpha = .99$. 'No Int. Consumption' removes intermediate consumption from the investor's utility function (see Section 5.2). 'Longer Call Wait' sets $\lambda_C = \frac{1}{4}$. 'Longer Distribution Wait' sets $\lambda_D = \frac{1}{8}$. 'Lower Risk Aversion' sets $\gamma = 2$. 'Lower Risk-Free Rate' sets $r = .01$. The premiums are listed in a 'Step-by-Step' fashion: each row gives the fraction of wealth the investor would trade to move from the economy listed above to the current economy. Thus in the standard calibration (first column) the investor would be willing to trade 5.5% of his or her wealth to move from a the 'Change Pledge' economy to the 'No Pledge Time' Economy. In doing so, the investor would change his or her pledge from a fraction 0.194 to a fraction 0.179 of total wealth. We discuss these numbers in Section 5.2.

Table 3: Liquidity Premiums and Pledges With Cycles

Panel A: Premiums, Low State

	No Cycle	Standard	$Cov(\nu,s)>0$	$Cov(\nu,s)<0$	$Cov(\rho_L,s)<0$	$Cov(\mu,s)>0$	$Cov(\mu,s)<0$	$Cov(\{\mu,\nu\},s)>0$	$Cov(\{\mu,\nu\},s)<0$
No Private Equity	-	-	-	-	-	-	-	-	-
Baseline Model	0.095	0.090	0.088	0.090	0.090	0.084	0.094	0.072	0.082
Change Pledge	0.005	0.005	0.005	0.005	0.005	0.004	0.006	0.004	0.006
No Pledge Time	0.055	0.056	0.048	0.062	0.052	0.055	0.053	0.044	0.056
Fully Liquid	0.303	0.306	0.300	0.368	0.350	0.361	0.312	0.273	0.294

Panel B: Premiums, High State

	No Cycle	Standard	$Cov(\nu,s)>0$	$Cov(\nu,s)<0$	$Cov(\rho_L,s)<0$	$Cov(\mu,s)>0$	$Cov(\mu,s)<0$	$Cov(\{\mu,\nu\},s)>0$	$Cov(\{\mu,\nu\},s)<0$
No Private Equity	-	-	-	-	-	-	-	-	-
Baseline Model	0.095	0.094	0.097	0.088	0.096	0.084	0.102	0.076	0.085
Change Pledge	0.005	0.005	0.006	0.004	0.005	0.005	0.005	0.005	0.004
No Pledge Time	0.055	0.053	0.057	0.047	0.055	0.046	0.058	0.046	0.048
Fully Liquid	0.303	0.306	0.366	0.301	0.355	0.326	0.346	0.306	0.260

Panel C: Pledges, Low State

	Standard	With Cycle	$Cov(\nu,s)>0$	$Cov(\nu,s)<0$	$Cov(\rho_L,s)<0$	$Cov(\mu,s)>0$	$Cov(\mu,s)<0$	$Cov(\{\mu,\nu\},s)>0$	$Cov(\{\mu,\nu\},s)<0$
Baseline Model	0.167	0.157	0.154	0.157	0.157	0.142	0.156	0.140	0.157
Change Pledge	0.194	0.185	0.172	0.190	0.183	0.169	0.181	0.162	0.192
No Pledge Time	0.179	0.170	0.159	0.176	0.169	0.156	0.167	0.151	0.179
Fully Liquid	1.185	1.185	0.593	1.778	1.728	1.482	0.889	0.889	1.482

Panel D: Pledges, High State

	Standard	With Cycle	$Cov(\nu,s)>0$	$Cov(\nu,s)<0$	$Cov(\rho_L,s)<0$	$Cov(\mu,s)>0$	$Cov(\mu,s)<0$	$Cov(\{\mu,\nu\},s)>0$	$Cov(\{\mu,\nu\},s)<0$
Baseline Model	0.167	0.165	0.170	0.157	0.167	0.140	0.177	0.144	0.170
Change Pledge	0.194	0.192	0.200	0.177	0.194	0.160	0.206	0.172	0.194
No Pledge Time	0.179	0.177	0.186	0.164	0.179	0.148	0.190	0.160	0.181
Fully Liquid	1.185	1.185	1.778	0.593	1.204	0.889	1.482	1.482	0.889

