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Research Paper 369 March 2016

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ISSN 1441-8010 www.qfrc.uts.edu.au

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Abstract: Game put and call options are defined and solved with an approach based on differential equations which completely bypasses their usual treatment as stochastic Dynkin games. The view is taken that when the writer of the option plans to cancel an American put or call at particular values of the underlying asset, then those assets function as two-sided barriers for the cancellable option. A game option results when the location of the barriers is chosen such that the value of the option is minimized. With elementary maximum and comparison principles from differential equations, optimal cancellation strategies can be readily found and interpreted graphically for perpetual options. An analogous treatment appears possible for finite time game options. An application of the approach to an American game CEV call and to callable stock loans is described.

Introduction

The holder of a standard American or European option acquires certain rights specified by the option contract while the writer of the option is obligated to service the option as written. A modification of the standard option results when the holder retains all rights, but the writer is allowed to cancel the option at all or restricted times by paying the holder an agreed upon penalty charge. When the writer plans to terminate the option at such a time that its value to the holder is minimized, then the resulting cancellable contract is known as the game or Israeli version of the original option.

When Kifer introduced game (Israeli) options in 2000 he realized that they could be formulated as a stochastic optimal stopping Dynkin game [1]. Just as the optimal stopping time formulation for standard American options leads to free boundary problems and their associated variational inequalities, it is known that the Dynkin games also lead to variational inequalities which can then be solved numerically. This approach is exploited in detail in the thesis of [2] for processes with jump diffusion, for the formulation and error analysis of binomial approximations for game put options [3], and for the analysis of perpetual and finite horizon game calls in [10]. It appears that without exception the stochastic view has been, and continues to be, the basis for developping and extending the Dynkin game approach to increasingly more complicated cancellable options such as games with heterogeneous beliefs [4].

It is well known that for standard American options on diffusion and jump processes the valuation of options can also be based on free boundary problems for PDEs and PIDEs which are derivable without explicitly using stochastic theory other than Ito's lemma. This alternate approach is explored in many texts (see, e.g. [5], [6]) and has led to a variety of numerical methods for pricing options and their mathematical analysis. In particular, the solution of such free boundary problems as a sequence of time-discrete problems has been shown to be an effective way to compute both prices and the associated early exercise boundaries [7]. It appears possible that numerical methods for game options can likewise be based on a sequence of time discrete free boundary problems which are derivable and solvable entirely with numerical methods for ordinary and partial differential equations without recourse to the stochastic tools of the Dynkin theory.

This exposition introduces the PDE approach to game options for perpetual options. They have loomed large up to now in the discussion of game options because the options can be explicitly calculated (see, e.g. [8], [9], [10]). We shall rederive selected earlier results with the tools of differential equations from the following points of view.

We assume that the value of the uncancelled option follows a Black Scholes equation. When the writer cancels the option at a given value of the underlying asset, then the resulting new option assumes the value of the penalty payment at that asset value. This makes the point of cancellation a barrier for the new option. If we assume that this barrier is a down and out and up and out barrier with the cancellation value as a rebate, then the options right and left of the barrier are straightforward to calculate. If, however, the point of cancellation is an interior point of an optimal exercise region for the writer, then the boundary of the exercise region must be determined which leads to free boundary problems. The options outside this region are again easy to visualize and calculate as two disjoint barrier options. The game option results when the writer's exercise region is chosen such that the resulting barrier options are a sharp lower bound on all cancellable options. In that case the game option will incur the lowest replication cost.

The identification of the game option is based on elementary maximum and comparison principles for differential equations. It uses the tools of the Riccati transformation, a variant of the Brennan Schwartz method (see [7]), to pin down free boundaries, determine optimal points and intervals for cancellation and find the game option. The diffusive nature of the Black Scholes equation is essential for this approach, but its analytic solution is not needed. This suggests that the PDE approach can also be applied to the inhomogeneous ordinary differential equation for the time discrete approximation of game options with finite expiration.

Numerical simulations for a perpetual game put on a dividend paying asset and for a perpetual capped American CEV call illustrate the PDE approach. We conclude with an application of this approach to rederiving the raw data which are given in [14] and used there for an extensive discussion of fair loan fees for several types of stock loans.

I. The general framework:

Let $u(x,t)$ denote the value of an option written on an asset x at time to expiry $t = T - \tau$ for real time τ. We assume that $u(x,t)$ is a solution of the pricing equation

$$\mathcal{L}_{BS}u(x,t) = f(x,t) \tag{1.1}$$

where $\mathcal{L}_{BS}$ denotes a differential operator of the form

$$\mathcal{L}_{BS}u \equiv a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u + d(x,t)u_t.$$

A Black Scholes equation with variable coefficients would be an example of equation (1.1). It is augmented with boundary conditions on fixed or free boundaries which define the option under consideration. In addition, the intrinsic value

$$u(x,0) = u_0(x)$$

is specified. We shall assume that the value $u(x,t)$ is computable, analytically or approximately, for $t \in (0,T]$.

Let $v(x,t,y)$ denote the value of the option which confers on its holder the same rights as u, but also allows the writer to cancel the option at any time t at any point y by paying the holder the amount

$$v(y,t,y) = H(y,t)$$

where

$$H(x,t) = u_0(x) + \delta(x,t), \quad \delta(x,t) > 0.$$

$\delta(x,t)$ is considered a penalty which the writer incurs by canceling the option at y. It is assumed that the writer will cancel only if the resulting option $v(x,t,y)$ decreases the value of u so that

$$v(x,t,y) \leq u(x,t) \quad \text{for all } x.$$

Let us define

$$D(t) = \{x : H(x,t) \leq u(x,t)\}.$$

Cancellation at time t can occur only at points $y \in D(t)$. The complement of $D(t)$ will contain the continuation region for v where it satisfies (1.1) and the boundary conditions imposed on u.

If the set $D(t)$ is not empty then there will exist at least one admissible cancellation strategy. Suppose that the writer will cancel u whenever the underlying asset falls into $D(t)$. Then the resulting $v(x,t,y)$ is given by

$$v(x,t,y) = v^0(x,t,y) \equiv \min\{u(x,t), H(x,t)\} \tag{1.2}$$

for $y \in D(t)$.

v^0 is continuous in x, satisfies the pricing equation and its boundary conditions outside of $D(t)$, and assumes the cancellation value

$$v(y,t,y) = H(y,t)$$

We observe from

$$\lim_{t\to 0} u(x,t) = u_0(x)$$

and $\delta(x,0) > 0$ that $D(t)$ will be empty close to expiration. If

$$u(x,t) < H(x,t) \quad \text{for } t \in [0,t^*) \text{ and all } x$$

and

$$u(x,t^*) = H(x,t^*) \quad \text{for at least one } x$$

then we shall set

$$v(x,t,y) = u(x,t), \quad t \in [0,t^*] \text{ for all } y \in (0,\infty).$$

If $D(t)$ is empty again for some $t^{**} > t^*$ then no cancellation occurs and $v(x,t,y)$ is assumed to satisfy (1.1) for $t > t^{**}$, the boundary conditions for u and the initial condition $v(x,t^{**},y)$.

Assume that $D(t) \neq \emptyset$ and that there exists a subset $D^*(t) \subset D(t)$ such that for $y^* \in D^*(t)$

$$v(x,t,y^*) \leq v(x,t,y) \quad \text{for all } x \text{ and all } y \in D(t)$$

then the value $v(x,t,y)$ is minimized for all x when the option is scheduled to be canceled whenver x reaches $y^* \in D^*(t)$.

We note that if $y_1^*, y_2^* \in D^*(t)$ then for all x

$$v(x,t,y_1^*) \leq v(x,t,y_2^*) \leq v(x,t,y_1^*)$$

Hence $v(x,t,y^*)$ is a well defined function of x.

Definition: If $y^* \in D^*(t)$ then $v(x,t,y^*)$ is the value of the game version of the option $u(x,t)$ and $D^*(t)$ is its exercise region.

Since $v(x,t,y^*)$ is the cheapest option achievable through cancellation it follows that the game option, if it exists, must necessarily satisfy

$$v(x,t,y^*) \leq v^0(x,t,y)$$

where v^0 is given by (1.2).

