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Abstract

We develop novel methods for efficient analytical solution of all types of partial time barrier options with both single and double exponential and time varying boundaries, and specifically to treat forward-starting partial double barrier options, which present the simplest non-trivial example of the multiple exponential time-varying barrier case. Our methods reduce the pricing of all barrier options with time-varying boundaries to the pricing of a single European option. We express our novel results solely in terms of European first and second order Gap options. We are motivated by similar structures appearing in Structural Credit Risk models for firm default.

Keywords: Exotic Options, Method of Images, Partial Time Double Barrier Options, Window Double Barrier Options, Partial-time barrier options, Credit Risk

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1. Introduction

Barrier options have grown in popularity for several decades, particularly in the over-the-counter (OTC) and foreign exchange (FX) markets, for a variety of reasons. For instance, for a given option payoff at expiry, the corresponding barrier option will be cheaper than the equivalent European (or vanilla) option while still offering an equivalent level of protection. Also, barrier options offer greater flexibility than traditional options. This is because they can be tailored to meet the needs of individual market participants be they hedgers or speculators. In particular, barrier options with barrier monitoring windows commencing at a forward-starting arbitrary date before option expiry allow adaptable monitoring and greater flexibility to manage volatility risks during specified periods. Monitoring windows restricted to some subset of the option lifetime allow for similar protection as standard equity options however at a reduced premium.

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Barrier options however have a wider significance than just for exotic options on equities. An important reason for studying barrier option structures, and indeed a major motivation for this study, stems from the potential application that barrier options have in structural credit risk models which use the value of a firm to determine time of firm default. The first structural credit-risk model was Merton (1974), which extended the seminal Black and Scholes (1973) to the value of the firm, modelled as a stochastic process driven by Geometric Brownian Motion. In this model default occurs at the time of 'debt servicing' (corresponding to option expiry) if the firm's asset value is insufficient to repay outstanding debt. The default boundary is just the face value of the debt. However defaults can occur at any time.

An extension of the structural approach relevant here was undertaken in Black and Cox (1976). This work assumed that default occurs as soon as the value of a firm's assets drop below a certain threshold. Such thresholds have natural interpretations as knock-out barriers on the firm's value. Structural models of credit default therefore naturally share features with equity options having knock-out barrier features.

The value of the outstanding debt triggering default, namely the 'credit default barrier' may naturally be considered to be exponential and time-varying due to the time-value of money, rather than constant. Multiple exponential and time-varying barrier levels may be interpreted as thresholds triggering rating upgrades, downgrades and default in the manner of CreditMetrics (see Gupton et al. (1997))[1].

Although we do not explicitly treat the problem of credit default modelling in this work due to its unique challenges, we note that credit downgrade/default and credit upgrade scenarios naturally give rise to lower and upper exponential and time-varying barrier levels for the value of a firm for which the current state of the art is Monte-Carlo simulation. The application to credit risk scenarios is an important motivation for examining techniques for the efficient treatment of barrier option structures with multiple exponential and time-varying lower and upper boundaries, of which the partial-time late monitoring double barrier scenario is the simplest non-trivial case. The methodology we employ here points a way to getting tractable solutions in a multi-barrier context when using proper Black-Scholes dynamics for the firm value.

1.1. Background

In the classic Black-Scholes model, there is a wide literature dealing with the pricing and hedging of barrier options. Most of the papers appearing in the literature have approached the problem of pricing barrier and double barrier options by using the so-called expectations or probabilistic approach and have also assumed constant fixed level barriers. Within this framework, it is relatively straightforward to price and hedge single barrier options, and valuation formulae have been in the literature for a long time.

[1]CreditMetrics uses constant thresholds, and a simplified Brownian-Motion type latent variable dynamics instead of Geometric Brownian Motion interpretable as the firm's value.

A short while after the seminal paper of Black and Scholes (1973), Merton (1973) gave the pricing formula for an option with a continuously monitored lower (constant) knock-out boundary, the so-called single knock-out barrier option. An extended treatment for various types of weakly path dependent options was presented in Goldman et al. (1979). Rich (1994) and Rubinstein and Reiner (1991) also tackled the pricing of European single barrier options, including the knock-in barrier calls and puts using discounted expectations under the Equivalent Martingale Measure (EMM).

For monitoring windows extending throughout the full lifetime of the option, the original paper pricing double-barrier options in the Black-Scholes framework is attributed to Kunitomo and Ikeda (1992), and later reported in Zhang (1998). Kunitomo and Ikeda (1992) gave prices for the standard knock-out call and put options where the barrier levels are exponentially time-varying.

Monitoring windows may be restricted to a subset of the life of the option. Formulae for such partial-time barrier options were first derived by Heynen and Kat (1994) in the case of single down-and-out or up-and-out barrier monitoring. This was also explored in Carr (1995). The methodology employed by these authors was complicated, utilising theorems on Gaussian first passage times and an array of complex integrations.

Other methods that have appeared in the literature in the case of double knock-out constant barrier levels are the the Fourier series solutions of Geman and Yor (1996) and Pelsser (2000), however with series coefficients which must be obtained using numerical integration. Hui (1997) also applied Fourier series techniques to the problem of partial-time barrier options.

Some authors have approached the problem of pricing options with barrier features from a discrete sampling perspective. Fusai et al. (2006) tackled the pricing of discrete barrier options in the Black-Scholes framework by reducing the valuation problem to a Wiener-Hopf equation which they solved analytically. Howison and Steinberg (2007) employed matched asymptotic expansions to discuss the 'continuity correction' needed to relate the prices of discretely sampled barrier options and their continuously-sampled equivalents. In contrast Hsiao (2012) found approximate barrier solutions in the forward-starting case by numerically solving partial differential equations after applying the Boundary Integral Method.

More recently Buchen and Konstandatos (2009) introduced an alternative approach for analytically pricing single and double barrier options with exponential and time-varying barrier levels, using what they refer to as the *Method of Images* (MOI).

This method should not be confused with other techniques with a similar name. However it is somewhat reminiscent of the well-known method of images for solving boundary value problems in theoretical physics. Pricing barrier options is generally more complex than solving terminal value problems because options with barrier monitoring windows must also satisfy boundary conditions. This is analogous with initial value problems being simpler than initial boundary value (IBV) problems for the heat equation in theoretical physics. The MOI tackles the problem of pricing options with simultaneously active upper and lower exponential time-varying barrier features in a novel way, by utilising what is called the *image solution operator*, which is related to the mathematical symmetries satisfied by solutions of the Black-Scholes PDE.

The basic method first appeared in Buchen (2001), where single flat-barriers were treated, and was extended in Konstandatos (2003), and later in Konstandatos (2008) to the flat double barriers case, where formulae for partial time single and double barrier options with flat boundaries were also derived. A discussion of the flat barriers case may also be found in Buchen (2012).

Standard methods of treating time-varying boundaries involve transforming an exponential time-varying barrier problem to the constant (*i.e.* flat) barrier case through a change of variables. This technique is generally applicable in the single exponentially time-varying barrier case. It is only applicable in the double exponential barrier or more generally multiple exponentially time-varying barrier cases when all the barrier levels are growing (or decaying) exponentially at the same rate. In the case when two exponential barriers are allowed to vary at different rates, this approach will no longer work because any transformation that flattens one barrier level will not flatten the other.

This difficulty was first resolved by Kunitomo and Ikeda (1992) utilising a result from *Sequential Analysis* attributed to T.W. Anderson. The problem was later solved in greater generality in Buchen and Konstandatos (2009). There a solution was demonstrated for pricing exponential time-varying barrier option problems for a general payoff function and for all permissible parameters when monitoring extends over the whole life of the option. This was done by taking advantage of the algebraic properties of the image solution operator. For any payoff it was demonstrated that the associated full-monitoring window double barrier knock-out option may always be reduced to pricing a corresponding path-independent terminal-value (TV) problem. This approach avoids the need for complicated expectations calculations against the joint density of the underlying stock price and the maximum and minimum processes. It reduces the problem to that of pricing a single European option.

In this paper we extend the analysis of Buchen and Konstandatos (2009) to consider the arbitrage free pricing of partial-time options with either a single lower or upper boundary, or conversely with both an upper and lower boundary, in both early-monitoring and late-monitoring cases, where the boundaries are exponential and time varying in the vein of Kunitomo and Ikeda (1992). The extension of the methods to analytically treat the late[2]-monitoring partial-time double-barrier options with exponential and time-varying boundaries is one of the main theoretical contributions of the paper.

[2]or indeed any forward-starting

Our approach allows the efficient representation of all option prices in terms of essentially one type of simple analytical instrument: the Gap option. As far as we know, closed form expressions for the early and late monitoring double barrier options with exponential and time-varying boundaries, and their representations solely in terms of Gap options, are new. The Gap options as required in our analysis (also referred to as *thresh-hold options*) are European options, quite similar to standard calls/puts in the first-order case, and compound options such as calls-on-calls, calls-on-puts etc in the second-order case, but where the exercise condition is decoupled from the strike price: the strike and exercise prices are allowed to differ. Our resulting representations are highly structured and symmetric, and allow efficient and less error prone coding for numerical evaluation.

1.2. Organisation of paper

The paper is organised as follows. Section 2 describes the basic framework for pricing barrier options. Both the PDE and EMM approaches are discussed, since a combination of both methods plays an important role in developing the integration-free technique alluded to above. Section 3 describes the image solution operator and its use in pricing both single and double barrier options with essentially arbitrary payoff functions and exponential time-varying barriers for the Black-Scholes model. Some of this material has already appeared in Buchen and Konstandatos (2009), and is included here without proof for completeness of the presentation and because it relies on calendar time t rather than time to expiry $\tau = T - t$, as was used in Buchen and Konstandatos (2009). This modification has necessitated a restatement of the basic results and theorems upon which we build for this work, but the proofs and lemmas that we rely on apply, however with slight modifications. In the modified framework, we proceed to present a new and simpler proof of the properties of the image solution of the BSPDE, as well as a simplified proof of the main result of Buchen and Konstandatos (2009) using discounted expectations under the EMM. The remaining sections present a range of applications including the main contributions of this paper, namely, a novel and unified account of single and double exponential barrier option pricing for full-time and partial-time monitoring windows. Included in this are explicit representations for partial-time early and late monitoring double barrier options with exponentially time varying boundaries. Section 7 presents numerical results in both graphical and tabular form. These were obtained by direct numerical evaluation in the computer packages Mathematica and Matlab. An appendix gives a description of the Gap options referred to in this paper.

2. The Model Framework

We will work with calendar time t, *i.e.* running forward, so that $T > t$ is taken to be an option expiry time. We will also be assuming a standard Black-Scholes economy where the non-dividend paying underlying asset X_t follows geometric Brownian motion of constant volatility σ, described by the stochastic differential equation (sde)

$$dX_t = rX_t\,dt + \sigma X_t\,dB_t \tag{1}$$

Here r is the the risk free interest rate assumed to be constant and B_t is a standard Brownian motion. It is elementary to add in a constant, continuous dividend yield if required.

Definition 2.1. *The Black-Scholes operator $\mathscr{L}$ is defined by*

$$\mathscr{L}V(x,\tau) = \frac{\partial V}{\partial t} - rV + rx\frac{\partial V}{\partial x} + \tfrac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} \tag{2}$$

The corresponding BS-PDE is defined to be $\mathscr{L}V = 0$.

