Terror, Tragedy & Vibrations

Using Mathematical Models in Engineering

ISBN: 1 86365 539 5

PREFACE

"Rationally, let it be said in a whisper, experience is certainly worth more than theory." ... Amerigo Vespucci (1451–1512)

Terror, Tragedy and Vibrations – Using Mathematical Models in Engineering examines three situations where mathematics plays a key role in engineering projects. In terms of the quotation from Vespucci, it provides the experience on which students can build their theoretical understanding. The video also highlights the importance of communication skills for professionals in the mathematical and engineering professions.

This booklet is designed to supplement the video by providing a range of problems for students to explore, both individually and in groups. The questions include traditional problems based on calculus and algebraic techniques. They also include questions where students can check their understanding of the language and terminology involved. Some questions involve group discussions, and others involve writing reports for people who might not understand the technical terminology.

The video is appropriate for introductory courses in calculus at senior high school or early university level to give examples of practical, realistic and useful applications of mathematical methods. It would also be a useful resource for engineering courses, and to promote engineering courses and the engineering profession.

How to use the video

- You can play the video right through to introduce and illustrate the variety of mathematical methods used in engineering. One of the questions from the booklet can be used to start a class discussion about the uses of mathematics in engineering.
- You can play one section of the video to introduce a particular topic in mathematics: motion under gravity, exponential functions or periodic functions. The class can work in small groups on one of the questions from that section of the booklet and another question can be set for homework.
- Towards the end of a course you can play the video right through for revision of the mathematical ideas or for a final discussion on mathematics in engineering. One of the remaining exercises from the booklet can be used to focus the discussion.

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1. Terror – *The Tower of Terror*

1.1 Here is a description of the motion of the car in *The Tower of Terror*. Copy out the sentences using words from the list to fill in the spaces (each word can be used more than once).

acceleration, height, parabola, tangents, time, gravity, velocity

We concentrate on the vertical direction and model how the car rises and falls under the influence of ________ . We plot ________ on the vertical axis and ________ on the horizontal axis. The graph of the motion looks like part of a ________ . The slope of the ________ to this graph show the vertical ________ at each time, positive for the first part of the trip and negative as the car falls back. Remember that ________ is the derivative of ________ with respect to ________ . When we plot ________ against ________ we get a straight line with gradient –9.8, the ________ due to ________ .

1.2 One of your colleagues shows you the following sentence and says that they don't understand it (from Dreamworld's *Our World*, No 30 Feb/Mar 1999): *Peaking at 119 metres high, passengers on The Giant Drop experience the exhilaration of falling from 39 storeys at the speed of gravity (up to 135 kilometres per hour).* Explain the mistake and re-write the sentence correctly so that your colleague can understand it.

1.3 Imagine you are designing the Tower of Terror ride. You want to shoot a car 110 metres upwards (the tower is 119 metres high, but the car starts 3 metres above the ground and you want 6 metres as a safety margin at the top). What initial vertical velocity will the car need to get to this height? If the curve of the rail turns horizontal speed into vertical speed with a loss of only 5%, what horizontal speed will the car need to reach? If the car can be accelerated over a distance of 40 metres, what (constant) acceleration will you need to design?

1.4 A Dreamworld newsletter writes about the tiger cubs born in 1999: *For most, a weight gain of more than 25 kilograms in five months would raise some concern. However, for Sita, Taj, Rama and Sultan, this increase is a healthy one. The tiger cubs, who weighed approximately 1.5 kilograms when born on October 23 last year, have been steadily gaining weight and presently tip the scales at around 28 to 30 kilograms.* Let $w(t)$ be the weight of a cub at time t months after birth. Write down an equation involving the derivative dw/dt expressing a weight gain of ‘25 kilograms in five months’. What information is given about the situation at time $t = 0$? Solve the equation to write down the function that expresses the weight of the cub in terms of time.

