

Contextuality and memory cost of simulation of Majorana fermions

Susane Calegari¹, Juani Bermejo-Vega², and Michał Oszmaniec¹

¹Center for Theoretical Physics, Polish Academy of Sciences, Al. Lotników 32/46, 02-668 Warszawa, Poland

²University of Granada, Granada, Spain

November 16, 2020

Contextuality has been reported to be a resource for quantum computation [1], analogous to non-locality which is a known resource for quantum communication and cryptography [2]. In recent work, Karanjai et al. [3] show that the presence of contextuality places a lower bound on the memory required to classically simulate quantum processes. In particular, they bound the memory cost of simulating Clifford operators. The method to derive lower bound works for a general family of classical simulations that can be framed as stochastic evolution in ontological models (hidden variable models): For instance, it generalizes the Gottesman-Knill [4] and Wigner function simulation methods. However, the techniques in [3] are limited to quantum processes that of closed sub-theories, i.e., where products of measurable quantities are measurable, which are a very limited class of quantum circuits. Consequently, it excludes processes where measurements are loyal.

In this work, we generalize this connection to non-closed circuit families with a broader proof of contextuality, namely, observable-based proof of contextuality [5]. We show that the presence of contextuality places new lower bounds on the memory cost for simulating restricted classes of quantum computation. We apply this result to the simulation of the restricted model of quantum computation based on the braiding of Ising anyons known as topological quantum computation (TQC) model [6]. This model is the first known scheme of magic states distillation [7], a leading paradigm in fault-tolerant quantum computing. It is also of fundamental interest in the study of quantum resources that power quantum computation, as it lies at the intersection of two classically simulable sub-theories: FLO and Clifford circuits. For the TQC model, we prove that the lower bound in the memory required in a simulation is $n \log_2 n$, where n is the number of fermionic modes. This bound is extended to fermionic linear optics (FLO), a fermionic analogous of bosonic linear optics.

Lower bound in the memory cost Since our goal is to simulate quantum statistics, the ontological model used for the classical simulation will reproduce the Born rule probabilities of a quantum sub-theory. In the classical simulation the density matrix is represented by a probability distribution $\mu_\rho(\lambda)$ over the state space Λ , and the unitary channels become stochastic maps, $\Gamma_U(\lambda'|\lambda)$, while the measurements become sub-stochastic maps, $\Gamma_O(\lambda', k|\lambda)$. After a measurement the probability distribution $\mu_\rho(\lambda)$ is updated to $\mu_{\rho'}(\lambda')$ with probability $Pr(O, k|\rho, \lambda) = \sum_{\lambda', \lambda} \Gamma_O(\lambda', k|\lambda) \mu_\rho(\lambda)$. The internal state $\lambda \in \Lambda$ contains all the information necessary to characterize the statistics of all measurements allowed in the sub-theory. The lower bound in the space com-

plexity is obtained by finding a lower bound in the size of the state space Λ required to simulate the sub-theory.

Thus, we define a *sub-theory* as a set of observables, channels, and states that one can use in a class of experiments. In a fixed sub-theory we consider a set of quantum states $S = \{\rho_i\}$ and the set of all observables \mathbb{O}_S that have at least one eigenstate in S . We show that if \mathbb{O}_S is contextual, then $\bigcap_s \text{supp}(\mu_{\rho_i}) = \emptyset$ for any simulation of this sub-theory. States that are not single-shot distinguishable [3] must be classically represented by probability distributions that have intersecting supports, i.e., these states must share at least one internal state λ . This limits the size of the state space. By considering the sub-theory defined by the set of quantum states S_{total} , we prove that the lower bound for the size of the state space of this sub-theory is $|\Lambda| \geq |S_{total}|/m$ where m is the cardinality of the largest set $S \subset S_{total}$ that has a corresponding set of observables \mathbb{O}_S that are non-contextual. Therefore, the memory cost of simulating this sub-theory is lower bounded by $\log_d(|\Lambda|) \geq \log_d(|S_{total}|) - \log_d(m)$.

Application to the TQC with Ising anyons and FLO model – In the TQC model with Ising anyons [6], the initial state can be mapped to other states by the use of unitary channels also known as braid gates, $U_{ij} = \exp(-\frac{\pi}{4} m_i m_j)$, and measurable observables, $X_{ij} = -i m_i m_j$, with eigenvalues ± 1 , where m_i are Majorana operators. Majorana operators obey the commutation rules $m_i m_j + m_j m_i = 2\delta_{ij} \mathbb{I}$ and $m_i^\dagger = m_i$ for any i, j .

We show that the lower bound in the memory cost for the simulation of Majorana fermions scales in $n \log_2 n$ in the number of fermionic modes. We also show that the scaling is optimal using the classical simulation methods in [4].

Quantum computation with fermionic linear optics (FLO) can be seen as a generalization of the TQC with Ising anyons, where the unitaries, called FLO gates, are not restricted to the $\pi/4$ angle [8]. We extend our results and prove that the lower bound in the memory required in an ϵ -approximate simulation of the FLO model is $n^2 \log_2(\frac{1}{\epsilon})$.

Thus, our work establishes a connection between contextuality and memory cost of classically simulating quantum circuits. We do so for the most minimal scheme of magic state distillation in a physically motivated setting. We develop new techniques to derive lower bounds in the memory cost of simulating physical sub-theories and apply them to fermions for the first time.

References

- [1] Mark Howard, Joel Wallman, Victor Veitch, and Joseph Emerson. Contextuality supplies the ‘magic’ for quantum computation. *Nature*, 510(7505):351–355, 2014.
- [2] Peter W Shor and John Preskill. Simple proof of security of the bb84 quantum key distribution protocol. *Physical review letters*, 85(2):441, 2000.
- [3] Angela Karanjai, Joel J Wallman, and Stephen D Bartlett. Contextuality bounds the efficiency of classical simulation of quantum processes. *arXiv preprint arXiv:1802.07744*, 2018.
- [4] Scott Aaronson and Daniel Gottesman. Improved simulation of stabilizer circuits. *Physical Review A*, 70(5):052328, 2004.
- [5] Samson Abramsky and Adam Brandenburger. The sheaf-theoretic structure of non-locality and contextuality. *New Journal of Physics*, 13(11):113036, 2011.
- [6] Sergey Bravyi. Universal quantum computation with the $\nu=5/2$ fractional quantum hall state. *Physical Review A*, 73(4):042313, 2006.
- [7] Sergey Bravyi and Alexei Kitaev. Universal quantum computation with ideal clifford gates and noisy ancillas. *Physical Review A*, 71(2):022316, 2005.
- [8] Sergey Bravyi. Lagrangian representation for fermionic linear optics. *arXiv preprint quant-ph/0404180*, 2004.