All European derivative prices satisfy the BS-PDE in the unrestricted asset price domain $x > 0$ for $t < T$, where $t = T$ is the option's expiry date in the future, with a specified terminal value $V(x,T) = f(x)$.
The function $f(x)$ is called the derivative's *payoff function*. Such terminal value (TV) problems are generally easy to solve. For example, a solution can be written down using the formula for the Fundamental Theorem of Asset Pricing

$$V(x,\tau) = e^{-r\tau}\,\mathbb{E}_Q\{f(X_T)|X_t = x\} \tag{3}$$

where $\mathbb{E}_Q$ is the expectation under the risk-neutral measure Q and $\tau = T - t$ is the time remaining to option expiry.
It is well established (see Harrison and Pliska (1981)) that Eq(3) gives the arbitrage free price of the derivative if and only if the conditional expectation is taken with respect to the Equivalent Martingale Measure Q, under which X_T has the representation

$$X_T = x\exp\{(r - \tfrac{1}{2}\sigma^2)\tau + \sigma B^Q_\tau\}\} \tag{4}$$

where B^Q_τ is a Q-Brownian motion. Equation(4) solves the stochastic differential equation (SDE) (1). Thus, X_T/x is the exponential of a Gaussian random variable and is therefore log-normally distributed.

Before we proceed we need some basic definitions. Knock-out double barrier options are similar to single barrier equivalents, but have two barrier levels, a lower knock-out barrier level and an upper knock-out barrier level. If the spot price of the underlying asset were to reach either barrier level before expiry then the option will expire worthless. Conversely, the knock-in double-barrier option will always expire worthless unless the spot price were to reach either barrier level some time before expiry.

Barrier monitoring need not extend extend over the whole lifetime of the option. A barrier option may have a period when the barrier monitoring window is *active*, so that the option will be knocked out only if the barrier is breached in this period. It follows that in the complementary period when the barrier monitoring window is inactive, hitting the barrier has no effect on the payoff at expiry.

We shall refer to a barrier option as being *full-period* if the barrier monitoring window is active throughout the whole lifetime of the option, otherwise we will refer to the barrier option as being a *partial-time* barrier option. An *early monitoring* partial barrier option is one in which the barrier monitoring window is active from the start of the option until some future date before the expiry time, whereas a *late monitoring* partial barrier option's barrier monitoring window begins some time after the option's inception, ending at the expiry date. Naturally, breaching the barrier for any time outside the barrier monitoring window for both the early and late monitoring partial barrier options will have no effect.

We now consider the boundary-value TBV for the down-and-out (D/O) barrier option, with a single exponential time varying lower boundary using $V(x,t)$ for the option price,and $f(x)$ for any option payoff function at expiry $t = T$.

Problem 2.2. *Single Down-and-out Barrier Option*

$$\begin{cases} \mathscr{L}V(x,t) &= 0; & t < T; \quad x > b(t) \\ V(x,T) &= f(x); & x > b(T) \\ V(b(t),t) &= 0; & t < T \end{cases}$$

where $b(t) = Be^{\beta t}$ *is an exponential time varying barrier.*

The other single barrier types are the down-and-in (D/I), up-and-out (U/O) and up-and-in (U/I) and satisfy similar TBV problems but with different domains and boundary conditions. In particular, the U/O barrier domain is below the upper boundary level $x < b(t)$ but is otherwise identical to problem 2.2; the knock-in barrier options have the same domains as their knock-out companions, but have zero payoff at time T and boundary condition $V(x,t) = V_0(x,t)$ at $x = b(t)$ where $V_0(x,t)$ is the price of a corresponding European option with payoff $f(x)$. That is, $V_0(x,t)$ satisfies the simple TV problem: $\mathscr{L}V_0 = 0$ for all $t < T$ with terminal value $V_0(x,T) = f(x)$.

The following PDE describes the boundary-value problem for a double knock-out barrier option with exponential barriers for any payoff function $f(x)$ at expiry $t = T$. In this case when the asset price x hits either the lower or upper exponential time varying barriers at any time prior to expiry, the option instantly expires worthless.

Problem 2.3. *Double Knock-out Barrier Option*

$$\begin{cases} \mathscr{L}V(x,t) &=& 0; & t<T, \quad a(t)<x<b(t) \\ V(x,T) &=& f(x); & a(T)<x<b(T) \\ V(x,t) &=& 0; & x=a(t),b(t); \quad t<T \end{cases}$$

where $a(t)=Ae^{\alpha t}$ and $b(t)=Be^{\beta t}$ are exponentially time-varying lower and upper barrier levels such that $a(t)<b(t)$ for all $t\leq T$.

3. Method of Images for Exponential Time-Varying Barriers

In this section we present the results from Buchen and Konstandatos (2009) which underpin our approach, expressed in calendar time t, rather than in terms of time to expiry $\tau=T-t$. We refer readers to Buchen and Konstandatos (2009) for a derivation of the image operator for time-varying barriers from the image operator for constant barriers, by use of symmetry properties of the Black-Scholes PDE. We present several extensions which will prove necessary later on.

3.1. Single Exponential Time-Varying Barrier Case

Definition 3.1. *Let $V(x,t)$ be any function of x and t. We define the image function of V with respect to the single exponential time-varying barrier $x=b(t)$ denoted $\mathscr{I}_{b(t)}\{V(x,t)\}$ by:*

$$\mathscr{I}_{b(t)}\{V(x,t)\}=(b(t)/x)^{q_\beta}\,V(b^2(t)/x,t) \tag{5}$$

where $q_\beta=2(r-\beta)/\sigma^2-1$.

Given any solution $V(x,t)$ of the BS-PDE, $\mathscr{I}_{b(t)}V(x,t)$ is also a solution. Buchen (2012) provides a demonstration in the constant barriers case.

Remark 3.2. We may think of $\mathscr{I}_{b(t)}$ as an *operator* mapping solutions $V(x,t)$ of the BS-PDE to so-called *image solutions* $\mathscr{I}_{b(t)}V(x,t)$ with with the following three properties. Denoting I as the identity operator:

1. $\mathscr{I}_{b(t)}$ is an involution. Namely, $\mathscr{I}^{-1}_{b(t)}=\mathscr{I}_{b(t)}$ or $\mathscr{I}^2_{b(t)}=I$.
2. When $x=b(t)$, $\mathscr{I}_{b(t)}V=V$ *i.e.* $(I-\mathscr{I}_{b(t)})V(b,t)=0$.
3. If $x\neq b(t)$, x and the image price $y=b^2(t)/x$ always lie on opposite sides of the barrier level $b(t)$.

The most important property of the image operator with exponential barriers is expressed in the following Lemma. We give an alternative proof by direct application of the Fundamental Theorem of Asset Pricing.

Lemma 3.3. *Let $T > t$ be a future expiry time and let $b(t) = Be^{\beta t}$ be a time-varying exponential barrier. Given that $V(x,t)$ is a solution of the Black-Scholes PDE with payoff $V(x,T) = f(x)$, then the image function $\mathscr{I}_{b(t)}\{V(x,t)\}$ with respect to the barrier level $x = b(t)$ is a solution of the Black-Scholes PDE with payoff $\mathscr{I}_{b(T)}\{f(x)\}$.*

Proof. By use of the Feynman-Kac Theorem, or alternatively from the Fundamental Theorem of Asset Pricing (Harrison and Pliska (1983)), we can represent any solution of of the Black-Scholes PDE for an option with payoff $V(x,T) = f(x)$ as the following Gaussian expectation for $t < T$:

$$V(x,t) = e^{-r\tau}\mathbb{E}_Q\{F(X_T)|X_t = x\}$$

where $\tau = T - t$, and where under the equivalent martingale measure Q,

$$X_T \stackrel{d}{=} xe^{\mu\tau+\sigma\sqrt{\tau}Z}$$

where $Z \sim \mathscr{N}(0,1)$ and $\mu = r - \frac{1}{2}\sigma^2$. With this representation, we can write the $t < T$ price of option $\overset{*}{V}(x,t) = \mathscr{I}_{b(t)}V(x,t)$ with payoff $\mathscr{I}_{b(T)}\{f(x)\}$ as:

$$\begin{aligned}
\overset{*}{V}(x,t) &= e^{-r\tau}\mathbb{E}_Q\left\{\left(\tfrac{b(T)}{X_T}\right)^{q_\beta} f\left(\tfrac{b^2(T)}{X_T}\right)\right\} \\
&= e^{-r\tau}\mathbb{E}_Q\left\{\left(\tfrac{b(T)}{x}\right)^{q_\beta} e^{-q_\beta\mu\tau-q_\beta\sigma\sqrt{\tau}Z} f\left(\tfrac{b^2(T)}{x}e^{-\mu\tau-\sigma\sqrt{\tau}Z}\right)\right\} \\
&= e^{-r\tau}\left(\tfrac{b(t)}{x}\right)^{q_\beta}\left(\tfrac{b(T)}{b(t)}\right)^{q_\beta} e^{-q_\beta\mu\tau} \times \\
&\qquad \mathbb{E}_Q\left\{e^{-q_\beta\sigma\sqrt{\tau}Z} f\left(\tfrac{b^2(t)}{x}\left(\tfrac{b^2(T)}{b^2(t)}\right)e^{-\mu\tau-\sigma\sqrt{\tau}Z}\right)\right\} \\
&= e^{-r\tau}\left(\tfrac{b(t)}{x}\right)^{q_\beta} e^{q_\beta(\beta-\mu)\tau} \times \\
&\qquad \mathbb{E}_Q\left\{e^{-q_\beta\sigma\sqrt{\tau}Z} f\left(\tfrac{b^2(t)}{x}e^{2(\beta-\mu)\tau-\sigma\sqrt{\tau}Z}\right)\right\}
\end{aligned}$$

where we have used $b(T)/b(t) = e^{\beta\tau}$. The last expectation above can be simplified using the Gaussian Shift Theorem: for any function $H(Z)$ of a Gaussian rv Z, we have that

$$\mathbb{E}\left\{e^{cZ}H(Z)\right\} = e^{\frac{1}{2}c^2}\mathbb{E}\{H(Z+c)\}.$$

We therefore obtain:

$$\overset{*}{V}(x,t) = e^{-r\tau}\left(\tfrac{b(t)}{x}\right)^{q_\beta} e^{q_\beta\lambda\tau}\mathbb{E}_Q\left\{f\left(\tfrac{b^2(t)}{x}e^{\nu\tau-\sigma\sqrt{\tau}Z}\right)\right\}$$

for $\lambda = q(\beta - \mu) + \frac{1}{2}q^2\sigma^2 \equiv 0$, and $\nu = 2\beta - \mu + q_\beta\sigma^2 \equiv \mu$.

For any Gaussian Z, $\mathbb{E}\{H(-Z)\} = \mathbb{E}\{H(Z)\}$. Thus replacing Z by $-Z$ we get:

$$\begin{aligned}
\overset{*}{V}(x,t) &= e^{-r\tau}\left(\frac{b(t)}{x}\right)^{q_\beta} \mathbb{E}_Q\left\{f\left(\tfrac{b^2(t)}{x}\, e^{\mu\tau+\sigma\sqrt{\tau}Z}\right)\right\} \\
&= \left(\frac{b(t)}{x}\right)^{q_\beta} V\left(\frac{b^2(t)}{x}, t\right)
\end{aligned}$$

since $V(x,t) = e^{-r\tau}\mathbb{E}_Q\{f(xe^{\mu\tau+\sigma\sqrt{\tau}Z})\}$. □

An elegant and short proof that $\mathscr{I}_{b(t)}\{V(x,t)\}$ is a solution of the Black-Scholes PDE whenever $V(x,t)$ is may be found in Buchen and Konstandatos (2009). Buchen (2012) contains proofs in the case of constant b.

Definition 3.4. *Let $\mathscr{PV}_t$ denote the present-value or pricing operator in a Black-Scholes economy. That is, $\mathscr{PV}_t$ operates on any function $f(x)$ of the stock price $x = X_T$, to produce the arbitrage free value of the derivative at time $t < T$.*

Corollary 3.5. *Given any European option payoff $f(x)$ at $t = T$, and exponential time varying barrier level $b(t)$,*

$$\mathscr{PV}_t\left\{\mathscr{I}_{b(T)} f(x)\right\} = \mathscr{I}_{b(t)}\left\{\mathscr{PV}_t\{f(x)\}\right\}$$

Namely, the pricing operator $\mathscr{PV}$ and the Image Operator $\mathscr{I}$ commute, when taking account different points in time. Corollary 3.5 is an alternative statement of Lemma 3.3. Along with its extension Lemma 4.3, it turns out to be crucial to pricing exponential time varying barrier options, particularly with forward-starting monitoring windows by allowing us to obtain present values of various image terms that appear in the analysis. The basic idea is to embed the restricted (domain) TBV problem for a single barrier, into an equivalent unrestricted TV problem, and the following theorem, termed the *Method Of Images* (MOI), shows precisely how this is done.