1.5 In the video, General Manager Len Shaw says: *At all stages of the design, construction and running of the ride, safety is a prime consideration. The Australian AS3533 Standard for amusement rides is the only genuine complete standard in the world.* What does AS3533 contain, and how does it help to ensure the safety of a ride? Write a short report on the standard for a client who has engaged you as a consultant engineer to help design and construct a new ride.

1.6 The Giant Drop is a ride at Dreamworld that shares the tower of the Tower of Terror. Here is part of a description from ‘The Giant Drop Fast Facts’: *Riders’ legs dangle freely from the open air 8-seat gondola as they plummet at speeds of up to 135 km/hour during the 119 metre (39 storey) free fall. They pull 4Gs and come close to terminal speed. A sophisticated magnetic braking system stops the gondolas metres from the ground. Freefall time is 5 seconds, the suspenseful winch to the top takes approximately 90 seconds. It could be the most exhilarating 100 seconds of your life.* Use Newton’s laws to check the details of the motion for this ride. It will help you to know that the magnetic braking system actually starts to work at a height of 45 metres, and that the term "freefall" in the description seems to be used rather loosely to include the whole descent.

1.7 *Work on this exercise with a partner.*
The description of The Giant Drop states that the riders ‘come close to terminal speed’ (or terminal velocity). What is terminal velocity? What causes terminal velocity? Do all falling objects have the same terminal velocity? Does the notion of terminal velocity contradict Newton’s laws? Would terminal velocity be different on other planets? You can use the following newspaper extract as an example (Sunday Telegraph, 26 May 1991):
A skydiver who survived after falling nearly 3000 m when her parachute malfunctioned said she would like to go jumping again, even though doctors strongly advised her against it. In what is being called a "miraculous" survival, Jill Shields, 31, plunged to earth last week, her fall broken only by tree branches and the patch of swamp mud she landed in. She has spinal bone fractures, a fractured pelvis, two broken ribs and several bruises. There are no signs of paralysis. Ms Shields said she does not remember much about her fall and thinks she blacked out after 300 metres. She hit the ground at

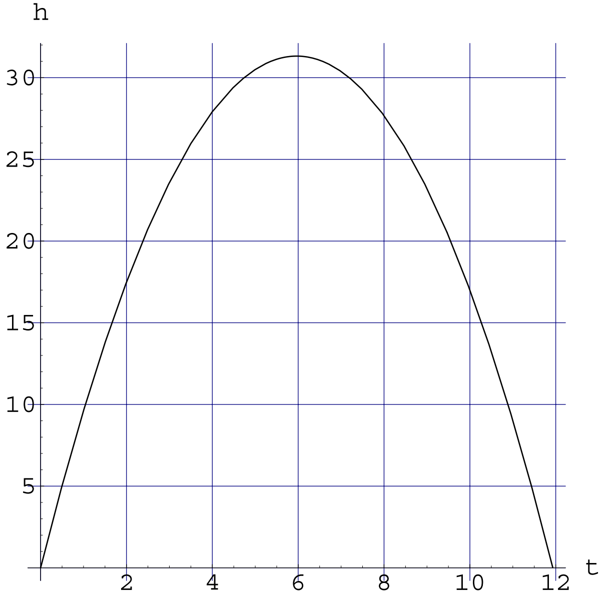
approximately 160 km/h, leaving a 37 cm-deep hole in the ground. She is in what doctors describe as a stable and alert condition, but they are amazed she is still alive.

1.8 In the video, presenter Natasha Mitchell says: *Acceleration is constant near the surface of the Earth, so the speed of fall depends only on the height or the time of the fall, but not on the car's mass. On other planets, the gravity and hence the acceleration are different.* The acceleration due to gravity (m/s^2) for bodies in the solar system is given as:

Mercury 3.31	Mars 3.61	Uranus 8.23	Moon 1.66
Venus 8.09	Jupiter 24.6	Neptune 11.2	
Earth 9.80	Saturn 10.4	Pluto 4.09	

A spanner is accidentally dropped by an astronaut from a height of 100 m and takes about seven and a half seconds to fall to the ground. On which body is the astronaut? How long would the spanner take to fall on the Moon and on Jupiter, the bodies with the least and greatest acceleration?