Whether there is a non-empty set $D^*(t)$, whether it consists of more than one point or contains a nonzero interval will depend on the type of option and its pay-off and penalty. However, we observe that if there is a sufficiently smooth curve $y(t) \in D(t)$ for $t \in (t^*, t^{**}]$, $t^{**} \leq T$ and the writer will cancel the option u at time t whenever the underlying reaches $y(t)$, then $y(t)$ is a barrier for the resulting option $v(x,t,y(t))$ because

$$v(y(t),t,y(t)) = H(y(t),t).$$

If we ASSUME that $y(t)$ is an isolated interface which separates continuation regions for v on $(0,y(t))$ and for v on $(y(t),\infty)$ then both of these barrier options with initial condition

$$v(x,t^*,y(t^*)) = u(x,t^*)$$

and subject to the boundary conditions given for u can be computed with standard (numerical) methods for Black Scholes type equations. One can then check whether

$$v(x,t,y(t)) \leq v^0(x,t,y(t)) \quad \text{for } t \in [t^*,T]$$

which is a necessary condition for $y(t) \in D^*(t)$.

However, $y(t)$ need not be an isolated interface. Instead it may belong to an exercise region in $D^*(t)$ where the writer will cancel the option. While this complicates the calculation of the option $v(x,t,y^*)$, the examples below will show that the assumption of an isolated interface can sometimes yield geometric insight which helps determine the game option and its exercise region $D^*(t)$.

The barrier options defining the game option will be found with a time discrete method of lines (MOL) approximation of the differential equations. It requires the solution of a sequence of free boundary problems, each of which may be thought of as a perpetual game option problem. Its solution is discussed in the following section for perpetual puts and calls where $f(x,t)$ in (1.1) is absent. Solutions will be found via the Riccati transformation on which the MOL solutions of [7] are based. The calculation of time discrete solutions with nonzeero $f(x,t)$ and their convergence to a time-varying game option have yet to be considered.

II. Game options for perpetual puts and calls

We assume that the pricing equation is the standard time independent Black Scholes equation

$$\mathcal{L}_{BS}u \equiv \frac{1}{2}\sigma^2 x^2 u'' + (r-q)xu' - ru = 0 \tag{1.3}$$

and that the penalty for cancellation is

$$H(x) = u_0(x) + \delta, \quad \delta > 0.$$

where u_0 is the intrinsic value of the option scaled so that the strike price is $K = 1$. We assume that $u(x)$ is known analytically or numerically so that the (now time independent) set $D(t)$ can be found. We assume it is not empty so that cancellation can occur.

Let $\tilde{y} \in D(t)$ be a point of maximum separation between $u(x)$ and $H(x)$. We shall show that $\tilde{y}$ necessarily belongs to the exercise region of $v(x,y^*)$ so that $v(\tilde{y},y^*) = H(\tilde{y})$.

Lemma 1. *If the game option $v(x,y^*)$ for a perpetual option $u(x)$ exists and $\tilde{y}$ is a point in $D(t)$ such that for all x*

$$u(x) - H(x) \leq u(\tilde{y}) - H(\tilde{y})$$

then $\tilde{y}$ necessarily is a point of exercise for $v(x,y^*)$.

Proof. Let us assume that $\tilde{y}$ belongs to a continuation region (a,b) for $v(x,y^*)$ while a and b are points of exercise. If we set

$$g(x) = u(x) - v(x,y^*) - (u(\tilde{y}) - H(\tilde{y}))$$

then $g(a) \leq 0$ and $g(b) \leq 0$. Moreover,

$$\mathcal{L}_{BS}g(x) = r(u(\tilde{y}) - H(\tilde{y})) \geq 0.$$

It follows from the maximum principle for (1.3) that $g(x)$ cannot be positive anywhere on $[a,b]$. This implies that

$$u(x) - v(x,y^*) \leq u(\tilde{y}) - H(\tilde{y})$$

so that $H(\tilde{y}) \leq v(\tilde{y},y^*)$. Since $v(x,y^*) \leq v^0(x,\tilde{y})$ it follows that $v(\tilde{y},y^*) = H(\tilde{y})$. $\square$

The point $\tilde{y}$ is considered known. If it separates continuation regions of $v(x, y^*)$ then, like $u(x)$, the barrier options $v(x, \tilde{y})$ are computable on $(0, \tilde{y})$ and $(\tilde{y}, \infty)$. The next lemma gives a necessary conditon for $v(x, \tilde{y})$ to be the game option $v(x, y^*)$.

Lemma 2. *Let $\tilde{y}$ be as defined in Lemma 1. Let $v(x, \tilde{y})$ be the option defined on $(0, \infty)$ obtained by assuming that $\tilde{y}$ is an isolated interface. If $v(x, \tilde{y}) < H(x)$ for $x \neq \tilde{y}$ and if $v(x, y^*)$ exists then*

$$v(x, y^*) = v(x, \tilde{y}) \quad \text{for all } x.$$

Proof. Since $v(x, y^*) \leq v(x, \tilde{y}) < H(x)$ for all $x \neq \tilde{y}$ it follows that $v(x, y^*)$ will not be canceled at any $x \neq \tilde{y}$. This implies that $v(x, y^*) = v(x, \tilde{y})$ because both solve the pricing equation on $(0, \tilde{y})$ and $(\tilde{y}, \infty)$ and the same boundary conditions at $\tilde{y}$ and as $y \to 0$ and $y \to \infty$.

Lemma 2 does not rule out the possibility that for cancellation at some point $\hat{y} \in D(t)$ the corresponding option $v(x, \hat{y})$ will fall below $v(x, \tilde{y})$ at some $x \in (0, \infty)$. The final lemma gives a sufficient condition for $v(x, \tilde{y})$ to be the game option. □

Lemma 3. *Let $v(x, \tilde{y})$ be the option of Lemma 2 and let $v(x, y)$ denote the barrier option obtained under the assumption that $y \in D(t)$ is an isolated interface for $v(x, y)$. If $v(x, \tilde{y}) \leq v(x, y)$ for all x and for all $y \in D(t)$ then*

$$v(x, y^*) = v(x, \tilde{y}).$$

Proof. $v(x, \tilde{y})$ will not be the game option if there is a point y in $D(t)$ such that $v(x, y) < v(x, \tilde{y})$ for some $x \in (0, \infty)$. If no such point exists then $v(x, \tilde{y})$ is the lowest value achievable through cancellation. □

We shall now consider some specific perpetual options.

Example 1. The perpetual game American put.

Let $u(x)$ denote the price of the perpetual American put satisfying (1.3) with boundary conditions

$$\begin{aligned} u(x) &= 1 - x, \quad 0 \leq x \leq s \\ u'(s) &= -1 \\ \lim_{x \to \infty} u(x) &= 0 \end{aligned}$$

where s is the early exercise boundary of the put. For constant parameters the analytic solution $\{u(x), s\}$ is known (see, e.g., [6]). It is not used below. However, we shall use the fact that $0 \geq u'(x) \geq -1$ for all x which can be established with the maximum principle for the differential equation satisfied by $u'(x)$ on $[s, \infty)$.

For the subsequent discussion of American barrier puts and their numerical solution it will be convenient to apply the Riccati transformation method of [7]. It is known that if

$u(X) = A > \max\{1-X, 0\}$ at some barrier X then the free boundary s of the corresponding barrier put is the largest root in $(0, X)$ of

$$\phi(x) \equiv R(x)(-1) + w(x) - (1-x) = 0 \tag{1.4a}$$

where

$$R' = b(x) - d(x)R - c(x)R^2, \quad R(X) = 0 \tag{1.4b}$$

$$w' = -c(x)R(x)w, \qquad w(X) = A \tag{1.4c}$$

with $b(x) = 1$, $c(x) = \frac{2r}{\sigma^2 x^2}$, $d(x) = -\frac{2(r-q)x}{\sigma^2 x^2}$. Once the initial value problems for R and w are solved and a root s of (1.4a) has been found, then the solution $u(x)$ and its derivative $u'(x)$ are linked through the affine (Riccati) transformation

$$u(x) = R(x)u'(x) + w(x) \tag{1.4d}$$

where $z(x) \equiv u'(x)$ is found over $[s, X]$ from the initial value problem

$$z'(x) = \frac{2}{\sigma^2 x^2}[(rR(x) - (r-q)x)z(x) + rw(x)], \quad z(s) = -1. \tag{1.4e}$$

We see by inspection that the solution of the Riccati equation (1.4b) is strictly negative on $(0, X)$. Moreover, it cannot cross the line $R_0(x) = -\gamma x$, where γ is the positive root of

$$\gamma = 1 + \frac{2(r-q)}{\sigma^2}\gamma - \frac{2r}{\sigma^2}\gamma^2. \tag{1.5}$$

This assures that $R_0(x) < R(x)$ so that $\lim_{x\to 0} R(x) = 0$. Since $w' > 0$ it also follows that $0 < w(x) \le w(X)$ on $(0, X]$.