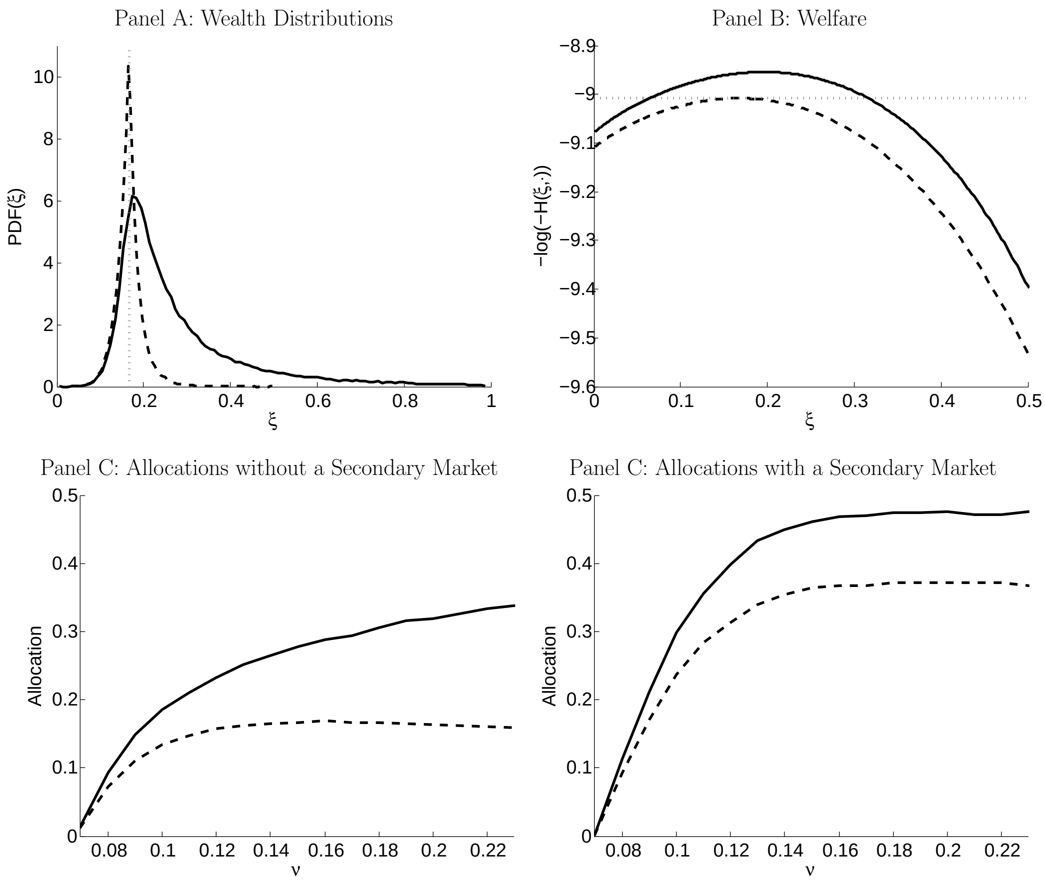
Panels A and B report welfare premiums (defined in Equation 11) while Panels C and D report optimal pledge amounts. Each row is references an economy with a different illiquidity fraction. 'No Private Equity' is a version of the model in which private equity does not exist (e.g. it is the classic Merton one-risky-asset model). 'Baseline Model' is the model of private equity described in Section 3 and Problem 1. 'Change Pledge' corresponds to Problem 4; 'No Pledge Time' corresponds to Problem 5; 'Fully Liquid' corresponds to Problem 6. Each column represents a change in a parameter or other economic condition. 'Standard' is our standard calibration, while 'With Cycle' sets $\lambda_C^{L,H}=\{1/3,2/3\}$ and $\lambda_C^{L,H}=\{1/6,2/6\}$ and is the reference point for all other changes in this table. $Cov(\nu,s)>0$ sets $\nu^{L,H}=\{.11,.19\}$. $Cov(\nu,s)<0$ sets $\nu^{L,H}=\{.19,.11\}$. $Cov(\rho_L,s)<0$ sets $\rho_L^{L,H}=\{.8,.2\}$. $Cov(\mu,s)>0$ sets $\mu^{L,H}=\{.07,.15\}$. $Cov(\mu,s)<0$ sets $\mu^{L,H}=\{.15,.07\}$. $Cov(\{\mu,\nu\},s)>0$ sets $\nu^{L,H}=\{.11,.19\}$ and $\mu^{L,H}=\{.07,.15\}$. $Cov(\{\mu,\nu\},s)<0$ sets $\nu^{L,H}=\{.19,.11\}$ and $\mu^{L,H}=\{.15,.07\}$. The premiums are listed in a 'Step-by-Step' fashion: each row gives the fraction of wealth the investor would trade to move from the economy listed above to the current economy. Thus in the standard calibration (first column) the investor would be willing to trade 5.5% of his or her wealth to move from a the 'Change Pledge' economy to the 'No Pledge Time' Economy. In doing so, the investor would change his or her pledge from a fraction 0.194 to a fraction 0.179 of total wealth. We discuss these numbers in Section 5.2.