1.9 The graph shows the height *h* metres of a rock as a function of time *t* seconds since it was thrown upwards on one of the bodies in the Solar System. Use the graph to estimate the initial velocity of the rock, and also the acceleration due to gravity, to find out where this is taking place.



2. Tragedy – *The Flight of UAL811*

2.1 Here is a description of the relationship between height and air pressure. Copy out the sentences using words from the following list to fill in the spaces (each word can be used more than once).

height, pressure, decreases, increases, dependent, independent, horizontal, vertical

At sea level, air pressure is about 100 kilopascals, but as height ________ the air pressure ________ . At the cruising ________ of an aircraft, the air ________ is only about 40 kilopascals. The graph shows air pressure on the ________ axis and height above sea level on the ________ axis. Since height is the ________ variable, we graph it on the ________ axis. Because ________ depends on ________ , it is the ________ variable, which we graph on the ________ axis.

2.2 *Work on this exercise with a partner.*
In the video we found that the relation between air pressure and height is given by the exponential function $P(h) = 100e^{-0.125h}$. Find out about the constant *e* that appears in this equation. What is its numerical value? What are some of its important properties? Why is it called *e*? What is the connection between *e* and logarithms? Prepare a written report of one or two pages on this topic, or a presentation that you can give to the rest of your class.

2.3 In the video, presenter Natasha Mitchell says: *We can fly comfortably because the aircraft cabin is pressurised. The design of the aircraft must allow for the difference in pressure between the inside and the outside of the cabin. Any hole in the cabin wall will cause a sudden drop in pressure in the cabin as the inside pressure equalises with the outside pressure.* Find two other examples of situations where difference in pressure needs to be considered in engineering design. Prepare a short report on one of these examples for a client who has hired you as a consulting engineer.

2.4 *Work on this exercise in a group of three or four people.*
If the relationship between two variables is $y = Ae^{kx}$, then taking logarithms of both sides gives $\ln y = \ln A + kx$.
This is a linear relationship with intercept ln *A* and slope *k*.
So, if the graph of $\ln y$ against *x* is a straight line, the relationship between *y* and *x* is exponential. This is another way of checking for an exponential relationship.

The data below show the air pressure at different heights above sea level. Does the graph of P against h seem to be exponential? Plot logarithms of the pressure values against height. Do you get a straight line? Find the equation of this straight line, giving the relationship between ln P and h. Then use this to find the equation relating P and h.

height h (km)	pressure P (kPa)
0	100.0
1	88.2
2	77.9
3	68.7
4	60.7
5	53.5
6	47.2
7	41.7
8	36.8
9	32.5
10	28.7

2.5 *Work on this exercise with a partner.*
If you ever go scuba diving, you will need to learn about the effect of depth on pressure. On the surface of the sea, the pressure is 1 atmosphere, but by the time you dive to a depth of 10 metres, the pressure has doubled. In fact, there is a linear relationship between the variables depth and pressure. Write down an equation that expresses the pressure P (in atmospheres) in terms of the depth d (metres). If you are using normal compressed air, you can dive safely up to a pressure of 3.5 atmospheres, how deep can you dive? Can you find out why the relationship between pressure and height is exponential, but between pressure and depth is linear?

2.6 A satellite has a radio-isotope power supply. The isotope decays exponentially, so the rate at which the power $P(t)$ is decreasing satisfies the equation $P'(t) = -0.005P(t)$, where t is the time in days since the satellite was launched. Initially, the power was 60 Watt.

(a) Find an expression for the power $P(t)$ as a function of t. Sketch a graph of this function.
(b) How much power will be available after one year? At what rate is the power decreasing at this time?
(c) What does the term *half-life* mean? What is the half-life of the satellite's power supply?
(d) The equipment aboard the satellite requires 15 Watt of power to operate. What is the operational life of the satellite?