By hypothesis $\phi(1) > 0$ and $\lim_{x\to 0}\phi(x) = w(0) - 1$. If $\phi(0) < 0$ then there will be a free boundary $s \in (0,1)$ such that $\phi(x) > 0$ in $(s, 1)$ and hence $\phi'(s) \ge 0$. From

$$\phi(x) = -R(x) + w(x) - (1-x)$$
$$\phi'(x) = -R'(x) + w'(x) + 1$$

and the equations (1.4b,c) we obtain algebraically that

$$\phi'(x) = \frac{2rR(x)}{\sigma^2 x^2}\phi(x) - \frac{2R(x)}{\sigma^2 x^2}[r - qx]. \tag{1.6}$$

$\phi(s) = 0$ and $\phi'(s) \ge 0$ can hold only if $(r - qs) \ge 0$ so that

$$s \le \min\left\{1, \frac{r}{q}\right\}$$

which is the familiar contact condition

$$\lim_{x\to s+} u''(x) \ge 0$$

for the early exercise boundary of the Black Scholes put. We note that (1.6) also rules out any other free boundary on $(0, s)$ so that s is unique. Moreover, $\phi(s) = 0$ implies that $R(s) = w(s) - (1 - s) < s - 1$. It then follows from

$$R'(s) = 1 + \frac{2R(s)}{\sigma^2 s^2}[(r-q)s - rR(s)] < 1 + \frac{2R(s)}{\sigma^2 s^2}[(r-q)s - r(s-1)]$$

that $R'(s) < 1$ for all r and $q > 0$ so that

$$\phi'(s) > w'(s) > 0.$$

Finally we observe that the solution w of (1.4c), and hence the function ϕ are strictly monotonely increasing with A. This implies that the free boundary s is strictly monotonely decreasing with A. Moreover, if $\{u(x), s\}$ denotes the value of the perpetual American put without a barrier and its early exercise boundary, then the free boundary for the barrier put with

$$w(X) = A < u(X)$$

will lie to the right of s. Thus the equations for R, w, and z need only be considered on $[s, X]$ where all coefficients are smooth and bounded. That makes R, w, and z smooth functions of their initial values so that $\phi(x)$ is a smooth function of X and A. $\phi'(s) \neq 0$ then makes $s(X)$ a continuous function of X and A so that $z'(x)$ is also a continuous function of X and A.

Let us now turn to the game put corresponding to the perpetual American put. We shall assume that q is sufficiently small so that for the penalty δ we have

$$r(1 + \delta) - q > 0. \tag{1.7}$$

For the cancellation pay-off

$$H(x) = \max\{1 - x, 0\} + \delta, \quad \delta > 0$$

the function $u(x) - H(x)$ takes on its maximum value at $\tilde{y} = 1$. If $\delta > u(1)$ then $u(x)$ will lie below $H(x)$ for all x and $D(t) = \emptyset$. The option will never be canceled and

$$v(x, y^*) = u(x).$$

If $\delta \leq u(1)$ then $D(t)$ is an interval containing $\tilde{y} = 1$. Cancellation at all points of $D(t)$ leads to the value function

$$v^0(x, y) = \min\{u(x), H(x)\}.$$

The game American put would have to lie on and below v^0.

According to Lemma 1 the point $\tilde{y} = 1$ is a cancellation point for $v(x, y^*)$ provided the game option exists. The barrier option $v(x, 1)$ of Lemma 2 on $(0, 1)$ satisfies

$$\begin{aligned} \mathcal{L}_{BS} v(x, 1) &= 0 \\ v(x, 1) &= 1 - x, \quad x \leq s(1) \\ v'(s(1), 1) &= -1 \\ v(1, 1) &= \delta, \end{aligned}$$

where $s(1)$ denotes the early exercise boundary for $v(x,1)$ in $(0,1)$. We observe that

$$\mathcal{L}_{BS}[v(x,1) - H(x)] = -qx + r(1+\delta) \quad \text{ on } (s(1),1)$$

For $r(1+\delta) - q > 0$ the function $v - H$ cannot have a non-negative maximum so that $v(x,1) < H(x)$ on $(0,1)$.

On $(1,\infty)$ the barrier option $v(x,1)$ is a solution of

$$\begin{aligned} \mathcal{L}_{BS} v(x,1) &= 0 \\ v(1,1) &= \delta, \quad \lim_{x\to\infty} v(x,1) = 0. \end{aligned}$$

The maximum principle can again be invoked to show that $v(x,1) < H(x)$ on $(1,\infty)$. If the game option is known to exist, say from the Dynkin theory, then by Lemma 2 the function $v(x,1)$ is the game option for the perpetual American put and $\tilde{y} = 1$ is the asset value where the writer will cancel the put.

For the game put Lemma 3 can be invoked to prove directly that $v(x,y^*) = v(x,1)$. We need to show that $v(x,1) \le v(x,y)$ for all $y \in D(t)$. For any $y \in D(t)$ with $y < 1$ the barrier option $v(x,y)$ will lie on or above $v(x,1)$ on $(0,y)$ because $v(y,y) - v(y,1) > 0$. On (y,∞) the option $v(x,y)$ coincides with $\gamma u(x)$ for some $\gamma < 1$. Since $u'(x) > -1$ this implies that $v(x,y) > H(x)$ on $(y,1]$ which assures that $v(x,y) \ge v(x,1)$ on $[1,\infty)$.

Suppose next that the put is canceled at some $y \in D(t)$ with $y > 1$. Then $v(x,y)$ satisfies

$$\begin{aligned} \mathcal{L}_{BS} v(x,y) &= 0 \quad \text{ on } (s(y),y) \\ v(y,y) &= \delta \end{aligned}$$

and

$$v(x,y) = 1 - x, \quad x \in [0, s(y)]$$
$$v'(s(y),y) = -1.$$

Since $u(y) \ge v(y,y)$ we know from (1.4a,b,c) that $s \le s(y)$. Let $\gamma = u(y) - \delta$ and consider

$$h(x) = (u(x) - \gamma) - v(x,y).$$

Then

$$\mathcal{L}_{BS} h(x) = r\gamma \ge 0$$

and

$$h'(s(y)) \ge 0, \quad h(y) = 0.$$

The maximum principle and the boundary data imply that $h(x) \le 0$ on $[s(y),y]$. In particular, it follows from $h(1) \le 0$ and $u' \le 0$ that

$$\delta \le v(1,y)$$

so that $s(y) \le s(1)$ and $v(s(1),y) \ge v(s(1),1)$. This guarantees that $v(x,y) \ge v(x,1)$ for all x an all $y \in D(t)$. Finally, suppose that cancellation occurs over an interval $[a,b] \subset D^*(t)$.

Then the above discussion shows that $v(x,1) \le v(x,a)$ on $[0,a]$ and $v(x,1) \le v(x,b)$ on (b,∞). Since always $v(x,1) < H(x)$ for $x \ne 1$ we see that $v(x,1) \le v(x,y^*)$ for all $y^* \in D^*(t)$. This makes $v(x,1)$ the game version of the perpetual American put.

We note that for constant parameters the perpetual put $u(x)$ and the two components of $v(x,1)$ have an analytic solution of the form

$$u(x) = c_1 x^{\alpha_+} + c_2 x^{\alpha_-} \tag{1.8}$$

where $\alpha_{+,-}$ are the positive and negative roots of

$$\frac{1}{2}\sigma^2\alpha(\alpha-1) + (r-q)\alpha - r = 0. \tag{1.9}$$

It is possible to compute $v(x,1)$ and $s(1)$ in closed form. For $q = 0$ in (1.3) the game option $v(x,1)$ is identified with the Dynkin game approach and computed analytically in [8] and [15]. The above discussion assures that $v(x,1)$ remains the perpetual game put for sufficiently small q. It is, however, the case that $v(x,1)$ ceases to be the game put when q is too large. A numerical evaluation of $u(x)$ and $v(x,1)$ over $(0,1)$ for two choices of q yields the following results:

Table 1: Numerical values for $u(x)$ and $v(x,1)$

q	δ	$u(1)$	$D(t)$	$v'(1-,1)$
.02	.1	.23670	[.792,1.81]	$-.5862$
.08	.1	.43648	[.543,2.73]	-1.500

$r = .03$, $\sigma = .2$.