Table 4: Liquidity Premiums for $N \geq 1$

	$N=1$	$N=2$	$N=\infty$
No Private Equity	-	-	-
Baseline Model	0.095	0.162	0.236
Change Pledge	0.005	0.012	0.040
No Pledge Time	0.055	0.064	0
Fully Liquid	0.303	0.235	0.192
Implied Return Premium	0.377	0.667	1.631

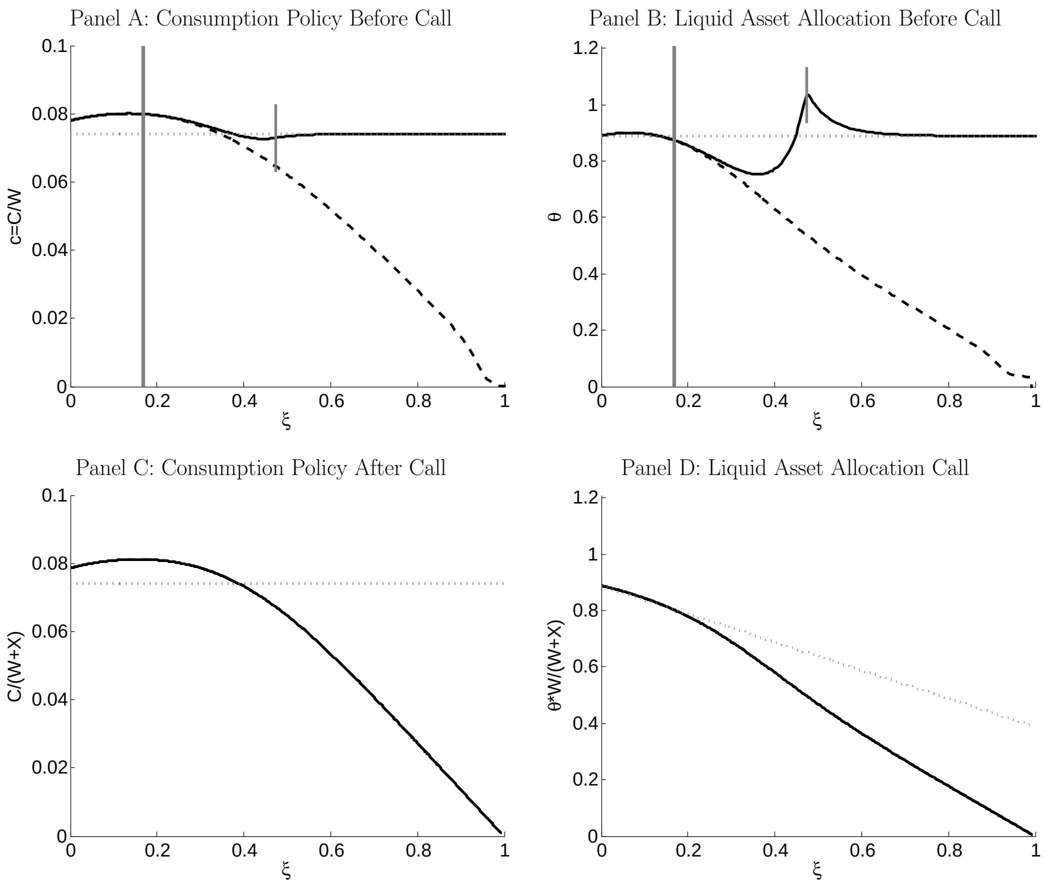
We report welfare premiums (defined in Equation 11) for economies with different numbers of private equity funds (N). Each row is references an economy with a different illiquidity fraction. 'No Private Equity' is a version of the model in which private equity does not exist (e.g. it is the classic Merton one-risky-asset model). 'Baseline Model' is the model of private equity described in Section 3 and Problem 1. 'Change Pledge' corresponds to Problem 4; 'No Pledge Time' corresponds to Problem 5; 'Fully Liquid' corresponds to Problem 6. The premiums are listed in a 'Step-by-Step' fashion: each row gives the fraction of wealth the investor would trade to move from the economy listed above to the current economy. Thus in the standard calibration (first column, $N=1$) the investor would be willing to trade 5.5% of his or her wealth to move from a the 'Change Pledge' economy to the 'No Pledge Time' Economy.