Theorem 3.6 (MOI for Exponential Time-Varying Boundaries)**.** *Let $U(x,t)$ solve the TV problem $\mathscr{L}U = 0$ with terminal value $U(x,T) = f(x)\mathbb{I}(x > b(T))$. Then the solution of the TBV problem $\mathscr{L}V = 0$ in $x > b(t)$ with $V(x,T) = f(x)$ and $V(b(t),t) = 0$ is given by*

$$V(x,t) = U(x,t) - \mathscr{I}_{b(t)}U(x,t) \tag{6}$$

for all $t > T$ and $x > b(t)$.

Proof. A proof using the algebraic properties of the image operator may be found in Buchen and Konstandatos (2009). We will now present a simplified proof below using several lemmas and discounted expectations under the EMM. □

Remark 3.7. Theorem 3.6 shows that the solution $V(x,t)$ of the restricted TBV problem for general down-and-out options with exponential time-varying boundaries, can be expressed in terms of the solution $U(x,t)$ of a related *unrestricted* TV problem. The related problem has its payoff function modified from $f(x)$ to $f(x)\mathbb{I}(x>b(T))$. We may therefore think of the related problem as a down-type *European* binary option which pays $f(x)$ only if the asset price is above the barrier level at expiry time T. A similar result holds for up-and-out options in the domain $x<b(t)$, where the modified payoff will be $f(x)\mathbb{I}(x<b(T))$.

Remark 3.8. To recover the method for flat barriers, simply set $\beta = 0$.

3.2. *Parity relations*

For any standard (European) option $V_0(x,t)$ with payoff $V_0(x,T) = f(x)$, there will be four corresponding barrier options, *viz* the down-and-out, down-and-in, up-and-out and up-and-in barrier options, all with the exponentially time varying boundary $x = b(t)$. It is not immediately apparent that all these barrier options are not in fact independent of each other. In fact, pricing just one of the four barrier options will immediately allow one to obtain the price of the remaining three. This result was first noted in Buchen (2001) for the case of constant (flat) barriers; it turns out that similar relations hold for exponentially time varying barriers as well.

The three parity relations connecting the four barrier option types are described below (in an obvious notation).

$$\begin{aligned} V_{do}(x,t)+V_{di}(x,t) &= V_0(x,t); \qquad x>b(t) \qquad (7)\\ V_{uo}(x,t)+V_{ui}(x,t) &= V_0(x,t); \qquad x<b(t) \qquad (8)\end{aligned}$$

While these two parity parity relations are well known, the next is rarely quoted.

$$\mathscr{I}_{b(t)}\{V_{di}(x,t)\} = V_{ui}(x,t) \qquad \text{or} \qquad \mathscr{I}_{b(t)}\{V_{ui}(x,t)\} = V_{di}(x,t) \qquad (9)$$

With the above parity relations and the image operator it is now possible to express the prices of *all four* exponential barrier options for any given payoff as in the following lemma.

Lemma 3.9. *Let* $\overset{*}{U}(x,t) = \mathscr{I}_{b(t)}\{U(x,t)\}$ *denote the image of* $U(x,t)$ *with respect to the barrier* $x = b(t)$*, where*

$U(x,t)$ is defined in Theorem 3.6 and similarly let $\overset{*}{V}_0(x,t) = \mathscr{I}_{b(t)}\{V_0(x,t)\}$. Then

$$
\begin{aligned}
V_{do}(x,t) &= U(x,t) - \overset{*}{U}(x,t) && (10)\\
V_{di}(x,t) &= V_0(x,t) - [U(x,t) - \overset{*}{U}(x,t)] && (11)\\
V_{ui}(x,t) &= \overset{*}{V}_0(x,t) + [U(x,t) - \overset{*}{U}(x,t)] && (12)\\
V_{uo}(x,t) &= [V_0(x,t) - \overset{*}{V}_0(x,t)] - [U(x,t) - \overset{*}{U}(x,t)] && (13)
\end{aligned}
$$

This lemma allows us to immediately price any barrier option of interest in terms of just two functions; $U(x,t)$ and $V_0(x,t)$ and their images. Furthermore, both U and V_0 are solutions of TV problems of the Black-Scholes PDE and are for many practical choices of payoff function $f(x)$, readily calculated. Recall that $V_0(x,t)$ is the $\mathscr{PV}$ of a European option with expiry T payoff $f(x)$, whilst $U(x,t)$ is the $\mathscr{PV}$ of a European option with expiry T payoff $f(x)\mathbb{I}(x>b(T))$. It should also be noted, with the inherent symmetries represented by the above parity relations, numerous alternate but equivalent formulations of the complete solution for all single exponential barrier options are possible.

4. Method of Images for Double Exponential Barriers

In order to make this work self-contained, we include in this section the main results of Buchen and Konstandatos (2009) extended for double exponential time varying barrier options, and translated to calendar time t rather than time-to-expiry τ. We refer the reader there for proofs of the results which are quoted below without proof. The rapid convergence of the doubly-infinite sums in Theorem 4.1 are also found there for arbitrary $f(x)$.

Lemma 4.3, Corollary 4.4, Lemma 4.5 and Lemma 4.6 are new and proofs are supplied.

We first clarify some notation. By the symbol $\mathscr{I}_{ab}$ we mean the composition of the two image operations $\mathscr{I}_a\mathscr{I}_b$, where $\mathscr{I}_b$ is carried out first and $\mathscr{I}_a$ second. That is, for any solution Φ of the Black-Scholes PDE:

$$\mathscr{I}_{ab}\{\Phi(x,t)\} = \mathscr{I}_a\{\mathscr{I}_b\{\Phi(x,t)\}\}$$

Similarly when we compose any sequence of n Image operations. Note that image operations with respect to different barriers (a,b) do not commute so the order of such image operations is important.

Theorem 4.1. *The solution of Problem 2.3 for an arbitrary payoff function $f(x)$ is given by*

$$V(x,t) = \mathscr{K}_{a(t)}^{b(t)}\{U(x,t)\} \tag{14}$$

where $U(x,t)$ solves the unrestricted TV problem for the BS-PDE:

$$\begin{cases} \mathscr{L}U(x,t) &= 0; \quad t<T, \quad x>0 \\ U(x,T) &= f(x)\mathbb{I}(a(T)<x<b(T)) \end{cases} \tag{15}$$

and where $\mathscr{K}_a^b$ is a doubly infinite sequence of image operators evaluated at time t, defined by any of the equivalent representations

$$\begin{aligned} \mathscr{K}_{a(t)}^{b(t)} &= (I-\mathscr{I}_{a(t)})\sum_{n=-\infty}^{\infty}\mathscr{H}^n_{a(t)b(t)} = (I-\mathscr{I}_{a(t)})\sum_{n=-\infty}^{\infty}\mathscr{H}^n_{b(t)a(t)} \\ &= (I-\mathscr{I}_{b(t)})\sum_{n=-\infty}^{\infty}\mathscr{H}^n_{b(t)a(t)} = (I-\mathscr{I}_{b(t)})\sum_{n=-\infty}^{\infty}\mathscr{H}^n_{a(t)b(t)} \end{aligned} \tag{16}$$

where for a given positive integer n, $\mathscr{H}^n_{ab} = \mathscr{I}_{ab}\mathscr{I}_{ab}\cdots\mathscr{I}_{ab}$, $(n$ ab-pairs$)$ and $\mathscr{H}^0_{a(t)b(t)} = I$, the identity operator.

The following lemma makes evaluation of the infinite sums above feasible.

Lemma 4.2. *The $2n-$fold image operator $\mathscr{H}^n_{ab}$ is equivalent to the double exponential image*

$$\mathscr{H}^n_{a(t)b(t)} = \mathscr{I}_{b(t)}\cdot\mathscr{I}_{(b(t)^{n+1}/a(t)^n)} \tag{17}$$

and for any integer $n>0$,

$$\mathscr{H}^{-n}_{a(t)b(t)} = \mathscr{H}^n_{b(t)a(t)} \tag{18}$$

Buchen and Konstandatos (2009) contains a proof by mathematical induction. The following lemma is an important extension of Corollary 3.5.

Lemma 4.3. *Let $J_n(t)$ denote any sequence of n Image operators at time $t\leq T$, with respect to exponential time-varying barriers. For example, $J_n(t) = \mathscr{I}_{a_na_{n-1}\cdots a_1}$ where the a_i are all positive exponential functions of time. Then*

$$\mathscr{P}\mathscr{V}_t\left[J_n(T)\{f(x)\}\right] = J_n(t)\{\mathscr{P}\mathscr{V}_t\left[f(x)\right]\} \tag{19}$$

That is, the $\mathscr{P}\mathscr{V}_t$ operator and the sequence of Image operators commute, however at different points in time.

Proof. We proceed by induction. The result is obviously true for $n=1$, as this is just Corollary 3.5. Next assume the result is true for some $n > 1$ and consider $J_{n+1}(t) = \mathscr{I}_{a_{n+1}} \cdot J_n(t)$. Then

$$\begin{aligned}
\mathscr{PV}_t\left[J_{n+1}(T)f(x)\right] &= \mathrm{PV}\left[\mathscr{I}_{a_{n+1}(T)} \cdot J_n(T)f(x)\right] \\
&= \mathscr{I}_{a_{n+1}(t)}\mathrm{PV}\left[J_n(T)f(x)\right] \quad \text{(by Cor 3.5)} \\
&= \mathscr{I}_{a_{n+1}(t)} \cdot J_n(t)\mathrm{PV}\left[f(x)\right] \quad \text{(by assumption)} \\
&= J_{n+1}(t)\mathrm{PV}\left[f(x)\right]
\end{aligned}$$

The result follows. □

In pricing the partial time double-barrier options in the next section, the following corollary will be required in the application of Theorem 4.1. We note that by definition $\mathscr{H}^n_{ab}$ consists of a sequence of n images taken with respect to barrier levels (a,b). Each result in Corollary 4.4 immediately follows as particular applications of Lemma 4.3.

Corollary 4.4. *Given any function of the underlying asset $f(x)$ and exponential time varying lower and upper barrier levels $(a(t),b(t))$, and for times $t < T$ in the Black-Scholes economy we have:*

$$\begin{aligned}
\mathscr{PV}_t\left\{\mathscr{H}^n_{a(T)b(T)}\left[f(x)\right]\right\} &= \mathscr{H}^n_{a(t)b(t)}\left\{\mathscr{PV}_t\left[f(x)\right]\right\} \\
\mathscr{PV}_t\left\{\mathscr{I}_{a(T)}\mathscr{H}^n_{a(T)b(T)}\left[f(x)\right]\right\} &= \mathscr{I}_{a(t)}\mathscr{H}^n_{a(t)b(t)}\left\{\mathscr{PV}_t\left[f(x)\right]\right\} \\
\mathscr{PV}_t\left\{\mathscr{I}_{b(T)}\mathscr{H}^n_{a(T)b(T)}\left[f(x)\right]\right\} &= \mathscr{I}_{b(t)}\mathscr{H}^n_{a(t)b(t)}\left\{\mathscr{PV}_t\left[f(x)\right]\right\}
\end{aligned}$$

Two further lemmas lead us ultimately to the MOI for double (exponential) barrier options.