For illustration we show in Fig. 1 the price of the barrier put $v(x,1)$ for $q = 0.08$ as well as the lines $1-x$ and $1+\delta-x$ when (1.7) does not hold. We observe that for sufficiently large q the barrier $v(x,1)$ will lie above $H(x)$ near $x=1$. Since the game option has to lie below $v^0(x,y)$, the barrier put $v(x,1)$ cannot be the game option on $(0,1)$. However, $v(x,1)$ still provides geometric insight.

When $v(x,1)$ rises above $H(x)$ for $x<1$ then there is a point $b<1$ where $v(x,1)$ recrosses $H(x)$ because $v(s(1),1) = 1-s(1) < H(s(1))$. For $y \in [b,1]$ consider now the problem

$$\mathcal{L}_{BS} v(x,y) = 0$$

$$v(y,y) = H(y)$$

$$v(x,y) = 1-x, \quad 0 < x \le s(y)$$

$$v'(s(y),y) = -1.$$

Since $v(x,b) = v(x,1)$ on $(0,b)$ we know that $v'(1,1) < -1$ and $v'(b,b) > -1$. Since $v'(y,y)$ is a continuous function of y we can use a bisection method to determine a point $\hat{y}$ such that

$v'(\hat{y},\hat{y}) = -1$. For any $y \in (b,\hat{y})$ the function $v(x,y)$ will lie on or above $v(x,\hat{y})$ on $(0,\hat{y})$, while for any $y \in (\hat{y},1)$ the function $v(x,y)$ will exceed $H(x)$ on $(\hat{y},y)$.

Thus, if we set

$$v(x,y^*) = \begin{cases} v(x,\hat{y}), & 0 < x < \hat{y} \\ H(x), & \hat{y} \le x < 1 \\ v(x,1), & x \ge 1 \end{cases}$$

then $v(x,y^*)$ is the game put. Its continuation regions are $(s(\hat{y}),\hat{y})$, $(1,\infty)$ and the exercise region is $[\hat{y},1]$. Cancellation at any other point will yield a function which lies on or above $v(x,\hat{y})$ for all x.

The bisection method applied with the data of Table 1 for $q = .08$ yields

$$\hat{y} = .7485, \quad s(\hat{y}) = .2895.$$

The corresponding $v(x,y^*)$ over $[0,1]$ and the perpetual option $u(x)$ over $[0,1]$ and the barrier option $v(x,1)$ are shown in Fig. 1.

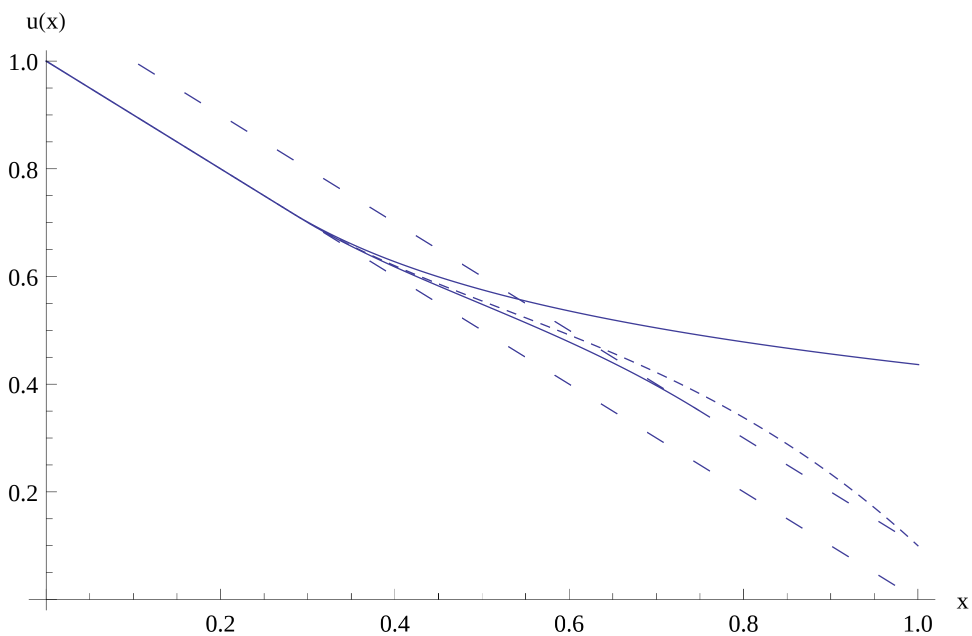


Figure 1: Perpetual American put and its corresponding game put over $[0,1]$
Upper solid curve: Perpetual put $u(x)$
Lower solid curve: Perpetual game put $v(x,y^*)$ over $[0,\hat{y}]$
Intermediate broken curve: Barrier put $v(x,1)$
The two straight lines denote the intrinsic value of $u(x)$ and the penalty pay-off

Example 2. The game option for a capped perpetual American call.

The steps taken above to characterize the game put are readily modified to find the game version of a capped perpetual American call. We shall only outline the approach.

Let $\{u(x), s\}$ be the value of a capped perpetual American call satisfying (1.3) and the boundary conditions

$$\begin{aligned} &u(0) = 0 \\ &u(x) = \min\{x, L\} - 1 \quad \text{if } \min\{s, L\} \le x \end{aligned}$$

and

$$u'(s) = 1 \qquad \text{if } s \le L$$

where $L - 1$ is the cap on u scaled so that its strike price is $K = 1$. (We remark that the existence of a cap has little influence on the subsequent analysis or the numerical solution of the problem.)

Let $\{v(x, X), s(X)\}$ denote the value of a capped call with the same early exercise conditions and the barrier condition

$$v(X, X) = A$$

at a given barrier $X < \min\{s(X), L\}$ for $A > \min\{X, L\} - 1$. We note that $v(x, X)$ would coincide with $u(x)$ on $[X, \infty)$ if we set

$$v(X, X) = u(X).$$

For ease of notation let us write $s \equiv s(X)$ for the free boundary of the barrier call. s depends on X and A and, as for the put, can be found with the Riccati transformation method as the first root to the right of $x = X$ of the following analog of equation (1.4a)

$$\phi(x) \equiv R(x) + w(x) - (x - 1) = 0 \tag{1.10}$$

where R and w are solutions of the equations (1.4b,c) for $x > X$. By inspection $R(x) > 0$ on (X, ∞). $w(x)$ is a positive decreasing function of x and increasing with A for $A > 0$. This implies that for fixed X the function $\phi(x)$ is an increasing function of A. We also observe that $R(x)$ cannot cross the line $R_0(x) = +\gamma x$ where γ is again the positive root of (1.5). It is straightforward to verify that $\gamma(q)$ as a function of dividend rate q satisfies $\gamma(0) = 1$ and $\gamma'(q) < 0$ for all $q \ge 0$. From $R(x) \le \gamma(q)x$ follows that

$$\phi(X) = A - (X - 1) > 0 \quad \text{and} \quad \phi(x) < (q - 1)x - 1 < 0$$

for sufficiently large x. Hence for all $q > 0$ the uncapped call will have a free boundary, while the capped call may or may not have a free boundary $s \in [X, L]$. In either case $\phi(x) > 0$ on $[X, \min\{s, L\})$, and if there is a free boundary s then $\phi'(s) \le 0$.

Using (1.10) and the differential equations (1.4b,c) we obtain the following analog of (1.6)

$$\phi'(x) = -\frac{2rR(x)}{\sigma^2 x^2}\phi(x) + \frac{2R(x)}{\sigma^2 x^2}[r - qx].$$

In view of $\phi'(s) \le 0$ a free boundary s can occur on $[X, L]$ only if $r - qs \le 0$ which is the familiar contact condition for a Black Scholes call.

The Riccati transformation yields the option price

$$v(x, X) = R(x)z(x) + w(x)$$

where $z(x) \equiv v'(x, X)$ is the solution of (1.4e) subject to the initial condition

$$z(s) = 1 \quad \text{if } s \le L$$

or

$$z(L) = \frac{(L-1) - w(L)}{R(L)} \quad \text{if } \phi(x) \text{ has no zero on } [X, L].$$

In this case we note from $\phi(L) = R(L) + w(L) - (L-1) > 0$ that $z(L) \le 1$ and that $z(L)$ is decreasing with A.