Figure 1: Wealth Distributions and Welfare



We present analysis of the economy with $N = 1$ under our standard calibration (Table 1). Panel A presents the distribution of ξ at three different stopping times. The dotted line is the value of ξ just after a pledge is made. The dashed line is the distribution of ξ at the time capital is called. the solid line is the distribution of ξ at the time capital is returned to investors. Panel B shows the value function H for three times: at the time of a pledge (dotted line), during the pledge to call wait ($S = C$), and during the call-to-return wait ($S = D$). Panel C presents the optimal pledge (dashed line) and the average simulated allocation (solid line) for different values of ν. Panel D does the same, but in an economy in which a secondary market with $\alpha = .99$ has been added.

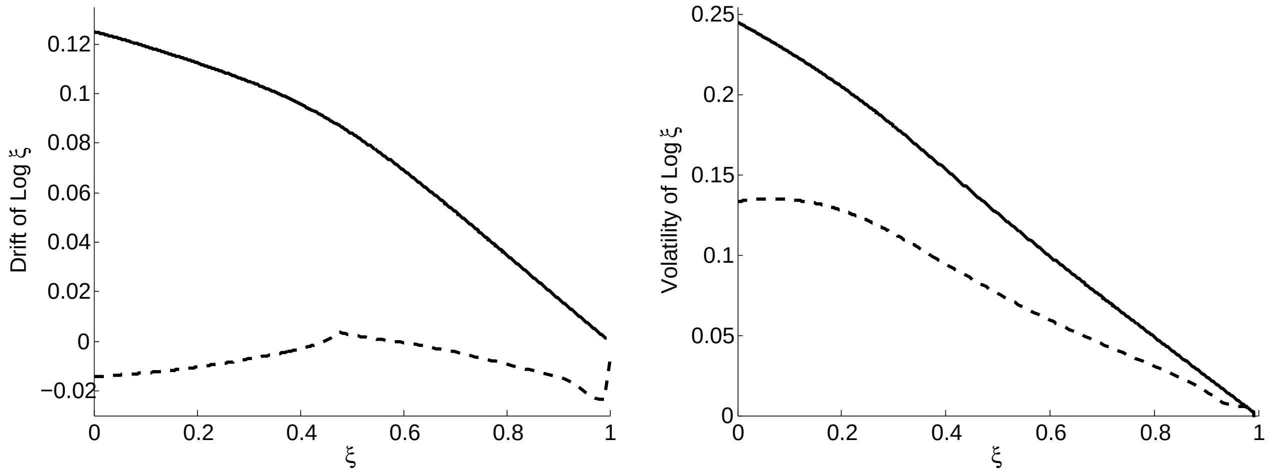
Figure 2: Consumption and Liquid Asset Allocation