Lemma 4.5. *Let $\lambda(t) = b(t)/a(t) = (B/A)e^{(\beta-\alpha)t}$. Then for any function f, given $s = \pm 1$,*

$$\mathscr{H}^n_{a(t)b(t)}\left\{f(x)\right\}\mathbb{I}(sx > sb(t)) = \mathscr{H}^n_{a(t)b(t)}\left\{f(x)\mathbb{I}(sx > s\lambda^{2n}b(t))\right\}$$

Proof. The proof depends on the image property

$$\mathscr{I}_{a(t)}\{g(x)\cdot\mathbb{I}(sx < sb(t))\} = \mathscr{I}_{a(t)}\{g(x)\}\cdot\mathbb{I}(sx > sa(t)^2/b(t))$$

We then find, by use of Lemma 4.2:

$$
\begin{aligned}
\mathscr{H}^n_{a(t)b(t)}\{f(x)\}\mathbb{I}(sx>sb(t)) &= \mathscr{I}_{b(t)}\left\{\mathscr{I}_{(b(t)^{n+1}/a^n)}\{f(x)\}\right\}\mathbb{I}(sx>sb(t)) \\
&= \mathscr{I}_b\left\{\mathscr{I}_{(b(t)^{n+1}/a^n)}\{f(x)\}\mathbb{I}(sx<sb(t))\right\} \\
&= \mathscr{I}_b\cdot\mathscr{I}_{(b(t)^{n+1}/a^n)}\left\{f(x)\mathbb{I}(sx>s\frac{b(t)^{2n+2}}{a(t)^{2n}b(t)x})\right\}
\end{aligned}
$$

and the result follows. □

The following lemma allows us to find explicit expressions for the sequence of time-varying images $\mathscr{H}^n_{ab}$ in the representations in Theorem 4.1.

Lemma 4.6. *Let $\Phi(x,t)$ be any arbitrary function of the underlying asset and time, and set $\lambda(t)=b(t)/a(t)$ for exponential time-varying barrier levels $(a(t),b(t))$. The explicit representations of the actions of the Images operator $\mathscr{I}$ and the multiple Images operator $\mathscr{H}^n_{ab}$ is given as follows.*

$$
\begin{aligned}
\mathscr{H}^n_{a(t)b(t)}\{\Phi(x,t)\} &= \lambda(t)^{np_n}(x/a(t))^{q_n}\Phi(\lambda^{2n}x,t) && (20)\\
&= \lambda(t)^{np_{n+1}}(x/b(t))^{q_n}\Phi(\lambda^{2n}x,t) && (21)\\
\mathscr{I}_{a(t)}\mathscr{H}^n_{a(t)b(t)}\{\Phi(x,t)\} &= \lambda(t)^{np_n}(a(t)/x)^{p_n}\Phi(\lambda^{2n}a(t)^2/x,t) && (22)\\
\mathscr{I}_{b(t)}\mathscr{H}^n_{a(t)b(t)}\{\Phi(x,t)\} &= \lambda(t)^{np_{n+1}}(b(t)/x)^{p_{n+1}}\Phi(\lambda^{2n}b^2/x,t) && (23)
\end{aligned}
$$

where

$$
\begin{aligned}
p_n &= nq_\beta-(n-1)q_\alpha; \qquad q_n=n(q_\beta-q_\alpha)\\
q_\alpha &= 2(r-\alpha)/\sigma^2-1; \qquad q_\beta=2(r-\beta)/\sigma^2-1
\end{aligned}
$$

Proof. Writing $a = a(t)$, $b = b(t)$, and $\lambda = \lambda(t)$ we have:

$$\begin{aligned}
\mathscr{H}^n_{ab}\{\Phi(x,t)\} &= \mathscr{I}_b \mathscr{I}_{b^{n+1}/a^n}\{\Phi(x,t)\} \quad \text{(by Lemma 4.2)} \\
&= \mathscr{I}_b \left\{ \left(\frac{b^{n+1}}{a^n x}\right)^{(n+1)q_\alpha - nq_\beta} \Phi(b^{2n+2}/(a^{2n}x),t) \right\} \\
&= \left(\frac{b}{x}\right)^{q_\beta} \left(\frac{b^{n+1}}{a^n b^2} x\right)^{(n+1)q_\beta - nq_\alpha} \Phi((b^{2n}/a^{2n})x,t) \\
&= \left(\frac{b^n}{a^n}\right)^{nq_\beta - (n-1)q_\alpha} \left(\frac{x}{a}\right)^{n(q_\beta - q_\alpha)} \Phi((b^{2n}/a^{2n})x,t)
\end{aligned}$$

and Eq(20) follows. Now using Eq(20), we have:

$$\begin{aligned}
\mathscr{I}_a \mathscr{H}^n_{ab}\{\Phi(x,t)\} &= \left(\frac{a}{x}\right)^{q_\alpha} \lambda^{np_n} \left(\frac{a^2}{ax}\right)^{q_n} \Phi(\lambda^{2n}a^2/x,t) \\
&= \lambda^{np_n} \left(\frac{a}{x}\right)^{q_\alpha + q_n} \Phi(\lambda^{2n}a^2/x,t)
\end{aligned}$$

and Eq(22) follows. Eq(21) and Eq(23) follow along similar lines.

□

Note that Eq(20) and Eq(21) are equivalent representations for the effect of $\mathscr{H}^n_{ab}$. We now state the Method of Images for double (exponential) barrier options.

Theorem 4.7 (Method of Images for Double Exponential Barriers)**.** *Let $U(x,t)$ solve the (unrestricted) TV problem for the BS-PDE $\mathscr{L}U = 0$, with terminal value*

$$U(x,T) = f(x)\mathbb{I}(a(T) < x < b(T))$$

Then the unique arbitrage free solution of Problem 2.3 can be expressed entirely in terms of $U(x,t)$ and in terms of previously defined parameters, is given explicitly by the doubly-infinite sum

$$V(x,t) = \sum_{n=-\infty}^{\infty} \lambda^{np_n} \left\{ \left(\frac{x}{a}\right)^{q_n} U(\lambda^{2n}x,t) - \left(\frac{a}{x}\right)^{p_n} U(a^2\lambda^{2n}/x,t) \right\} \tag{24}$$

Proof. The proof follows the steps in Buchen and Konstandatos (2009) using the results of Lemma 4.6 and Theorem 4.7. We note the following equivalent representations of the double barrier price. Let $U_{a,n}(x,t) = (x/a)^{q_n}U(\lambda^{2n}x,t)$

and $U_{b,n}(x,t) = (x/b)^{q_n} U(\lambda^{2n} x, t)$. Then

$$
\begin{aligned}
V(x,t) &= \sum_{n=-\infty}^{\infty} \lambda^{n p_n} \left[U_{a,n}(x,t) - \mathscr{I}_a\{U_{a,n}(x,t)\} \right] \\
&= \sum_{n=-\infty}^{\infty} \lambda^{n p_{n+1}} \left[U_{b,n}(x,t) - \mathscr{I}_b\{U_{b,n}(x,t)\} \right]
\end{aligned}
$$

With these representations and property (2) of remark 3.2, it becomes immediately clear how the zero boundary conditions at $x = a(t)$ and $x = b(t)$ are satisfied. Since by construction the sum is a solution of the Black-Scholes PDE given that U and its mathematical image are solutions of the Black Scholes PDE. By invoking uniqueness of solutions for Linear Parabolic PDEs (subject to reasonable restrictions on $f(x)$) the result follows. □

Note that Eq(24) prices a double exponential barrier option for an arbitrary payoff function for a monitoring window extending over the whole life of the option. This representation and separate application of Lemma 4.4 and Lemma 4.6 are necessary prerequisites for pricing the partial-time double barrier options with late monitoring. To do the latter we need to determine $U(x,t)$ as described in Theorem 4.7 which depends on $f(x)$. We carry out this task in the next section.

5. Full-Window Barrier Options

In this section we derive prices for barrier options whose barrier monitoring windows extend over the full lifetime of the option for standard calls and puts. As indicated in the introduction, our methods allow us to express all results solely in terms of the First-Order Gap Options described in the Appendix AppendixA.1, and are included for completeness.

5.1. Single Barrier Down-and-out Calls and Puts

We start by pricing the standard down-and-out call barrier option, V_{DOC} with a single exponential barrier $x = b(t) = Be^{\beta t}$, over $[t,T]$. This price satisfies problem 2.2 with payoff function $f(x) = (x-k)^+$, where k is the strike price of the option.

By application of Theorem (3.6), we have to determine the $t < T$ price $U(x,t)$ of the European option, with T payoff:

$$
U(x,T) = (x-k)^+ \mathbb{I}(x > b(T)) \equiv (x-k)\mathbb{I}(x > k'); \quad k' = k \vee b(T)
$$

where $x \vee y = \max(x,y)$ is the maximum between two values.

This may be statically replicated for all $t \leq T$ in terms of the first-order gap options from Section AppendixA.1:

$$U(x,t) = \mathscr{G}^{+}_{k'}(x,\tau;k); \quad \tau = T-t$$

It follows from Theorem (3.6) that the $t<T$ price of V_{DOC} is given by:

$$V_{DOC}(x,t) = \mathscr{G}^{+}_{k'}(x,\tau;k) - (b(t)/x)^{q_\beta}\mathscr{G}^{+}_{k'}(b^2(t)/x,\tau;k) \tag{25}$$

Similarly, the related price $U(x,t)$ for the down-and-out put barrier option, V_{DOP} has $t=T$ payoff:

$$U(x,T) = (k-x)^{+}\mathbb{I}(x>b(T)) \equiv -(x-k)\left[\mathbb{I}(x<k') - \mathbb{I}(x<b(T))\right]$$

with $t<T$ price:

$$U(x,t) = -\mathscr{G}^{-}_{k'}(x,\tau;k) + \mathscr{G}^{-}_{b(T)}(x,\tau;k)$$

It follows that the price of the down-and-out barrier put is:

$$\begin{aligned} V_{DOP}(x,t) \quad &= \quad -\mathscr{G}^{-}_{k'}(x,\tau;k) + (b(t)/x)^{q_\beta}\mathscr{G}^{-}_{k'}(\tfrac{b^2(t)}{x},\tau;k) \\ &\quad\quad + \mathscr{G}^{-}_{b(T)}(x,\tau;k) - (b(t)/x)^{q_\beta}\mathscr{G}^{-}_{b(T)}(\tfrac{b^2(t)}{x},\tau;k) \end{aligned} \tag{26}$$

Note that $V_{DOP}(x,t) \equiv 0$ when $k<b(T)$, as expected.

Prices for the other barrier options (U/O, D/I, U/I) can be obtained using the parity relations described in Lemma 3.9.

5.2. *Double Barrier Options*

In this section we reproduce the prices for the standard double barrier call and put options with simultaneous lower and upper exponential and time varying boundaries $(a(t),b(t)) = (Ae^{\alpha t}, Be^{\beta t})$. Our approach is that of Buchen and Konstandatos (2009), although they were originally priced in Kunitomo and Ikeda (1992). We reproduce these results not only for completeness but also because the analysis will be required in derivation of the late-monitoring double-barrier option prices in Section 6.4.

We will denote the call barrier option, with a double barrier window with exponential boundaries, over $[t,T]$ as V_{DBC}. The corresponding put will be denoted V_{DBP}. To price the call option of strike price k, we need only apply Theorem 4.7 with the specific payoff function $f(x) = (x-k)^{+}$.