3. Vibrations – *The Anzac Bridge Cables*

3.1 Here is a description of the motion of the cables on the Anzac Bridge. Copy out the sentences using words from the following list to fill in the spaces (each word can be used more than once).

harmonic, displacement, sine, derivative, time, vibration, differential

Take one of the cables holding the deck of the bridge. When it moves, it vibrates: what mathematical model can we use to express the amount of ________ at the centre of the cable? A graph of ________ at the centre of the cable against ________ has the shape of a ________ curve. When we graph the ________ against ________ or ________ we don't get a straight line. However, if we graph the second ________ against the ________ , we do get a straight line. From this we can write down a ________ equation and find the function that gives ________ in terms of ________ . This type of motion is known as simple ________ motion.

3.2 *Work on this exercise with a partner.*
The Millennium Bridge is a 320-metre footbridge over the river Thames in London. It was built at a cost of $A46 million, but was closed on its opening weekend in June 2000 due to a serious design fault which caused the bridge to swing dangerously when pedestrians walked across it. Find out more about this bridge (*Bad Vibrations*, New Scientist, 8 July 2000 might be a good place to start). Prepare a short presentation for your class on the problem.

3.3 *Work on this exercise in a group of three or four people.*
Read the following definition of simple harmonic motion (from *Chambers Science and Technology Dictionary*, 1991). Prepare a short presentation for your class on this type of motion, using slides, handouts or other materials. In your presentation, you should explain all the terms used in the definition and illustrate some of the examples the dictionary has given (or maybe use some others).
Simple Harmonic Motion: The motion of a particle (or system) for which the force on the particle is proportional to its distance from a fixed point and is directed towards the fixed point. The particle executes an oscillatory motion about the point. The motion satisfies the equation $(d^2x/dt^2) = -\omega^2 x$ where x is the displacement of the particle and ω is a constant for the motion.

The majority of small amplitude oscillatory motions are simple harmonic, e.g., the oscillations of a mass suspended by a spring, the swing of a pendulum, the vibrations of a violin string, the oscillations of atoms or molecules in a solid, the oscillations of air as a sound wave passes. When such a motion takes place in a resistive medium, e.g. air, the oscillations die away with time; the motion is then said to be *damped*.

3.4 *Work on this exercise with a partner.*
In the video, RTA Engineer Peter Wellings says: *Every engineering project has its problems. Some of them are technical, and to solve them you need a good technical base, including knowledge of mathematical models. Others are broader: community consultation, environmental concerns or safety issues. These problems need communication skills and an ability to work with and motivate other people.* Select an engineering project and investigate what community, environmental and safety problems had to be considered. Discuss how these problems were solved, and what skills were needed by the project engineers.

3.5 *Work on this exercise in a group of three or four people.*
In the video, Natasha Mitchell describes how to set up a mathematical model of a physical situation: *The plan of attack is to find some straight-line relationship between variables, or more commonly between derivatives and variables. From this linear relationship, we can find an equation with derivatives in it, called a differential equation. From this, we can find the function.* What is a differential equation (or DE), and how can you solve one? How can you check that a specific function is a solution of a DE? Give two examples of DEs that occur in engineering, and show how to solve one of them.

3.6 The volume V (litres) of air in the lungs of a person at rest can be described as a function of time t (seconds) by the trigonometric function $V(t) = 0.6 - 0.5\cos 0.25\pi t$.

(a) How long does a breathing cycle take for this person? Graph the function $V(t)$ over one period. What are the minimum and maximum volumes of air in the lungs?

(b) Use calculus to find $V'(t)$ and $V''(t)$. Explain what these functions represent physically.

(c) Prepare a table of values of the functions $V(t)$ and $V''(t)$ for $t = 0, 1, \ldots, 8$. Graph $V''(t)$ against $V(t)$. What type of graph do you get? Can you find its equation?

(d) Use your results from (b) to write $V''(t)$ in terms of $V(t)$. Since this equation involves a (second) derivative, it is called a differential equation. Compare your differential equation with your result from (d).

4. Equations from Data – *Harder Examples*

Use a computer package for these questions, if possible.