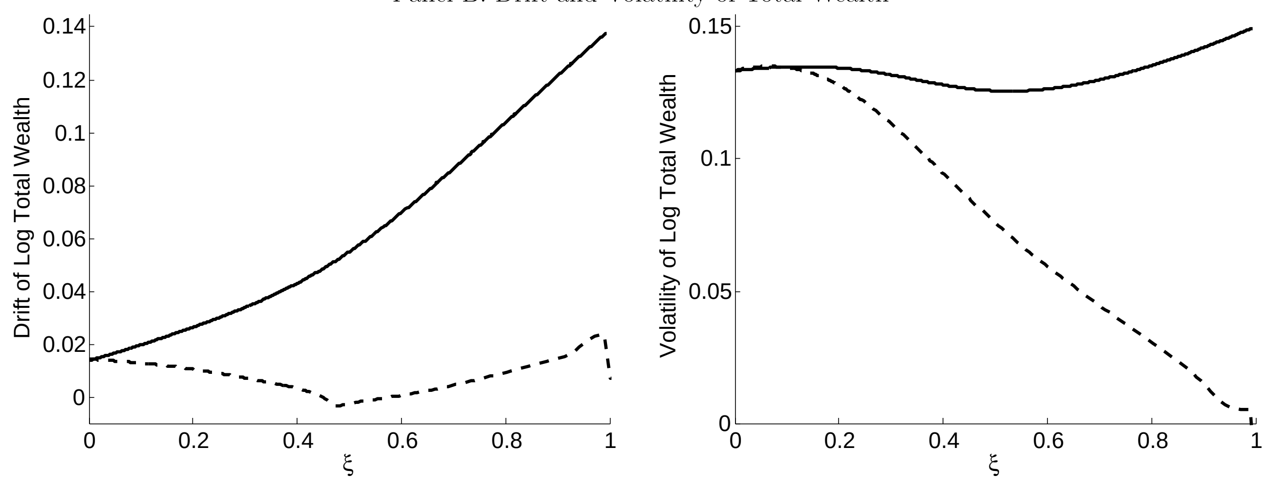
We present analysis of the economy with $N = 1$ under our standard calibration (Table 1). For each panel, the dashed line is for an economy in which there is no strategic default – the cost of not honoring a pledge is infinite. The solid line is for an economy in which strategic default is possible and the cost is to lose access to private equity thereafter. The dashed and dotted lines overlap in Panels B and C. In all panels, the dotted line is the Merton one-asset economy baseline policy. The small solid vertical lines in Panels A and B represent the point of strategic default: is ξ is above this point, the investor will not honor his or her pledge.

Figure 3: Drift and Volatility of Wealths

Panel A: Drift and Volatility of ξ



Panel B: Drift and Volatility of Total Wealth



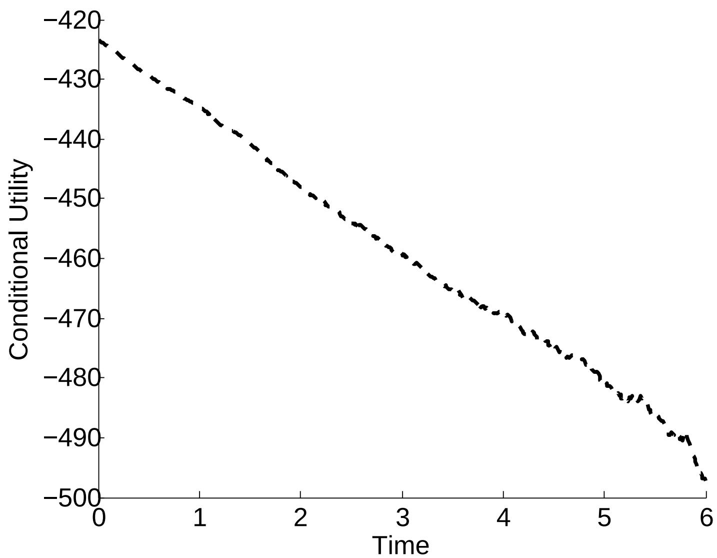
We present analysis of the economy with $N = 1$ under our standard calibration (Table 1). For each panel, the dashed line is the value the drift or volatility takes during the pledge-to-call wait, and the solid line is the value the drift or volatility takes during the call-to-return wait. Total wealth and ξ are defined in Section 3.3.