Thus we must first price a standard European option with

$$
\begin{aligned}
U(x,T) &= (x-k)^{+}\mathbb{I}(a(T)<x<b(T)) \\
&\equiv (x-k)\left[\mathbb{I}(x>k')-\mathbb{I}(x>b(T))\right]; \quad k'=k\vee a(T)
\end{aligned}
$$

This is replicable in terms of first order gap options for all $t \le T$, as follows:

$$
U(x,t)=\mathscr{G}^{+}_{k'}(x,\tau;k)-\mathscr{G}^{+}_{b(t)}(x,\tau;k) \tag{27}
$$

It follows that the $t<T$ double-barrier call price is given by:

$$
\begin{aligned}
V_{DBC}(x,t)=\sum_{n=-\infty}^{\infty}\lambda^{np_n}\Bigg\{&\left(\frac{x}{a}\right)^{q_n}\left[\mathscr{G}^{+}_{k'}(\lambda^{2n}x,\tau;k)-\mathscr{G}^{+}_{b(T)}(\lambda^{2n}x,\tau;k)\right] \\
&-\left(\frac{a}{x}\right)^{p_n}\left[\mathscr{G}^{+}_{k'}(\tfrac{\lambda^{2n}a^2}{x},\tau;k)-\mathscr{G}^{+}_{b(T)}(\tfrac{\lambda^{2n}a^2}{x},\tau;k)\right]\Bigg\}
\end{aligned} \tag{28}
$$

Similarly, the standard double-barrier put price is:

$$
\begin{aligned}
V_{DBP}(x,t)=\sum_{n=-\infty}^{\infty}\lambda^{np_n}\Bigg\{&\left(\frac{x}{a}\right)^{q_n}\left[-\mathscr{G}^{-}_{k'}(\lambda^{2n}x,\tau;k)+\mathscr{G}^{-}_{a(T);k}(\lambda^{2n}x,\tau)\right] \\
&-\left(\frac{a}{x}\right)^{p_n}\left[-\mathscr{G}^{-}_{k'}(\tfrac{\lambda^{2n}a^2}{x},\tau;k)+\mathscr{G}^{-}_{a(T)}(\tfrac{\lambda^{2n}a^2}{x},\tau;k)\right]\Bigg\}
\end{aligned} \tag{29}
$$

where $k'=k\wedge b(T)$ where $x\wedge y=\min(x,y)$ is the minimum between two values.

6. Partial-Time Barrier Options

In this section we apply the theorems for the previous sections to the pricing of various partial-time barrier options with exponential boundaries. The time-horizon of the options we consider will have two future dates, T_1 and T_2 where $T_1<T_2$. For early monitoring partial-time barrier options, the barrier window is taken to be the interval $[t,T_1]$; while for late monitoring partial-time barrier options, the barrier window is taken to be the interval $[T_1,T_2]$. In both cases the option payoff is made at time T_2. To simplify notation we will adhere to the convention

$$
a_i=a(T_i), \quad b_i=b(T_i), \quad i=1,2
$$

6.1. *Early Monitoring Partial-Time Barrier Options*

6.1.1. *Single Exponential Barrier Partial-Time Calls and Puts*

Here we derive prices for the partial-time, down-and-out call barrier option, V_{DOC}^{PT} with a single exponential boundary at $x = b(t)$ with monitoring over $[0, T_1]$. As there is no barrier window over $[T_1, T_2]$, V_{DOC}^{PT} may be thought of as a barrier option over $[t, T_1]$, satisfying problem 2.2 with payoff $f(x) = C_k(x, \tau)$, where $C_k(x, \tau)$ denotes the price of a strike k European call option with time $\tau = (T_2 - T_1)$ remaining to expiry.

We apply Theorem (3.6) and use the first-order Gap option representation of a vanilla call-price (Eq(A.2)), namely $C_k(x, \tau) = G_k^+(x, \tau; k)$. We simply have to determine the $t < T_1$ price $U(x,t)$ of the European option with T_1 payoff:

$$U(x, T_1) = \mathscr{G}_k^+(x, \tau; k)\mathbb{I}(x > b_1)$$

This derivative may be statically replicated for all $t \leq T_1$ in terms of the second-order gap options defined in Section AppendixA.2. Thus,

$$U(x,t) = \mathscr{G}_{b_1,k}^{++}(x, \tau_{1,2}; k)$$

It follows from Theorem (3.6) that the $t < T_1$ price for the down-and-out partial-time early monitoring exponential barrier call option is:

$$V_{DOC}^{PTE}(x,t) = \mathscr{G}_{b_1 k}^{++}(x, \tau_{1,2}; k) - (b(t)/x)^{q_\beta} \mathscr{G}_{b_1 k}^{++}(\tfrac{b^2(t)}{x}, \tau_{1,2}; k) \tag{30}$$

An application of Eq(10) now allows us to determine the formulae for the down-and-in, up-and-in and up-and-out partial-time early monitoring exponential barrier call options as well:

$$\begin{aligned} V_{DIC}^{PTE}(x,t) &= \mathscr{G}_k^+(x, \tau_1; k) - \mathscr{G}_{b_1 k}^{++}(x, \tau_{1,2}; k) \\ &\quad + (b(t)/x)^{q_\beta} \mathscr{G}_{b_1 k}^{++}(\tfrac{b^2(t)}{x}, \tau_{1,2}; k) \qquad (31) \\ V_{UIC}^{PTE}(x,t) &= (b(t)/x)^{q_\beta} \mathscr{G}_k^+(\tfrac{b^2(t)}{x}, \tau_1) + \mathscr{G}_{b_1 k}^{++}(x, \tau_{1,2}; k) \\ &\quad - (b(t)/x)^{q_\beta} \mathscr{G}_{b_1 k}^{++}(\tfrac{b^2(t)}{x}, \tau_{1,2}; k) \qquad (32) \end{aligned}$$

$$\begin{aligned} V_{UOC}^{PTE}(x,t) &= \mathscr{G}_k^+(x, \tau_1; k) - (b(t)/x)^{q_\beta} \mathscr{G}_k^+(\tfrac{b^2(t)}{x}, \tau_1; k) \\ &- \mathscr{G}_{b_1 k}^{++}(x, \tau_{1,2}; k) + (b(t)/x)^{q_\beta} \mathscr{G}_{b_1 k}^{++}(\tfrac{b^2(t)}{x}, \tau_{1,2}; k) \qquad (33) \end{aligned}$$

Similarly, the $t < T_1$ prices for the partial-time early monitoring exponential barrier put options may also be determined:

$$
\begin{aligned}
V^{PTE}_{DOP}(x,t) &= -\mathscr{G}^{+-}_{b_1 k}(x,\tau_{1,2};k) + (b(t)/x)^{q_\beta}\mathscr{G}^{+-}_{b_1 k}(\tfrac{b^2(t)}{x},\tau_{1,2};k) && (34)\\
V^{PTE}_{DIP}(x,t) &= -\mathscr{G}^{-}_{k}(x,\tau_1;k) + \mathscr{G}^{+-}_{b_1 k}(x,\tau_{1,2};k) \\
&\qquad - (b(t)/x)^{q_\beta}\mathscr{G}^{+-}_{b_1 k}(\tfrac{b^2(t)}{x},\tau_{1,2};k) && (35)\\
V^{PTE}_{UIP}(x,t) &= -(b(t)/x)^{q_\beta}\mathscr{G}^{-}_{k}(\tfrac{b^2(t)}{x},\tau_1;k) \\
&\quad -\mathscr{G}^{+-}_{b_1 k}(x,\tau_{1,2};k) + (b(t)/x)^{q_\beta}\mathscr{G}^{+-}_{b_1 k}(\tfrac{b^2(t)}{x},\tau_{1,2};k) && (36)\\
V^{PTE}_{UOP}(x,t) &= -\mathscr{G}^{-}_{k}(x,\tau_1;k) + (b(t)/x)^{q_\beta}\mathscr{G}^{-}_{k}(\tfrac{b^2(t)}{x},\tau_1;k) \\
&\quad +\mathscr{G}^{+-}_{b_1 k}(x,\tau_{1,2};k) - (b(t)/x)^{q_\beta}\mathscr{G}^{+-}_{b_1 k}(\tfrac{b^2(t)}{x},\tau_{1,2};k) && (37)
\end{aligned}
$$

6.2. Double Exponential Barrier Partial-Time Calls and Puts

We consider lower and upper exponential and time varying boundaries $(a(t),b(t)) = (Ae^{\alpha t}, Be^{\beta t})$. To price the partial-time double barrier call and put options with early monitoring, again note that there is no barrier monitoring over $[T_1,T_2]$. The call option V^{PT}_{DBC} may therefore be thought of as a double-barrier option over $[t,T_1]$ with payoff function at time T_1 being a European call option with time $\tau = T_2 - T_1$ remaining to expiry, provided the option is within the dual exponential barrier windows. Identifying the representation of the call price in terms of first-order the Gap option (Eq(A.2)) with $\xi = k$ and $s = +1$ for the exercise condition, we express the time T_1 value as $f(x) = \mathscr{G}^{+}_{k}(x,\tau;k)$.
By application of Theorem (4.7), we only need to determine the $t < T_1$ price $U(x,t)$ of the European option, with T_1 payoff:

$$
\begin{aligned}
U(x,T_1) &= \mathscr{G}^{+}_{k}(x,\tau;k)\mathbb{I}(a_1 < x < b_1)\\
&= \mathscr{G}^{+}_{k}(x,\tau;k)\,[\mathbb{I}(x > a_1) - \mathbb{I}(x > b_1)]
\end{aligned}
$$

since $a_1 < b_1$ by assumption. This can be statically replicated for all $t \leq T_1$ in terms of two second-order gap options, from Section AppendixA.2:

$$U(x,t) = \mathscr{G}^{++}_{a_1 k}(x,\tau_{1,2};k) - \mathscr{G}^{++}_{b_1 k}(x,\tau_{1,2};k)$$

It follows from Theorem (4.7) that the $t < T_1$ price is given by:

$$
\begin{aligned}
V_{DBC}^{PTE}(x,t) &= \sum_{n=-\infty}^{\infty} \lambda^{np_n} \left\{ \left(\frac{x}{a}\right)^{q_n} \left[\mathscr{G}_{a_1 k}^{++}(\lambda^{2n}x, \tau_{1,2};k) - \mathscr{G}_{b_1 k}^{++}(\lambda^{2n}x, \tau_{1,2};k) \right] \right. \\
&- \left. \left(\frac{a}{x}\right)^{p_n} \left[\mathscr{G}_{a_1 k}^{++}(\tfrac{a^2\lambda^{2n}}{x}, \tau_{1,2};k) - \mathscr{G}_{b_1 k}^{++}(\tfrac{a^2\lambda^{2n}}{x}, \tau_{1,2};k) \right] \right\}
\end{aligned} \tag{38}
$$

Similarly, the partial-time double barrier put with early monitoring has $t < T_1$ price:

$$
\begin{aligned}
V_{DBP}^{PTE}(x,t) &= \sum_{n=-\infty}^{\infty} \lambda^{np_n} \left\{ \left(\frac{x}{a}\right)^{q_n} \left[\mathscr{G}_{a_1 k}^{--}(\lambda^{2n}x, \tau_{1,2};k) - \mathscr{G}_{b_1 k}^{--}(\lambda^{2n}x, \tau_{1,2};k) \right] \right. \\
&- \left. \left(\frac{a}{x}\right)^{p_n} \left[\mathscr{G}_{a_1 k}^{--}(\tfrac{a^2\lambda^{2n}}{x}, \tau_{1,2};k) - \mathscr{G}_{b_1 k}^{--}(\tfrac{a^2\lambda^{2n}}{x}, \tau_{1,2};k) \right] \right\}
\end{aligned} \tag{39}
$$

6.3. Late Monitoring Partial Time-Barrier Options

When we have a late-monitoring barrier window extending over $[T_1, T_2]$, it follows that over the complementary interval $[t, T_1]$ we have a simple European option without a barrier. We consider one upper exponential and time varying boundary $b(t) = Be^{\beta t}$ with monitoring over $[T_1, T_2]$ in this section.

6.3.1. Partial-time Down-and-out call and put options with late monitoring

The partial-time, down-and-out call with late monitoring V_{DOC}^{PTL} has T_1 price which corresponds to a down-and-out barrier option with time $\tau = T_2 - T_1$ to expiry, provided we begin above the barrier at time T_1.

