One-shot quantum state redistribution and quantum Markov chains

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Quantum state redistribution holds an important place in the theory of quantum communication. This task is well understood in the asymptotic and i.i.d. setting [7] and has a rate of communication given by quantum conditional mutual information (QCMI). Some achievable rates have also been been established in the one-shot setting in Refs. [3, 6, 2]. However, we do not know of an explicit optimal or near-optimal expression for the one-shot rate. The problem of characterizing the rate remains an important open problem in one-shot quantum information theory as well in other areas such as quantum communication complexity [9], given the applications of one-shot quantum state redistribution. In fact, a near-optimal bound on the one-shot rate for a classical version of this task was only recently provided in Ref. [1], where the rate was shown to be $\mathrm{D}^{\epsilon}_{\max}(P_{XYZ}\|Q_{XYZ}) + \mathrm{O}(\log(1/\epsilon))$, where P_{XYZ} is the joint distribution of XYZ, Z is the random variable to be redistributed, Q_{XYZ} is the Markov chain given by the distribution $Q_{XYZ}(xyz) \coloneqq P_{X|Y}(x)P_Y(y)P_{Z|Y}(z)$, and $\mathrm{D}^{\epsilon}_{\max}$ denotes *smooth max-relative entropy*. $\mathrm{D}^{\epsilon}_{\max}$ is a one-shot version of the relative entropy that is well known to be associated to source compression tasks (both classical and quantum).

Is it possible that one-shot quantum state redistribution could also be characterized by a quantity similar to that in the classical case? A powerful counterexample implicit in Ref. [5] (see also Ref. [8, Section VI]) definitively rules out the immediate generalization of the classical bound. The work presents an example where the quantum conditional mutual information $\mathrm{I}(R:C\,|\,B)_\psi$ of a quantum state ψ^{RBC} is significantly smaller than the regularised version of the distance $\min_{\sigma^{RBC}} \mathrm{D}(\psi^{RBC}\|\sigma^{RBC})$ of the state ψ from quantum Markov chains, i.e., from σ^{RBC} such that $\mathrm{I}(R:C\,|\,B)_\sigma = 0$, where D denotes relative entropy. Since regularised max-relative entropy equals relative entropy, a direct quantum analogue of the classical bound cannot characterize the communication rate for one-shot quantum state redistribution.

An alternative expression for QCMI is as follows (see Ref. [4, Lemma 1]):

$$\mathrm{I}(R:C\,|\,B)_\psi \quad = \min_{\sigma^{RBC}\,:\,\mathrm{I}(R:C\,|\,B)_\sigma=0,\ \sigma^{RB}=\psi^{RB}} \Big(\mathrm{D}\big(\psi^{RBC}\big\|\sigma^{RBC}\big) - \mathrm{D}\big(\psi^{BC}\big\|\sigma^{BC}\big)\Big)\ . \tag{1}$$

This equation connects QCMI and quantum Markov chains. Is there a one-shot protocol for quantum state redistribution with communication given by a one-shot analogue of the expression in Eq. (1)? Such a protocol would connect quantum state redistribution and quantum Markov chains in an operational way and open avenues for application in settings where quantum Markov chains play an important role.

In this work, we present a protocol that achieves the above goal, with communication cost that resembles that in Eq. (1) with the two instances of D replaced by $\mathrm{D}^{\epsilon}_{\max}$ and $\mathrm{D}^{\epsilon}_{\mathrm{H}}$ appropriately, where $\mathrm{D}^{\epsilon}_{\mathrm{H}}$ denotes *smooth hypothesis testing relative entropy*. Given a quantum state ϕ^{RBC}, we identify a natural subset of Markov extensions of ϕ^{RB}, denoted by $\mathsf{ME}^{\epsilon,\phi}_{R-B-C}$, and establish the following theorem.

Theorem 1. *For any pure quantum state $|\psi\rangle^{RABC}$, the quantum communication cost of redistributing the register C from Alice (who initially holds AC) to Bob (who initially holds B) with error $\mathrm{O}(\sqrt{\epsilon}\,)$ is*

$$\frac{1}{2}\inf_{\psi'\in\mathsf{B}^{\epsilon}(\psi^{RBC})}\ \inf_{\sigma^{RBC}\in\mathsf{ME}^{\epsilon,\psi'}_{R-B-C}} \left[\mathrm{D}_{\max}\big(\psi'^{RBC}\|\sigma^{RBC}\big) - \mathrm{D}^{\epsilon}_{\mathrm{H}}\big(\psi'^{BC}\|\sigma^{BC}\big)\right] + \mathrm{O}\left(\log\frac{1}{\epsilon}\right)\ .$$

The communication cost of our protocol is smaller than the prior achievable communication costs in Refs. [2, 3]. When ψ^{RBC} is classical, the communication cost reduces to a one-shot version of the classical conditional mutual information. This cost is similar to the near-optimal communication cost in Ref. [1]. The key technique underlying the protocol is a reduction procedure using embezzling quantum states, that allows us to use the protocol of Ref. [2] as a subroutine. In the special case where ψ^{RBC} is a quantum Markov chain, our protocol leads to near-zero communication as a black-box. This feature, which requires embezzling states, was not known for the previous protocols.

References

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