Figure 4: Timing Risk

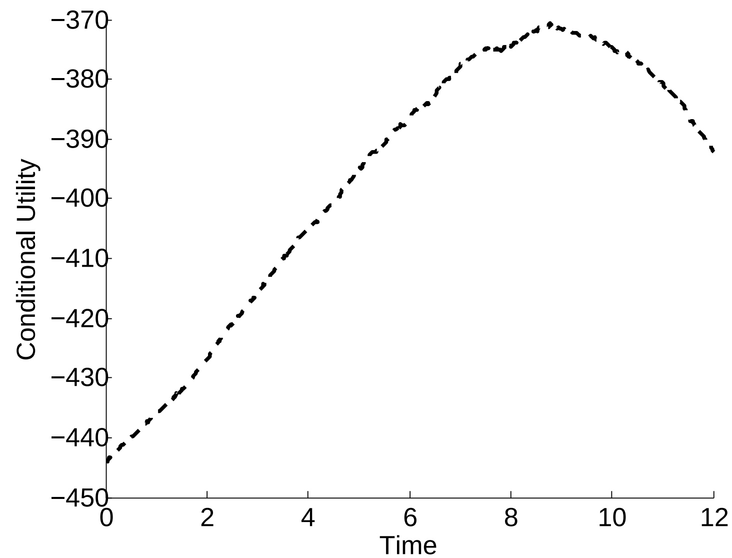
Panel A: Premiums

	λ_C	λ_D	Both
$M = 1$ (Standard)	-	-	-
$M = 2$	-0.00029	0.039	0.039
$M = 4$	-0.00044	0.070	0.072
$M = 8$	-0.00053	0.087	0.093
$M = 16$	-0.00057	0.097	0.104
$M = 32$	-0.00060	0.100	0.110