Turning to the game option, let $\{u(x), s\}$ now stand for the capped perpetual American call with exercise boundary s, if s exists in $[X, L]$. Suppose that the call can be canceled by the writer at any time and asset value x by paying

$$H(x) = \max\{\min\{x, L\} - 1, 0\} + \delta$$

where $\delta > 0$ is the cancellation penalty. We shall assume that $0 \le u'(x) \le 1$ and that $u(1) \ge \delta$ so that $D(t) \ne \emptyset$. The point $\tilde{y}$ of maximum separation between u and H is $\tilde{y} = 1$. If the call is canceled at $\tilde{y}$ then $v(x, 1)$ is defined by two uncoupled barrier options satisfying (1.3) on $(0, 1)$ and $(1, \infty)$ and the boundary conditions

$$\begin{aligned} v(0, 1) &= 0 \\ v(1, 1) &= \delta \end{aligned}$$

for $v(x, 1)$ on $(0, 1)$, and

$$\begin{aligned} v(1, 1) &= \delta \\ v(x, 1) &= \min\{x, L\} - 1, \quad x > s(1) \end{aligned}$$

and

$$v'(s(1), 1) = 1 \qquad \text{if } s(1) < L$$

for $v(x, 1)$ on $(1, \infty)$. From the monotonicity with respect to A we know that $s(1) < s$ if $s < L$.

If $v(x, 1) < H(x)$ for all $x \ne 1$ and the game option $v(x, y^*)$ is known to exist, then $v(x, y^*) = v(x, 1)$ and $y^* = 1$ is the only exercise point.

If we set $g(x) = v(x, 1) - H(x)$ then on $(0, 1)$ we see that

$$\mathcal{L}_{BS} g(x) = r\delta > 0$$

$$g(0) < g(1) = 0.$$

By the maximum principle $g(x)$ cannot have a non-negative maximum on $(0,1)$ so that

$$v(x,1) < H(x) \text{ on } (0,1).$$

Hence $v(x,y^*)$ cannot have a point of cancellation in $(0,1)$. By inspection we see that

$$v(x,1) = \delta\frac{u(x)}{u(1)}.$$

On $(1,L)$ we see that

$$\mathcal{L}_{BS}g(x) = -(r-q)x + r(x-1+\delta) = qx + \delta r - r$$

$$g(1) = 0, \quad g(\min\{s(1),L\}) < 0.$$

If $q+\delta r > r$ then we again have the conclusion that $g(x) < 0$ on $(1, \min\{s1, L\})$. However, if $q+\delta r < r$ this conclusion need not hold and $v(x,1)$ can conceivably rise above $H(x)$ near $x = 1$.

If $v(x,1)$ rises above $H(x)$ for $x \in (1,b)$ then $v'(1,1) \geq 1$ and $v'(b,1) \leq 1$. Suppose that $v(x,1)$ has a free boundary $s(1) < L$. For all $y \in [1,b]$ the barrier function $v(x,y)$ with boundary condition

$$v(y,y) = H(y)$$

will yield an exercise boundary $s(y) \leq s(1)$ because $v(y,1) \geq H(y)$, which in turn implies that $v(x,y) \leq v(x,1)$ on $(y, s(1))$. The latter conclusion also holds if the cap is reached by $v(x,1)$ or both.

We assume here without proof that $v'(y,y)$ is a continuous function of y. Then the bisection method can be applied on $[1,b]$ to find a $\hat{y}$ such that

$$v'(\hat{y},\hat{y}) = 1.$$

The maximum principle applied to $\mathcal{L}_{BS}v'(x,\hat{y}) = 0$ shows that $v'(x,\hat{y}) < 1$ on $(\hat{y}, s(\hat{y}))$ so that $v(x,\hat{y}) < H(x)$ for $x > \hat{y}$. Hence the value of the game option corresponding to the capped perpetual call is

$$v(x,y^*) = \begin{cases} v(x,1), & x \in (0,1) \\ H(x), & x \in [1,\hat{y}] \\ v(x,\hat{y}), & x \in (\hat{y}, \min\{s(\hat{y}), L\} \\ \min\{x,L\} - 1, & x > \min\{s(\hat{y}), L\}. \end{cases}$$

The exercise region is the interval $D^*(t) = [1, \hat{y}]$.

Since it makes little difference for a numerical method whether parameters are constant we shall illustrate the process of pricing a game call as barrier options by considering a capped American CEV perpetual call. It is representative of a perpetual option with asset dependent volatility for which no convenient analytic solution is available. For a survey of CEV options we refer to [11].

For a capped CEV call $C(S,t)$ with strike price K written on an asset S the pricing equation (1.3) for $x = S/K$ and $u(x) = C/K$ becomes

$$\frac{1}{2}(\sigma x^{\beta})^2 x^2 u''(x) + (r-q)xu'(x) - ru(x) = 0 \tag{1.11}$$

where σ is a constant which will depend on the strike price K. (1.11) corresponds to a standard Black Scholes equation with volatility (σx^{β}) for $\beta \geq -1$.

An American call capped at $x = L$ is subject to the boundary conditions

$$u(0) = 0$$

and

$$u(s) = s - 1$$

$$u'(s) = 1$$

if there is an early exercise boundary $s < L$. Otherwise the fixed boundary condition

$$u(L) = L - 1 \tag{1.12}$$

is imposed.

We shall compute the game option corresponding to the capped call for the following parameters, chosen to accentuate the features of the PDE approach,

$$\beta = .1, \quad \sigma = .2, \quad r = .05, \quad q = .01, \quad \delta = .2, \quad L = 7.8$$

The capped call $u(x)$ can be computed with the method of lines with the above x-dependent volatility. It turns out that $u(x)$ has no free boundary $[0, 7.8]$ so that (1.12) applies. The value of the capped call at $x = 1$ is

$$u(1) = .6401$$

so that $D(t) = \{x : u(x) \geq \max\{x-1, 0\} + \delta\} \neq \emptyset$.

The barrier option $v(x,1)$ also has no free boundary on $[1, L]$ and therefore must satisfy $v(1,1) = \delta$ and (1.12). The numerical solution yields

$$v'(1,1) = 1.6422$$

so that $v(x,1)$ will rise above $H(x)$ near $x = 1$. With bisection we find that

$$v(2.187, 2.187) = 1.387, \quad v'(2.187, 2.187) = .999958$$

and that there is an early exercise boundary

$$s(2.187) = 7.6528.$$

The game option corresponding to the capped perpetual CEV call is

$$v(x, y^*) = \begin{cases} v(x,1), & 0 < x < 1 \\ x - 1 + \delta, & 1 < x < 2.187 \\ v(x, 2.187), & 2.187 < x < 7.6528 \\ x - 1, & 7.6528 < x < 7.8. \end{cases}$$

where on $(0,1)$ the function $v(x,1)$ solves (1.11) subject to

$$v(0,1) = 0, v(1,1) = \delta = .2.$$

The function $v(x, 2.187)$ solves (1.11) subject to free boundary conditions at both endpoints of the continuation interval which turn out to be

$$v(2.187, 2.187) = 1.387, \quad v'(2.187, 2.187) \cong 1$$

$$v(7.6528, 2.187) = 6.6528, \quad v'(7.6528, 2.187) \cong 1.$$

The capped call $u(x)$, and the barrier options $v(x,1)$, $v(x, 2.187)$ are shown in Fig. 2 for $0 < x < 3$.