We can express this as:

$$
\begin{aligned}
V_{DOC}^{PTL}(x,T_1) &= \left[\mathscr{G}_{k'}^{+}(x,\tau;k) - (b_1/x)^{q_\beta} \mathscr{G}_{k'}^{+}(b_1^2/x,\tau;k) \right] \mathbb{I}(x > b_1) \\
&= \mathscr{G}_{k'}^{+}(x,\tau;k)\mathbb{I}(x > b_1) - \mathscr{I}_{b_1}\left[\mathscr{G}_{k'}^{+}(x,\tau;k)\mathbb{I}(x < b_1) \right]
\end{aligned}
$$

where $\tau = (T_2 - T_1)$ and $k' = k \vee b_2$. Note that this late-monitoring barrier option is the one designated type-$B2$ in Heynen and Kat (1994).

Applying Theorem 3.6, we may therefore statically replicate the option price for $t < T_1$ in terms of second order gap options:

$$
\begin{aligned}
V_{DOC}^{PTL}(x,t) &= \mathscr{G}_{b_1 k'}^{++}(x,\tau_{1,2};k) - \mathscr{I}_{b(t)} \mathscr{G}_{b_1 k'}^{-+}(x,\tau_{1,2};k) \\
&= \mathscr{G}_{b_1 k'}^{++}(x,\tau_{1,2};k) - (b(t)/x)^{q_\beta} \mathscr{G}_{b_1 k'}^{-+}(\tfrac{b^2(t)}{x},\tau_{1,2};k)
\end{aligned} \tag{40}
$$

With $k' = k \vee b_2$ again, the late-monitoring partial time down-and-out put has T_1 price

$$
\begin{aligned}
V^{PTL}_{DOP}(x, T_1) &= \left[-\mathscr{G}^{-}_{k'}(x, \tau; k) + \mathscr{G}^{-}_{b_1}(x, \tau; k) \right] \mathbb{I}(x > b_1) \\
&\qquad \mathscr{I}_{b_1} \left[(\mathscr{G}^{-}_{k'}(x, \tau; k) - \mathscr{G}^{-}_{b_1}(x, \tau; k)) \mathbb{I}(x < b_1) \right]
\end{aligned}
$$

Theorem 3.6 allows us to statically replicate the $t < T_1$ price as:

$$
\begin{aligned}
V^{PTL}_{DOP}(x, t) &= -\mathscr{G}^{+-}_{b_1 k'}(x, \tau_{1,2}; k) + \mathscr{G}^{+-}_{b_1 b_1}(x, \tau_{1,2}; k) \\
&\qquad \mathscr{I}_{b(t)} \left[\mathscr{G}^{--}_{b_1 k'}(x, \tau_{1,2}; k) - \mathscr{G}^{--}_{b_1 b_1}(x, \tau_{1,2}; k) \right] \\
&= -\mathscr{G}^{+-}_{b_1 k'}(x, \tau_{1,2}; k) + \mathscr{G}^{+-}_{b_1 b_1}(x, \tau_{1,2}; k) \\
\left(\frac{b(t)}{x} \right)^{q_\beta} &\qquad \left[\mathscr{G}^{--}_{b_1 k'}\left(\frac{b^2(t)}{x}, \tau_{1,2}; k\right) - \mathscr{G}^{--}_{b_1 b_1}\left(\frac{b^2(t)}{x}, \tau_{1,2}; k\right) \right]
\end{aligned} \tag{41}
$$

6.4. Late monitoring partial-time double barrier options

As the late monitoring partial-time double barrier option presents several technical issues not previously met, we present a detailed analysis in this section for the partial time late monitoring call option. Despite the apparent complexity, the approach is the same as in earlier calculations. The complexity arises because Theorem 14 is not directly applicable due to the late-monitoring window. The extension of the analysis to this situation is one of the main theoretical contributions of this work.

As with the early-monitoring case, we again consider lower and upper exponential and time varying boundaries $(a(t), b(t)) = (Ae^{\alpha t}, Be^{\beta t})$. However the monitoring now occurs over $[T_1, T_2]$

The partial-time double-barrier call option with late monitoring V^{PTL}_{DBC} has T_1 price corresponding to a double barrier option with time $\tau = (T_2 - T_1)$ to expiry, provided the underlying asset falls within the barrier window at time T_1. Otherwise, the option would be immediately knocked-out. Using the representation of the double barrier solution given by Eq(14) from Theorem 4.1, and Eq(27) from the representation of the double-barrier call option solution over a full-monitoring window, we may express the T_1 price as:

$$
V^{PTL}_{DBC}(x, T_1) = \overbrace{\mathscr{K}^{b_1}_{a_1} \{\Phi(x, \tau)\}}^{\text{DB Call}} \mathbb{I}(a_1 < x < b_1)
$$

Using Eq(27) from the representation of the double-barrier call option price with full monitoring applied over $[T_1, T_2]$, we identify $k' = k \vee a_2$ in the following:

$$\Phi(x,\tau) = \mathscr{G}^{+}_{k'}(x,\tau;k) - \mathscr{G}^{+}_{b_2}(x,\tau;k)$$

We are able to write that

$$V^{PTL}_{DBC}(x,T_1) = \mathscr{K}^{b_1}_{a_1}\{\Phi(x,\tau)\}\,\mathbb{I}(x>a_1) - \mathscr{K}^{b_1}_{a_1}\{\Phi(x,\tau)\}\,\mathbb{I}(x>b_1) \tag{42}$$

This follows since by assumption $a_1 < b_1$, and the identity $\mathbb{I}(a_1 < x < b_1) \equiv \mathbb{I}(x>a_1) - \mathbb{I}(x>b_1)$. Using the first and fourth representations in Theorem 4.1 to expand the operators $\mathscr{K}$ in Eq(42), we have:

$$\begin{aligned} V^{PTL}_{DBC}(x,T_1) &= \sum_{n=-\infty}^{\infty}\left[\mathscr{H}^n_{a_1b_1} - \mathscr{I}_{a_1}\mathscr{H}^n_{a_1b_1}\right]\{\Phi(x,\tau)\}\,\mathbb{I}(x>a_1) \\ &\quad - \sum_{n=-\infty}^{\infty}\left[\mathscr{H}^n_{a_1b_1} - \mathscr{I}_{b_1}\mathscr{H}^n_{a_1b_1}\right]\{\Phi(x,\tau)\}\,\mathbb{I}(x>b_1) \end{aligned}$$

Now using Lemma 4.5, with $\lambda_1 = b_1/a_1$, it follows that:

$$\begin{aligned} V^{PTL}_{DBC}(x,T_1) &= \sum_{n=-\infty}^{\infty}\mathscr{H}^n_{a_1b_1}\left\{\Phi(x,\tau)\mathbb{I}(x>\lambda_1^{2n}a_1)\right\} \\ &\quad - \sum_{n=-\infty}^{\infty}\mathscr{I}_{a_1}\mathscr{H}^n_{a_1b_1}\left\{\Phi(x,\tau)\mathbb{I}(x<\lambda_1^{2n}a_1)\right\} \\ &\quad - \sum_{n=-\infty}^{\infty}\mathscr{H}^n_{a_1b_1}\left\{\Phi(x,\tau)\mathbb{I}(x>\lambda_1^{2n}b_1)\right\} \\ &\quad + \sum_{n=-\infty}^{\infty}\mathscr{I}_{b_1}\mathscr{H}^n_{a_1b_1}\left\{\Phi(x,\tau)\mathbb{I}(x<\lambda_1^{2n}b_1)\right\} \end{aligned}$$

then by use of Lemma 4.6 we find that for $t < T_1 < T_2$:

$$
\begin{aligned}
V^{PTL}_{DBC}(x,t) &= \mathscr{P}\mathscr{V}_t\left[V^{PTL}_{DBC}(x,T_1)\right] \\
&= \sum_{n=-\infty}^{\infty} \mathscr{H}^n_{a(t)b(t)}\left\{\mathscr{P}\mathscr{V}_t\left[\Phi(x,\tau)\mathbb{I}(x>\lambda_1^{2n}a_1)\right]\right\} \\
&- \sum_{n=-\infty}^{\infty} \mathscr{I}_{a(t)}\mathscr{H}^n_{a(t)b(t)}\left\{\mathscr{P}\mathscr{V}_t\left[\Phi(x,\tau)\mathbb{I}(x<\lambda_1^{2n}a_1)\right]\right\} \\
&- \sum_{n=-\infty}^{\infty} \mathscr{H}^n_{a(t)b(t)}\left\{\mathscr{P}\mathscr{V}_t\left[\Phi(x,\tau)\mathbb{I}(x>\lambda_1^{2n}b_1)\right]\right\} \\
&+ \sum_{n=-\infty}^{\infty} \mathscr{I}_{b(t)}\mathscr{H}^n_{a(t)b(t)}\left\{\mathscr{P}\mathscr{V}_t\left[\Phi(x,\tau)\mathbb{I}(x<\lambda_1^{2n}b_1)\right]\right\}
\end{aligned}
$$

The present values inside the infinite sums are readily calculated. We write $r_1 = \lambda_1^{2n}a_1$ and $s_1 = \lambda_1^{2n}b_1$. From the definition of the Second Order Gap Option payoffs from Section A.3, we have for the first sum:

$$
\mathscr{P}\mathscr{V}_t\left[\Phi(x,\tau)\mathbb{I}(x>\lambda_1^{2n}a_1)\right] = \mathscr{G}^{++}_{r_1k'}(x,\tau_{1,2};k) - \mathscr{G}^{++}_{r_1b_2}(x,\tau_{1,2};k)
$$

and similarly for the remainder:

$$
\begin{aligned}
&V^{PTL}_{DBC}(x,t) = \\
&\quad \sum_{n=-\infty}^{\infty} \mathscr{H}^n_{a(t)b(t)}\left\{\mathscr{G}^{++}_{r_1k'}(x,\tau_{1,2};k) - \mathscr{G}^{++}_{r_1b_2}(x,\tau_{1,2};k)\right\} \\
&\quad - \sum_{n=-\infty}^{\infty} \mathscr{I}_{a(t)}\mathscr{H}^n_{a(t)b(t)}\left\{\mathscr{G}^{-+}_{r_1k'}(x,\tau_{1,2};k) - \mathscr{G}^{-+}_{r_1b_2}(x,\tau_{1,2};k)\right\} \\
&\quad - \sum_{n=-\infty}^{\infty} \mathscr{H}^n_{a(t)b(t)}\left\{\mathscr{G}^{++}_{s_1k'}(x,\tau_{1,2};k) - \mathscr{G}^{++}_{s_1b_2}(x,\tau_{1,2};k)\right\} \\
&\quad + \sum_{n=-\infty}^{\infty} \mathscr{I}_{b(t)}\mathscr{H}^n_{a(t)b(t)}\left\{\mathscr{G}^{-+}_{s_1k'}(x,\tau_{1,2};k) - \mathscr{G}^{-+}_{s_1b_2}(x,\tau_{1,2};k)\right\}
\end{aligned}
$$

We find explicit expressions for the sequences of images by several applications of Lemma 4.6, to finally write the $t < T_1$ solution as:

$$
\begin{aligned}
&V^{PTL}_{DBC}(x,t) = \\
&\sum_{n=-\infty}^{\infty} \lambda^{np_n} \left(\frac{x}{a}\right)^{q_n} \left\{\mathscr{G}^{++}_{r_1 k'}(\lambda^{2n}x, \tau_{1,2}; k) - \mathscr{G}^{++}_{r_1 b_2}(\lambda^{2n}x, \tau_{1,2}; k)\right\} \\
&- \sum_{n=-\infty}^{\infty} \lambda^{np_n} \left(\frac{a}{x}\right)^{p_n} \left\{\mathscr{G}^{-+}_{r_1 k'}(\lambda^{2n}a^2 x, \tau_{1,2}; k) - \mathscr{G}^{-+}_{r_1 b_2}(\lambda^{2n}a^2 x, \tau_{1,2}; k)\right\} \\
&- \sum_{n=-\infty}^{\infty} \lambda^{np_{n+1}} \left(\frac{x}{b}\right)^{q_n} \left\{\mathscr{G}^{++}_{s_1 k'}(\lambda^{2n}x, \tau_{1,2}; k) - \mathscr{G}^{++}_{s_1 b_2}(\lambda^{2n}x, \tau_{1,2}; k)\right\} \\
&+ \sum_{n=-\infty}^{\infty} \lambda^{np_{n+1}} \left(\frac{b}{x}\right)^{p_{n+1}} \left\{\mathscr{G}^{-+}_{s_1 k'}(\tfrac{\lambda^{2n}b^2}{x}, \tau_{1,2}; k) - \mathscr{G}^{-+}_{s_1 b_2}(\tfrac{\lambda^{2n}b^2}{x}, \tau_{1,2}; k)\right\}
\end{aligned}
\tag{43}
$$

where $a = a(t)$, $b = b(t)$, $\lambda = \lambda(t) = b(t)/a(t)$, $\lambda_1 = \lambda(T_1)$, $\tau_i = T_i - t$ for $i = 1,2$, $k' = k \vee a_2$; $(a_i, b_i) = (a(T_i), b(T_i))$ for $i = 1,2$ and (p_n, q_n) as defined in Lemma 4.6.