4.1 *Work on this question with a partner.*
Sometimes, rather than an equation, you only have function values. Show how to estimate the derivative of a function using central differences or some other method. Use the table of values for a function $s(t)$ that describes distance s metres travelled in t minutes by an animal.

time t	0	1	2	3	4	5	6	7	8	9	10	11
dist s	0	2	5	10	19	28	34	38	41	43	44	44

4.2 The differential equation that models an object falling under the influence of gravity is obtained from Newton's second law of motion (mass × acceleration = force). If m is the mass of the object, v is the velocity and g is the acceleration due to gravity, the equation is $m(dv/dt) = mg$. We are taking our zero point from the object's initial position, so that g and v are positive in the downward direction.

(a) Solve the differential equation with $g = 9.8 \text{ ms}^{-2}$.
(b) If a skydiver takes 12 seconds to freefall to the ground, what was his initial height?
(c) Calculate a table of values showing the distance fallen by t seconds, for t = 0 to 12. Show these values on a graph.
(d) The table below shows the actual distance fallen (by a dummy with mass 65 kg) from time t = 0 to 12. Compare these values with those you found in part (c). Why are the values different?

time t (s)	distance x (m)
0	0
1	4.9
2	18.9
3	42.1
4	73.8
5	111.6
6	153.6
7	198.7
8	246.3
9	296.0
10	346.9
11	399.0
12	452.0

(e) Draw a plot of the data from part (d) and compare with your previous graph. Describe the differences.

(f) At some velocities, air resistance on a falling body is proportional to the square of the velocity and acts in the opposite direction to the velocity. Explain carefully why the equation that describes the motion of the dummy is $m\frac{dv}{dt} = mg - kv^2$.

(g) Use central differences (or an equivalent graphical method) to estimate values of velocity and then acceleration for the data in (d). Make reasonable guesses for the missing values at the first and last time points.

(h) If you plot the acceleration dv/dt against v^2, you should get a straight line with intercept 9.8. From the slope of this line, find the value of k for this particular falling body.

(i) If you have studied the appropriate techniques, solve the differential equation from (f) by first separating variables and then using partial fractions. Check your solution by comparing values with those in the table in (d).

4.3 Lead can be absorbed into the body from the environment. Here is a table of values to show how lead is absorbed into the blood when the lead levels in the environment are constant.

time t (years)	lead in blood x (mg)	lead in tissues z (mg)
0	0	0
1	4.2	4.6
2	7.4	8.5
3	9.7	11.8
4	11.4	14.7
5	12.7	17.2
6	13.6	19.4
7	14.3	21.2
8	14.9	22.8
9	15.2	24.2
10	15.5	25.4
11	15.7	26.4
12	15.9	27.3

(a) Plot the lead levels in the blood against time. What sort of relationship does the graph show?

(b) Use central differences (or an equivalent graphical method) to estimate values of the rate of uptake of lead into the blood, that is, the derivative dx/dt.

(c) Find a straight-line relationship between one of the variables and the derivative by trying various plots. Use this to write down a differential equation that describes the level of lead in the blood as a function of time.
(d) Solve the differential equation to find the function that describes the amount of lead in the blood at time t years. What happens to the lead in the blood as time tends to infinity?
(e) Lead is absorbed from the blood to the body tissues at a different rate. Repeat the process above to find the relationship between time and the amount of lead in body tissues. Again, examine what happens as time tends to infinity.

4.4 Kirstie has a problem with her weight. She has kept a record of her weight in kilograms over the last two years, making a measurement on the first day of every month:

month t	weight w	month t	weight w
0	60	12	60
1	65	13	65
2	69	14	69
3	70	15	70
4	69	16	69
5	65	17	65
6	60	18	60
7	55	19	55
8	51	20	51
9	50	21	50
10	51	22	51
11	55	23	55

(a) Plot Kirstie's weight w against time t. What type of mathematical curve would describe her weight over time? Write down the equation of the function $w(t)$.
(b) Use central differences (or an equivalent graphical method) to estimate the derivative dw/dt and the second derivative d^2w/dt^2.
(c) Find a straight-line relationship between the variables and derivatives by trying various plots. Use the equation of this line to write down a differential equation satisfied by $w(t)$.
(d) Show that the function you wrote down in (b) is in fact a solution to the differential equation.