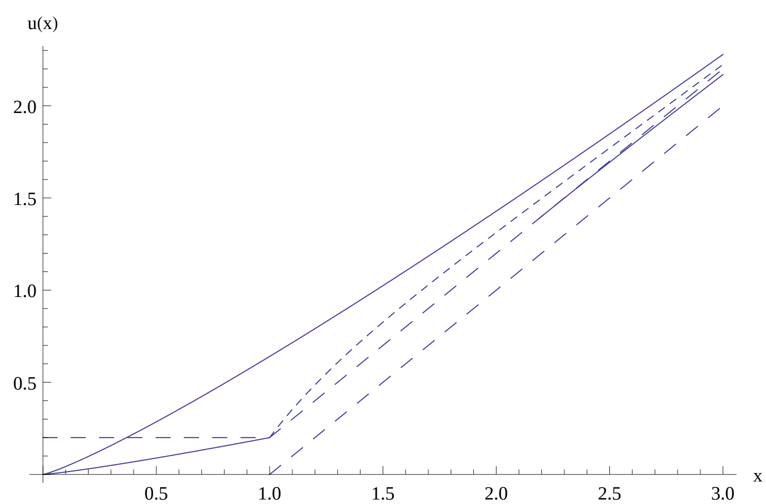


Figure 2: Capped perpetual CEV call and its game option counterpart
Upper solic curve: Capped CEV call $u(x)$
Lower solid curve: Game option $v(x,1) = v(x,y^*)$ on $(0,1)$
Game option $v(x, 2.187) = v(x, y^*)$ on $(2.187, 7.6528)$
Intermediate broken curve: $v(x,1)$
Cancellation region $[1, 2.187]$
The straight lines denote the intrinsic value of $u(x)$ and the penalty pay-off

III. Fair loan fees for stock loans

As an application of a game call let us consider the pricing of certain stock loans considered in [14]. We shall summarize the contracts discussed there.

Suppose a borrower obtains at real time $t = 0$ a loan Q from a lender and secures the loan with a stock of value S_0 at that time. This asset value is assumed to follow geometric Brownian motion to that

$$dS = (r - q)Sdt + \sigma SdW.$$

The risk free interest rate r, the dividend rate q and the volatility σ are assumed constant. In addition the lender charges a loan service fee c. The loan agreement is assumed to hold for all $t > 0$ and interest on the loan will accrue at a rate γ so that the loan balance at time t is $e^{\gamma t}Q$. During the life of the loan the dividends paid on S go to the lender. As explained in [14] the loan rate γ may exceed the interest rate r.

If $C(S, t)$ denotes the value of the loan agreement, then it can be modeled with the Black Scholes equation

$$\frac{1}{2}\sigma^2 S^2 C_{SS} + (r - q)SC_S - rC + C_t = 0 \tag{1.13}$$

(see, e.g., [13]). It is assumed valid for all $t > 0$.

Three different types of stock loans are compared in [14] which reflect repayment options and obligations for the borrower.

i) In a so-called non-recourse collateralized loan the borrower has the option to redeem the asset $S(t)$ at any time t by repaying the current loan balance $e^{\gamma t}Q$ to the lender.

At redemption the value of the non-recourse loan is

$$C(S, t) = \max\{S - e^{\gamma t}Q, 0\} \tag{1.14}$$

so that $\max\{S - e^{\gamma t}Q, 0\}$ is the intrinsic value of C which implies that

$$C(0, t) = 0.$$

As in a standard American call, $C(S, t)$ cannot fall below its intrinsic value while its value is maximized at exercise at $S(t)$ if $V_S(S(t), t) = 1$. Thus $C(S, t)$ corresponds to an American call with time dependent strike price

$$K = e^{\gamma t}Q,$$

and an early exercise boundary $S(t)$ where the value matching and smooth pasting conditions

$$C(S(t), t) = S(t) - e^{\gamma t}Q \tag{1.15}$$

$$C_S(S(t), t) = 1. \tag{1.16}$$

hold. Should the asset value fall below the loan balance, the borrower would not redeem his collateral.

ii) To guard against eroding asset values the lender can impose a margin call requirement. Different types of margin calls are possible. In [14] a cancellation of the contract is considered when the underlying asset falls to a specified fraction of the loan balance so that

$$C(\pi Q e^{\gamma t}, t) = 0 \quad \text{for some } \pi < 1.$$

The early exercise conditions (1.15,1.16) remain in effect.

iii) Alternatively, the lender can cancel the loan at any time by paying the borrower the amount

$$H(S,t) = \max\{S - e^{\gamma t} Q, 0\} + \delta Q e^{\gamma t}$$

where $\delta Q e^{\gamma t}$ is the penalty for terminating the contract. It is assumed that the lender will call in the loan at a time and asset value which minimizes the value $C(S,t)$ of the loan option at all S. The resulting loan contract is termed a callable loan and corresponds to a game call option. The conditions (1.15, 1.16) remain valid.

At the inception at $t = 0$ of one of these three loans the lender receives the asset with spot value S_0 and a loan service fee $c(S_0, Q)$, while the borrower receives the loan amount Q and the contract $C(S,t)$. The fair value of the service fee is defined as

$$c(S_0, Q) = C(S_0, 0) + Q - S_0.$$

The financial implications for lender and borrower of these loans is the subject of an exhaustive study in [14]. It is based in part on stochastic arguments and relies heavily on analytic solutions of perpetual options. Here we shall rederive the data of [14] with the PDE approach presented above. It largely bypasses the need for closed form solutions and appears applicable with little change to loans with finite maturity.

It will be convenient to scale the calls for (1.13) by setting

$$S = e^{\gamma t} Q x, \quad U(x,t) = \frac{C(S,t)}{Q}$$

Then

$$U_x(x,t) = C_S(S,t) e^{\gamma t}, \quad U_{xx}(x,t) = C_{SS}(S,t)(e^{\gamma t})^2 Q$$
$$U_t(x,t) = \gamma C_S(S,t) e^{\gamma t} x + \frac{C_t(S,T)}{Q}$$

so that the Black Scholes equation becomes

$$\frac{1}{2}\sigma^2 x^2 U_{xx} + (r - \gamma - q) x U_x - rU + U_t = 0. \tag{1.17}$$

The early exercise conditions for all three loan contracts are

$$U(s(t),t) = e^{\gamma t}(s(t) - 1), \quad U_x(s(t),t) = e^{\gamma t}$$

where $s(t)$ is the early exercise boundary for U in the $x - t$ plane.

For the non-recourse loan we have

$$U(0,t) = 0. \tag{1.18}$$

For the loan with margin call the boundary condition is

$$U(\pi,t) = 0, \tag{1.19}$$

while for a callable loan the cancellation payment for $U(x,t)$ becomes

$$\hat{H}(x,t) = \max\{e^{\gamma t}(x-1,0)\} + \delta e^{\gamma t}. \tag{1.20}$$

By inspection we see that all three problems have a separation of variables solution

$$U(x,t) = u(x)e^{\gamma t}, \quad s(t) = s_0 e^{\gamma t}$$

where

$$\frac{1}{2}\sigma^2 x^2 u''(x) + (r-\gamma-q)xu'(x) - (r-\gamma)u = 0 \tag{1.21}$$

$$u(s_0) = s_0 - 1, \quad u'(s_0) = 1$$

and

$$\hat{H}(x,t) = \hat{H}(x)e^{\gamma t} \quad \text{ with } \hat{H}(x) = \max\{x-1,0\} + \delta.$$

Thus the non-recourse loan can be solved as a perpetual American call, the margin call loan is a perpetual call with down and out barrier at $x = \pi$, while the problem of pricing the callable loan has formally been converted to a game option for a perpetual American call.

Equation (1.21) is a (time independent) Black Scholes equation with an effective interest rate $\tilde{r} = r - \gamma$. Its presence in (1.21) is due entirely to the separation of variables solution technique. It has no financial significance but $\tilde{r} < 0$ does influence the numerical solution of (1.21) as discussed below.

If $\tilde{r} > 0$ then we have a standard perpetual American call so that the solution for the non-recourse and for the margin call loan are readily available analytically or numerically. Moreover, the barrier option $v(x,1)$ subject to the early exercise conditions for a call and the interface condition

$$v(1,1) = \delta$$

will be the game option on $(1,\infty)$ corresponding to the call $u(x)$ provided $v'(1,1) \leq 1$.

Numerical values for the fair stock loan fee of the three loans are given in [14] for two sets of financial parameters. We choose a few representative values from [14] and compare them with their method of lines (MOL) counterparts.

Case 1: Stock loan fees for loans with positive effective interest rate $\tilde{r}$

The following data apply in Case 1:

$$r = .05, \quad \gamma = .02, \quad q = .015, \quad \sigma = .15, \quad S_0 = 100, \quad \pi = .8, \quad \delta = .2$$

The following numerical results are obtained with the method of lines [7].

Table 2: Fair loan fees $c(S_0, Q)$ and early exercise boundary $S(t)$

Q	Loan Type	$c(S_0, Q)$ [14]	MOL $c(S_0, Q)$	MOL $S(0)$
80	i	24.0118	24.0118	248.48
	ii	14.2848	14.2848	233.06
	iii	13.8014	13.8576	232.29
100	i	39.5872	39.5872	316.61
	ii	20.8271	20.8271	291.33
	iii	19.8911	20.0000	290.36

where type i) is the non-recourse loan, ii) is the margin call loan and iii) is the callable loan.