Remark 6.1. We note a counter-intuitive aspect of the solution. There is no barrier monitoring for $t < T_1$. However the time-varying barriers have been continued into the interval $[t, T_1]$ as if the time-varying barriers were *still operating*.

6.5. Partial time double barrier put option with late monitoring

Following similar reasoning as for the call version, the price of the partial time late monitoring exponential double barrier put may also be obtained. Omitting the details, the result is:

$$
\begin{aligned}
&V^{PTL}_{DBP}(x,t) = \\
&\sum_{n=-\infty}^{\infty} \lambda^{np_n} \left(\frac{x}{a}\right)^{q_n} \left\{\mathscr{G}^{--}_{r_1 k'}(\lambda^{2n}x, \tau_{1,2}; k) - \mathscr{G}^{--}_{r_1 a_2}(\lambda^{2n}x, \tau_{1,2}; k)\right\} \\
&- \sum_{n=-\infty}^{\infty} \lambda^{np_n} \left(\frac{a}{x}\right)^{p_n} \left\{\mathscr{G}^{+-}_{r_1 k'}(\tfrac{\lambda^{2n}a^2}{x}, \tau_{1,2}; k) - \mathscr{G}^{+-}_{r_1 a_2}(\tfrac{\lambda^{2n}a^2}{x}, \tau_{1,2}; k)\right\} \\
&- \sum_{n=-\infty}^{\infty} \lambda^{np_{n+1}} \left(\frac{x}{b}\right)^{q_n} \left\{\mathscr{G}^{--}_{s_1 k'}(\lambda^{2n}x, \tau_{1,2}; k) - \mathscr{G}^{--}_{s_1 a_2}(\lambda^{2n}x, \tau_{1,2}; k)\right\} \\
&+ \sum_{n=-\infty}^{\infty} \lambda^{np_{n+1}} \left(\frac{b}{x}\right)^{p_{n+1}} \left\{\mathscr{G}^{+-}_{s_1 k'}(\tfrac{\lambda^{2n}b^2}{x}, \tau_{1,2}; k) - \mathscr{G}^{+-}_{s_1 a_2}(\tfrac{\lambda^{2n}b^2}{x}, \tau_{1,2}; k)\right\}
\end{aligned}
\tag{44}
$$

where now $k' = k \wedge b_2 = \min(k, b_2)$, and where otherwise the remaining symbols are as in Eq(43).

n	Term	Cumulative Sum
-3.0000	0.0000	0.0000
-2.0000	0.0000	0.0000
-1.0000	0.0000	0.0000
0.0000	28.0073	28.0073
1.0000	-5.2027	22.8046
2.0000	0.0000	22.8046
3.0000	0.0000	22.8046

(46)

Table 1: Evaluation of terms in Eq(43) for the partial-time late monitoring double barrier call. 'Term' is the value of the n-th term under the summation in Eq(43), and 'Cumulative Sum' gives the approximate price.

7. Computations

In this section we present numerical computations for our formulae from Section 6. As will be demonstrated, convergence for the doubly-infinite sums in our formulae is very rapid. We consider the following choice of parameters representing expiry times, risk-free rates and stock price volatilities.

ParameterValues

T_1	T_2	x	k	r	σ
$1/12$	$1/6$	1000	1000	0.05	0.20

(45)

Namely, we consider one-month early and late partial-barrier monitoring windows $t \in [0, T_1]$ and $t \in [T_1, T_2]$ respectively struck at k.

In (46) we demonstrate the numerical evaluation of Eq(43) with the above parameters, and for the choices $A = 850; B = 1150; \alpha = -0.0150; \beta = 0.0150$. 'Term' is the value of the n-th term in the summation in Eq(43), and 'Cumulative Sum' gives the cumulative approximation of the price.

In general the numerical evaluation of our formulae only requires a few terms on either side of the $n = 0$ term.

Figure 1 displays the results of computations for the partial-time early monitoring double-barrier call (labelled EM) given by Eq(38) and the late monitoring double barrier call Eq(43) (labelled LM) as a function of the current stock price x for the above choices for the other parameters. For comparison the dashed line indicates the standard double barrier call price given by Eq(28), with upper and lower monitoring windows extending over the full life of the option $t \in [0, T_2]$. Figure 2 displays the equivalent results for the puts given by Eq(39), Eq(44) and Eq(29) respectively.

As expected, for all values of x the restriction of the monitoring to a subset of the option lifetime increases option values, with both the early and late monitoring values lying wholly above the dashed lines. The largest effect occurs

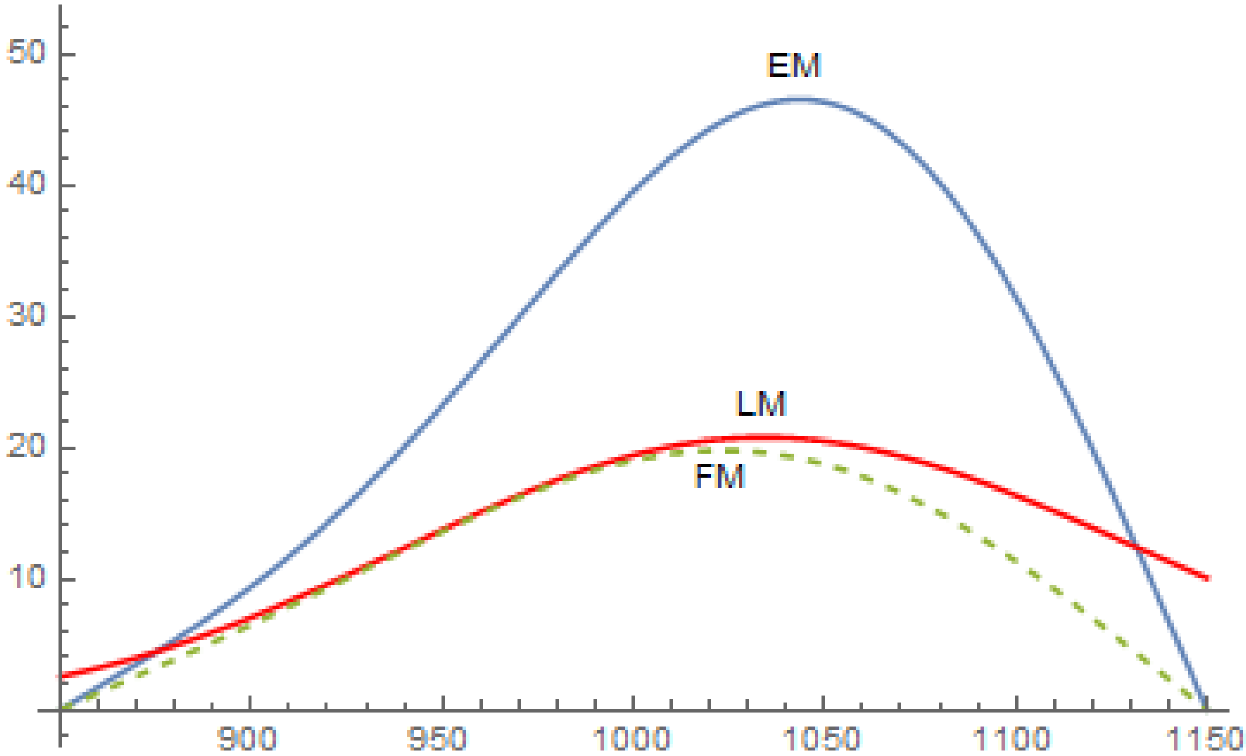

Figure 1: Eq(38), (43) and (28)

in the case of the Early Monitoring (EM) windows.

We conclude our numerical evaluations by considering different pairs of values for (A,B) and several choices of (α,β) for exponential barrier levels as given in (47) labelled (a) to (e). Numerical computations for these choices of the exponential parameters in the corresponding columns are provided in the tables. They are chosen to coincide with those used in Kunitomo and Ikeda (1992) and Buchen and Konstandatos (2009) after accounting for notational differences, where corresponding tables in the case of full-window monitoring for Eq(28) and Eq(29) may also be found.

	(a)	(b)	(c)	(d)	(e)
α	-0.010	-0.010	0	$+0.010$	$+0.015$
β	-0.015	$+0.010$	0	-0.010	$+0.010$

(47)

For each table of numerical computations below column (a) represents the numerical results for exponentially decaying barriers, column (e) represents exponentially increasing barriers. Columns (b) and (d) represent diverging and

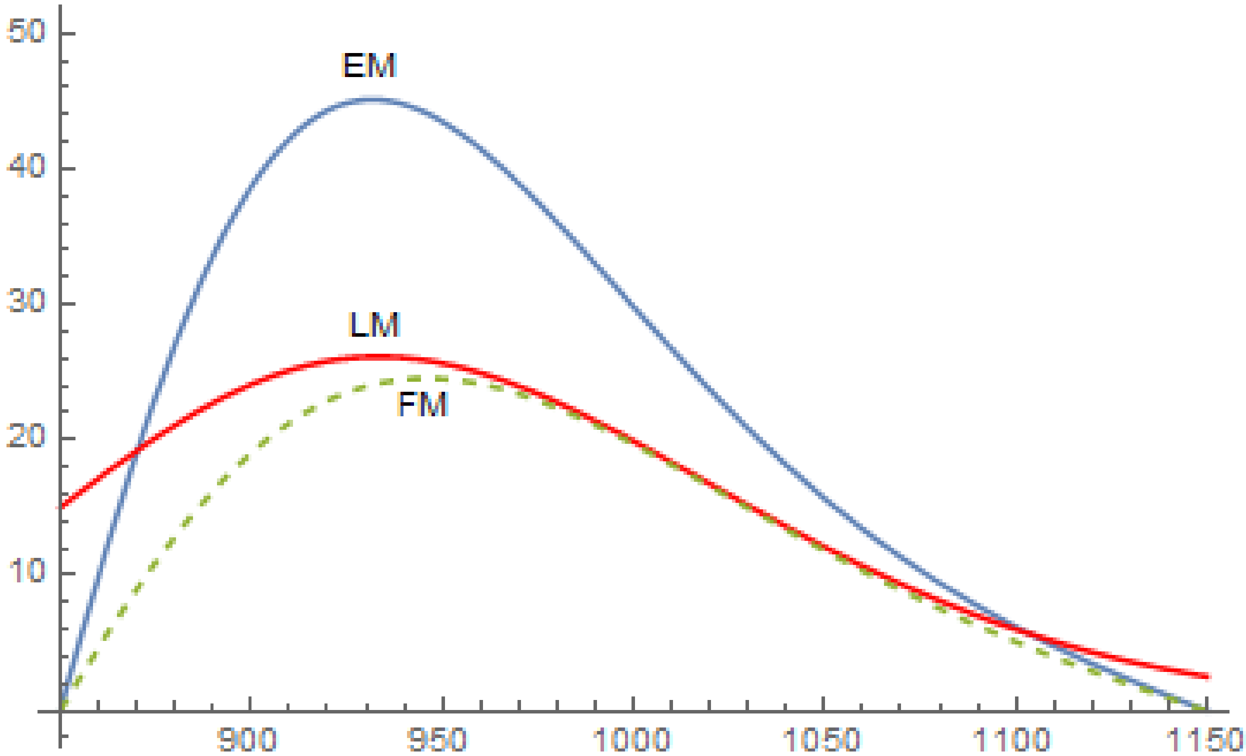

Figure 2:

converging barrier levels respectively. Column (c) corresponds to constant (*i.e.* 'flat' or non - time-varying) barrier levels. The case $(A = 0,\ B = \infty)$ corresponds to standard vanilla options, namely without any barriers. We evaluated our formulae for the choice of parameters in 45. The tables illustrate the effects of the time-varying barriers compared to the constant barriers in (c). We included tables for both the calls and puts for completeness.