Panel B: Pledge to Call

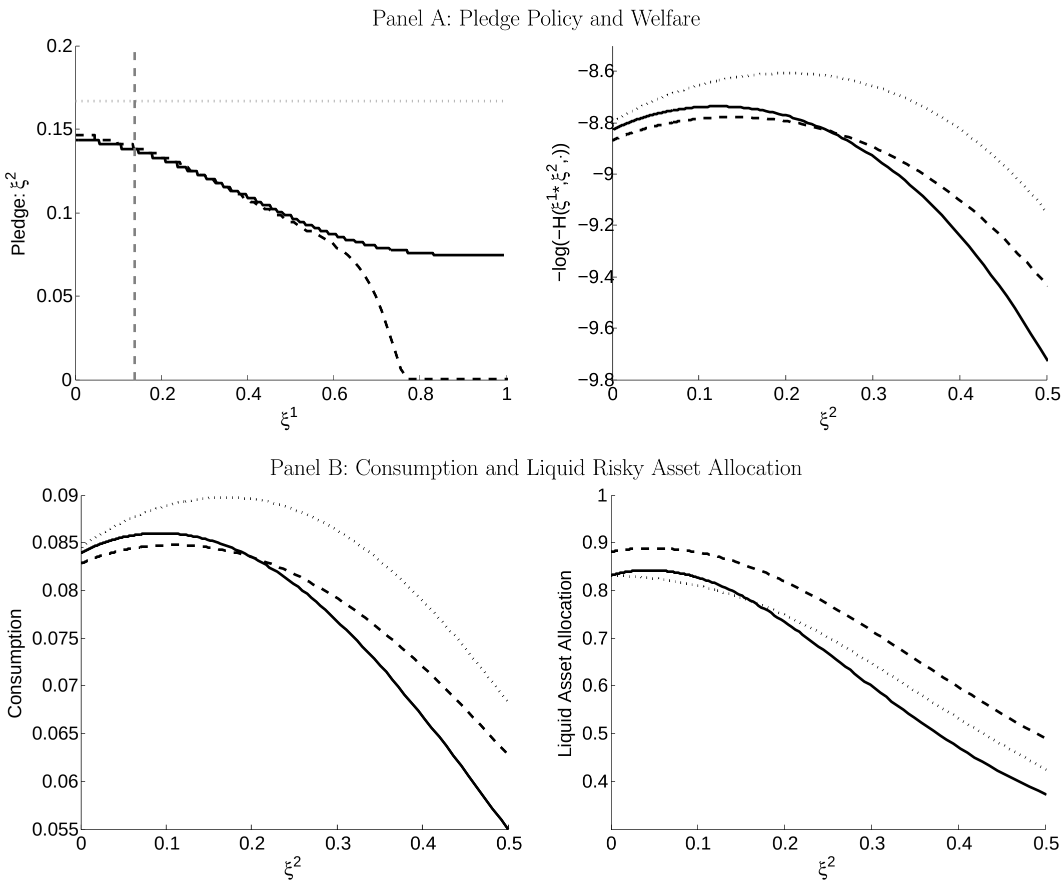


Panel C: Call to Distribution



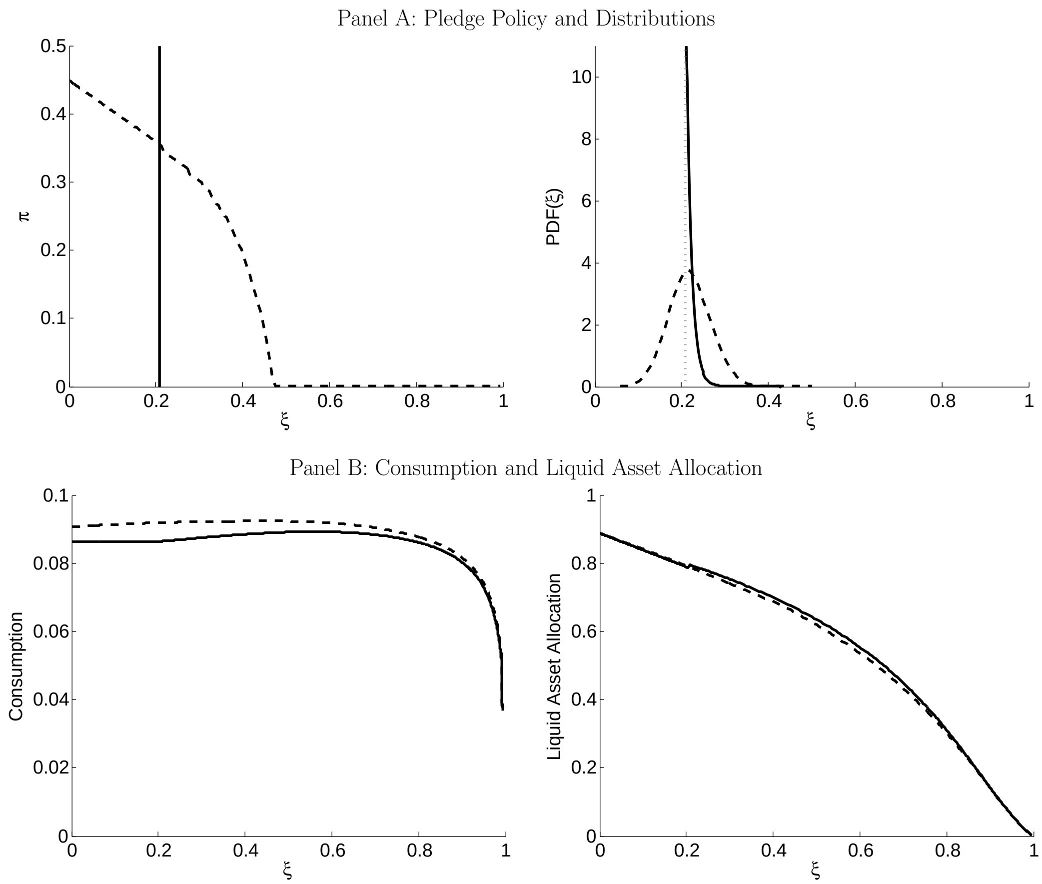
Panel A presents the premiums associated with the M-clock economies, and Panels B and C present the results of the simulation of average ex-post condition utility values, both described in Section 5.1.

Figure 5: $N = 2$ Welfare and Policies



We present analysis of the economy with $N = 2$. Panel A (left side) presents the optimal pledge policy for fund 2, given the current allocation to fund 1. The grey vertical line is the optimal value of ξ^1 (Ξ^*), and the dotted line in the optimal value of a pledge in the $N = 1$ economy. The dashed line shows the optimal pledge amount before fund 1 has called capital, and the solid line shows the optimal pledge value afterwards. Panel A (right side) presents the value function as ξ^2 varies and ξ^1 is held at Ξ^*. The dashed line shows the value before either fund 1 or 2 has called capital. The solid line shows the value after fund 1 has called but before fund 2 has called. The dotted line shows the value after both have called. Panel B presents the consumption policy and the liquid risky asset allocation, both as a fraction of total wealth, for the same three states of the economy.

Figure 6: $N = \infty$ Welfare and Policies



We present analysis of the economy with $N = \infty$. Panel A (left side) presents the optimal pledge policy for the standard economy (Problem 7, dashed line). The dotted line is the minimum value of ξ in the economy in which the pledge can be changed (Problem 8, solid line). Panel A (right side) shows the distribution of ξ in the same two economies, simulated as described in Appendix A. Here the dotted line is the minimum value of ξ in the 'Change Pledge' economy and the solid line the PDF of ξ in that economy. The dashed line is the PDF of xi in the standard economy. Panel B presents the consumption policy and the liquid risky asset allocation, both as a fraction of total wealth.