The MOL values are obtained numerically with the Riccati transformation on a grid fine enough so that no discernible discretization error is expected. The singularity in the coefficients of the equations (1.4) at $x = 0$ is avoided by introducing the artificial diffusion

$$\max\left\{\frac{1}{2}\sigma^2 x^2, 10^{-6}\right\}$$

into equation (1.21). For positive effective interest rates $\tilde{r}$ this regularization has no observable influence.

The reason for the slight difference in the game option values of loan type iii) is not known. However, it is known that the game call should be exercised by the lender at the strike price so that $c(100, 100) = 100v(1, 1) = 20$. We also remark that $v'(1+, 1) = .9558$ so that $v(x, 1)$ is indeed the game option value.

Case 2: Stock loan fees for loans with negative effective interest rate $\tilde{r}$

The following data apply in Case 2:

$$r = .05, \quad \gamma = .07, \quad q = 0, \quad \sigma = .15, \quad S_0 = 100, \quad \pi = .8, \quad \delta = .2$$

Since $\tilde{r} = -.02$ the perpetual call (1.21) for the non-recourse loan has a negative effective interest rate which can make its solution non-unique. This problem is resolved on probabilistic grounds in [12], but can also be solved via the analytic solution of (1.21) and simple financial considerations.

The difficulty arises because equation (1.21) has the general solution

$$u(x) = c_1 x^{\alpha_+} + c_2 x^{\alpha_-}, \tag{1.22}$$

where $\alpha_\pm$ are the two roots of

$$\frac{1}{2}\sigma^2\alpha^2 + \left(\tilde{r} - q - \frac{1}{2}\sigma^2\right)\alpha - \tilde{r} = 0,$$

provided the roots are distinct. For the data of this example we obtain the two values

$$\alpha_+ = 1.777777, \quad \alpha_- = 1.$$

Thus from (1.22) we see that $u(0) = 0$ for all c_1, c_2 so that only two boundary conditions are available to pin down the three unknowns c_1, c_2 and s.

The quadratic formula for the roots $\alpha_\pm$ can be written in the form

$$\alpha_\pm(\tilde{r}) = \frac{\sigma^2 - \left(\tilde{r} - q + \frac{\sigma^2}{2}\right) + \sqrt{\left(\tilde{r} - q + \frac{\sigma^2}{2}\right)^2 + 2\sigma^2 q}}{\sigma^2}$$

so that

$$\alpha_\pm(r) = 1 - A \pm \beta|A|$$

where

$$A = \left(\tilde{r} - q + \frac{\sigma^2}{2}\right) \quad \text{and} \quad \beta = \frac{\sqrt{A^2 + 2\sigma^2 q}}{|A|} \geq 1.$$

Thus for $\sigma\tilde{r}q \neq 0$ we see that

$$\alpha_+(\tilde{r}) > 1 \quad \text{and} \quad \alpha_-(\tilde{r}) < 1.$$

There is no financial reason that the stock loan should vary discontinuously when the loan rate γ approaches the interest rate r. This implies that the correct form of the analytic solution (1.22) for a call should be

$$u(x) = cx^{\alpha_+}$$

since otherwise u' would blow up as $x \to 0$. It is now simple to verify that the value of the non-recourse loan, in scaled variables, is given by

$$u(x) = \frac{s}{\alpha_+}\left(\frac{x}{s}\right)^{\alpha_+}, \quad s = \frac{\alpha_+}{\alpha_+ - 1} \tag{1.23}$$

which makes the fair service fee for the non-recourse stock loan

$$c(S_0, Q) = Qu\left(\frac{S_0}{Q}\right) + Q - S_0.$$

The numerical solution of (1.21) with the MOL equations (1.4) on $(0, \infty)$ for $\tilde{r} < 0$ does not cope well with the ill-posedness of the problem. For this reason the numerical results for the non-recourse loan are obtained from (1.23).

On the other hand, the stock loan with margin call only needs the solution $u(x)$ on $[\pi, \infty)$ which is readily found analytically or with the Riccati approach.

From the PDE values of the margin call loan ii) we find that

$$u(1) = .1002 < \delta = .2.$$

Moreover, the analytic or numerical solution shows that $u'(x)$ for the margin call loan satisfies

$$0 \leq u'(x) \leq 1 \quad \text{on } (.8, s_0).$$

This implies that

$$u(x) \leq H(x)$$

Table 3: Fair loan fees and early exercise boundary at $t = 0$, $r = .05$, $\gamma = .07$, $q = 0$, $S_0 = 100$, $\pi = .8$, $\delta = .2$

Q	Loan Type	$c(S_0, Q)$ [14]	MOL $c(S_0, Q)$	MOL $S(0)$
80	i	15.1764	15.1769*	182.86*
	ii	1.9376	1.9376	119.62
	iii		8.0155	149.53
100	i	29.5716	29.5722*	228.57*
	ii	10.0193	10.0193	149.53
	iii		20.000	186.91

Starred values are obtained from (1.21).

for all x, and hence that the lender would not call back the margin call loan. For this reason the game option Type iii) fees are not listed in [14].

We continue with some comments and numerical results for stock loans with finite time to expiry. When scaled by the loan amount Q, the loan value $U(x, t)$ will satisfy (1.17) with value at expiry at time T given as

$$U(x, T) = \max\{(x - 1)e^{\gamma T}, 0\}$$

and the early exercise condition

$$U(s(t), t) = (s(t) - 1)e^{\gamma t}, \quad U_x(s(t), t) = e^{\gamma t}.$$

For loan types i) and ii) we have the boundary conditions

i) $U(0, t) = 0$

ii) $U(.8, t) = 0$

Numerical results can be obtained for both types of stock loans with the time-discrete method of lines as described in detail in [7]. Only the positive interest rate r appears in (1.17) for which the numerical results are stable with respect to the time and space discretization employed for the method of lines.

Table 4 lists the computed loan fees at real time $t = 0$ for the two sets of parameters of Tables 2 and 3.

It is apparent that the perpetual loan values give limited guidance to the fair loan fees for a finite time loan.

Fig. 3 shows the early exercise boundaries $S(t)/Q = e^{\gamma t}s(t)$ for the margin call loans over a thirty year life span. $s(t)$ is readily provided by the MOL-Riccati solution method.

For both cases we set: $\lim_{t\to T} s(t) = \max\left\{1, \frac{(r-\gamma)}{q}e^{\gamma t}\right\}$ which is implied by equation (1.21) as $t \to T$.

Table 4: Fair loan fees of non-recourse and margin call stock loans at $t = 0$, $S_0 = 100$ and expiration in T years

Case	Type	T	Q = 80	Q = 100	Q = 120	
1	i)	20	16.5688	29.1119	43.3403	
1	i)	50	22.1984	36.9425	52.8761	
1	i)	∞	24.0118	39.5872	56.3043	
2	i)	20	5.0746	16.3084	30.9215	
2	i)	50	8.3633	20.8528	35.9055	
2	i)	∞	15.1769	29.5722	45.6624	from (1.23)
1	ii)	20	12.7931	19.6781	24.3385	
1	ii)	50	14.2068	20.7680	24.5988	
1	ii)	∞	14.2848	20.8271	24.6129	
2	ii)	20	1.9257	10.0068	21.7028	
2	ii)	50	1.9376	10.0194	21.7055	
2	ii)	∞	1.9376	10.0193	21.7054	

Case 1: $r = .05, \gamma = .02$, $q = .015$, $\sigma = .15$
Case 2: $r = .05$, $\gamma = .07$, $q = 0$, $\sigma = .15$
Type i): non-recourse loan
Type ii): margin call loan with barrier at $S = .8Qe^{\gamma t}$.