A	B	(a)	(b)	(c)	(d)	(e)
0	∞	36.7473	36.7473	36.7473	36.7473	36.7473
400	1600	36.7473	36.7473	36.7473	36.7473	36.7473
500	1500	36.7473	36.7473	36.7473	36.7473	36.7473
600	1400	36.7473	36.7473	36.7473	36.7473	36.7473
700	1300	36.7451	36.7454	36.7453	36.7452	36.7454
800	1200	36.3525	36.3908	36.3759	36.3605	36.3908
850	1150	33.9672	34.1649	34.0872	34.0076	34.1648
900	1100	24.9851	25.4928	25.2862	25.0775	25.4802
930	1070	14.8154	15.3785	15.1185	14.8579	15.2903
950	1050	6.27181	6.69242	6.4541	6.21764	6.51759

Table 2: Partial Time Double Exponential Barrier Call Eq(38) over $[0,T_2]$; early monitoring in $[0,T_1]$. Column: (a) represents exponentially decaying barriers; (e) represents exponentially increasing barriers; (b) and (d) have diverging and converging barrier levels respectively; (c) corresponds to constant barrier levels.

A	B	(a)	(b)	(c)	(d)	(e)
0	∞	28.4486	28.4486	28.4486	28.4486	28.4486
400	1600	28.4486	28.4486	28.4486	28.4486	28.4486
500	1500	28.4486	28.4486	28.4486	28.4486	28.4486
600	1400	28.4486	28.4486	28.4486	28.4486	28.4486
700	1300	28.4486	28.4486	28.4486	28.4486	28.4486
800	1200	28.4313	28.4313	28.4303	28.4293	28.4288
850	1150	27.8417	27.8424	27.8193	27.7955	27.784
900	1100	22.3234	22.3449	22.2073	22.0677	22.0187
930	1070	13.5001	13.5947	13.3803	13.1647	13.1509
950	1050	5.63792	5.79734	5.59391	5.39183	5.44798

Table 3: Partial Time Double Exponential Barrier Put Eq(39) over $[0,T_2]$; early monitoring in $[0,T_1]$. Columns (a) – (e) as above.

A	B	(a)	(b)	(c)	(d)	(e)
0	∞	36.7473	36.7473	36.7473	36.7473	36.7473
400	1600	36.7473	36.7473	36.7473	36.7473	36.7473
500	1500	36.7468	36.7469	36.7469	36.7468	36.7469
600	1400	36.7257	36.7295	36.7281	36.7265	36.7295
700	1300	36.2272	36.2977	36.2706	36.2421	36.2977
800	1200	30.5697	31.1141	30.9007	30.6815	31.1141
850	1150	21.6931	22.6221	22.2536	21.8809	22.622
900	1100	9.43168	10.3582	9.98134	9.60793	10.3491
930	1070	3.25419	3.83824	3.58057	3.33087	3.78988
950	1050	0.754799	1.00638	0.877944	0.759433	0.945573

Table 4: Partial Time Double Exponential Barrier Call Eq(43) over $[0,T_2]$; forward-starting monitoring $[T_1,T_2]$. Columns (a) – (e) as above.

A	B	(a)	(b)	(c)	(d)	(e)
0	∞	28.4486	28.4486	28.4486	28.4486	28.4486
400	1600	28.4486	28.4486	28.4486	28.4486	28.4486
500	1500	28.4486	28.4486	28.4486	28.4486	28.4486
600	1400	28.4486	28.4486	28.4486	28.4486	28.4486
700	1300	28.446	28.446	28.4458	28.4455	28.4454
800	1200	27.4567	27.4567	27.4043	27.3495	27.3213
850	1150	22.6446	22.6452	22.4396	22.229	22.1224
900	1100	11.3368	11.3534	11.0248	10.696	10.5482
930	1070	4.13872	4.19812	3.94328	3.6943	3.6282
950	1050	1.00811	1.07483	0.943772	0.821954	0.821282

Table 5: Partial Time Double Exponential Barrier Put Eq(44) over $[0,T_2]$; forward-starting monitoring $[T_1,T_2]$. Columns (a) – (e) as above.

8. Conclusion

In contrast to other methods our approach allowed us to directly obtain pricing formulae for all the exponential and time-varying single and double barrier options we considered without the need for a single integration against an equivalent martingale measure nor the complicated joint density of the underlying stock price and the maximum and minimum. Our analysis was conducted directly in the original asset and time variables without the need for either variable transformations, or transformations into Fourier space.

In the single exponential and time varying barrier case, the prices of all the related barrier options we considered with either full monitoring or partial monitoring barrier windows whether with knock-out or knock-in boundaries, are related via a set of image function parity relations which we presented. The use of the parity relations allowed all the option prices to be expressed solely in terms of portfolios created from a *single* first-order European Gap option instrument and it's mathematical images with respect to the exponentially time-varying barrier.

We provided proofs for a new set of lemmas allowing us to extend the results of Buchen and Konstandatos (2009) for simultaneous lower and upper exponential and time-varying barrier levels to where the monitoring window extends to a subset of the lifetime of the option. They allowed the efficient analytical treatment of the single and double barrier options with late-monitoring windows, and also required a second-order variant of the European Gap option instruments which may be interpreted as generalised compound options. It is a rather remarkable result that all the barrier options we considered, regardless of the monitoring window, have prices which can be expressed solely in terms of a single family of *Gap options*.

The intermediate result of the decomposition of the partial barrier and double barrier option prices into first and second order Gap options is essentially *model independent*, and as far as we know are new.

Our approach may be readily used to treat any sequence of late-starting double exponential and time-varying barrier monitoring windows. Although we did not treat the problem of credit default modelling explicitly, we note that credit default modelling naturally requires lower and upper exponential and time-varying barrier levels on the value of a firm. The techniques and results of this work particularly in the late monitoring double barrier case point a way to getting tractable solutions in a multi-barrier context for credit default modelling when using proper Black-Scholes dynamics in contrast to modelling firm value using a simple Wiener process as the latent variable driving default.

AppendixA. Gap Options

Many exotic options can be priced entirely in terms of simpler 'building block' derivatives. For example Buchen (2004) demonstrated such decompositions for some examples of path-independent dual-expiry options. More details can be found in Konstandatos (2008) or more recently in Buchen (2012). We undertake a similar approach here, however our analysis requires so-called gap options, for which the the option payoff is decoupled from the exercise condition.

First order Gap options essentially standard European calls and puts with the exception that their strike and exercise prices are allowed to differ. Second order gap options are are compound first order Gap options, and may be considered as simple generalisations of the standard compound options of Geske (1979) and of the framework of Buchen (2004). Gap options have also been referred in the literature as *threshold options*. These building blocks are the only ones needed to construct the prices of all the exotic barrier options considered in this paper.

AppendixA.1. First-order Gap Options

We define the first-order gap option of exercise price ξ and strike price k, denoted by $\mathscr{G}^s_\xi(x,\tau;k)$, where $\tau = (T-t)$, as the price of a European derivative security with expiry T payoff:

$$\mathscr{G}^s_\xi(x,0;k) = (x-k)\mathbb{I}(sx > s\xi) \tag{A.1}$$

Here $s = \pm 1$ indicates the type of gap option: *e.g.* $s = +1$ indicates an 'up-type' gap option with exercise condition $x > \xi$. Conversely $s = -1$ indicates a 'down-type' with exercise condition $x < \xi$. The *gap* is defined to be $|\xi - k|$, the absolute difference between the exercise and strike prices. Under Black-Scholes dynamics the price of such instruments for $t < T$ is expressible in terms of the uni-variate normal distribution function:

$$\mathscr{G}_\xi^s(x,\tau;k) = x\mathscr{N}(sd_\xi) - ke^{-r\tau}\mathscr{N}(sd'_\xi) \tag{A.2}$$

where

$$\left[d_\xi, d'_\xi\right] = [\log(x/\xi) + (r \pm \tfrac{1}{2}\sigma^2)\tau]/\sigma\sqrt{\tau}$$

Clearly, a vanilla call of strike price k and time $\tau = (T-t)$ remaining to expiry has zero gap and price given by $C_k(x,\tau) = \mathscr{G}_k^+(x,\tau;k)$. Similarly, a corresponding vanilla put with zero gap has price $P_k(x,\tau) = -\mathscr{G}_k^-(x,\tau;k)$.

AppendixA.2. Second-order Gap Options

To define the required higher-order building blocks, consider the scenario of two dates T_1, T_2 with $T_1 < T_2$ and let $\tau_i = (T_i - t)$ and $\tau = (T_2 - T_1)$. The second-order gap option $\mathscr{G}_{\xi_1\xi_2}^{s_1s_2}(x,\tau_1,\tau_2;k)$ is defined such that at time T_1 it pays a first order gap option $\mathscr{G}_{\xi_2}^{s_2}(x,\tau;k)$ with time τ remaining to expiry, provided the stock price at time T_1 is either above or below some expiry price ξ_1. The second-order gap option's T_1 payoff is therefore:

$$\mathscr{G}_{\xi_1\xi_2}^{s_1s_2}(x,0,\tau;k) = \mathscr{G}_{\xi_2}^{s_2}(x,\tau;k)\cdot\mathbb{I}(s_1x > s_1\xi_1) \tag{A.3}$$

where $s_i = \pm 1$.

It is also useful to write down the payoff of this second-order gap option at time T_2. From (A.1) this payoff is

$$G_{\xi_1\xi_2}^{s_1s_2}(x_1,x_2;k) = (x_2 - k)\mathbb{I}(s_1x_1 > s_1\xi_1)\mathbb{I}(s_2x_2 > s_2\xi_2) \tag{A.4}$$

where $x_i = X(T_i)$. Thus the second-order gap option requires its holder to buy one unit of the underlying asset at time T_2 for k (dollars), but only if the asset price at time T_1 is above (or below) the exercise price ξ_1 *and* if the asset price at time T_2 is above (or below) the exercise price ξ_2. There are four different types of second-order gap options corresponding to the choice of signs for (s_1, s_2).

To simplify notation somewhat, we shall write $\tau_{1,2}$ for the pair (τ_1, τ_2). The price of second-order gap option, under Black Scholes dynamics, is readily expressible in terms of the bi-variate normal distribution as

$$\mathscr{G}_{\xi_1\xi_2}^{s_1s_2}(x,\tau_{1,2};k) = x\mathscr{N}(s_1d_1, s_2d_2; s_1s_2\rho) - ke^{-r\tau_2}\mathscr{N}(s_1d'_1, s_2d'_2; s_1s_2\rho) \tag{A.5}$$

where $\tau_i = T_i - t$, $\rho = \sqrt{\tau_1/\tau_2}$ and

$$\left[d_i, d_i'\right] = [\log(x/\xi_i) + (r \pm \tfrac{1}{2}\sigma^2)\tau_i]/\sigma\sqrt{\tau_i}$$

A proof is omitted however the result follows under discounted expectations. Second-order gap options are examples of generalised compound options.

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