ANSWERS

1.1 gravity, height, time, parabola, tangents, velocity, velocity, height, time, velocity, time, acceleration, gravity.

1.2 "Speed of gravity" doesn't make sense, since gravity causes constant acceleration. Use "under the influence of gravity". "Up to 135 km/hr" is maximum speed reached while falling.

1.3 Using $v^2 = u^2 + 2as$ with $a = -9.8$, $s = 110$ when $v = 0$ gives $u = 46.4$ m/s as the initial vertical velocity. Horizontal velocity needs to be 48.9 m/s to allow for 5% loss. Applying the same equation, we need $v = 48.9$, $u = 0$ and $s = 40$, giving $a = 29.9$ m/s^2, about 3 Gs.

1.4 $\frac{dw}{dt} = 5, w(0) = 1.5, w(t) = 5t + 1.5$

1.6 Use downward direction as positive. During freefall in the first 74m, riders are subject to 1G (1 gravity) and reach a velocity of 38 m/s (use $v^2 = u^2 + 2as$), ie 137 km/hr. Then magnets decelerate the riders at about 4G, ie $a = -40$ m/s^2, so the speed is reduced to zero in about 18 metres, then 27 m slow descent.

1.8 Using $s = ut + \frac{1}{2}at^2$ with $s = 100$, $u = 0$ and $t = 7.5$ gives $a = 3.56$, so Mars or Mercury. On Mars, the actual time of fall would be 7.4 s, on Mercury it would be 7.8 s.

1.9 Initial velocity $u = 10.3$ m/s and $a = -1.7$ m/s^2, so must be on Moon.

2.1 increases, decreases, height, pressure, vertical, horizontal, independent, horizontal, pressure, height, dependent, vertical.

2.4 $\ln P = 4.605 - 0.125h$, so $P = 100\exp(-0.125h)$.

2.5 $P = 0.1d + 1$, 25 m. Air is compressible but water is not.

2.6 (a) $P(t) = 60\exp(-0.005t)$

(b) 9.7 Watt, –0.048 Watt/day

(c) Time until power is halved, 139 days

(d) Two half-lives, ie 277 days

3.1 vibration, displacement, time, sine, derivative, time, displacement, derivative, displacement, differential, displacement, time, harmonic.

3.6 (a) Period 8 seconds, Minimum 0.1 L, maximum 1.1 L.

(b) $V'(t) = 0.125\pi \sin 0.25\pi t$ is the velocity of air into the lungs, $V''(t) = 0.03125\pi^2 \cos 0.25\pi t$ is the acceleration.

(c) Linear graph

(d) $V''(t) = 0.0625\pi^2[0.6 - V(t)]$ $= 0.37 - 0.62V(t)$

4.2 (a) $v = 9.8t$

(b) Using $h = 4.9t^2$, 705.6 m

(c) Parabola, vertex at the origin.

(d) Values less than those in (c) due to air resistance.

(e) Starts as a parabola, as in (c), but becomes more linear.

(g) Note that $v = 0$ and $a = 9.8$ when $t = 0$.

(h) $\frac{dv}{dt} = 9.8 - 0.0033v^2$, $k = 0.215$

(i) $$v = \sqrt{\frac{mg}{k}}\;\frac{1 - \exp(-2t\sqrt{gk/m})}{1 + \exp(-2t\sqrt{gk/m})}$$

4.3 (a) Exponential relationship.

(c) Linear between x and dx/dt, $\frac{dx}{dt} = 4.99 - .306x = .306(16.3 - x)$

(d) $x = 16.3(1 - e^{-0.306t})$. As t increases, x tends to 16.3 mg.

(e) $z = 33(1 - e^{-0.15t})$. As t increases, z tends to 33 mg.

4.4 (a) Sine curve, $w = 60 + 10\sin(\pi t/6)$

(c) Linear between d^2w/dt^2 and w, $d^2w/dt^2 = -4w + 60$