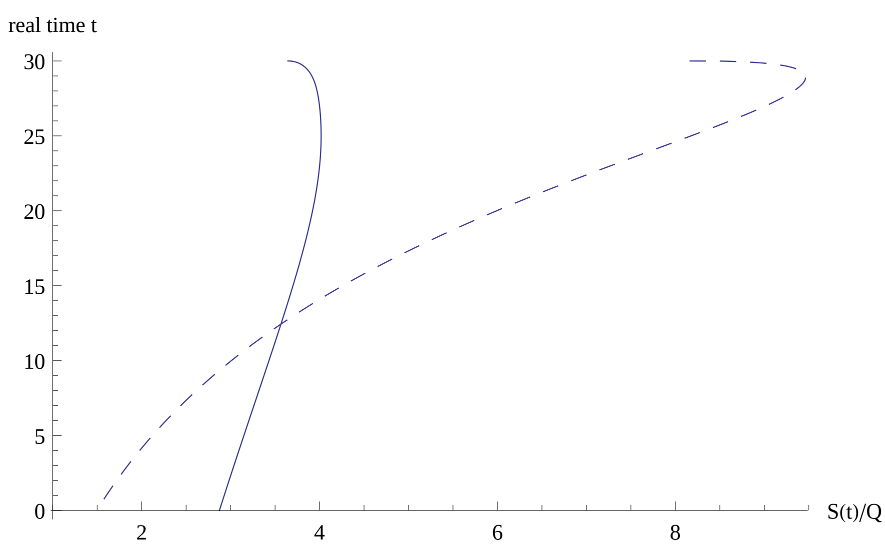
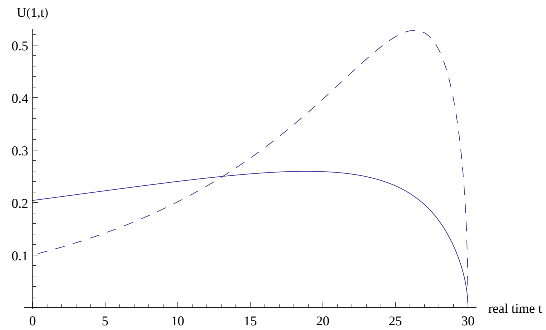


Figure 3: Early exercise boundaries for 30 year margin call stock loans.
Solid curve: Case 1 margin call early exercise boundary $S(t)/Q$
Dashed curve: Case 2 margin call early exercise boundary $S(t)/Q$

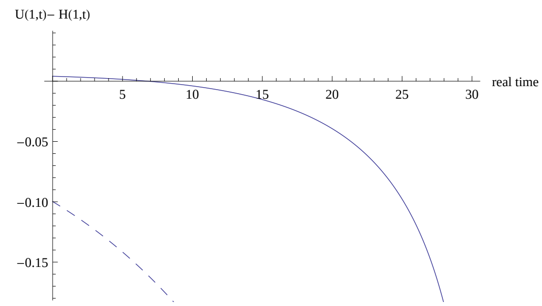
Fig. 4a shows the loan value $U(1,t)$ as a function of real time t for two 30 year stock loans with margin call at $x = .8$. We recall that

$$QU(x,t) = C(S,t) \quad \text{where } S = Qe^{\gamma t}x$$

so that $x = 1$ corresponds to the strike price of the loan. The value of $U(1,t)$ is not monotone with time for either margin call loan.



(a) Scaled loan value $= U(1,t)$ for the 30 year margin call loan over the life of the loan.
Solid curve: Case 1
Broken curve: Case 2.
Fig. 4a



(b) $F(t) = U(1,t) - \delta e^{\gamma t}$ for the 30 year margin call loan over the life of the loan.
Solid curve: Case 1
Broken curve: Case 2.
Fig. 4b

Figure 4

Fig. 4b shows the difference $F(t)$ between $U(1,t)$ and the (scaled) cancellation pay-off

$$H(1,t)/Q = \delta e^{\gamma t}, \quad \delta = .2, \quad t \in [0,30].$$

When $F(t) > 0$ then $D(t) \neq \emptyset$. Calling the margin call loan will decrease the value of $C(S,t)$ and is advantageous to the lender. For Case 1 cancellation would be beneficial for $t \in [0, 6.8]$. The game option reflecting optimal cancellation remains to be determined. For Case 2 we have $D(t) = \emptyset$ for all $t \in [0,30]$ and no cancellation should occur.

We conclude with some preliminary comments on game options for puts and calls with finite expiration. As in Section 1 let t denote the time to expiry at T.

If the writer of the option is allowed to cancel it only at discrete times $\{t_n\}$ in the interval $[0,T]$, then it seems reasonable to assume that cancellation will occur whenever the underlying falls into the set $D(t_n)$ so that the option jumps from $u(x, t_n-)$ to

$$v^0(x, t_n+, y) = Pu(x, t_n) \equiv \min\{u(x, t_n-), H(x, t_n)\}$$

where P will be called a projection.

v^0 becomes the initial condition for $u(x,t)$ over the next time interval when no cancellation is allowed. This cancellation schedule will decrease the value of u, when there is no time available to wait until the underlying reaches the optimal value for cancellation. However, this jump will cause discontinuities in the option value and its derivatives which will need to be analysed and resolved numerically.

The problem would seem to be mitigated for unconstrained cancellation at all times. The barrier condition will arise smoothly at time t^* at the value $\tilde{y}$ of maximum separation of u and v^0 and evolve continuously. We expect that a time discrete MOL approximation of the option with projection after each Δt-time step will mirror this behavior. In fact, we conjecture that this MOL solution with projection will converge to the game version of the original put and call. To test this conjecture we consider the perpetual put and its game version in Example 1. The perpetual put is approximated with a long-term but finite time American put. This put is then solved with a time-discrete MOL method combined with projection after each time step. This calculation is essentially identical with that for a standard American put and does not include a search for $\hat{y}$ at any time.

Fig. 5 shows the game option $v(x, y^*)$ of Fig. 1 and the MOL solution $v^0(x, T)$ at real time $\tau = 0$ over $(0, 1)$. To plotting accuracy both option values coincide over $(0, 1)$ and both have the same optimal exercise boundary $s(\hat{y}) = .2895$ for the holder. Thus in this case the solution of the MOL equations with projection provides a good approximation to the value of the game option at $\tau = 0$.

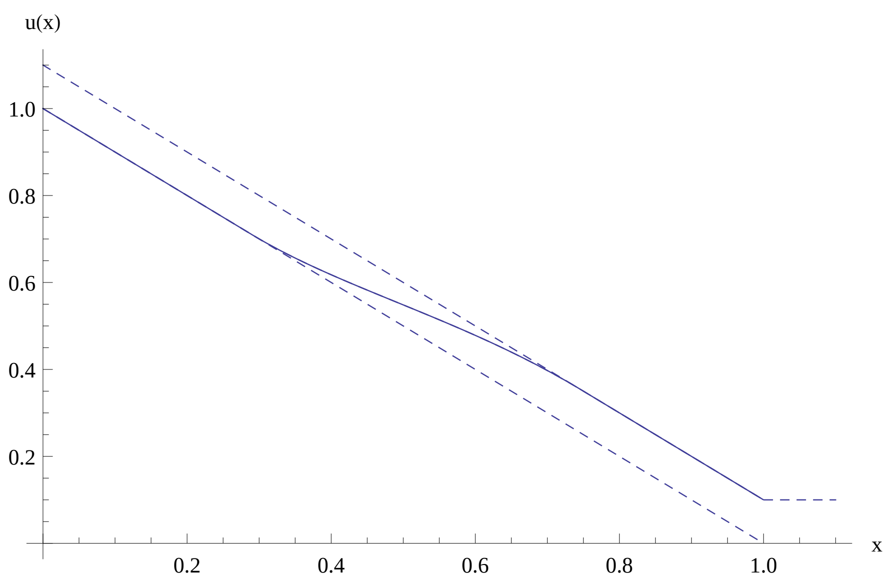


Figure 5: Game option for a perpetual put and its approximation with a cancellable long term finite time put
Solid curve: MOL solution $v^0(x, T)$ with projection after each time step
Broken curve: perpetual game option $v(x, y^*)$ of Fig. 1.

To plotting accuracy both curves coincide. $r = .03$, $q = .08$, $\sigma = .2$, $\delta = .1$
Put expiration $T = 100$, 10 000 MOL steps with $\Delta t = 10^{-2}$. The spacial discretization for the MOL equations is sufficiently fine that all errors are due to the time discretization.

In other MOL simulations with projection for finite time puts and calls the resulting $\{v^0(x, t_n)\}$ is readily found and and stable with respect to changes in Δt. If it can be proven that the time discrete solutions with projection do indeed converge to the game version of the American option as $\Delta t \to 0$ then the game option would be as easy to compute as the original option.

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